UDC 517 . 9

THE CARTAN – MONGE GEOMETRIC APPROACH TO THE CHARACTERISTIC METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST AND HIGHER ORDERS

ГЕОМЕТРИЧНИЙ ПІДХІД КАРТАНА – МОНЖА ДО МЕТОДУ ХАРАКТЕРИСТИК ДЛЯ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ЧАСТИННИМИ ПОХІДНИМИ ПЕРШОГО ТА ВИЩИХ ПОРЯДКІВ

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We develop the Cartan – Monge geometric approach to the characteristic method for nonlinear partial differential equations of the first and higher orders. The Hamiltonian structure of characteristic vector fields related with nonlinear partial differential equations of the first order is analyzed, the tensor fields of special structure are constructed for defining characteristic vector fields naturally related with nonlinear partial differential equations of higher orders.

Розвинуто геометричний підхід Картана – Монжа до методу характеристик для нелінійних диференціальних рівнянь з частинними похідними першого та вищих порядків. Досліджено гамільтонову структуру характеристичних векторних полів, пов'язаних із нелінійними диференціальними рівняннями з частинними похідними першого порядку, та побудовано тензорні поля зі спеціальною структурою для визначення характеристичних полів, природно пов'язаних із нелінійними диференціальними рівняннями з частинними похідними вищих порядків.

1. Introduction: geometric backgrounds of the classical characteristic method. The characteristic method [1–4] proposed in XIX century by A. Cauchy was very nontrivially developed by G. Monge, having introduced the geometric notion of a characteristic surface, related with partial differential equations of the first order. The latter, being augmented with a very important notion of characteristic vector fields, appeared to be fundamental [4, 5–7] for the characteristic method, whose main essence consists in bringing about the problem of studying solutions to our partial differential equation to an equivalent one of studying some set of ordinary differential equations. This way of reasoning succeeded later in a development of the Hamilton–Jacobi theory, making it possible to describe a wide class of solutions to first order partial differential equations of the form

$$H(x;u,u_x)=0, \tag{1.1}$$

where $H\in C^2(\mathbb{R}^{n+1}\times\mathbb{R}^n;\mathbb{R})$, $||H_x||\neq 0$, is called a Hamiltonian function and $u\in C^2(\mathbb{R}^n;\mathbb{R})$ is an unknown function to be found. The equation (1.1)) is endowed still with a boundary value condition,

$$u|_{\Gamma_\varphi}=u_0, \tag{1.2}$$

with $u_0\in C^1(\Gamma_\varphi;\mathbb{R})$ defined on some smooth almost everywhere hypersurface

$$\Gamma_\varphi:=\{x\in\mathbb{R}^n:\varphi(x)=0,\ \ ||\varphi_x||\neq 0\}, \tag{1.3}$$

where $\varphi\in C^1(\mathbb{R}^n;\mathbb{R})$ is some smooth function on $\mathbb{R}^n$.

Following Monge's ideas, let us introduce the characteristic surface $S_H\subset\mathbb{R}^{n+1}\times\mathbb{R}^n$ as

$$S_H:=\{(x;u,p)\in\mathbb{R}^{n+1}\times\mathbb{R}^n:H(x;u,p)=0\}, \tag{1.4}$$

where we put, by definition, $p:=u_x\in\mathbb{R}^n$ for all $x\in\mathbb{R}^n$. The characteristic surface (1.4) was effectively described by Monge within his geometric approach by means of the so-called Monge cones $K\subset T(\mathbb{R}^{n+1})$ and their duals $K^*\subset T^*(\mathbb{R}^{n+1})$ [4, 6]. The corresponding differential-geometric analysis of this Monge scenario was later done by E. Cartan, who reformulated [4, 8] the geometric picture drown by Monge by means of the related compatibility conditions on dual Monge cones and the notion of an integral submanifold $\Sigma_H\subset S_H$ naturally assigned to special vector fields on the characteristic surface S_H. In particular, Cartan had introduced on S_H the differential 1-form

$$\alpha^{(1)}:=du-<p,dx>, \tag{1.5}$$

where $<\cdot,\cdot>$ is the usual scalar product in $\mathbb{R}^n$, and demanded its vanishing along the dual Monge cones $K^*\subset T^*(\mathbb{R}^{n+1})$, concerning the corresponding integral submanifold imbedding mapping

$$\pi:\Sigma_H:\rightarrow S_H. \tag{1.6}$$

This means that the 1-form

$$\pi^*\alpha_1^{(1)}:=du-<p,dx>|_{\Sigma_H}\Rightarrow 0 \tag{1.7}$$

for all points $(x;u,p)\in\Sigma_H$ of a solution surface Σ_H defined in such a way that $K^*=T^*(\Sigma_H)$. The obvious corollary from the condition (1.7) is the second Cartan condition

$$d\pi^*\alpha_1^{(1)}=\pi^*d\alpha_1^{(1)}=<dp,\wedge dx>|_{\Sigma_H}\Rightarrow 0. \tag{1.8}$$

These two Cartan's conditions (1.7) and (1.8) should be still augmented with the characteristic surface S_H invariance condition for the differential 1-form $\alpha_2^{(1)} \in \Lambda^1(S_H)$,

$$\alpha_2^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{1.9}$$

The conditions (1.7), (1.8) and (1.9), when imposed on the characteristic surface $S_H \subset \mathbb{R}^{n+1} \times \times\mathbb{R}^n$, make it possible to construct the proper characteristic vector fields on S_H, whose suitable characteristic strips [4, 6] generate the sought solution surface Σ_H. Thereby, having solved the corresponding Cauchy problem related with the boundary value conditions (1.2) and (1.3) for these characteristic vector fields, considered as ordinary differential equations on S_H, one can construct a solution to our partial differential equation (1.1). And what is interesting, this solution in many cases can be represented [1, 9] in exact functional-analytic Hopf – Lax type form. The latter is a natural consequence from the related Hamilton – Jacobi theory, whose main ingredient consists in proving the fact that the solution to our equation (1.1) is exactly the extremal value of some Lagrangian functional, naturally associated [2, 7, 10] with a given Hamiltonian function.

Below we will construct the proper characteristic vector fields for partial differential equations of the first order (1.1) on the characteristic surface S_H, generating the solution surface Σ_H as suitable characteristic strips related to the boundary conditions (1.2) and (1.3), and next generalize the Cartan – Monge geometric approach to partial differential equations of the second and higher orders.

2. The characteristic vector field method: first order partial differential equations. Consider on the surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ a characteristic vector field $K_H : S_H \to T(S_H)$ in the form

$$\left.\begin{aligned} \frac{dx}{d\tau} &= a_H(x;u,p) \\ \frac{dp}{d\tau} &= b_H(x;u,p) \\ \frac{du}{d\tau} &= c_H(x;u,p) \end{aligned}\right\} := K_H(x;u,p), \tag{2.1}$$

where $\tau \in \mathbb{R}$ is a suitable evolution parameter and $(x;u,p) \in S_H$. Since, owing to the Cartan – Monge geometric approach, conditions (1.7), (1.8) and (1.9) hold along the solution surface Σ_H, we can satisfy them, applying the interior differentiation $i_{K_H}\colon \Lambda(S_H) \to \Lambda(S_H)$ [10 – 12] to the corresponding differential forms $\alpha_1^{(1)}$ and $d\alpha_1^{(1)}$,

$$i_{K_H}\alpha_1^{(1)} = 0, \quad i_{K_H} d\alpha_1^{(1)} = 0.$$

As a result of simple calculations one finds that

$$\begin{gathered} c_H = < p, a_H >, \\ \beta^{(1)} := < b_H, dx > - < a_H, dp > |_{S_H} = 0 \end{gathered} \tag{2.2}$$

for all points $(x; u, p) \in S_H$. The obtained 1-form $\beta^{(1)} \in \Lambda^1(S_H)$ must be, evidently, compatible with the defining invariance condition (1.9) on S_H. This means that there exists a scalar function $\mu \in C^1(S_H; \mathbb{R})$ such that the condition

$$\mu\alpha_2^{(1)} = \beta^{(1)}$$

holds on S_H. This gives rise to the following final relationships:

$$a_H = \mu\frac{\partial H}{\partial p}, \quad b_H = -\mu\left(\frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u}\right),$$

which, together with the first equality of (2.2) complete the search for the structure of the characteristic vector fields $K_H : S_H \to T(S_H)$,

$$K_H = \left(\mu\frac{\partial H}{\partial p}; \left\langle p, \mu\frac{\partial H}{\partial p}\right\rangle, -\mu\left(\frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u}\right)\right)^{\intercal}.$$

Now we can pose a suitable Cauchy problem for the equivalent set of ordinary differential equations (2.1) on S_H as follows:

$$\frac{dx}{d\tau} = \mu\frac{\partial H}{\partial p} : x\big|_{\tau 0} = x_0(x) \in \Gamma_\varphi, \quad x\big|_{\tau=t(x)} = x \in \mathbb{R}^n\backslash\Gamma_\varphi,$$

$$\frac{du}{d\tau} = \left\langle p, \mu\frac{\partial H}{\partial p}\right\rangle \; : u|_{\tau=0} = u_0(x_0(x)), \;\; u\big|_{\tau=t(x)} = u(x), \tag{2.3}$$

$$\frac{dp}{d\tau} = -\mu\left(\frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u}\right) : p\big|_{\tau=0} = \frac{\partial u_0(x_0(x))}{\partial x_0},$$

where $x_0(x) \in \Gamma_\varphi$ is the intersection point of the corresponding vector field orbit, starting at a fixed point $x \in \mathbb{R}^n\backslash\Gamma_\varphi$, with the boundary hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ at the moment of "time"$\tau = t(x) \in \mathbb{R}$. As a result of solving the corresponding "inverse"Cauchy problem (2.3) one finds the following exact functional-analytic expression for a solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ to the boundary-value problem (1.2) and (1.3):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \bar{\mathcal{L}}(x; u, p)d\tau, \tag{2.4}$$

where, by definition,

$$\bar{\mathcal{L}}(x; u, p) := \left\langle p, \mu\frac{\partial H}{\partial p}\right\rangle$$

for all $(x; u, p) \in S_H$. If the Hamiltonian function $H : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}$ is nondegenerate, that is $H\mathrm{ess}H := \det(\partial^2 H/\partial p\partial p\} \neq 0$ for all $(x; u, p) \in S_H$, then the first equation of (2.3) can be solved with respect to the variable $p \in \mathbb{R}^n$ as

$$p = \psi(x, \dot{x}; u)$$

for $(x, \dot{x}) \in T(\mathbb{R}^n)$, where $\psi : T(\mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ is some smooth mapping. This gives rise to the following canonical expression of the Lagrangian function

$$\mathcal{L}(x, \dot{x}; u) := \bar{\mathcal{L}}(x; u, p)|_{p=\psi(x,\dot{x};u)},$$

and to the resulting solution (2.4),

$$u(x) = u_0(x_0(x)) + \int\limits_0^{t(x)} \mathcal{L}(x, \dot{x}; u) d\tau. \tag{2.5}$$

The functional-analytic form (2.5) is already proper for constructing its equivalent Hopf – Lax type form, being very important for finding so called generalized solutions [1, 5, 13] to the partial differential equation (1.1). This aspect of the Cartan – Monge geometric approach we suppose to analyze in detail elsewhere.

3. The characteristic vector field method: second order partial differential equations. Assume we are given a second order partial differential equation

$$H(x; u, u_x, u_{xx}) = 0, \tag{3.1}$$

where the solution is $u \in C^2(\mathbb{R}^n; \mathbb{R})$ and the generalized "Hamiltonian"function satisfies $H \in c \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n); \mathbb{R})$. Putting $p^{(1)} := u_x, p^{(2)} := u_{xx},\ x \in \mathbb{R}^n$, one can construct within the Cartan – Monge generalized geometric approach the characteristic surface

$$S_H := \left\{ (x; u, p^{(1)}, p^{(2)}) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n) : H(x; u, p^{(1)}, p^{(2)}) = 0 \right\} \tag{3.2}$$

and a suitable Cartan's set of differential one- and two-forms:

$$\begin{gathered} \alpha_1^{(1)} := du - < p^{(1)}, dx > |_{\Sigma_H} \Rightarrow 0, \\ d\alpha_1^{(1)} := < dx, \wedge dp^{(1)} > |_{\Sigma_H} \Rightarrow 0, \\ \alpha_2^{(1)} := dp^{(1)} - < p^{(2)}, dx > |_{\Sigma_H} \Rightarrow 0, \\ d\alpha_2^{(1)} := < dx, \wedge dp^{(2)} > |_{\Sigma_H} \Rightarrow 0, \end{gathered} \tag{3.3}$$

vanishing on a corresponding solution submanifold $\Sigma_H \subset S_H$. The set of differential forms (3.3) should be augmented with the characteristic surface S_H invariance differential 1-form

$$\alpha_3^{(1)} := dH|_{S_H} \Rightarrow 0, \tag{3.4}$$

vanishing, respectively, an the characteristic surface S_H.

Let the characteristic vector field $K_H : S_H \to T(S_H)$ on S_H be given by the expressions

$$\left.\begin{aligned} \frac{dx}{d\tau} &= a_H(x; u, p^{(1)}, p^{(2)}) \\ \frac{du}{d\tau} &= c_H(x; u, p^{(1)}, p^{(2)}) \\ \frac{dp^{(1)}}{d\tau} &= b_H^{(1)}(x; u, p^{(1)}, p^{(2)}) \\ \frac{dp^{(2)}}{d\tau} &= b_H^{(2)}(x; u, p^{(1)}, p^{(2)}) \end{aligned}\right\} := K_H(x; u, p^{(1)}, p^{(2)}), \tag{3.5}$$

for all $(x; u, p^{(1)}, p^{(2)}) \in S_H$. To find the vector field (3.5) it is necessary to satisfy the Cartan compatibility conditions in the following geometric form:

$$\begin{gathered} i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \end{gathered} \tag{3.6}$$

where, as above, $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ is the internal differentiation of differential forms along the vector field $K_H : S_H \to T(S_H)$. As a result of conditions (3.6) one finds that

$$\begin{gathered} c_H = < p^{(1)}, a_H >, \quad b_H^{(1)} = < p^{(2)}, a_H >, \\ \beta_1^{(1)} := < a_H, dp^{(1)} > - < b_H^{(1)}, dx > |_{S_H} \Rightarrow 0, \\ \beta_2^{(1)} := < a_H, dp^{(2)} > - < b_H^{(2)}, dx > |_{S_H} \Rightarrow 0, \end{gathered} \tag{3.7}$$

being satisfied on S_H identically. The conditions (3.7) must be augmented still with the characteristic surface invariance condition (3.4). Notice now that $\beta_1^{(1)} = 0$ owing to the second condition of (3.7) and the third condition of (3.3). Thus, we need now to make compatible the basic scalar 1-form (3.4) with the vector-valued 1-form $\beta_2^{(1)} \in \Lambda(S_H) \otimes \mathbb{R}^n$. To do this let us construct, making use of the $\beta_2^{(1)}$, the following parametrized set of, respectively, scalar 1-forms:

$$\beta_2^{(1)}[\mu] := < \bar{\mu}^{(1|0)} \otimes a_H, dp^{(2)} > - < b_H^{(2)}, \bar{\mu}^{(1|0)} \otimes dx > |_{S_H} \Rightarrow 0, \tag{3.8}$$

where $\bar{\mu}^{(1|0)} \in C^1(S_H; \mathbb{R}^n)$ is any smooth vector-valued function on S_H. The compatibility condition for (3.8) and (3.4) gives rise to the next relationships:

$$\begin{gathered} \bar{\mu}^{(1|0)} \otimes a_H = \frac{\partial H}{\partial p^{(2)}}, \\ < \bar{\mu}^{(1|0)}, b_H^{(2)} > = -\left(\frac{\partial H}{\partial x} + p^{(1)}\frac{\partial H}{\partial u} + \left\langle \frac{\partial H}{\partial p^{(1)}}, p^{(2)} \right\rangle\right), \end{gathered} \tag{3.9}$$

holding on S_H. Take now a dual vector function $\mu^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ such that $<\mu^{(1|0)}, \bar{\mu}^{(1|0)}>=1$ for all points of S_H. Then from (3.9) one easily finds that

$$a_H = \left\langle \mu^{(1|0)}, \frac{\partial H}{\partial p^{(2)}} \right\rangle,$$
$$b_H^{(2)} = -\mu^{(1|0),*} \otimes \left(\frac{\partial H}{\partial x} + p^{(1)} \frac{\partial H}{\partial u} + \left\langle \frac{\partial H}{\partial p^{(1)}}, p^{(2)} \right\rangle \right). \tag{3.10}$$

Combining now the first two relationships of (3.7) with the found above relations (3.10) we get a final form for the characteristic vector field (3.5),

$$K_H = \Big(a_H; < p^{(1)}, a_H >, < p^{(2)}, a_H >, -\mu^{(1|0),*} \otimes$$
$$\otimes \left(\frac{\partial H}{\partial x} + p^{(1)} \frac{\partial H}{\partial u} + \left\langle \frac{\partial H}{\partial p^{(1)}}, p^{(2)} \right\rangle \right) \Big)^{\intercal}, \tag{3.11}$$

where $a_H = <\mu^{(1|0)}, \partial H/\partial p^{(2)}>$ and $\mu^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ is some smooth vector-valued function on S_H. Thereby, we can construct, as before, solutions to our partial second order differential equation (3.1) by means of solving the equivalent Cauchy problem for the set of ordinary differential equations (3.5) on the characteristic surface S_H.

4. The characteristic vector field method: partial differential equations of higher orders. Consider a general nonlinear partial differential equation of higher order $m \in \mathbb{Z}_+$,

$$H(x; u, u_x, u_{xx}, ..., u_{mx}) = 0, \tag{4.1}$$

where it is assumed that $H \in C^2(\mathbb{R}^{n+1} \times (\mathbb{R}^n)^{\otimes m(m+1)/2}; \mathbb{R})$. Within the generalized Cartan – Monge geometric characteristic method, we need to construct the related characteristic surface S_H as

$$S_H := \Big\{ (x; u, p^{(1)}, p^{(2)}, ..., p^{(m)}) \in$$
$$\in \mathbb{R}^{n+1} \times (\mathbb{R}^n)^{\otimes m(m+1)/2} : H(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)}) = 0 \Big\}, \tag{4.2}$$

where we put $p^{(1)} := u_x \in \mathbb{R}^n$, $p^{(2)} := u_{xx} \in \mathbb{R}^n \otimes \mathbb{R}^n$, ..., $p^{(m)} \in (\mathbb{R}^n)^{\otimes m}$ for $x \in \mathbb{R}^n$. The corresponding solution manifold $\Sigma_H \subset S_H$ is defined naturally as the integral submanifold of the following set of one- and two-forms on S_H :

$$\begin{aligned}
\alpha_1^{(1)} &:= du- < p^{(1)}, dx > |_{\Sigma_H} \Rightarrow 0, \\
d\alpha_1^{(1)} &:= < dx, \wedge dp^{(1)} > |_{\Sigma_H} \Rightarrow 0, \\
\alpha_2^{(1)} &:= dp^{(1)}- < p^{(2)}, dx > |_{\Sigma_H} \Rightarrow 0, \\
d\alpha_2^{(1)} &:= < dx, \wedge dp^{(2)} > |_{\Sigma_H} \Rightarrow 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_m^{(1)} &:= dp^{(m-1)}- < p^{(m)}, dx > |_{\Sigma_H} \Rightarrow 0, \\
d\alpha_m^{(1)} &:= < dx, \wedge dp^{(m)} > |_{\Sigma_H} \Rightarrow 0,
\end{aligned} \tag{4.3}$$

vanishing on Σ_H. The set of differential forms (4.3) is augmented with the determining characteristic surface S_H invariance condition

$$\alpha_{m+1}^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{4.4}$$

Proceed now to constructe the characteristic vector field $K_H : S_H \to T(S_H)$ on the hypersurface S_H within the developed above generalized characteristic method. Take the expressions

$$\left.\begin{aligned}
\frac{dx}{d\tau} &= a_H(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)}) \\
\frac{du}{d\tau} &= c_H(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)} \\
\frac{dp^{(1)}}{d\tau} &= b_H^{(1)}(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)}) \\
\frac{dp^{(2)}}{d\tau} &= b_H^{(2)}(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)}) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{dp^{(m)}}{d\tau} &= b_H^{(m)}(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)})
\end{aligned}\right\} := K_H(x; u, p^{(1)}, p^{(2)}), \tag{4.5}$$

for $(x; u, p^{(1)}, p^{(2)}, ..., p^{(m)}) \in S_H$ and satisfy the corresponding Cartan compatibility conditions in the following geometric form:

$$
\begin{aligned}
& i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \\
& i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& i_{K_H}\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0.
\end{aligned}
\tag{4.6}
$$

As a result of suitable calculations in (4.6) one gets the following expressions:

$$
\begin{aligned}
& c_H = < p^{(1)}, a_H >, \quad b_H^{(1)} = < p^{(2)}, a_H >, \\
& \beta_1^{(1)} := < a_H, dp^{(1)} > - < b_H^{(1)}, dx > |_{S_H} \Rightarrow 0, \\
& \beta_2^{(1)} := < a_H, dp^{(2)} > - < b_H^{(2)}, dx > |_{S_H} \Rightarrow 0, \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& \beta_m^{(1)} := < a_H, dp^{(m)} > - < b_H^{(m)}, dx > |_{S_H} \Rightarrow 0,
\end{aligned}
\tag{4.7}
$$

being satisfied on S_H identically.

It is now easy to see that all of 1-forms $\beta_j^{(1)} \in \Lambda^1(S_H)\otimes(\mathbb{R}^n)^{\otimes j}, \; j = \overline{1, m-1}$, are vanishing identically on S_H owing to the relationships (4.3). Thus, as a result, we obtain the only relationship

$$
\beta_m^{(1)} := < a_H, dp^{(m)} > - < b_H^{(m)}, dx > |_{S_H} \Rightarrow 0, \tag{4.8}
$$

which should be compatibly combined with that of (4.4). To do this suitably with the tensor structure of the 1-forms (4.8), we take a smooth tensor function $\bar{\mu}^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ on S_H and construct the parametrized set of scalar 1-forms

$$
\beta_m^{(1)}[\mu] := < \bar{\mu}^{(m-1|0)} \otimes a_H, dp^{(m)} > - < b_H^{(m)}, \bar{\mu}^{(m-1|0)} \otimes dx > |_{S_H} \Rightarrow 0, \tag{4.9}
$$

which can be now identified with the 1-form (4.4). This gives rise right away to the relationships

$$
\begin{aligned}
& \bar{\mu}^{(m-1|0)} \otimes a_H = \frac{\partial H}{\partial p^{(m)}}, \\
& < \bar{\mu}^{(m-1|0)}, b_H^{(m)} > = -\left(\frac{\partial H}{\partial x} + p^{(1)}\frac{\partial H}{\partial u} + \left\langle \frac{\partial H}{\partial p^{(1)}}, p^{(2)} \right\rangle + \ldots + \left\langle \frac{\partial H}{\partial p^{(m-1)}}, p^{(m)} \right\rangle\right),
\end{aligned}
\tag{4.10}
$$

holding on S_H. Now we can take a dual tensor-valued function $\mu^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ on S_H such that $< \mu^{(m-1|0)}, \bar{\mu}^{(m-1|0)} > = 1$ for all points of S_H. Then from (4.10) we easily get the sought unkown expressions

$$a_H = \left\langle \mu^{(m-1|0)}, \frac{\partial H}{\partial p^{(m)}} \right\rangle,$$
$$b_H^{(m)} = -\mu^{(1|0),*} \otimes \left(\frac{\partial H}{\partial x} + p^{(1)} \frac{\partial H}{\partial u} + \left\langle \frac{\partial H}{\partial p^{(1)}}, p^{(2)} \right\rangle + \ldots + \left\langle \frac{\partial H}{\partial p^{(m-1)}}, p^{(m)} \right\rangle \right). \tag{4.11}$$

The obtained above result (4.11), combined with suitable expressions from (4.7), gives rise to the following final form for the characteristic vector field (4.5):

$$K_H = \Bigg(a_H; < p^{(1)}, a_H >, < p^{(2)}, a_H >, \ldots, < p^{(m)}, a_H >,$$
$$- \mu^{(m-1|0),*} \otimes \left(\frac{\partial H}{\partial x} + p^{(1)} \frac{\partial H}{\partial u} + \right.$$
$$\left. + \left\langle \frac{\partial H}{\partial p^{(1)}}, p^{(2)} \right\rangle + \ldots + \left\langle \frac{\partial H}{\partial p^{(m-1)}}, p^{(m)} \right\rangle \right) \Bigg)^{\intercal},$$

where $a_H = < \mu^{(m-1|0)}, \partial H / \partial p^{(m)} >$ and $\mu^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ is some smooth tensor-valued function on S_H. The resulting set (4.5) of ordinary differential equations on S_H allows to construct exact solutions to our partial differential equation (4.1) in a suitable functional-analytic form, being often very useful for analyzing its properties important for applications. On these and related questions we plan to stop in detail elsewhere later.

5. Acknowledgements. The authors are very grateful to their friends and collegues from Dept. of Mathematics at AGH, Krakow, for helpful and constructive discussions of the problems treated in the article.

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Received 25.09.2006