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Zoran H. Perić, Marko Milojković, Dragan Antić,
Aleksandra Z. Jovanović
Faculty of Electronic Engineering, University of Nis, Serbia
e-mail: peric@elfak.ni.ac.yu

Adaptive uniform polar quantization

A simple and complete analysis for an optimal uniform polar quantizer with respect to mean-square error (MSE) as efficient quantization technique for any number of points N (Fixed-Rate) is given. Conditions for the optimality of the polar quantizer and all main equations for phase partitions and optimal number of levels are presented. Improved performance over product polar quantization and scalar uniform quantization proposed in the literature is afforded by allowing a variable number of phases at each magnitude level.

Key words: *uniform polar quantization, method of Lagrange multipliers, optimization.*

Introduction

Polar quantization techniques as well as their applications in areas such as computer holography, discrete Fourier transform encoding, image processing and communications have been studied extensively in the literature. Synthetic Aperture Radars (SARs) images can be represented in polar format (i.e., magnitude and phase components). In the case of MSE quantization of a symmetric two-dimensional source, polar quantization gives the best result in the field of the implementation [1]. The motivation behind this work is to maintain high accuracy of phase information that is required for some applications such as interferometry and polarimetry, without loosing massive amounts of magnitude information [1, 2].

Problem of the optimal uniform quantization, even for the simplest case, which is uniform scalar quantization, is rather actual nowadays [4]. If we apply the Gaussian quantizer on an arbitrary source we can take advantage of the central limit theorem and the known structure of an optimal scalar quantizer for a Gaussian random variable to code a general process by first filtering it in order to produce an approximately Gaussian density, then scalar-quantizing the result, and finally, inverse-filtering it to recover the original. Various processing techniques, when applied to non-Gaussian sources with memory, produce sequences that are «approximately» independent and Gaussian [7]. In previous works about polar quantization [1, 3] only product uniform quantization was

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always considered ($\boldsymbol{N} = P \times L$). That optimization approximated granular distortion as:

$$D_g = \frac{r_{\max}^2}{12L^2} + \frac{2\pi^2}{3P^2}.$$

SAR images can be represented in polar format (i.e., magnitude and phase components). In the case of MSE quantization of a symmetric two-dimensional source, polar quantization gives the best result in the field of the implementation. Polar quantization consists of separate but uniform magnitude and phase N level quantization, so that rectangular coordinates of the source (x, y) are transformed into the polar coordinates in form: $r = (x^2 + y^2)^{1/2}$, $\phi = \tan^{-1}(y/x)$ where r represents magnitude and ϕ is a phase. The optimal uniform polar quantization (OUPQ) is very similar to the original uniform polar quantization (UPQ) except the fact that the number of the regions for the phase angle varies depending on the result of magnitude quantization. In other words each concentring ring in quantization pattern allows to have a different number of partitions in the phase quantizer (P_i) when r is in the i-th magnitude ring. Their implementation remains simple requiring only two scalar quantizers and lookup table of the P_i. One UPQ must satisfy the constraint $\sum_{i=1}^{L} P_i = N$ in order to use all of N regions for the quantization.

In this paper polar quantizers are designed and analysed under additional constraint that each scalar quantizer is a uniform one. This restriction has the following advantages over optimal polar quantization: the implementation is simple, and no compressor-expander pair is needed. Adaptive uniform polar quantization can be used in ADPCM systems.

Optimal uniform polar quantization

Uniform scalar quantization was considered in previous papers as a uniform quantization in magnitude and phase where the number of points was constant. The transformed probability density function for the Gaussian source takes the following form:

$$f(r,\phi) = \frac{1}{2\pi\sigma^2} \cdot r e^{\frac{-r^2}{2\sigma^2}} = \frac{f(r)}{2\pi}. \tag{1}$$

We consider uniform polar quantizer of L magnitude levels and P_i phase reconstruction levels on a magnitude reconstruction level m_i, $1 \le i \le L$. In order to find a truly optimal quantizer we have to minimize the distortion, so we proceed as follows.

Magnitude decision levels and reconstruction levels are given as:

$$r_i = (i-1)\Delta,\ 1 \le i \le L+1,$$
$$m_i = (i-1/2)\Delta,\ 1 \le i \le L,$$

where r_i is defined in range $1 \le i \le L$ ($0 < r_1 < r_2 < \ldots < r_L < r_{L+1} = r_{\max}$) and m_i in ($0 < < m_1 < m_2 < \ldots < m_L$). Next, we make a partition of each magnitude ring into P_i phase subpartitions. Let $\phi_{i,j}$ and $\phi_{i,j+1}$ be two phase decision levels, and let $\psi_{i,j}$ be the j-th phase reconstruction level for the i-th magnitude ring, $1 \le j \le P_i$.

Then: $\phi_{i,j} = (j-1)\pi / P_i$, and $\psi_{i,j} = (2j-1)\pi / P_i$.

Total distortion D is a combination of granular and overload distortions, $D = D_g + D_o$. Granular distortion D_g [2] takes the following form:

$$D_g = \sum_{i=1}^{L} \sum_{j=1}^{P_i} \int_{\phi_{i,j}}^{\phi_{i,j+1}} \int_{r_i}^{r_{i+1}} [r^2 + m_i^2 - 2rm_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi . \tag{2}$$

After the integration over ϕ and the reordering, (2) becomes as follows:

$$D_g(P_1, \cdots, P_L) = \sum_{i=1}^{L} \int_{r_i}^{r_{i+1}} [r^2 + m_i^2 - 2rm_i \sin c(\frac{\pi}{P_i})] f(r) dr \quad \text{(where } \sin c(x) = \sin(x)/x\text{)} \tag{3}$$

and overload distortion D_o is described as:

$$D_o = \sum_{j=1}^{P_i} \int_{\phi_{L,j}}^{\phi_{L,j+1}} \int_{r_{\max}}^{\infty} [r^2 + m_L^2 - 2rm_L \cos(\phi - \psi_{L,j})] \cdot \frac{f(r)}{2\pi} dr d\phi . \tag{4}$$

We use [5]: $\frac{\sin(x)}{x} = 1 - 0{,}16605x^2 + \varepsilon(x) \approx 1 - \frac{1}{6}x^2 + \varepsilon(x)$ and after this approximation D_g becomes as follows:

$$D_g = \sum_{i=1}^{L} (\int_{r_i}^{r_{i+1}} (r - m_i)^2 f(r) dr + \int_{r_i}^{r_{i+1}} \frac{1}{3P_i^2} \pi^2 r m_i f(r) dr. \tag{5}$$

Thus, we practically define separate amplitude and phase distortions as follows:

$$D_g = D_g^L + D_g^\theta$$

and for very large N, asymptotically, last equation can be given as:

$$D_g = \frac{r_{\max}^2}{12L^2} + \sum_{i=1}^{L} \frac{(\Delta\theta_i)^2}{12} m_i^2 f(m_i). \tag{6}$$

The minimization of the function $D_g(\boldsymbol{P})$ for fixed number of magnitude levels L constrained by total number of reconstruction points N is formulated in this way: minimize $D_g(\boldsymbol{P})$ under the constraints:

$$g_0(P_1, P_2, \cdots, P_L) = N - \sum_{i=1}^{L} P_i \ge 0 \tag{7}$$

$$g_i(P_i) = P_i \geq 0; \quad 1 \leq i \leq L.$$

Before describing the minimization procedure, we prove that the problem of minimization of the $D_g(\boldsymbol{P})$ is a convex programming problem. This follows directly from Lemma 1.

Lemma 1: Function $D_g(\boldsymbol{P})$ is convex and constraints $g_0(\boldsymbol{P})$ and $g_i(\boldsymbol{P}_i)$ form the convex set .

Proof of Lemma 1: To prove that the function $D_g(\boldsymbol{P})$ is convex and constraints $g_0(\boldsymbol{P})$ and $g_i(\boldsymbol{P}_i)$ form the convex set it is sufficient to prove that Hessian matrices of the following functions: $D_g(\boldsymbol{P})$, $-g_0(\boldsymbol{P})$, $-g_i(P_i)$, $1 \leq i \leq L$ are positive semi-definite [6, p. 27].

Conditions that satisfy the optimal solution for mentioned problem will be seeked using the method of Lagrange multipliers, as: $J = D_g + \lambda \sum P_i$ where λ represents Lagrangian multiplier.

From $\dfrac{\partial J}{\partial P_i} = 0$ we obtain $\dfrac{\partial J}{\partial P_i} = -\dfrac{2\pi^2}{3(P_i)^3} m_i \displaystyle\int_{r_i}^{r_{i+1}} rf(r)dr + \lambda$.

The optimization problem for polar quantizer can be formulated in this way: it is necessary to find partial derivations of $D_g(\boldsymbol{P})$. It follows from (10) that

$$\frac{\partial D_g}{\partial P_i} = \frac{2\pi^2}{3(P_i)^3} m_i \int_{r_i}^{r_{i+1}} rf(r)dr,$$

while the second partial derivation is $\dfrac{\partial^2 D_g}{\partial P_i \partial P_j} = \begin{cases} 2\dfrac{\pi^2}{(P_i)^4} m_i \displaystyle\int_{r_i}^{r_{i+1}} rf(r)dr, \; i = j, \\ 0, \; i \neq j. \end{cases}$

It can be easily concluded that: $\dfrac{\partial^2 D_g}{\partial P_i \partial P_j} \geq 0.$

For Hessian matrix it obviously holds:

$$\begin{bmatrix} \dfrac{\partial^2 D_g}{\partial P_1 \partial P_1} \cdots \dfrac{\partial^2 D_g}{\partial P_1 \partial P_L} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial^2 D_g}{\partial P_L \partial P_1} \cdots \dfrac{\partial^2 D_g}{\partial P_L \partial P_L} \end{bmatrix} \geq 0,$$

while for the constraints we have:

$$-\frac{\partial^2 g_0}{\partial P_i \partial P_j}=0\,,\ -\frac{\partial^2 g_i}{\partial P_i \partial P_j}=0,\quad 1\le i\le L\quad \begin{bmatrix} \dfrac{\partial^2 g_0}{\partial P_1\partial P_1}\cdots\dfrac{\partial^2 g_0}{\partial P_1\partial P_L} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial^2 g_0}{\partial P_L\partial P_1}\cdots\dfrac{\partial^2 g_0}{\partial P_L\partial P_L} \end{bmatrix}=0\,,\ 0\le i\le L\,.$$

This completes the proof and it is completely proved that $D_g(\boldsymbol{P})$ is a convex function of $\boldsymbol{P}$.

After applying method of Lagrange multipliers, we have:

$$J=\sum_{i=1}^{L}\Big(\int_{r_i}^{r_{i+1}}(r-m_i)^2 f(r)dr+\int_{r_i}^{r_{i+1}}\frac{1}{3P_i^2}\pi^2 r m_i f(r)dr\Big)+\lambda\sum_{i=1}^{L}P_i\,.$$

Then:

$$P_{iopt}=N\frac{\sqrt[3]{m_i\int_{r_i}^{r_{i+1}} rf(r)dr}}{\sum_{j=1}^{L}\sqrt[3]{m_j\int_{r_j}^{r_{j+1}} rf(r)dr}};\quad 1\le i\le L\quad \text{for fixed } N. \tag{8}$$

This is the exact result without asymptotical analysis. Our goal is to find $\boldsymbol{r}_{\max}$, $\boldsymbol{L}_{opt}$, and $(\boldsymbol{P}_{iopt})_{1\le i\le L}$ for which D_g is minimal.

For Gaussian source (after transformed into the polar coordinates) where $f(r)=\frac{r}{\sigma^2}e^{\frac{-r^2}{2\sigma^2}}$, optimal number of points gets a new form:

$$P_i\approx\frac{Nm_i r_{\max}/\sigma^2}{3(1-e^{\frac{-r_{\max}^2}{6\sigma^2}})L}e^{\frac{-m_i^2}{6\sigma^2}}\,. \tag{9}$$

These analyses yield to the resulting granular distortion asymptotically

$$D_g=\frac{r_{\max}^2}{12L^2}+\frac{9\pi^2L^2\sigma^4}{N^2r_{\max}^2}(1-e^{\frac{-r_{\max}^2}{6\sigma^2}})^3. \tag{10}$$

The optimal number of levels problem can be solved analytically only for the asymptotical analysis as it is suggested: from the condition $\frac{\partial D_g}{\partial L}=0$ we come to the optimal solution L_{opt}

$$L_{opt} = \frac{r_{\max} / \sigma}{\sqrt[4]{108\,(1 - e^{\frac{-r_{\max}^2}{6\sigma^2}})^3}} \sqrt{\frac{N}{\pi}} \,. \tag{11}$$

Now, we finally have the equation for the optimal distortion of the uniform polar quantizer:

$$D_g^{opt} = \frac{\sqrt{3}\pi\sigma^2}{N}(1 - e^{\frac{-r_{\max}^2}{6\sigma^2}})^{3/2} \,. \tag{12}$$

Especially interesting is to compare optimal uniform polar quantization with optimal product polar quantization using proposed method of the optimization. Optimal product granular distortion is then:

$$D_g^{prod\ \ opt} = \frac{r_{\max}\ \sigma\pi\ \sqrt{2}}{6\,N} \,, \tag{13}$$

$$G(i) = 10\log(\frac{D_g^{prod}(i)}{D_g^{opt}(i)}) = 10\log(\frac{r_{\max}(i)\sqrt{2}}{3\sqrt{3\sigma_i}\,(1 - e^{\frac{-r_{\max}^2(i)}{6\sigma_i^2}})^{3/2}}) \,, \tag{14}$$

where $r_{\max} = 2\sigma\sqrt{\ln L}$ [4].

Now we have:

$$L = \frac{2\sqrt{\ln\sqrt{N}}}{\sqrt[4]{108\,(1 - N^{-\frac{1}{3}})}} \sqrt{\frac{N}{\pi}} \,, \tag{15}$$

$$D_g^{skal} = \frac{\Delta^2}{12} = \left(\frac{2r_{\max}}{\sqrt{N}}\right)^2 \frac{1}{12} \,. \tag{16}$$

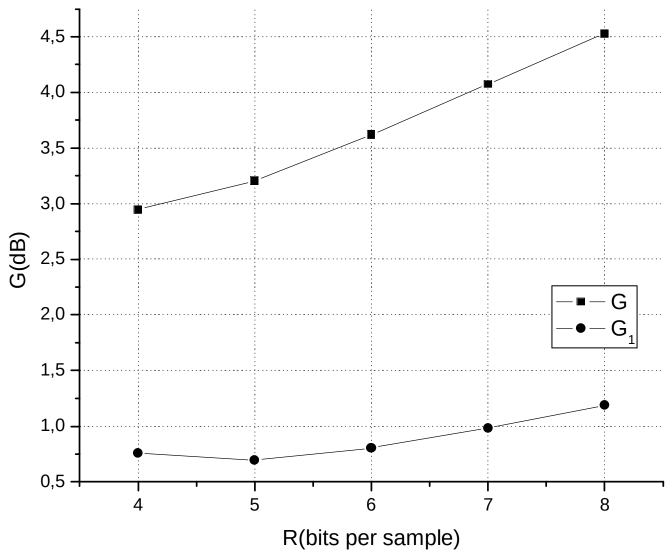
Gains over scalar and optimal product quantization are as follows:

$$G = 10\log\frac{D_g^{skal}}{D_g^{opt}} = 10\log\left(\frac{8\ln\sqrt{N}}{3\sqrt{3}\pi(1 - L^{-\frac{2}{3}})^{\frac{3}{2}}}\right) \tag{17}$$

and

$$G_1 = 10\log\frac{D_g^{prod}}{D_g^{opt}} = 10\log\left(\frac{2\sqrt{2\ln L}}{3\sqrt{3}(1-L^{-\frac{2}{3}})^{-\frac{3}{2}}}\right). \quad (18)$$

Figure shows the gains for different number of bits per sample.



Gain [dB] as a function of number of bits per sample

Application of Adaptive Uniform Polar Quantization:

Short-time pdf of speech segments are described by Gaussian pdf [8]. This paper addresses potential improvements achievable by means of joint quantization of two consecutive samples (x, y) referred to as two-dimensional quantization (2-D quantization) over the scalar quantization. Also a transform coding scheme known as spectral phase coding (SPC) is a reliable technique for coding a nonstationary or large dynamic range discrete-time series into a digital form. SPC is essentially a polar format representation of the discrete Fourier Transform (DFT) of a random phase time series. SPC utilises the discrete Fourier Transform and a two-dimensional quantizer to obtain its reliable characteristics.

Also it may apply optimal uniform polar quantization at Adaptive Differential Pulse Code Modulation (ADPCM). In ADPCM systems it utilises uniform scalar quantization [9]. We give a gain which can be achieved if we use optimal uniform polar quantizer in ADPCM systems.

Conclusion

We introduced optimal uniform polar quantization through simple and complete analysis by constructing an optimal uniform polar MSE quantizer for sources with circularly symmetrical probability density.

We also gave an equation for optimal number of points for different levels and for optimal number of levels. The equations for optimal uniform polar distortion D_g^{opt} are provided.

Numerical results confirm the potentialities of such an approach. They show that gain based on OUPQ method application is (2,9–4,5) dB over scalar quantization and (0,7–1,9) dB over optimal product quantization. When polar quantization is in use, distortion can be reduced by applying OUPQ method of the optimization. Quantizers described here are simple for the application and realization.

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