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Construction of Constitutive Relationships for Simple in Noll's Sense Materials with Viscoelastic-Viscoplastic Behavior

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Within the class of simple in Noll's sense materials, the media with viscoelastic-viscoplastic behavior have been singled out, whose arbitrary deformations and types of symmetry in properties have been expressed by general constitutive relationships, in which the long-term fading memories of the deformation and time histories take place, and the approaches to their specialization have been developed.

Keywords: constitutive relationships, viscoelastic-viscoplastic materials, long-term fading memory.

The major problem of the mechanics of a deformable solid body is the development of methods for constructing physically grounded, mathematically rigorous constitutive relationships that allow not only describing, but also predicting, at various levels of accuracy, the behavior of a wide class of materials existing in nature in a broad range of variation of the conditions of their deformation.

Despite the achievements in using the phenomenological approach to the construction of constitutive relationships and a large number of proposed models [1–10], at present, this approach does not allow for a complete solution to the above problem, particularly when applied to arbitrary deformations and types of symmetry in the properties of viscoelastic-viscoplastic materials.

The fact that the theory of simple in Noll's sense materials (hereinafter referred to as simple materials, media, continua), is still general enough to include practically all known purely mechanical phenomenological models of material deformation that are governed by the principle of a specimen, and the success in constructing constitutive relationships for simple elastic, viscoelastic, and elasto-plastic continua by the methods of rational continuum mechanics [11, 12] testify to a great potential for using this approach in developing constitutive relationships for viscoelastic-viscoplastic media.

In this paper, the media with viscoelastic-viscoplastic behavior have been distinguished within the class of simple materials [11], whose arbitrary deformations and types of symmetry in properties are expressed by constructed general constitutive relationships in which the long-term fading memories of the deformation and time histories take place, and the approaches to their specialization have been developed.

Let us single out simple viscoelastic-viscoplastic materials by postulating the following basic properties:

(i) stresses depend on the path shape in the tensor strain space (deformation history) and on the history of traversing this deformation history in time (time history);

(ii) the time history memory of the materials within the active and passive deformation fades in time;

(iii) the independent of time memory of the deformation history within the active deformation fades along the length of the path in the tensor strain space;

(iv) the total strains can in some way be divided into elastic and plastic components;

(v) a certain yield criterion is true;

(vi) a certain law of yielding is fulfilled.

Hereafter, in considering scalar or tensor functions ψ at the present moment and in the past, it will be convenient to characterize the past moment t' by the positive value $s \equiv t - t'$ [11], where t is the present moment of time. The history of the function ψ up to the moment t will be defined by ψ^t, its value being $\psi^t(s)$:

$$\psi^t = \psi^t(s) \equiv \psi(t-s).$$

Here t is fixed and $s \geq 0$. For every t, the history of ψ^t is defined over $[0, \infty)$.

We describe the behavior of a viscoelastic-viscoplastic material by a general constitutive relationship for a simple material [11]:

$$\sigma^{\mathbf{R}} = \mathbf{G}(\mathbf{C}^t), \tag{1}$$

where σ is the Cauchy stress tensor, $\sigma^{\mathbf{R}}$ is defined by $\sigma^{\mathbf{R}} = \mathbf{R}^T \sigma \mathbf{R}$, $\mathbf{R}$ is the rotation tensor in multiplicative decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ of the deformation gradient $\mathbf{F}$, $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors of the deformation, respectively, $\mathbf{R}^T$ is the transpose of $\mathbf{R}$, $\mathbf{C}^t$ is the history of the right Cauchy–Green tensor, and $\mathbf{G}$ denotes a mapping of histories $\mathbf{C}^t$ onto symmetric tensors.

Proceeding from the first key property of viscoelastic-viscoplastic materials and using the data from [13, 14], Eq. (1) can be presented as

$$\sigma^{\mathbf{R}} = \mathbf{G}(\mathbf{C}^t) = \mathbf{G}(\mathbf{C}^{\xi}; \xi^t), \tag{2}$$

where $\mathbf{C}^{\xi}$ is the deformation history of the of the right Cauchy–Green tensor, ξ is the arc length along the strain path determined according to [15], and ξ^t is the time history of traversing $\mathbf{C}^{\xi}$ or simply the time history.

Later throughout this text we shall consider the processes of deformation as those starting at a certain reference moment of time t_0 from an unstressed and unstrained reference configuration κ_0 assuming that the active process begins with the onset of the deformation process, unloading is absent, and $\mathbf{C}^{\xi}$ and ξ^t are smooth continuous parameter functions differentiated as many times as necessary.

Let us assume that viscoelastic-viscoplastic materials have a long-term fading in time memory (hereinafter referred to as the fading memory), and this memory represents a property that can be mathematically expressed using the function of the simple material response.

Having taken the history of $\mathbf{C}^{\xi}$ in (2) to be constant, we vary ξ^t. For this family of the deformation processes, Eq. (2) takes the following form:

$$\sigma^{\mathbf{R}} = \overline{\mathbf{G}}(\xi^t). \tag{3}$$

Basing on relationship (3), consider the difference between the static response and all other responses. Just as f^t designates the history up to the moment t of the arbitrary function f over $(-\infty, +\infty)$, we designate the history of the constant function f, whose value always equals to a, by a^c:

$$a^c(s) \equiv a, \qquad 0 \le s \le \infty. \tag{4}$$

Thus, $\xi(t)^c$ represents a constant history (or a history constant) corresponding to the current value $\xi(t)$ of the arc length ξ along the strain path for point in reference configuration $\mathbf{X}$ in the history $\mathbf{C}^{\xi}$. In order to enable consideration of the static case outlined here, just as it was done by the author of [11], we assume that if ξ^t is the history belonging to the domain D_1 of the response $\overline{\mathbf{G}}$ definition, then for each s over $[0, \infty)$ the constant history $(\xi^t(s))^c$ also belongs to D_1. The value $\overline{\mathbf{G}}(\xi(t)^c)$ of the response $\overline{\mathbf{G}}$ represents the stresses corresponding to being at rest in the state obtained from κ_0 during deformation along the path $\mathbf{C}^{\xi}$ whose arc length equals to $\xi(t)$.

In an elastoplastic material, particularly with a fixed $\mathbf{C}^{\xi}$ history, the stresses are always static for all ξ^t in D_1 [13]:

$$\sigma^{\mathbf{R}} = \sigma_s^{\mathbf{R}} = \mathbf{G}(\mathbf{C}^t) = \overline{\mathbf{G}}(\xi^t) = \mathbf{G}(\mathbf{C}^{\xi}) = \overline{\mathbf{G}}(\xi(t)^c) = \overline{\mathbf{g}}(\xi(t)), \tag{5}$$

where σ_s is the value of the static stress.

The main idea of the fading memory in a viscoelastic-viscoplastic material is that when the history ξ^t is close to the constant history $\xi(t)^c$, the stresses $\overline{\mathbf{G}}(\xi^t)$ are close to the static stresses. In other words, a small deviation from the constant history $\xi(t)^c$ induces the stresses, which are only slightly different from those in an elastoplastic material that correspond to $\xi(t)^c$. We specify the notions of "smallness" and "closeness" with the help of topology. When the topologies are introduced, we can speak of continuity in precise terms and formulate a precise and general axiom of continuity as an essential condition for the fading memory: the response $\overline{\mathbf{G}}$ is continuous in each constant history $\xi(t)^c$ in D_1.

Just as it was done by the author of [11], we consider real functions that are summable with respect to some Lebesque–Stieltjes measure μ on a real line R and assume that the following relationship is true [11]

$$||m|| \equiv \sqrt{\int_R |m(s)|^2\, d\mu}, \tag{6}$$

where $|\cdot|$ and $||\cdot||$ are, respectively, the norm and the semi-norm.

The measure μ is generated by a real non-decreasing function β in a well-known manner

$$\beta(s-0)=\beta(s), \qquad \mu\{[a,b)\}=\beta(b)-\beta(a) \tag{7}$$

for all real values of a and b. We consider only those histories that represent the functions set over $[0,\infty)$ and assume [11] that the past only makes a finite contribution to the semi-norms of the bounded histories. Let us call the measure μ an obliviating measure, if [11]

$$\beta(s)\equiv 0 \quad \text{at} \quad s\le 0, \qquad \lim_{s\to\infty}\beta(s)=M<\infty. \tag{8}$$

This implies that transferring any interval of the line infinitely far in the past reduces its measure to zero:

$$\lim_{c\to\infty}\mu\{[a+c,\, b+c)\}=0. \tag{9}$$

We call semi-norm (6) calculated from the measure satisfying condition (8) m history memory, which corresponds to this measure. The collection of m histories, for which the semi-norm $||m||$ is finite, forms a functional space, which is a subspace of the space of all the histories measurable with respect to μ. This subspace is called here the space of histories with finite memory. It includes all the bounded measurable histories and, in particular, all constant histories $\xi(t)^c$. Just as the author of [11] did it, we assume that a certain obliviating measure μ has been established once and for all keeping in mind that our results depend on the choice of this measure. Suppose that the definition domain D_1 of the response $\overline{\mathbf{G}}$ from (3) is a connected subset of the space of histories with a finite memory with respect to μ.

Consider the materials, which satisfy the axiom of continuity for the topology obtained on the basis of the obliviating measure, and give the following definition.

Definition. A viscoelastic-viscoplastic material has a weak fading memory if it satisfies the axiom of continuity, with the discontinuity determined using the obliviating measure:

$$\sigma^{\mathbf{R}} = \overline{\mathbf{G}}(\xi^t) = \overline{\mathbf{g}}(\xi(t)) + \mathbf{o}(1) \quad \text{for} \quad \|\xi^t - \xi(t)^c\| \to 0. \tag{10}$$

Thus, on condition that the memory of the difference of the history ξ^t and constant history $\xi(t)^c$ is rather small, the stresses are close to those in an elastoplastic material corresponding to $\xi(t)$.

In particular, the remainder term in (10) identically equals to zero in an elastoplastic material. That is why, for (10) to hold true, the obliviating measure should be such that

$$\|\xi^t - \xi(t)^c\| = 0. \tag{11}$$

Inversely, if, according to this definition of the memory, relationship (10) is true with the remainder term equal to zero, the material is elastoplastic. The function β, which defines a obliviating measure of this kind, is a single jump at $s = 0$.

Each time we assume the material to have a weak fading memory, we choose some function β.

Definition [11]. Let ξ^t be a time history. Then the time history $\xi^t_{\{t_0\}}$ defined by

$$\xi^t_{\{t_0\}} = \begin{cases} \xi(t_0), & 0 \le s \le t - t_0, \\ \xi^{t_0}(s - (t - t_0)), & s > t - t_0, \end{cases}$$

is called the static continuation of the given time history.

Using the notion of static continuation, the phenomenon of stress relaxation in a viscoelastic-viscoplastic material is modeled based on the assertion that, if some neighborhood of the particle X has been maintained in the state of rest for quite a long time, the stresses in X approach the value they would have had if this neighborhood had always been in the state of rest.

Stress Relaxation Theorem. For any fixed moment t and any history ξ^t in D_1, the history of static continuation $\xi^t_{\{t_0\}}$ also belongs to D_1, and the limit $\overline{\mathbf{G}}(\xi^t_{\{t_0\}})$ at $t_0 \to -\infty$ exists and represents static stresses corresponding to $\xi(t)$:

$$\lim_{t_0 \to -\infty} (\overline{\mathbf{G}}(\xi^t_{\{t_0\}})) = \overline{\mathbf{G}}(\xi(t)^c) \equiv \overline{\mathbf{g}}(\xi(t)). \tag{12}$$

A similar theorem for viscoelastic materials was proved in [11] with some limitations. Analysis has shown that this proving is also true for the above case.

We take the history memory ξ^t in the form proposed by Coleman and Noll [11]:

$$\|\xi^t\|^2 = B|\xi(t)|^2 + \int_0^\infty k(s)|\xi^t(s)|^2\, ds, \tag{13}$$

where B is a positive constant. We call the function k a obliviator or an influence function.

Similarly to the way it was done in [11], we construct approximations that are higher than (10). To this end, assume that the principle of the fading memory of the nth order is as follows: for static history $\xi(t)^c$, the response $\overline{\mathbf{G}}$ is n times Frechet-differentiable. Then

$$\sigma^{\mathbf{R}} = \overline{\mathbf{G}}(\xi^t) = \overline{\mathbf{g}}(\xi(t)) + \sum_{i=1}^{n} \overline{\mathbf{G}}_i(\xi^t - \xi(t)^c) + \mathbf{o}(||\xi^t - \xi(t)^c||^n), \qquad (14)$$

where $\overline{\mathbf{G}}_i$ are the bounded homogeneous polynomial mappings of the ith degree dependent on the variable $\xi(t)$ at a fixed $\mathbf{C}^{\xi}$. In the Frechets expansion, the above mapping is replaced by the sum of simpler mappings with an error tending to zero faster than the nth degree of the memory $||\xi^t - \xi(t)^c||$ of the difference between the true history ξ^t and the corresponding constant history $\xi(t)^c$.

The viscoelastic-viscoplastic materials considered herein exhibit the long-term fading memories of the deformation and time histories on the active deformation. These two types of the fading memory are independent and are governed by different laws of fading. With a constant strain value and varying time, the material considered shows the long-term memory of the time history fading in time, whereas the long-term fading memory of the deformation history is absent. During passive deformation, the material has the fading memory alone (viscoelastic behavior).

If we assume that the material has the fading memory of the first order, then Eq. (14) approximates the deviations from the stresses in an elastoplastic material with the help of the bounded linear functional. The collection of all the histories with the finite memory forms the Hilbert space, and, according to the Frechet–Riss theorem, each bounded linear functional in the Hilbert space admits presentation in the form of a scalar product. Assume that the fading memory of the Coleman–Noll type is being considered, then, according to (13), we obtain

$$\sigma^{\mathbf{R}} = \overline{\mathbf{g}}(\xi(t)) + \int_0^{\infty} h(s)\mathbf{K}(\xi(t), s)[\xi^t(s) - \xi(t)]ds + \mathbf{o}(||\xi^t - \xi(t)^c||), \qquad (15)$$

where the kernel $\mathbf{K}$ is the second-rank tensor such that

$$\int_0^{\infty} |\mathbf{K}(\xi(t), s)|^2 \, ds < \infty.$$

If we truncate the correction term, we obtain a relationship independent of the reference system, which can be used at large deformations for describing the behavior of a viscoelastic-viscoplastic material with a fading memory and arbitrary symmetry of properties.

Constitutive relationships for simple hardening elastoplastic materials with a long-term fading memory of the deformation history based on Eq. (5) were constructed elsewhere [16–18].

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