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Generalization of the H.A. Schwarz Theorem on Stability of Minimal Surfaces

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We proved two theorems on stability of minimal submanifolds in a Riemannian space, which can be included in a regular family of minimal submanifolds.

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1. Introduction

H.A. Schwarz proved stability of a minimal surface in 3-dimensional Euclidean space E^3, when this minimal surface could be included in a regular family of minimal surfaces [1]. It follows from this theorem that every compact domain on a minimal surface $z = z(x^1, x^2)$ is stable.

Notice, that the question of minimal surface stability was considered in [3–9]. The existence and applications of stable minimal surfaces were given in [10–16].

Here we give the generalizations of this theorem for the cases of minimal hypersurfaces in a Riemannian space and for 2-dimensional surfaces in 4-dimensional Riemannian space.

Let F^n be a minimal submanifold with boundary Γ in a Riemannian manifold V^N. We consider some submanifold Φ^n with the same boundary Γ, which is close to F^n in the class C^1.

We say that a compact domain D with nonempty boundary Γ on a minimal submanifold F^n is stable, if for all submanifolds Φ^n with the same boundary Γ, close to D in the class C^1 but different from D, the volume $Vol(\Phi^n)$ is grater than the volume of D

$$Vol(\Phi^n) > Vol(D).$$

Theorem 1. *If a simple connected compact domain D on an orientable minimal hypersurface F^n in the Riemannian manifold V^{n+1} can be included in a regular family of minimal hypersurfaces, then this domain D is stable.*

Theorem 2. *Let F^2 be an orientable minimal surface in 4-dimensional Riemannian manifold. Let a simple connected compact domain D on F^2 can be included in a 2-parametric regular family of minimal surfaces with integrable distribution of normal planes. Then this domain D is stable.*

Later we construct a 2-parametric family of stable minimal surfaces in Euclidean space E^4 with nonintegrable distribution of normal planes. From another side, there exists a nonstable minimal surface in the Euclidean space E^4, which can be included in the regular family of minimal surfaces. In this case the distribution of normal planes is nonintegrable, too. This example shows that the second condition in Th. 2 is essential.

2. Minimal Hypersurface

Later under F^n we understand the simple connected compact domain D.

Let F^n be included in the regular family of minimal hypersurfaces $F^n(t)$ such that $F^n(0) = F^n$. We introduce on F^n a coordinate system with the coordinates $y^1, \dots, y^n$. With the help of orthogonal trajectories to the family $F^n(t)$ we construct a coordinate system with the coordinates $y^1, \dots, y^{n+1}$ in some neighborhood of F^n. Every $F^n(t)$ corresponds to the equation $y^{n+1} = const$. The metric of the space V^{n+1} takes the following form:

$$ds^2 = \sum_{i=1}^{n} a_{ij} dy^i dy^j + a_{n+1,n+1}(dy^{n+1})^2, \tag{1}$$

where all coefficients depend on all coordinates as regular functions of the class C^1. Later $1 \le i, j \le n$. Denote $a_{n+1,n+1} = h, \quad y^{n+1} = t$.

Lemma 1. *The coefficients L_{ij} of the second quadratic form of $F^n(t)$ have the following form:*

$$L_{ij} = -\frac{1}{2h}\frac{\partial a_{ij}}{\partial t}, \quad i, j = 1, \dots, n. \tag{2}$$

This lemma is well known (see, for example, [2]).

Denote $a = |a_{ij}|$.

Lemma 2. *If every $F^n(t)$ is a minimal hypersurface, then*

$$\frac{\partial a}{\partial t} = 0. \tag{3}$$

For a minimal hypersurface the mean curvature H is equal to zero, and we have

$$H = \frac{1}{n} L_{ij} a^{ij} = 0,$$

where a^{ij} are the elements of the inverse matrix to $||a_{ij}||$. As a consequence of (2) and (3), we obtain

$$a^{ij} \frac{\partial a_{ij}}{\partial t} = 0.$$

For simplicity we denote $\frac{\partial a_{ij}}{\partial t} = a'_{ij}$. Introduce also the following vectors:

$$l_i = (a_{1i}, \dots, a_{ni}), \quad l'_i = (a'_{1i}, \dots, a'_{ni}).$$

Later we write these vectors in the form of columns and denote the determinant by [ ]. We have evidently

$$\frac{\partial a}{\partial t} = [l'_1, l_2, \dots, l_n] + \dots + [l_1, \dots, l'_n] = a \sum_{i,j}^{n} a^{ij} \frac{\partial a_{ij}}{\partial t} = 0.$$

Let Φ^n be some hypersurface, which is close to F^n in the class C^1. In this case Φ^n has one-to-one projection on F^n and in the correspondent points its tangent spaces are close. We can write the representation of Φ^n in the evident form:

$$y^{n+1} = f(y^1, \dots, y^n)$$

with the condition $f|_\Gamma = 0$. Denote later $\frac{\partial f}{\partial y^i} = f_i$. The first quadratic form $dl^2 = b_{ij} dy^i dy^j$ of Φ^n can be calculated with the help of metric form of V^{n+1}

$$dl^2 = \sum_{i,j}^{n} (a_{ij} dy^i dy^j + h^2 f_i f_j dy^i dy^j).$$

Hence

$$b_{ij} = a_{ij} + h^2 f_i f_j.$$

Introduce the vectors

$$a_i = (a_{1i}, \dots, a_{ni}), \quad m = (f_1, \dots, f_n).$$

Later these vectors are written in the form of columns. We have

$$|b_{ij}| = [a_1 + h^2 f_1 m, a_2 + h^2 f_2 m, \ldots, a_n + h^2 f_n m] = a + h^2 \sum_i^n [a_1, \ldots, m, \ldots, a_n] f_i,$$

where in sum the vector m stays on the i-th place. Taking the decomposition of every determinant in sum, we obtain

$$|b_{ij}| = a(1 + h^2 \sum_{i,j=1}^n f_i f_j a^{ij}).$$

But the matrix a^{ij} is positively determined, so

$$\sum_{i,j=1}^n f_i f_j a^{ij} \geq 0, \tag{4}$$

and the equality can be only in the case when all $f_i = 0$. Therefore, $f = const$. But $f|_\Gamma = 0$. Hence, $f = 0$. If we put a condition that Φ^n is different from F^n, then there exists some subset, where in (4) we have strong inequality. Denote by G the domain of the coordinates $y^1, \ldots, y^n$. Now we can calculate the volume of Φ^n and compare it with the volume of F^n

$$Vol(\Phi^n) = \int_G \sqrt{|b_{ij}(y^1, \ldots, f)|} dy^1 \ldots dy^n > \int_G \sqrt{a(y^1, \ldots, f)} dy^1 \ldots dy^n$$

$$= \int_G \sqrt{a(y^1, \ldots, 0)} dy^1 \ldots dy^n = Vol(F^n).$$

Hence, F^n is the stable minimal hypersurface.

The reviewer remarked that in the paper by H. Rosenberg [17] there were some statements close to the ones of Th. 1. But in the paper there was indicated only a weak stability. Besides, the consideration was too short and therefore not clear enough.

3. Minimal Surface in a 4-Dimensional Riemannian Space

Let F^2 be a minimal surface in the Riemannian 4-dimensional space V^4. We suppose that F^2 is included in a 2-parametric regular family of minimal surfaces $F^2(t, \tau)$ in some neighborhood D such that $F^2(0, 0) = F^2$. We say that it is the first family. Through every point in the neighborhood D of F^2 there goes one and only one surface from this family. Therefore, at this point the normal plane is determined, and we have a distribution of normal planes.

By the conditions of Th. 2, the distribution of these normal planes is integrable. So, there exists the second family of the surfaces which are orthogonal to the surfaces from the first family. With the help of these two families, in the same way we can construct the coordinates in the considered neighborhood. We take the coordinate system y^1, y^2 on the surface F^2 and take the surface N_0 from the second family, which goes through some point $p_0 \in F^2$. We introduce the coordinates y^3, y^4 on the surface N_0. So, if a point $p \in D$, then through this point p there goes one surface from the second family, which intersects with F^2 at the point with coordinates y^1, y^2 as well as one surface from the first family, which intersects with N_0 at the point with coordinates y^3, y^4. Hence the point p has coordinates $y^1, \dots, y^4$.

Later the Latin indexes have the value 1 or 2, and the Greek ones 3 or 4, respectively. Then the first quadratic form of V^4 will be

$$ds^2 = \sum_{i,j=1}^{2} a_{ij} dy^i dy^j + \sum_{\alpha,\beta=3}^{4} a_{\alpha\beta} dy^\alpha dy^\beta,$$

where all coefficients depend on $y^1, \dots, y^4$.

Now let some surface Φ^2 be close to F^2 and have the same boundary. We can represent Φ^2 in the following form

$$y^\alpha = f^\alpha(y^1, y^2), \quad \alpha = 3, 4,$$

and $f^\alpha = 0$ on the boundary. Denoting the metric of Φ^2 by $dl^2 = b_{ij} dy^i dy^j$ we obtain

$$b_{ij} = a_{ij} + a_{\alpha\beta} y^\alpha_{,i} y^\beta_{,j},$$

where $y^\alpha_{,i}$ are the derivatives with respect to coordinate y^i. Let $F^n(t,\tau)$ be a minimal surface, which goes through the point with coordinates y^1, y^2, y^3, y^4 on the surface Φ^2.

Lemma 3. *Determinant a of the first quadratic form of $F^2(t,\tau)$ does not depend on y^3 and y^4*

$$\frac{\partial a}{\partial y^3} = \frac{\partial a}{\partial y^4} = 0.$$

Let $\xi_k = \{\xi^\alpha_k\}, \quad k = 1, 2$, be an orthogonal basis of normal plane of $F^2(t,\tau)$ and L^k_{ij} be the coefficients of the second quadratic forms of $F^2(t,\tau)$ with respect to this basis. Following the definition of the second quadratic forms (see [2]), we have two equations for $\sigma = 3, 4$

$$y^\sigma_{,ij} + \bar{\Gamma}^\sigma_{\mu\nu} y^\mu_{,i} y^\nu_{.j} = L^k_{ij} \xi^\sigma_k,$$

where $\bar{\Gamma}^{\sigma}_{\mu\nu}$ are the Christoffel symbols of the metric of V^4. Here $y^{\sigma}_{,ij}$ are the second covariant derivatives of the function y^{σ} with respect to the metric of $F^2(t,\tau)$. We notice that every surface of this kind has the representation

$$y^3 = const, \quad y^4 = const.$$

Hence

$$y^{\sigma}_{,i} = 0, \quad y^{\sigma}_{,ij} = \frac{\partial^2 y^{\sigma}}{\partial y^i \partial y^j} - \Gamma^k_{ij} y^{\sigma}_{,k} = 0, \quad \sigma = 3,4,$$

where Γ^k_{ij} are the Christoffel symbols of the metric of $F^n(t,\tau)$.

Besides, $y^i_{,j} = 0$. So we have

$$\bar{\Gamma}^{\sigma}_{ij} = L^k_{ij}\xi^{\sigma}_k.$$

From the expressions of the Christoffel symbols we obtain

$$\frac{1}{2}a^{\sigma\beta}\left(\frac{\partial a_{\beta i}}{\partial y^j} + \frac{\partial a_{\beta j}}{\partial y^i} - \frac{\partial a_{ij}}{\partial y^{\beta}}\right) = L^k_{ij}\xi^{\sigma}_k.$$

But $a_{\alpha i} = 0$. Therefore

$$-\frac{1}{2}a^{\sigma\beta}\frac{\partial a_{ij}}{\partial y^{\beta}} = L^k_{ij}\xi^{\sigma}_k.$$

For a minimal surface we have

$$L^k_{ij}a^{ij} = 0, \quad k = 1,2.$$

Therefore, we have the system of equations

$$\frac{\partial a}{\partial y^3}a^{33} + \frac{\partial a}{\partial y^4}a^{34} = 0,$$

$$\frac{\partial a}{\partial y^3}a^{34} + \frac{\partial a}{\partial y^4}a^{44} = 0.$$

From here Lemma 3 follows.

Now we have

$$|b_{ij}| = \begin{vmatrix} a_{11} + a_{\alpha\beta}y^{\alpha}_{,1}y^{\beta}_{,1}, & a_{12} + a_{\gamma\sigma}y^{\gamma}_{,1}y^{\sigma}_{,2} \\ a_{21} + a_{\alpha\beta}y^{\alpha}_{,2}y^{\beta}_{,1}, & a_{22} + a_{\gamma\sigma}y^{\gamma}_{,2}y^{\sigma}_{,2} \end{vmatrix}.$$

Denote by $y^{\beta|i} = y^{\beta}_{,k}a^{ki}$ and

$$p^{\alpha\beta} = \begin{vmatrix} y^{\alpha}_{,1}, & y^{\beta}_{,1} \\ y^{\alpha}_{,2}, & y^{\beta}_{,2} \end{vmatrix}.$$

Then the expression of $|b_{ij}|$ can be transformed to the following one:

$$|b_{ij}| = a(1 + a_{\alpha\beta}(y^{\beta|2}y^{\alpha}_{,2} + y^{\alpha}_{,1}y^{\beta|1}) + \frac{1}{2a}a_{\alpha\beta}a_{\gamma\sigma}p^{\alpha\gamma}p^{\beta\sigma}).$$

Denote by $grady^{\alpha}$ the gradient of the function y^{α} with respect to coordinates y^1, y^2 and the metric of $F^2(t,\tau)$. So we obtain

$$|b_{ij}| = a[1 + a_{\alpha\beta}(grady^{\alpha}, grady^{\beta}) + \frac{1}{2a}(a_{33}a_{44} - (a_{34})^2)(p^{34})^2].$$

Here the brackets () at the second member in the right side denote the scalar product in the metric $a_{ij}dy^i dy^j$ at a point of $F^2(t,\tau)$. It is clear that the third term is nonnegative. Let us denote

$$A = (grady^3)^2, \quad B = (grady^3, grady^4), \quad C = (grady^4)^2.$$

The second term in the expression of $|b_{ij}|$ has the form

$$a(Aa_{33} + 2Ba_{34} + Ca_{44}).$$

We have evidently

$$AC - B^2 \geq 0, \quad a_{33}a_{44} - (a_{34})^2 \geq 0.$$

Under these conditions the expression $T = Aa_{33} + 2Ba_{34} + Ca_{44} \geq 0$.

Hence $|b_{ij}| \geq a$. If there is an equality here, then $p^{34} = 0$. In this case there exist some functions $\theta(y^1, y^2)$ and $\phi^{\alpha}(\theta)$ such that

$$y^{\alpha} = \phi^{\alpha}(\theta), \quad \alpha = 3, 4.$$

Under this condition the expression T has the form

$$T = |grad\theta|^2 \phi^{\alpha}\phi^{\beta}a_{\alpha\beta}.$$

So, from $T = 0$ we conclude that $y^{\alpha} = const, \alpha = 3, 4$. But Φ^2 is different from F^2. Therefore, we have some subset, where $|b_{ij}| > a$.

But a depends neither on y^3, nor on y^4. Therefore $Vol(\Phi^2) > Vol(F^2)$. Theorem 2 is proved.

4. One Example

Now we construct a 2-parametric family of minimal surfaces in E^4 with the nonintegrable distribution of normal planes.

Denote by x^k the coordinates in E^4 and $z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4$. Consider the family of minimal surfaces in E^4 , which are given as level surfaces of an analytical function of two complex variables

$$f(z_1, z_2) = c_1 + ic_2.$$

We have two real equations

$$\Phi_1 = Ref(z_1, z_2) = c_1,$$

$$\Phi_2 = Imf(z_1, z_2) = c_2.$$

It is a well-known fact that this surface is minimal and it is a holomorphic curve in E^4. Every compact domain is an absolutely minimized area. Normal plane is determined by the following vectors:

$$X_i = grad\Phi_i, \quad i = 1, 2.$$

Then the condition of integrability of distribution of normal planes has the following form

$$\nabla_{X_2} X_1 - \nabla_{X_1} X_2 = \lambda_1 X_1 + \lambda_2 X_2 \tag{5}$$

with some coefficients λ_k. We take the particular example

$$f = z_1 z_2 + z_1^2 + z_2^2.$$

Then evidently we obtain

$$\Phi_1 = x_1 x_3 - x_2 x_4 + x_1^2 - x_2^2 + x_3^2 - x_4^2,$$

$$\Phi_2 = x_2 x_3 + x_1 x_4 + 2x_1 x_2 + 2x_3 x_4.$$

Consequently,

$$\begin{aligned} grad\Phi_1 &= (x_3 + 2x_1, -x_4 - 2x_2, x_1 + 2x_3, -x_2 - 2x_4), \\ grad\Phi_2 &= (x_4 + 2x_2, x_3 + 2x_1, x_2 + 2x_4, x_1 + 2x_3). \end{aligned} \tag{6}$$

For the simplicity of notation denote $\Phi_1 = \Phi, \quad \Phi_2 = \Psi$. By calculation we obtain the matrices of the second derivatives for the functions Φ and Ψ

$$||\frac{\partial^2 \Phi}{\partial x_i \partial x_j}|| = \begin{pmatrix} 2, & 0, & 1, & 0 \\ 0, & -2, & 0, & -1 \\ 1, & 0, & 2, & 0 \\ 0, & -1, & 0, & -2 \end{pmatrix}, \quad ||\frac{\partial^2 \Psi}{\partial x_i \partial x_j}|| = \begin{pmatrix} 0, & 2, & 0, & 1 \\ 2, & 0, & 1, & 0 \\ 0, & 1, & 0, & 2 \\ 1, & 0, & 2, & 0 \end{pmatrix}. \tag{7}$$

Let us denote the second derivatives of functions, for example, Φ, by Φ_{ij}. Introduce the notation

$$\nabla_i = \sum_j (\Phi_{ij}\Psi_j - \Psi_{ij}\Phi_j). \tag{8}$$

With the help of (5) - (8) we obtain the system of equations for λ_i

$$\begin{aligned}
\nabla_1 &= 10x_2 + 8x_4 = \lambda_1(x_3 + 2x_1) + \lambda_2(x_4 + 2x_2),\\
\nabla_2 &= -10x_1 - 8x_3 = \lambda_1(-x_4 - 2x_2) + \lambda_2(x_3 + 2x_1),\\
\nabla_3 &= 8x_2 + 10x_4 = \lambda_1(x_1 + 2x_3) + \lambda_2(x_2 + 2x_4),\\
\nabla_4 &= -8x_1 - 10x_3 = \lambda_1(-x_2 - 2x_4) + \lambda_2(x_1 + 2x_3).
\end{aligned}$$

From the first two equations we have

$$\lambda_1 = -\frac{80(x_2x_3 - x_1x_4)}{(x_1 + 2x_3)^2 + (x_2 + 2x_4)^2}.$$

From the last two equations we find

$$\lambda_1 = -\frac{80(x_2x_3 - x_1x_4)}{(2x_1 + x_3)^2 + (2x_2 + x_4)^2}.$$

These expressions are different, so the system does not have any solution.
Hence, the distribution of normal planes is nonintegrable.

5. Minimal Surfaces in E^4 with Nonparametric Representation

Let the minimal surface $F^2 \subset E^4$ be given in the form

$$x_3 = u(x_1, x_2),$$

$$x_4 = v(x_1, x_2).$$

Later we denote derivatives in the form of u_i, u_{ij}. The functions u and v of a minimal surface satisfy two differential equations (see, for example, [10])

$$u_{11}(1 + u_2^2 + v_2^2) - 2u_{12}(u_1u_2 + v_1v_2) + u_{22}(1 + u_1^2 + v_1^2) = 0,$$

$$v_{11}(1 + u_2^2 + v_2^2) - 2v_{12}(u_1u_2 + v_1v_2) + v_{22}(1 + u_1^2 + v_1^2) = 0. \tag{9}$$

It is easy to construct the family of minimal surfaces $F^2(c_1, c_2)$

$$x_3 = u + c_1,$$

$$x_4 = v + c_2, \quad c_i = const.$$

A normal plane is determined by vectors X_1, X_2

$$X_1 = (u_1, u_2, -1, 0),$$

$$X_2 = (v_1, v_2, 0, -1).$$

The condition of integrability of the distribution of normal planes is represented by the following system of equations:

$$\begin{aligned} u_{11}v_1 + u_{12}v_2 - v_{11}u_1 - v_{12}u_2 &= \lambda_1 u_1 + \lambda_2 v_1, \\ u_{12}v_1 + u_{22}v_2 - v_{12}u_1 - v_{22}u_2 &= \lambda_1 u_2 + \lambda_2 v_2, \\ 0 &= -\lambda_1 + 0\lambda_2, \\ 0 &= 0\lambda_1 - \lambda_2. \end{aligned} \tag{10}$$

From here we have $\lambda_1 = \lambda_2 = 0$. Hence the condition (10) has the form

$$\begin{aligned} u_{11}v_1 + u_{12}v_2 &= v_{11}u_1 + v_{12}u_2, \\ u_{12}v_1 + u_{22}v_2 &= v_{12}u_1 + v_{22}u_2. \end{aligned} \tag{11}$$

Therefore, by **Theorem 2** *the minimal surface in E^4 at nonparametric representation is strongly stable if it satisfies the system of equations (11).*

The reviewer proposed to construct an example of minimal surface which would satisfy the system (9),(11).

To construct this example we put

$$u = \alpha(x_1) + \beta(x_2), \quad v = \xi(x_1) + \eta(x_2).$$

Then the system (9),(11) has the following form:

$$\begin{aligned} \alpha''(1 + \beta'^2 + \eta'^2) + \beta''(1 + \alpha'^2 + \xi'^2) &= 0, \\ \xi''(1 + \beta'^2 + \eta'^2) + \eta''(1 + \alpha'^2 + \eta'^2 &= 0, \\ \alpha''\xi' - \xi''\alpha' &= 0, \\ \beta''\eta' - \eta''\beta' &= 0, \end{aligned}$$

where $'$ (prime) denotes the derivatives of function α or $\beta, \ldots$ with respect to their arguments. From the third and forth equations we obtain

$$\alpha = C_1\xi + C_2, \quad \beta = C_3\eta + C_4,$$

where C_i are constants. After substitution α and β into the first and the second equations we conclude that $C_1 = C_3$ and the second equation can be rewritten in the form of equation with separate arguments

$$\frac{\xi''}{1+\xi'^2 a^2} = -\frac{\eta''}{1+\eta'^2 a^2} = k,$$

where $k = const, a = \sqrt{1+C_1^2}$. By integration we obtain the equation of minimal surface

$$u = \sqrt{a^2-1}v, \quad v = \frac{1}{ka^2} ln \frac{\cos(kax_2+d_2)}{\cos(kax_1+d_1)},$$

where d_i are constants. It is evidently that the surface is not determined on the whole plane x_1, x_2.

In [7] M.J. Micallef proved the following **Corollary 5.1** *A complete stable minimal surface in E^4, which is an entire graph, is holomorphic.* He indicated that in [10] R. Osserman constructed the examples of entire two-dimensional minimal graphs in E^4, which were not holomorphic with respect to any orthogonal complex structure on E^4. These graphs are unstable by Cor. 5.1. So, on this surface there exist the unstable domains. One of the Osserman surfaces has the following representation:

$$x_3 = u = \frac{1}{2}\cos\frac{x_2}{2}(e^{x_1} - 3e^{-x_1}),$$

$$x_4 = v = -\frac{1}{2}\sin\frac{x_2}{2}(e^{x_1} - 3e^{-x_1}).$$

It is possible to include this surface in the family of minimal surfaces. The distribution of normal planes is not integrable, because the equations (11) for this surface are not satisfied. Therefore, the condition of integrability of the distribution of normal planes in Th. 2 is essential.

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