Український математичний вісник
Том 5 **(2008)**, № 4, 470 – 487

Tangent spaces to metric spaces and to their subspaces

Oleksiy A. Dovgoshey

(*Presented by V. Ya. Gutlyanskii*)

Abstract. We investigate tangent spaces and metric space valued derivatives at points of a general metric spaces. The conditions under which two different subspaces of the metric space have isometric tangent spaces in the common point of these subspaces are determinated.

2000 MSC. 54E35.

Key words and phrases. Metric spaces, tangent spaces.

1. Introduction. Tangent metric spaces

The recent achievements in the metric space theory are closely related to some generalizations of the differentiation. The concept of the upper gradient [9, 10, 12], Cheeger's notion of differentiability for Rademacher's theorem in certain metric measure spaces [5], the metric derivative in the studies of metric space valued functions of bounded variation [1, 4] and the Lipshitz type approach in [8] are interesting and important examples of such generalizations. A very interesting technical tool to develop a theory of a differentiation in metric separable spaces is the fact that every separable metric space admits an isometric embedding into the dual space of a separable Banach space. It provides a linear structure, and so a differentiation, for separable metric spaces, see for example a rather complete theory of rectifiable sets and currents on metric spaces in [2, 3].

These generalizations of the differentiability usually lead to nontrivial results only for the assumption that metric spaces have "sufficiently many" rectifiable curves. In almost all mentioned approaches we see that theories of differentiations in metric spaces involve an induced linear structure that is able to use the classical differentiations in the linear normed spaces.

Received 11.09.2008

ISSN 1810 – 3200.

A new, intrinic, notion of differentiabililty for the mappings between the general metric spaces was produced by O. Dovgoshey and O. Martio in [7]. A basic technical tool in [7] is tangent spaces to an arbitrary metric space X at a point $a \in X$ that was defined as a factor space of a family of sequences of points $x_n \in X$ which converge to a. This approach makes possible to define a metric space valued derivative of functions $f : X \to Y$, X and Y are metric spaces, as a mapping between tangent spaces to X at the point a and, respectively, to Y at the point $f(a)$. The analysis of general properties of tangent spaces and of metric space valued derivatives is the main purpose of the present paper.

Let (X, d) be a metric space and let a be point of X. Fix some sequence $\tilde{r}$ of positive real numbers r_n which tend to zero. In what follows this sequence $\tilde{r}$ be called a *normalizing sequence*. Let us denote by $\tilde{X}$ the set of all sequences of points from X.

Definition 1.1. *Two sequences* $\tilde{x}, \tilde{y} \in \tilde{X}$, $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ *and* $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ *are mutually stable* (*with respect to a normalizing sequence* $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$) *if there is a finite limit*

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{1.1}$$

We shall say that a family $\tilde{F} \subseteq \tilde{X}$ is *self-stable* (w.r.t. $\tilde{r}$) if every two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is *maximal self-stable* if $\tilde{F}$ is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

A standard application of Zorn's lemma leads to the following

Proposition 1.1. *Let* (X, d) *be a metric space and let* $a \in X$. *Then for every normalizing sequence* $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ *there exists a maximal self-stable family* $\tilde{X}_a = \tilde{X}_{a,\tilde{r}}$ *such that* $\tilde{a} := \{a, a, \dots\} \in \tilde{X}_a$.

Note that the condition $\tilde{a} \in \tilde{X}_a$ implies the equality

$$\lim_{n\to\infty} d(x_n, a) = 0$$

for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ which belongs to $\tilde{X}_a$.

Consider a function $\tilde{d} : \tilde{X}_a \times \tilde{X}_a \to \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.1). Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \le \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}_a$. Hence $(\tilde{X}_a, \tilde{d})$ is a pseudometric space.

Definition 1.2. *The pretangent space to the space X at the point a w.r.t. a normalizing sequence $\tilde{r}$ is the metric identification of the pseudometric space $(\tilde{X}_{a,\tilde{r}}, \tilde{d})$.*

Since the notion of pretangent space is basic for the present paper, we remaind this metric identification construction.

Define a relation $\sim$ on $\tilde{X}_a$ by $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then $\sim$ is an equivalence relation. Let us denote by $\Omega_a = \Omega_{a,\tilde{r}} = \Omega^X_{a,\tilde{r}}$ the set of equivalence classes in $\tilde{X}_a$ under the equivalence relation $\sim$. It follows from general properties of pseudometric spaces, see, for example, [11, Chapter 4, Th. 15], that if ρ is defined on Ω_a by

$$\rho(\alpha, \beta) := \tilde{d}(\tilde{x}, \tilde{y}) \tag{1.2}$$

where $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then ρ is the well-defined metric on Ω_a. The metric identification of $(\tilde{X}_a, \tilde{d})$ is, by definition, the metric space (Ω_a, ρ).

Remark that $\Omega_{a,\tilde{r}} \neq \emptyset$ because the constant sequence $\tilde{a}$ belongs to $\tilde{X}_{a,\tilde{r}}$, see Proposition 1.1.

Let $\{n_k\}_{k\in\mathbb{N}}$ be an infinite, strictly increasing sequence of natural numbers. Let us denote by $\tilde{r}'$ the subsequence $\{r_{n_k}\}_{k\in\mathbb{N}}$ of the normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ and let $\tilde{x}' := \{x_{n_k}\}_{k\in\mathbb{N}}$ for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}$. It is clear that if $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}'). \tag{1.3}$$

If $\tilde{X}_{a,\tilde{r}}$ is a maximal self-stable (w.r.t. $\tilde{r}$) family, then, by Zorn's lemma, there exists a maximal self-stable (w.r.t. $\tilde{r}'$) family $\tilde{X}_{a,\tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\} \subseteq \tilde{X}_{a,\tilde{r}'}.$$

Denote by $\mathrm{in}_{\tilde{r}'}$ the mapping from $\tilde{X}_{a,\tilde{r}}$ to $\tilde{X}_{a,\tilde{r}'}$ with $\mathrm{in}_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$. It follows from (1.2) that after metric identifications $\mathrm{in}_{\tilde{r}'}$ pass to an isometric embedding em': $\Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$ under which the diagram

$$\begin{array}{ccc} \tilde{X}_{a,\tilde{r}} & \xrightarrow{\mathrm{in}_{\tilde{r}'}} & \tilde{X}_{a,\tilde{r}'} \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle p'} \\ \Omega_{a,\tilde{r}} & \xrightarrow{\mathrm{em}'} & \Omega_{a,\tilde{r}'} \end{array} \tag{1.4}$$

is commutative. Here p and p' are metric identification mappings, $p(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\}$ and $p'(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{a,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}, \tilde{y}) = 0\}$.

Let X and Y be two metric spaces. Recall that a map $f : X \to Y$ is called an *isometry* if f is distance-preserving and onto.

Definition 1.3. *A pretangent* $\Omega_{a,\tilde r}$ *is tangent if* em'*:* $\Omega_{a,\tilde r} \to \Omega_{a,\tilde r'}$ *is an isometry for every* $\tilde r'$*.*

To verify the correctness of this definition, we must prove that if $\tilde X^{(1)}_{a,\tilde r'}$ and $\tilde X_{a,\tilde r'}$ are two distinct maximal self-stable families such that the double inclusion

$$\tilde X_{a,\tilde r'} \supseteq \left\{\tilde x' : \tilde x \in \tilde X_{a,\tilde r}\right\} \subseteq \tilde X^{(1)}_{a,\tilde r'} \tag{1.5}$$

holds and $em' : \Omega_{a,\tilde r} \to \Omega_{a,\tilde r'}$ is an isometry, then $em'_1 : \Omega_{a,\tilde r} \to \Omega^{(1)}_{a,\tilde r'}$ is also an isometry, where $\Omega^{(1)}_{a,\tilde r'}$ is the metric identification of $\tilde X^{(1)}_{a,\tilde r'}$. Indeed, it is clear that if $\tilde x = \{x_n\}_{n\in\mathbb N} \in \tilde X_{a,\tilde r}$, $\tilde y = \{y_k\}_{k\in\mathbb N} \in \tilde X_{a,\tilde r'}$ and

$$\lim_{k\to\infty} \frac{d(y_k, x_{n_k})}{r_{n_k}} = 0,$$

then there is $\tilde z \in \tilde X_{a,\tilde r}$ with $\tilde z' = \tilde y$. Consequently, since em' is an isometry and (1.4) is commutative, the mapping $\mathrm{in}_{r'} : \tilde X_{a,\tilde r} \to \tilde X_{a,\tilde r'}$ is surjective, i.e.,

$$\tilde X_{a,\tilde r'} = \left\{\tilde x' : \tilde x \in \tilde X_{a,\tilde r}\right\}.$$

Hence, by (1.5), we obtain the inclusion $\tilde X^{(1)}_{a,\tilde r'} \supseteq \tilde X_{a,\tilde r'}$. It implies the equality $\tilde X^{(1)}_{a,\tilde r'} = \tilde X_{a,\tilde r'}$ because $\tilde X_{a,\tilde r'}$ is maximal self-stable. Hence $\mathrm{em}'_1 = \mathrm{em}'$, so that em'_1 is an isometry.

These arguments give the following proposition.

Proposition 1.2. *Let* X *be a metric space with a marked point* a*,* $\tilde r$ *a normalizing sequence and* $\tilde X_{a,\tilde r}$ *a maximal self-stable family with pretangent space* $\Omega_{a,\tilde r}$*. The following statements are equivalent.*

(*i*) $\Omega_{a,\tilde r}$ *is tangent.*

(*ii*) *For every subsequence* $\tilde r'$ *of the sequence* $\tilde r$ *the family* $\{\tilde x' : \tilde x \in \tilde X_{a,\tilde r}\}$ *is maximal self-stable w.r.t.* $\tilde r'$*.*

(*iii*) *A function* $\mathrm{em}' : \Omega_{a,\tilde r} \to \Omega_{a,\tilde r'}$ *is surjective for every* $\tilde r'$*.*

(*iv*) *A function* $\mathrm{in}'_r : \tilde X_{a,\tilde r} \to \tilde X_{a,\tilde r'}$ *is surjective for every* $\tilde r'$*.*

Now we introduce an equivalence relation for the classification of normalizing sequences.

Definition 1.4. *Let* X *be a metric space with a marked point* a*. Two normalizing sequences* $\tilde r$ *and* $\tilde t$ *are equivalent at the point* a*,* $\tilde r \approx \tilde t$*, if the logical equivalence*

$$(\tilde F \text{ is self-stable w.r.t. } \tilde r) \Longleftrightarrow (\tilde F \text{ is self-stable w.r.t. } \tilde t)$$

is true for every $\tilde F \subseteq \tilde X$ *with* $\tilde a \in \tilde F$*.*

A normalizing sequence $\tilde{r}$ will be called *confluented in a point* a if there exists an one-point pretangent space $\Omega_{a,\tilde{r}}$ (it certainly implies that all pretangent $\Omega_{a,\tilde{r}}$ are one-point).

Theorem 1.1. *Let* (X, d) *be a metric space with a marked point* a *and let* $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$, $\tilde{t} = \{t_n\}_{n\in\mathbb{N}}$ *be two normalizing sequences which are equivalent at the point* a. *Then at least one of the following statements holds.*

(i) *There is a real number* $c > 0$ *such that*

$$\lim_{n\to\infty} \frac{r_n}{t_n} = c. \tag{1.6}$$

(ii) *The sequences* $\tilde{r}$ *and* $\tilde{t}$ *are confluented in the point* a.

Proof. Suppose that both sequences $\tilde{r}$ and $\tilde{t}$ are not confluented in a. Then there are $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ from $\tilde{X}$ such that

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{a}) = \lim_{n\to\infty} \frac{d(x_n, a)}{r_n} > 0 \quad \text{and} \quad \tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{a}) = \lim_{n\to\infty} \frac{d(y_n, a)}{t_n} > 0 \tag{1.7}$$

where $\tilde{a} = (a, a, \dots)$. If $\tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{a}) > 0$ or $\tilde{d}_{\tilde{t}}(\tilde{x}, \tilde{a}) > 0$, then we obtain

$$0 < \frac{\tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{a})}{\tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{a})} = \lim_{n\to\infty} \frac{t_n}{r_n} < \infty$$

or, respectively,

$$0 < \frac{\tilde{d}_{\tilde{t}}(\tilde{x}, \tilde{a})}{\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{a})} = \lim_{n\to\infty} \frac{r_n}{t_n} < \infty,$$

i.e., Statement (i) holds. Now observe that the equalities

$$\tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{a}) = \tilde{d}_{\tilde{t}}(\tilde{x}, \tilde{a}) = 0 \tag{1.8}$$

lead to a contradiction because (1.7) and (1.8) imply

$$0 = \lim_{n\to\infty} \frac{r_n}{t_n} = \infty.$$

Thus if Statement (i) does not hold, then at least one of the sequences $\tilde{r}$ and $\tilde{t}$ is confluented. We claim that if $\tilde{r}$ or $\tilde{t}$ is confluented, then both $\tilde{r}$ and $\tilde{t}$ are confluented. Indeed, if $\tilde{r}$ is confluented and we have a finite limit

$$\tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{a}) = \lim_{n\to\infty} \frac{d(y_n, a)}{t_n} \neq 0 \tag{1.9}$$

for $\tilde{y} = \{y_n\}_{n\in\mathbb{N}} \in \tilde{X}$, then $\tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{a}) = 0$ because $\tilde{t} \approx \tilde{r}$ and $\tilde{r}$ is confluented in a. Write

$$y_n^* := \begin{cases} y_n & \text{if } n \text{ is odd} \\ a & \text{if } n \text{ is even} \end{cases} \tag{1.10}$$

for every $n \in \mathbb{N}$ and put $\tilde{y}^* := \{y_n^*\}_{n\in\mathbb{N}}$. Then we obtain $\tilde{d}_{\tilde{r}}(\tilde{y}^*, \tilde{a}) = 0$. Thus the family

$$\tilde{F} := \{\tilde{y}, \tilde{y}^*, \tilde{a}\}$$

is self-stable w.r.t. $\tilde{r}$. Since $\tilde{r} \approx \tilde{t}$, this family also is self-stable w.r.t $\tilde{t}$. Consequently there is a finite limit

$$\tilde{d}_{\tilde{t}}(\tilde{a}, \tilde{y}^*) = \lim_{n\to\infty} \frac{d(y_n^*, a)}{t_n}.$$

Hence, by (1.9) and (1.10), we obtain

$$0 \neq \lim_{n\to\infty} \frac{d(y_{2n+1}^*, a)}{t_{2n+1}} = \lim_{n\to\infty} \frac{d(y_{2n}^*, a)}{t_{2n}} = 0.$$

This contradiction shows that $\tilde{t}$ is confluented if $\tilde{r}$ is confluented.

Hence Statement (i) holds if Statement (ii) does not hold, and the theorem follows. □

Remark 1.1. It is clear that if there is $c > 0$ such that (1.6) holds, then normalizing sequences $\tilde{r}$ and $\tilde{t}$ are equivalent at all points of an arbitrary metric space.

Proposition 1.3. *Let (X, d) be a metric space with a marked point a. The following propositions are equivalent.*

- (i) *The point a is an isolated point of the metric space X.*
- (ii) *Every two normalizing sequences are equivalent at the point a.*
- (iii) *All normalizing sequences are confluented in a.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are trivial. To prove $(ii) \Rightarrow (i)$ suppose that the relation

$$\tilde{r} \approx \tilde{t} \tag{1.11}$$

holds for every two normalizing $\tilde{r}$ and $\tilde{t}$ but there is $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}$ such that $\lim_{n\to\infty} d(x_n, a) = 0$ and $d(x_n, a) > 0$ for all $n \in \mathbb{N}$. Let $\tilde{x}' = \{x_{n_k}\}_{k\in\mathbb{N}}$ be an infinite subsequence of $\tilde{x}$ with

$$\lim_{k\to\infty} \frac{d(x_{n_k}, a)}{d(x_k, a)} = 0. \tag{1.12}$$

Write

$$\tilde{r} := \{d(x_k, a)\}_{k\in\mathbb{N}}, \qquad \tilde{t} := \{d(x_{n_k}, a)\}_{k\in\mathbb{N}}.$$

Now, by the construction, both $\tilde{r}$ and $\tilde{t}$ are not confluented and, moreover, (1.12) imply that the negation of (1.6) is true for all $c > 0$. Hence, by Theorem 1.1, $\tilde{r}$ and $\tilde{t}$ are not equivalent at the point a, contrary to (1.11). Thus $(ii) \Rightarrow (i)$ is true.

If a is not an isolated point of X, then there is a sequence $\tilde{b} = \{b_n\}_{n\in N} \in \tilde{X}$ such that $\lim_{n\to\infty} d(a, b_n) = 0$ and $d(a, b_n) \neq 0$ for all $n \in \mathbb{N}$. Consider the normalizing sequence $\tilde{r} = \{r_n\}_{n\in N}$ with $r_n := d(a, b_n)$. It follows immediately from (1.1) that $\tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{b}) = 1$ where $\tilde{a}$ is the constant sequence $\{a, a, \dots\}$. The application of Zorn's lemma shows that there is a maximal self-stable family $\tilde{X}_{a,\tilde{r}}$ such that $\tilde{a}, \tilde{b} \in \tilde{X}_{a,\tilde{r}}$. Then the metric identification of the pseudometric space $(\tilde{X}_{a,\tilde{r}}, \tilde{d})$ has at least two points. Consequently we also have $(iii) \Rightarrow (i)$. □

2. Metric space valued derivatives. Definition and general properties

Let (X_i, d_i), $i = 1, 2$, be metric spaces with marked points $a_i \in X_i$ and $\tilde{r}_i = \{r_n^{(i)}\}_{n\in\mathbb{N}}$ normalizing sequences and $\tilde{X}^i_{a_i,\tilde{r}_i}$ maximal self-stable families with correspondent pretangent spaces $\Omega_{a_i,\tilde{r}_i}$. For functions $f : X_1 \to X_2$ define the mappings $\tilde{f} : \tilde{X}_1 \to \tilde{X}_2$ as

$$\tilde{f}(\tilde{x}) = \{f(x_i)\}_{i\in\mathbb{N}} \quad \text{for} \quad \tilde{x} = \{x_i\}_{i\in\mathbb{N}} \in \tilde{X}_1. \tag{2.1}$$

Definition 2.1. *A function $f : X_1 \to X_2$ is differentiable w.r.t. the pair $(\tilde{X}^1_{a_1,\tilde{r}_1}, \tilde{X}^2_{a_2,\tilde{r}_2})$ if the following conditions are satisfied:*

(i) $\tilde{f}(\tilde{x}) \in \tilde{X}^2_{a_2,\tilde{r}_2}$ *for every* $\tilde{x} \in \tilde{X}^1_{a_1,\tilde{r}_1}$;

(ii) The implication $\big(\tilde{d}_{\tilde{r}_1}(\tilde{x}, \tilde{y}) = 0\big) \Longrightarrow \big(\tilde{d}_{\tilde{r}_2}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) = 0\big)$ *is true for all* $\tilde{x}, \tilde{y} \in \tilde{X}^1_{a_1,\tilde{r}_1}$, *where*

$$\tilde{d}_{\tilde{r}_1}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \frac{d_1(x_n, y_n)}{r_n^{(1)}},$$

$$\tilde{d}_{\tilde{r}_2}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) = \lim_{n\to\infty} \frac{d_2(f(x_n), f(y_n))}{r_n^{(2)}}.$$

Remark 2.1. Note that Condition (i) of Definition 2.1 implies the equality $f(a_1) = a_2$.

Let $p_i : \tilde{X}^i_{a_i,\tilde{r}_i} \to \Omega_{a_i,\tilde{r}_i}$, $i = 1, 2$, be metric identification mappings.

Definition 2.2. *A function $D^*f : \Omega_{a_1,\tilde{r}_1} \to \Omega_{a_2,\tilde{r}_2}$ is a metric space valued derivative of $f : X_1 \to X_2$ at the point $a_1 \in X_1$ (or, in short, a derivative of f) if f is differentiable w.r.t. $(\tilde{X}^1_{a_1,\tilde{r}_1}, \tilde{X}^2_{a_2,\tilde{r}_2})$ and the following diagram*

$$\begin{array}{ccc} \tilde{X}^1_{a_1,\tilde{r}_1} & \xrightarrow{\ \tilde{f}\ } & \tilde{X}^2_{a_2,\tilde{r}_2} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \Omega_{a_1,\tilde{r}_1} & \xrightarrow{\ D^*f\ } & \Omega_{a_2,\tilde{r}_2} \end{array} \tag{2.2}$$

is commutative.

In this section we establish some common properties of the metric space valued derivatives.

Let us show that the metric space valued derivative is unique if exists. Indeed, suppose that diagram (2.2) is commutative with $D^*f = D_1^*f$ and with $D^*f = D_2^*f$. Let $\beta \in \Omega_{a_1,\tilde{r}_1}$. Since p_1 is a surjection, there is $\tilde{x}_1 \in \tilde{X}^1_{a_1,\tilde{r}_1}$ such that $\beta = p_1(\tilde{x}_1)$. From the commutativity of (2.2) we obtain

$$D_1^*(\beta) = D_1^*(p_1(\tilde{x}_1)) = p_2(\tilde{f}(\tilde{x}_1)) = D_2^*(p_1(\tilde{x}_1)) = D_2^*(\beta),$$

i.e., $D_1^* = D_2^*$.

The following proposition shows that the Chain Rule remains valid for the metric space valued derivatives.

Proposition 2.1. *Let X_i be metric spaces with marked points $a_i \in X_i$ and $\tilde{r}_i$ normalizing sequences and $\tilde{X}^i_{a_i,\tilde{r}_i}$ maximal self-stable families with pretangent spaces $\Omega_{a_i,\tilde{r}_i}$, $i = 1, 2, 3$.*

Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be differentiable functions, f w.r.t. the pair $(\tilde{X}^1_{a_1,\tilde{r}_1}, \tilde{X}^2_{a_2,\tilde{r}_2})$ and g w.r.t. $(\tilde{X}^2_{a_2,\tilde{r}_2}, \tilde{X}^3_{a_3,\tilde{r}_3})$.

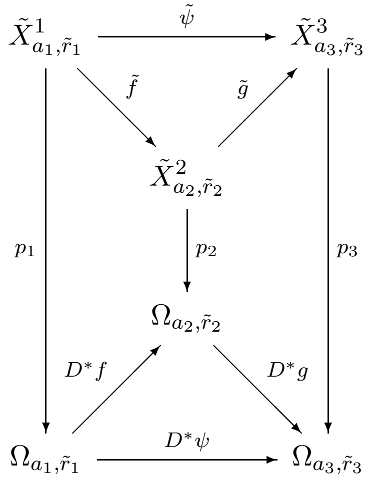
Then the superposition $\psi = g \circ f$ is differentiable w.r.t. $(\tilde{X}^1_{a_1,\tilde{r}_1}, \tilde{X}^3_{a_3,\tilde{r}_3})$ and

$$D^*(\psi) = (D^*g) \circ (D^*f). \tag{2.3}$$

Proof. The differentiability of ψ is an immediate consequence of the differentiability of f and g, see Definition 2.1. To prove (2.3) note that

$$p_2 \circ \tilde{f} = (D^*f) \circ p_1 \text{ and } p_3 \circ \tilde{g} = (D^*g) \circ p_2,$$

see the following diagram



(2.4)

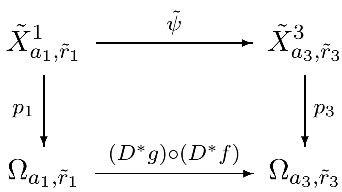
where $\tilde{\psi} = \tilde{g} \circ \tilde{f}$. Consequently we have

$$p_3 \circ \tilde{\psi} = p_3 \circ (\tilde{g} \circ \tilde{f}) = (p_3 \circ \tilde{g}) \circ \tilde{f} = ((D^*g) \circ p_2) \circ \tilde{f}$$
$$= (D^*g) \circ (p_2 \circ \tilde{f}) = (D^*g) \circ (D^*f) \circ p_1,$$

that is

$$p_3 \circ \tilde{\psi} = (D^*g \circ D^*f) \circ p_1.$$

Hence the diagram



is commutative. The uniqueness of the derivative $D^*\psi$ and Definition 2.2 imply (2.3). □

Proposition 2.2. *Let (X, d) and (Y, ρ) be metric spaces, $a \in X$ and $b \in Y$ marked points in these spaces, and $f : X \to Y$ a function such that $f(a) = b$. Suppose for every maximal self-stable family $\tilde{X}_{a,\tilde{r}} \subseteq \tilde{X}$ there is a maximal self-stable family $\tilde{Y}_{b,\tilde{t}} \subseteq \tilde{Y}$ such that f is differentiable w.r.t. the pair $(\tilde{X}_{a,\tilde{r}}, \tilde{Y}_{b,\tilde{t}})$. Then f is continuous at the point a.*

Proof. We may suppose that a is not an isolated point of X. Let $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}$ be a sequence such that

$$\lim_{n\to\infty} d(x_n, a) = 0 \quad \text{and} \quad d(x_n, a) \neq 0$$

for all $n \in \mathbb{N}$. Then there is a maximal self-stable family $\tilde{X}_{a,\tilde{r}} \supseteq \{\tilde{a}, \tilde{x}\}$ where $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ is a normalizing sequence with $r_n := d(x_n, a)$ for $n \in \mathbb{N}$. Hence, by the supposition, there exists a normalizing sequence $\tilde{t} = \{t_n\}_{n\in\mathbb{N}}$ for which the limit

$$\lim_{n\to\infty} \frac{\rho(f(x_n), b)}{t_n}$$

is finite. Consequently we have $\lim_{n\to\infty} \rho(f(x_n), b) = 0$ because

$$\lim_{n\to\infty} t_n = 0.$$

Hence the function f is continuous at the point a. □

3. Tangent spaces to subspaces of metric spaces

Let (X, d) be a metric space with a marked point a, let Y and Z be subspaces of X such that $a \in Y \cap Z$ and let $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ be a normalizing sequence.

Definition 3.1. *The subspaces Y and Z are tangent equivalent at the point a w.r.t. $\tilde{r}$ if for every $\tilde{y}_1 = \{y_n^{(1)}\}_{n\in\mathbb{N}} \in \tilde{Y}$ and every $\tilde{z}_1 = \{z_n^{(1)}\}_{n\in\mathbb{N}} \in \tilde{Z}$ with finite limits*

$$\tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{y}_1) = \lim_{n\to\infty} \frac{d(y_n^{(1)}, a)}{r_n} \quad \textit{and} \quad \tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{z}_1) = \lim_{n\to\infty} \frac{d(z_n^{(1)}, a)}{r_n}$$

there exist $\tilde{y}_2 = \{y_n^{(2)}\}_{n\in\mathbb{N}} \in \tilde{Y}$ and $\tilde{z}_2 = \{z_n^{(2)}\}_{n\in\mathbb{N}} \in \tilde{Z}$ such that

$$\lim_{n\to\infty} \frac{d(y_n^{(1)}, z_n^{(2)})}{r_n} = \lim_{n\to\infty} \frac{d(y_n^{(2)}, z_n^{(1)})}{r_n} = 0.$$

We shall say that Y and Z are strongly tangent equivalent at a if Y and Z are tangent equivalent at a for all normalizing sequences $\tilde{r}$.

Let $\tilde{F} \subseteq \tilde{X}$. For a normalizing sequence $\tilde{r}$ we define a family $[\tilde{F}]_Y = [\tilde{F}]_{Y,\tilde{r}}$ by the rule

$$(\tilde{y} \in [\tilde{F}]_Y) \Leftrightarrow ((\tilde{y} \in \tilde{Y}) \;\&\; (\exists\, \tilde{x} \in \tilde{F} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0)). \tag{3.1}$$

Note that $[\tilde{F}]_Y$ can be empty for some nonvoid families $\tilde{F}$ if the set $X \setminus Y$ is “big enough”.

Proposition 3.1. *Let Y and Z be subspaces of a metric space X and let $\tilde{r}$ be a normalizing sequence. Suppose that Y and Z are tangent equivalent (w.r.t. $\tilde{r}$) at a point $a \in Y \cap Z$. Then following statements hold for every maximal self-stable (in $\tilde{Z}$) family $\tilde{Z}_{a,\tilde{r}}$.*

(*i*) *The family* $[\tilde{Z}_{a,\tilde{r}}]_Y$ *is maximal self-stable* (*in* $\tilde{Y}$) *and we have the equalities*

$$[[\tilde{Z}_{a,\tilde{r}}]_Y]_Z = \tilde{Z}_{a,\tilde{r}} = [\tilde{Z}_{a,\tilde{r}}]_Z. \tag{3.2}$$

(*ii*) *If* $\Omega^Z_{a,\tilde{r}}$ *and* $\Omega^Y_{a,\tilde{r}}$ *are metric identifications of* $\tilde{Z}_{a,\tilde{r}}$ *and, respectively, of* $\tilde{Y}_{a,\tilde{r}} := [\tilde{Z}_{a,\tilde{r}}]_Y$, *then the mapping*

$$\Omega^Z_{a,\tilde{r}} \ni \alpha \longmapsto [\alpha]_Y \in \Omega^Y_{a,\tilde{r}} \tag{3.3}$$

is an isometry. Furthermore, if $\Omega^Z_{a,\tilde{r}}$ *is tangent, then* $\Omega^Y_{a,\tilde{r}}$ *also is tangent.*

Proof. (*i*) Let $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}_{a,\tilde{r}} := [\tilde{Z}_{a,\tilde{r}}]_Y$. Then, by (3.1), there exist $\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_{a,\tilde{r}}$ such that

$$\tilde{d}_{\tilde{r}}(\tilde{y}_1, \tilde{z}_1) = \tilde{d}_{\tilde{r}}(\tilde{y}_2, \tilde{z}_2) = 0. \tag{3.4}$$

Since $\tilde{z}_1$ and $\tilde{z}_2$ are mutually stable, $\tilde{y}_1$ and $\tilde{y}_2$ also are mutually stable. Consequently, $\tilde{Y}_{a,\tilde{r}}$ is self-stable. The similar arguments show that $[\tilde{Y}_{a,\tilde{r}}]_Z$ is also self-stable. Moreover, since

$$[[\tilde{Z}_{a,\tilde{r}}]_Y]_Z = [\tilde{Y}_{a,\tilde{r}}]_Z \supseteq \tilde{Z}_{a,\tilde{r}},$$

the maximality of $\tilde{Z}_{a,\tilde{r}}$ implies the first equality in (3.2). The second one also simply follows from the maximality of $\tilde{Z}_{a,\tilde{r}}$. It still remains to prove that $\tilde{Y}_{a,\tilde{r}}$ is a maximal self-stable subset of $\tilde{Y}$. Let $\tilde{Y}^m_{a,\tilde{r}}$ be a maximal self-stable family in $\tilde{Y}$ such that $\tilde{Y}^m_{a,\tilde{r}} \supseteq \tilde{Y}_{a,\tilde{r}}$. Then $[\tilde{Y}^m_{a,\tilde{r}}]_Z$ is self-stable and $[\tilde{Y}^m_{a,\tilde{r}}]_Z \supseteq \tilde{Z}_{a,\tilde{r}}$. Since $\tilde{Z}_{a,\tilde{r}}$ is maximal self-stable, the last inclusion implies the equality $[\tilde{Y}^m_{a,\tilde{r}}]_Z = \tilde{Z}_{a,\tilde{r}}$. Using this equality and (3.2) we obtain

$$\tilde{Y}^m_{a,\tilde{r}} = [[\tilde{Y}^m_{a,\tilde{r}}]_Z]_Y = [\tilde{Z}_{a,\tilde{r}}]_Y := \tilde{Y}_{a,\tilde{r}},$$

i.e., $\tilde{Y}_{a,\tilde{r}}$ is maximal self-stable.

(*ii*) Let $\alpha \in \Omega^Z_{a,\tilde{r}}$ and let $\tilde{z} \in \tilde{Z}_{a,\tilde{r}}$ such that $\tilde{z} \in \alpha$. It follows from (3.1) that

$$[\alpha]_Y = \{\tilde{y} \in \tilde{Y} : \tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{z}) = 0\}. \tag{3.5}$$

The last equality implies that function (3.3) is distance-preserving. In addition, using (3.5) we see that

$$[[\alpha]_Y]_Z = \alpha \quad \text{and} \quad [[\beta]_Z]_Y = \beta$$

for every $\alpha \in \Omega^Z_{a,\tilde{z}}$ and every $\beta \in \Omega^Y_{z,\tilde{r}}$. Consequently function (3.3) is bijective. To prove that $\Omega^Y_{a,\tilde{r}}$ is tangent if $\Omega^Z_{a,\tilde{r}}$ is tangent we can use Statement (*ii*) of Proposition 1.2 and Statement (*i*) of Proposition 3.1. □

Corollary 3.1. *Let Y and Z be subspaces of a metric space X. Suppose that Y and Z are tangent equivalent at a point $a \in Y \cap Z$ w.r.t. a normalizing sequence $\tilde{r}$ and that there exists a unique maximal self-stable (in $\tilde{Z}$) family $\tilde{Z}_{a,\tilde{r}} \ni \tilde{a}$. Then $\tilde{Y}_{a,\tilde{r}} := [\tilde{Z}_{a,\tilde{r}}]_Y$ is a unique maximal self-stable family (in $\tilde{Y}$) which contains $\tilde{a}$.*

Proof. Let $Y^*_{a,\tilde{r}} \ni \tilde{a}$ be a maximal self-stable family in $\tilde{Y}$. Then, by Statement (i) of Proposition 3.1, $[Y^*_{a,\tilde{r}}]_Z$ is maximal self-stable (in $\tilde{Z}$). Since $\tilde{a} \in [Y^*_{a,\tilde{r}}]$, we have $[Y^*_{a,r}]_Z = \tilde{Z}_{a,r}$. Hence, by (3.2),

$$Y^*_{a,r} = [[Y^*_{a,r}]_Z]_Y = [\tilde{Z}_{a,\tilde{r}}]_Y = \tilde{Y}_{a,\tilde{r}}.$$

□

Let Y be a subspace of a metric space (X,d). For $a \in Y$ and $t > 0$ we denote by

$$S^Y_t = S^Y(a,t) := \{y \in Y : d(a,y) = t\}$$

the sphere (in the subspace Y) with the center a and the radius t. Similarly for $a \in Z \subseteq X$ and $t > 0$ define

$$S^Z_t = S^Z(a,t) := \{z \in Z : d(a,z) = t\}.$$

Write

$$\varepsilon_a(t,Z,Y) := \sup_{z \in S^Z_t} \inf_{y \in Y} d(z,y) \tag{3.6}$$

and

$$\varepsilon_a(t) = \varepsilon_a(t,Z,Y) \vee \varepsilon_a(t,Y,Z). \tag{3.7}$$

Theorem 3.1. *Let Y and Z be subspaces of a metric space (X,d) and let $a \in Y \cap Z$. Then Y and Z are strongly tangent equivalent at the point a if and only if*

$$\lim_{t \to 0} \frac{\varepsilon_a(t)}{t} = 0. \tag{3.8}$$

Proof. Suppose that limit relation (3.8) holds. Let $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ be a normalizing sequence and $\tilde{z} = \{z_n\}_{\in \mathbb{N}} \in \tilde{Z}$ be a sequence with a finite limit

$$\tilde{d}_{\tilde{r}}(\tilde{a},\tilde{z}) = \lim_{n \to \infty} \frac{d(a,z_n)}{r_n}.$$

To find $\tilde{y} = \{y_n\} \in \tilde{Y}$ such that

$$\tilde{d}_{\tilde{r}}(\tilde{y},\tilde{z}) = 0 \tag{3.9}$$

note that we can take $\tilde{y} = \tilde{a}$ if $\tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{z}) = 0$. Hence, without loss of generality, we suppose

$$0 < \tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{z}) = \lim_{n\to\infty} \frac{d(z_n, a)}{r_n}. \tag{3.10}$$

It follows from (3.7) and (3.8) that

$$\lim_{t\to 0} \frac{\varepsilon_a(t, Z, Y)}{t} = 0. \tag{3.11}$$

Inequality (3.10) implies that there is $n_0 \in \mathbb{N}$ such that $d(z_n, a) > 0$ if $n \geq n_0$. Write for every $n \in \mathbb{N}$

$$t_n := \begin{cases} 1 & \text{if } n < n_0 \\ d(z_n, a) & \text{if } n \geq n_0. \end{cases} \tag{3.12}$$

The definition of $\varepsilon_a(t, Z, Y)$ implies that for every $n \in \mathbb{N}$ there is $y_n \in Y$ with

$$d(z_n, y_n) \leq \varepsilon_a(t_n, Z, Y) + t_n^2. \tag{3.13}$$

Put $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ where y_n are points in Y for which (3.13) holds. Now using (3.10)–(3.12) we obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{d(z_n, y_n)}{r_n} &\leq \lim_{n\to\infty} \frac{d(z_n, a)}{r_n} \limsup_{n\to\infty} \frac{d(z_n, y_n)}{t_n} \\ &\leq \tilde{d}_{\tilde{r}}(\tilde{z}, \tilde{a}) \limsup_{n\to\infty} \frac{\varepsilon_a(t_n, Z, Y) + t_n^2}{t_n} = 0. \end{aligned}$$

Consequently, $\lim_{n\to\infty} \frac{d(z_n, y_n)}{r_n} = 0$, i.e., (3.9) holds. Similarly, we can prove that for every $\tilde{y} \in \tilde{Y}$ with a finite $\tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{a})$ there is $\tilde{z} \in \tilde{Z}$ such that $\tilde{d}_{\tilde{r}}(\tilde{z}, \tilde{y}) = 0$. Hence if (3.8) holds, then Y and Z are strongly tangent equivalent at the point a.

Suppose now that (3.8) does not hold. More precisely, we shall assume that

$$\limsup_{n\to\infty} \frac{\varepsilon_a(t, Z, Y)}{t} > 0.$$

Then there is a sequence $\tilde{t}$ of positive numbers t_n with $\lim_{n\to\infty} t_n = 0$ and there is $c > 0$ such that for every $n \in \mathbb{N}$ there exists $z_n \in S^Z(a, t_n)$ for which

$$\inf_{y\in Y} d(z_n, y) \geq ct_n = cd(a, z_n). \tag{3.14}$$

Let us denote by $\tilde{z}$ the sequence of points $z_n \in Z$ which satisfy (3.14). Take the sequence $\tilde{t} = \{t_n\}_{n\in\mathbb{N}}$ as a normalizing sequence. Then, by (3.14), we obtain

$$\limsup_{n\to\infty} \frac{d(z_n, y_n)}{t_n} \geq c > 0$$

for every $\tilde{y} = \{y_n\}_{n\in\mathbb{N}} \in \tilde{Y}$. Consequently, Y and Z are not strongly tangent equivalent at the point a. □

Consider now the case where $Z = X$. Let (X, d) be a metric space and let $a \in Y \subseteq X$. If $\tilde{X}_{a,\tilde{r}} \subseteq \tilde{X}$ and $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{Y}$ are maximal self-stable families and $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{X}_{a,\tilde{r}}$, then there is a unique isometric embedding $E_{m_Y} : \Omega^Y_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ such that the following diagram

$$\begin{array}{ccc} \tilde{Y}_{a,\tilde{r}} & \xrightarrow{\text{in}} & \tilde{X}_{a,\tilde{r}} \\ {\scriptstyle p_Y}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \Omega^Y_{a,\tilde{r}} & \xrightarrow{E_{m_Y}} & \Omega_{a,\tilde{r}} \end{array} \tag{3.15}$$

is commutative. Here $\Omega_{a,\tilde{r}}$ and $\Omega^Y_{a,\tilde{r}}$ are pretangent spaces corresponding to $\tilde{X}_{a,r}$ and, respectively, to $\tilde{Y}_{a,\tilde{r}}$, p_Y and p are appropriate metric identification maps and $\text{in}(\tilde{y}) = \tilde{y}$ for all $\tilde{y} \in \tilde{Y}_{a,\tilde{r}}$.

Corollary 3.2. *Let (X, d) be a metric space, let Y be a subspace of X and let $a \in Y$. The following conditions are equivalent.*

- (i) *For every maximal self-stable $\tilde{X}_{a,\tilde{r}} \subseteq \tilde{X}$ there is a maximal self-stable $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{Y}$ such that $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{X}_{a,\tilde{r}}$ and the embedding $E_{m_Y} : \Omega^Y_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ is an isometry.*
- (ii) *The equality*

$$\lim_{t\to 0} \frac{\varepsilon_a(t, X, Y)}{t} = 0$$

holds.

- (iii) *X and Y are strongly tangent equivalent at the point a.*

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ follows from Theorem 3.1. To prove $(iii) \Rightarrow (i)$ note that

$$[\tilde{X}_{a,\tilde{r}}]_Y = \tilde{Y} \cap \tilde{X}_{a,\tilde{r}}$$

for every maximal self-stable $\tilde{X}_{a,\tilde{r}}$ if X and Y are strongly tangent equivalent at the point a. Consequently, (iii) implies (i) because mapping (3.3) is an isometry.

Now suppose that Condition (i) is satisfied. To prove $(i) \Rightarrow (iii)$ it is sufficient to show that for every maximal self-stable $\tilde{X}_{a,\tilde{r}}$ and every $\tilde{x}_0 \in \tilde{X}_{a,\tilde{r}}$ there is $\tilde{y}_0 \in \tilde{Y}$ such that

$$\tilde{d}_{\tilde{r}}(\tilde{x}_0, \tilde{y}_0) = 0. \tag{3.16}$$

Let $\alpha = p(\tilde{x}_0)$. Since E_{m_Y} is an isometry, E_{m_Y} is surjective. Thus $E_{m_Y}^{-1}(\alpha) \neq \emptyset$. The last relation and

$$p_Y^{-1}(E_{m_Y}^{-1}(\alpha)) \neq \emptyset$$

are equivalent because p_Y also is surjective. Since

$$E_{m_Y} \circ p_Y = p \circ \mathrm{in}$$

and $p^{-1}(\alpha) = \{\tilde{x} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{x}_0) = 0\}$, we have

$$\begin{aligned}\emptyset \neq p_Y^{-1}(E_{m_Y}^{-1}(\alpha)) &= \mathrm{in}^{-1}(p^{-1}(\alpha)) \\ &= \mathrm{in}^{-1}(\{\tilde{x} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}_0, \tilde{x}) = 0\}) \\ &= \{\tilde{y} \in \tilde{Y} : \tilde{d}_{\tilde{r}}(\tilde{x}_0, \tilde{y}) = 0\},\end{aligned}$$

that implies (3.16) with some $\tilde{y}_0 \in \tilde{Y}$. □

Obviously, Condition (ii) of Corollary 3.2 is satisfied if Y is a dense subset of X. Therefore, we have the following.

Corollary 3.3. *Let (X, d) be a metric space and let Y be a dense subspace of X. Then X and Y are strongly tangent equivalent at all points $a \in Y$, in particular, the pretangent spaces to X and to Y are pairwise isometric for all normalizing sequences at every point $a \in Y$.*

Consider now some examples.

The following result was proved in [6]. Let $X = \mathbb{R}$ or $X = \mathbb{C}$ or $X = \mathbb{R}^+ = [0, \infty)$ and let

$$d(x, y) = |x - y|$$

for all $x, y \in X$.

Proposition 3.2. *Each pretangent space $\Omega^X_{0,\tilde{r}}$ (to X at the point 0) is tangent and isometric to (X, d) for every normalizing sequence $\tilde{r}$.*

Using Theorem 3.1 and Proposition 3.2 we can easily obtain future examples of tangent spaces to some subspaces of the Euclidean space E^n. The first example will be examined in details.

Example 3.1. Let $F : [0,1] \to E^n,\ n \geq 2$, be a simple closed curve in the Euclidean space E^n, i.e., F is continuous and $F(0) = F(1)$ and

$$F(t_1) \neq F(t_2)$$

for every two distinct points $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| \neq 1$. We can write F in the coordinate form

$$F(t) = (f_1(t), \dots, f_n(t)), \qquad t \in [0,1].$$

Suppose that all functions $f_i,\ 1 \leq i \leq n$, are differentiable at a point $t_0 \in (0,1)$ and

$$F'(t_0) = (f_1'(t_0), \dots, f_n'(t_0)) \neq (0, \dots, 0).$$

(In the case $t_0 = 0$ or $t_0 = 1$ we must use the one-sided derivatives.) We claim that each pretangent space to the subspace $Y := F([0,1]) \subseteq E^n$ at the point $a = F(t_0)$ is tangent and isometric to $\mathbb{R}$ for every normalizing sequence $\tilde{r}$. Indeed, by Proposition 3.1 and by Proposition 3.2, it is sufficient to show that Y is strongly tangent equivalent to the straight line

$$Z = \big\{(z_1(t), \dots, z_n(t)) : (z_1(t), \dots, z_n(t)) \\ = F'(t_0)(t - t_0) + F(t_0),\ t \in \mathbb{R}\big\}$$

at the point $a = F(t_0)$.

The classical definition of the differentiability of real functions shows that limit relation (3.8) holds with these Y and Z. Hence, by Theorem 3.1, Y and Z are strongly tangent equivalent at the point $a = F(t_0)$.

Example 3.2. Let $f_i : [-1,1] \to \mathbb{R},\ i = 1, \dots, n$, be functions such that $f_1(0) = \cdots = f_n(0) = c$ where $c \in \mathbb{R}$ is a constant. Suppose all f_i have a common finite derivative b at the point 0, $f_1'(0) = \cdots = f_n'(0) = b$. Write

$$a = (0, c) \quad \text{and} \quad X = \bigcup_{i=1}^{n} \{(t, f_i(t)) : t \in [-1,1]\},$$

i.e., X is an union of the graphs of the functions f_i. Let us consider X as a subspace of the Euclidean plane E^2. Then each pretangent space $\tilde{X}_{a,\tilde{r}}$ to the space X at the point a is tangent and isometric to $\mathbb{R}$.

Example 3.3. Let f_1, f_2 be two functions from the precedent example. Put

$$X = \big\{(x, y) : f_1(x) \wedge f_2(x) \leq y \leq f_1(x) \vee f_2(x),\ x \in [-1,1]\big\},$$

i.e., X is the set of points which lie between the graphs of the functions f_1 and f_2. Then each pretangent space $\tilde{X}_{a,\tilde{r}}$ to X at the point $a = (0, c)$ is tangent and isometric to $\mathbb{R}$.

Example 3.4. Let α be a positive real number. Write

$$X = \{(x, y, z) \in E^3 : \sqrt{y^2 + z^2} \leq x^{1+\alpha},\ x \in \mathbb{R}^+\},$$

i.e., X can be obtained by the rotation of the plane figure $\{(x, y) \in E^2 : 0 \leq y \leq x^{1+\alpha},\ x \in \mathbb{R}^+\}$ around the real axis. Then each pretangent space $\tilde{X}_{a,\tilde{r}}$ to X at the point $a = (0, 0, 0)$ is tangent and isometric to $\mathbb{R}^+$.

Example 3.5. Let $U \subseteq \mathbb{C}$ be an open set and let $F : U \to E^n,\ n \geq 2$, be an one-to-one continuous function,

$$F(x, y) = (f_1(x, y), \ldots, f_n(x, y)), \quad (x, y) \in U,$$

and let (x_0, y_0) be a marked point of U. Suppose that all f_i are differentiable at the point (x_0, y_0) and that the rank of the Jakobian matrix of F equals two at this point. Write

$$X = F(U), \quad a = F(x_0, y_0).$$

Consider the parametrized surface X as a subspace of E^n. Then every pretangent space $\Omega^X_{a,\tilde{r}}$ is tangent and isometric to $\mathbb{C}$.

Acknowledgments. The author thanks the Department of Mathematics and Statistics of the University of Helsinki for the comfortable setting in the May–June 2008 when he began to work with the initial version of this paper. This work also was partially supported by the Ukrainian State Foundation for Basic Researches, Grant Φ 25.1/055.

References

[1] L. Ambrosio, *Metric space valued functions of bounded variation* // Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **17** (1990), N 3, 439–478.

[2] L. Ambrosio, B. Kircheim, *Rectifiable sets in metric and Banach spaces* // Math. Ann., **318** (2000), N 3, 527–555.

[3] L. Ambrosio, B. Kircheim, *Currents in mertic spaces* // Acta Math. **185** (2000), N 2, 1–80.

[4] L. Ambrosio, P. Tilli, *Topics on Analysis in Metric Spaces,* Cambridge University Press, Cambridge, 2004.

[5] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces* // GAFA, Geom. func. anal. **9** (1999), 428–517.

[6] O. Dovgoshey, F. Abdullayev, M. Küçükaslan, *Tangent metric spaces to convex sets on the plane* // Reports in Math., Helsinki Univ., **481** (2008), 31 p.

[7] O. Dovgoshey, O. Martio, *Tangent spaces to metric spaces* // Reports in Math., Helsinki Univ., **480** (2008), 20 p.

[8] P. Hajlasz, *Sobolev spaces on an arbitrary metric space* // Potential Anal. **5** (1996), 403–415.

[9] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, NewYork, 2001.

[10] J. Heinonen, P. Koskela, *From local to global in quasiconformal structures* // Proc. Natl. Acad. Sci. USA **93** (1996), 554–556.

[11] J. L. Kelley, *General Topology*, D. Van Nostrand Company, Princeton, 1965.

[12] N. Shanmugalingam, *Newtonian spaces: an extention of Sobolev spaces to metric measure spaces* // Rev. Mat. Iberoamericana **16** (2000), 243–279.

Contact information

Oleksiy A. Dovgoshey

Institute of Applied Mathematics
and Mechanics of NASU,
R. Luxemburg str. 74,
Donetsk 83114,
Ukraine
E-Mail: aleksdov@mail.ru,
dovgoshey@iamm.ac.donetsk.ua