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Kähler Geometry and Burgers' Vortices

We study the Navier-Stokes and Euler equations of incompressible hydrodynamics. Taking the divergence of the momentum equation leads, as usual, to a Poisson equation for the pressure: in this paper we study this equation in two spatial dimensions using Monge-Ampère structures. In two dimensional flows where the Laplacian of the pressure is positive, a Kähler geometry is described on the phase space of the fluid; in regions where the Laplacian of the pressure is negative, a product structure is described. These structures can be related to the ellipticity and hyperbolicity (respectively) of a Monge-Ampère equation. We then show how this structure can be extended to a class of canonical vortex structures in three dimensions.

Keywords: *Monge-Ampère equations, Navier-Stokes equations, Euler equations,*

1. Introduction

A considerable literature exists on the dynamics of vortex tubes, particularly on the topic of the Burgers' vortex (Burgers 1948). In an influential paper that contains substantial references, Moffatt, *et al.* (1994) coined the simile *Burgers' vortices are the sinews of turbulence* and thus identified the heart of the problem; that is, these filament-like vortices stitch together the large-scale anatomy of vortical dynamics. Despite the twisting, bending and tangling they undergo, they appear to be the preferred states of Navier-Stokes turbulent flows. The purpose of this paper is to investigate the enduring subject of turbulence in the light of the recent advances made in the geometry of Kähler manifolds. We believe that evidence exists that suggests that turbulent vortical dynamics may be governed by geometric principles.

The incompressible Navier-Stokes equations, in two dimensions, are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\rho}\nabla P = \nu\nabla^2\boldsymbol{u}\,, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0\,. \tag{2}$$

Here, $\boldsymbol{u}(\boldsymbol{x},t)$ is the fluid velocity, the pressure and density of the fluid are denoted by $P(\boldsymbol{x},t)$ and $\rho(\boldsymbol{x},t)$ respectively, ∇ is the gradient operator and ν is the viscosity; in the inviscid case when $\nu = 0$ we have the Euler equations. The constraint imposed by the incompressibility condition

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3}$$

is very severe. It means that the convective derivative of the density vanishes. In turn this means that an initially homogeneous (constant density) fluid remains constant for all time;

$$\rho(\boldsymbol{x},0) = \rho(\boldsymbol{x},t) = \text{constant}.$$

Hereafter this density is taken as unity. Moreover, when (3) is applied across (1) it demands that velocity derivatives and the pressure are related by a Poisson equation

$$-\nabla^2 P = u_{i,j}u_{j,i}\,, \tag{4}$$

where ∇^2 is the Laplace operator (and the summation convention is used with $i, j = 1, 2$).

Burgers' vortices are examples of a *two-and-a-half-dimensional flow*, which can be defined by the class of velocity fields written as (Gibbon *et al.* (1999))

$$\boldsymbol{u}(x, y, z, t) = \{u_1(x, y, t),\ u_2(x, y, t),\ z\gamma(x, y, t)\}\ . \tag{5}$$

This flow is linear in z in the $\hat{\boldsymbol{k}}$-direction; thus it is stretching (or compressing) in that direction but is linked dynamically to its cross-sectional part. The nomenclature refers to the fact that it is neither fully two- nor three-dimensional but lies somewhere in-between[1] Its components must also satisfy the divergence-free condition

$$u_{1,x}(x, y, t) + u_{2,y}(x, y, t) + \gamma(x, y, t) = 0\,. \tag{6}$$

The class of velocity fields in equation (5), first used in Ohkitani & Gibbon (2000), is a more general classification of Burgers-type solutions and contains the specific form of the Burgers vortex solutions used in Moffatt *et al.* (1994). Included in (5) are the Euler solutions of Stuart (1987, 1991), in which u_1 and u_2 are also linear in x (say) leaving the dependent variables to be functions of y and t. Then stretching can occur in two directions thereby producing sheet-like vortical solutions.

[1]In the case of the three-dimensional Euler equations data can become rough very quickly; our manipulations in this paper are therefore purely formal. In fact it has been shown numerically in Ohkitani & Gibbon (2000) and analytically in Constantin (2000) that solutions of the type in (5) can become singular in a finite time, which is consistent with observations that vortex tubes have finite life-times; the singularity is not real in the full three-dimensional Euler sense as it has infinite energy but indicates that the flow will not sustain the structure (5) for more than a finite time. For the possibility of a real Euler singularity see Kerr (1983) and Kerr (2005).

The differences between the three-dimensional and two-dimensional Navier-Stokes equations are fundamental because the vortex stretching term $\omega \cdot \nabla \boldsymbol{u}$ in the equation for vorticity is present in the former but absent in the latter. Nevertheless, Lundgren (1982) has shown that for two-and-a-half-dimensional flows of the type

$$u_1 = -\tfrac{1}{2}x\gamma(t) + \psi_y \; u_2 = -\tfrac{1}{2}y\gamma(t) - \psi_x \; u_3 = z\gamma(t) \tag{7}$$

can be mapped into solutions of the two-dimensional Navier-Stokes equations with $\psi(x, y, t)$ as a stream function.

To investigate the geometric structure behind these solutions requires certain technical tools; these are outlined in §2 of this paper. The constraint in equation (4) is the basis of our geometric arguments, and because it is true for both the Navier-Stokes and Euler equations, the conclusions reached in this paper are valid for both cases. It is, of course, to be expected that any geometric structure should be independent of viscosity. From now on when we refer to the Navier-Stokes equations it should be implicitly understood that the Euler equations are also included. The Kähler structure for the two-dimensional Navier-Stokes equations is described in §3 and then formulated for two-and-a-half-dimensional Navier-Stokes flows in §4. Our results show that the necessary condition on the pressure for a Kähler structure to exist in two spatial dimensions (with time entering only as a parameter) for the two-dimensional Navier-Stokes equations is $\nabla^2 P > 0$. This constraint is highly restrictive: by no means all two-dimensional Navier-Stokes flows would conform to it. More promising is the equivalent condition for two-and-a-half dimensional solutions of type (7). Theorem 1 in §4 shows that these two-and-a-half-dimensional solutions have an underlying Kähler structure if $\nabla^2 P$ has a very large negative lower bound, thus associating a wide set of 'thin' solutions with the Kähler property. While the existence of a negative finite lower bound suggests some work still needs to be done, this result implies that preferred vortical thin sets have a connection with a

Kähler geometric structure that deserves further study. A different line of enquiry by Gibbon (2002) has shown that the three-dimensional Euler equations has a quaternionic structure in the dependent variables.

The work of Roubtsov & Roulstone (1997, 2001) showed how Kähler structures arise in atmosphere and ocean dynamics. The dynamics of cyclones and anti-cyclones, and ocean eddies, is strongly constrained by the rotation of the Earth, and this feature is key to the ubiquity of almost-complex structures on the phase space of so-called *balanced models* (see McIntyre & Roulstone (2002)). Central to this work is the fact that equations of Monge-Ampère type govern the balance between the wind and pressure distributions. In the context of balanced models, almost-complex structures are the signature of the slowly-evolving, large-scale weather systems in our atmosphere (and in the large-scale eddies in the oceans). The solutions considered in this paper represent the ideal cases of straight tubes or flat sheets; in reality, as indicated in the first paragraph of this section, these vortical objects constantly undergo processes of bending and tangling. Speculatively, it is possible that once this process is underway, solutions move from living on a Kähler manifold in two complex dimensions to other complex manifolds of a higher dimension, although this is a much more difficult mathematical problem to address and further results are presented by Roulstone *et al.* (2008).

This present work emerged from the observation that (4), when used in the context of atmospheric dynamics and modified by a term representing the rotation of the Earth, is often studied as part of a system of balance conditions for the fluid velocity $\boldsymbol{u}$ when the pressure field is given (and thus it is often considered a generalization of the notion of geostrophic balance). In the case of incompressible flows, this leads to a Monge-Ampère equation for a stream function (Charney 1955), and this was the trigger for our current investigation.

2. Differential Forms and Monge–Ampère Equations

In this section we prepare some tools that enable us to study certain partial differential equations arising in incompressible Navier-Stokes flows from the point-of-view of differential geometry. An introduction to the application of some basic elements of exterior calculus to the study of partial differential equations, with application to fluid dynamics, can be found in McIntyre & Roulstone (2002). Here, we shall draw largely on Lychagin *et al.* (1993) and Banos (2002).

A Monge–Ampère equation is a second order partial differential equation, which, for instance in two variables, can be written as follows:

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + E = 0, \tag{8}$$

where A, B, C and D are smooth functions of $(x, y, \phi, \phi_x, \phi_y)$. This equation is elliptic if

$$AC - 4B^2 - DE > 0. \tag{9}$$

In dimension n, a Monge–Ampère equation is a linear combination of the minors of the hessian matrix[1] of ϕ. We shall refer to such equations as *symplectic* Monge–Ampère equations when the coefficients A, B, C and D are smooth functions of $(x, y, \phi_x, \phi_y) \in T^*\mathbb{R}^2$; i.e. they are smooth functions on the quotient bundle $J^1\mathbb{R}^2/J^0\mathbb{R}^2$, where $J^1\mathbb{R}^2$ denotes the manifold of 1-jets on $\mathbb{R}^2$.

2.1. **Monge–Ampère operators.**

Lychagin (1979) has proposed a geometric approach to these equations, using differential forms on the cotangent space (i.e. the phase space). The idea is to associate with a form[2] $\omega \in \bigwedge^n(T^*\mathbb{R}^n)$, where $\bigwedge^n$ denotes the

[1]We denote by hess(ϕ) the determinant of the hessian matrix of ϕ. For example, in two variables, hess(ϕ) $= \phi_{xx}\phi_{yy} - \phi_{xy}^2$.

[2]The use of the Greek letters ω and Ω is common in differential geometry; these symbols should not be confused with the fluid vorticity vector $\boldsymbol{\omega}$.

space of differential n-forms on $T^*\mathbb{R}^n$, the Monge–Ampère equation $\triangle_\omega = 0$, where $\triangle_\omega : C^\infty(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)$ is the differential operator defined by

$$\triangle_\omega(\phi) = (\mathrm{d}\phi)^*\omega\,,$$

and $(\mathrm{d}\phi)^*\omega$ denotes the restriction of ω to the graph of the differential $\mathrm{d}\phi : \mathbb{R}^n \to T^*\mathbb{R}^n$ of ϕ. A form $\omega \in \bigwedge^n(T^*\mathbb{R}^n)$ is said to be effective if $\omega \wedge \Omega = 0$, where Ω is the canonical symplectic form on $T^*\mathbb{R}^n$. Then the so called Hodge-Lepage-Lychagin theorem tells us that this correspondence between Monge–Ampère equations and effective forms is one to one. For instance, the Monge–Ampère equation (8) is associated with the effective form

$$\omega = A\mathrm{d}p\wedge\mathrm{d}y + B(\mathrm{d}x\wedge\mathrm{d}p - \mathrm{d}y\wedge\mathrm{d}q) + C\mathrm{d}x\wedge\mathrm{d}q + D\mathrm{d}p\wedge\mathrm{d}q + E\mathrm{d}x\wedge\mathrm{d}y,$$

where (x, y, p, q) is the symplectic system of coordinates of $T^*\mathbb{R}^2$, and on the graph of $\mathrm{d}\phi$, $p = \phi_x$ and $q = \phi_y$. So, for example, if we pull-back the one-form $\mathrm{d}p$ to the base space, we have

$$\mathrm{d}p = \phi_{xx}\mathrm{d}x + \phi_{xy}\mathrm{d}y,$$

and then

$$\mathrm{d}p \wedge \mathrm{d}q = \mathrm{hess}(\phi)\mathrm{d}x \wedge \mathrm{d}y,$$

where we have also used the skew symmetry of the wedge product.

2.2. **Monge–Ampère structures.** The geometry of Monge-Ampère equations in n variables can be described by a pair

$$(\Omega, \omega) \in \bigwedge^2(T^*\mathbb{R}^n) \times \bigwedge^n(T^*\mathbb{R}^n)$$

such that

(1) Ω is symplectic; that is, nondegenerate ($\Omega \wedge \Omega \neq 0$) and closed ($\mathrm{d}\Omega = 0$)
(2) ω is effective; that is, $\omega \wedge \Omega = 0$.

Such a pair is called a Monge–Ampère structure. In four dimensions (that is $n = 2$), this geometry can be either complex or real and this distinction coincides with the usual distinction between elliptic and hyperbolic, respectively, for differential equations in two variables. Indeed, when $\omega \in \bigwedge^2(T^*\mathbb{R}^2)$ is a non-degenerate 2-form ($\omega \wedge \omega \neq 0$), one can associate with the Monge–Ampère structure $(\Omega, \omega) \in \bigwedge^2(T^*\mathbb{R}^2) \times \bigwedge^2(T^*\mathbb{R}^2)$ the tensor I_ω defined by

$$\frac{1}{\sqrt{|\mathrm{pf}(\omega)|}}\omega(\cdot,\cdot) = \Omega(I_\omega \cdot, \cdot)$$

where $\mathrm{pf}(\omega)$ is the pfaffian of ω: $\omega \wedge \omega = \mathrm{pf}(\omega)(\Omega \wedge \Omega)$. Thus, for the effective form ω associated with the Monge–Ampère equation (8), the $\mathrm{pf}(\omega)$ coincides with (9). This tensor is either an almost complex structure or an almost product structure:

(1) $\triangle_\omega$ is elliptic $\Leftrightarrow \mathrm{pf}(\omega) > 0 \Leftrightarrow I_\omega^2 = -Id$
(2) $\triangle_\omega$ is hyperbolic $\Leftrightarrow \mathrm{pf}(\omega) < 0 \Leftrightarrow I_\omega^2 = Id$

and it is integrable if and only if

$$\mathrm{d}\left(\frac{1}{\sqrt{|\mathrm{pf}(\omega)|}}\omega\right) = 0. \tag{10}$$

Given a pair of two-forms (Ω, ω) on $T^*\mathbb{R}^n$, such that $\omega \wedge \Omega = 0$, then by fixing the volume form in terms of Ω, we can define a pseudo-riemannian metric g_ω in terms of the quadratic form

$$g_\omega(X,Y) = \frac{\iota_X\Omega \wedge \iota_Y\omega + \iota_Y\Omega \wedge \iota_X\omega}{\Omega \wedge \Omega} \wedge \pi^*(vol), \quad X, Y \in T\mathbb{R}^n, \tag{11}$$

where vol is the volume form on $\mathbb{R}^n$ and $\pi : T^*\mathbb{R}^n \mapsto \mathbb{R}^n$. We can now identify our Monge–Ampère equation given by ω with an almost Kähler structure given by the triple $(\mathbb{R}^n, g_\omega, I_\omega)$ via

$$\omega(X,Y) \equiv g_\omega(I_\omega X, Y). \tag{12}$$

One can go further and in particular, in $\mathbb{R}^4$, one can show how a natural hyper-Kähler structure emerges by identifying points in $\mathbb{R}^4$ with quaternions $\ell \in \mathbb{H}$. This structure was utilized by Roubtsov & Roulstone (1997, 2001) in their description of nearly geostrophic models of meteorological flows.

3. Two-dimensional Navier-Stokes flows

If the flow described by (1) is two-dimensional, and if the fluid is incompressible, then we can represent the velocity by

$$\boldsymbol{u} = \boldsymbol{k} \times \nabla\psi, \tag{13}$$

where $\psi(x, y, t)$ is a stream function and $\boldsymbol{k}$ is the local unit vector in the vertical. If we substitute this for the velocity in (4), we get

$$\nabla^2 P = -2(\psi_{xy}^2 - \psi_{xx}\psi_{yy}) . \tag{14}$$

This is an equation of Monge–Ampère type (cf. (8)) for ψ, given $\nabla^2 P$, and it is an elliptic equation if

$$\nabla^2 P > 0 \tag{15}$$

(cf. (8) and (9) with $E = \nabla^2 P, D = -2, A = B = C = 0$; see also Larchevêque 1990, 1993). We use, once again, the usual notation for coordinates on $T^*\mathbb{R}^2$, $p = \psi_x, q = \psi_y$, and then we can express (14) geometrically on the graph of $\mathrm{d}\psi$ via

$$\omega_{2d} \equiv \nabla^2 P\, \mathrm{d}x \wedge \mathrm{d}y - 2\mathrm{d}p \wedge \mathrm{d}q; \quad \triangle_{\omega_{2d}} = 0. \tag{16}$$

In these coordinates $\Omega \equiv \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q$, and on the graph of $\mathrm{d}\psi$

$$\triangle_\Omega = 0, \tag{17}$$

which says simply that $\psi_{xy} = \psi_{yx}$. Equations (16) and (17) define an almost complex structure, $I_{\omega_{2d}}$, on $T^*\mathbb{R}^2$, given in coordinates by

$$I_{ik} = \frac{1}{\sqrt{2\nabla^2 P}} \Omega_{ij}\omega_{jk}.$$

That is

$$I_{\omega_{2d}} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

with $\nabla^2 P = 2\alpha^2$. This almost complex structure is integrable (cf. (10)) in the special case

$$\nabla^2 P = constant\,. \tag{18}$$

Recall that time is merely a parameter here. When P satisfies (18), we can introduce the coordinates $\mathcal{X}, \mathcal{Y}$, and a two-form $\omega_{\mathcal{X}Y}$

$$\mathcal{X} = x - i\alpha^{-1}q, \quad \mathcal{Y} = y + i\alpha^{-1}p, \quad \omega_{\mathcal{X}Y} = \mathrm{d}\mathcal{X}\wedge \mathrm{d}\mathcal{Y}, \tag{19}$$

then (14) together with (17) are equivalent to

$$\triangle_{\omega_{\mathcal{X}Y}} = 0. \tag{20}$$

To summarize, the graph of ψ is a complex curve in $(T^*\mathbb{R}^2, I_{\omega_{2d}})$. This is the basis for a Kähler description of the incompressible two-dimensional Navier-Stokes equations. The condition (15) will certainly not be satisfied by all two-dimensional Navier-Stokes flows. However, with the aid of Lundgren's transformation (Lundgren 1982), we find that the Kähler structure can be extended to a class of two-and-a-half-dimensional flows, as designated in §1, for which this condition is less restrictive.

4. A result for two-and-a-half dimensional flows

At this point it is appropriate to work with the two-and-a-half-dimensional Burgers solutions introduced in §1 in equations (5), (6) and (7). Based on the results of the last section, we shall prove a more realistic result for two-and-a-half-dimensional flows in Theorem 1.

Lundgren (1982) made a significant advance when he showed that the class of three-dimensional Navier-Stokes solutions

$$u_1(x,y,t) = -\tfrac{1}{2}\gamma(t)x + \psi_y\,; \qquad u_2(x,y,t) = -\tfrac{1}{2}\gamma(t)y - \psi_x \tag{21}$$

$$u_3(x,y,t) = z\gamma(t) + \phi(x,y,t) \tag{22}$$

under the limited conditions of a constant strain $\gamma(t) = \gamma_0$, can be mapped back to the two-dimensional Navier-Stokes equations under a stretched co-ordinate transformation; see also Majda (1986), Majda & Bertozzi (2002), Saffman (1993), and Pullin & Saffman (1998). In (21), $\psi = \psi(x, y, t)$ is a two-dimensional stream function. This idea was extended by Gibbon *et al.* (1999) to a time-dependent strain field $\gamma = \gamma(t)$ with the inclusion of a scalar $\phi(x, y, t)$ in (22). The class of solutions in (21), which are said to be of *Burgers-type*, is generally thought to represent the observed tube-sheet class of solutions in Navier-Stokes turbulent flows (Moffatt *et al.* (1994) and Vincent & Meneguzzi (1994)).

Depending upon the sign of $\gamma(t)$ the vortex represented by (21) either stretches in the z-direction and contracts in the horizontal plane, which is the classic Burgers vortex tube, or vice-versa, which produces a Burgers' vortex shear layer or sheet. Thus γ, which can be interpreted as the aggregate effect of other vortices in the flow, acts an externally imposed strain function or 'puppet master', and can switch a vortex between the two extremes of these two topologies as we discussed in §1.

This class of solutions is connected to the results of §§2 and 3 through the following theorem, which is the main result of this section, and of the paper[1]:

Theorem 1. *If a two-and-a-half-dimensional Burgers-type class of solutions has a Laplacian of the pressure that is bounded by* $\nabla_3^2 P > -\frac{3}{2}\gamma^2$ *then any associated underlying two-dimensional Navier-Stokes flow is of Kähler type.*

Proof. To prove this theorem we first need two Lemmas. Firstly let $\boldsymbol{u} = (u_1, u_2, u_3)$ be a candidate velocity field solution of the three-dimensional Navier-Stokes equations taken in the form

(23)
$$u_1 = u_1(x, y, t), \quad u_2 = u_2(x, y, t), \quad u_3 = z\gamma(x, y, t) + \phi(x, y, t).$$

[1]The notation used in this section is: ∇ is the two-dimensional gradient and ∇_3 is the three-dimensional gradient. ∇^2 and ∇_3^2 are the two- and three-dimensional Laplacians respectively (to avoid confusion with the symbol $\triangle$ in §2).

with z appearing only in u_3. With this velocity field the total derivative is now

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + (z\gamma + \phi)\frac{\partial}{\partial z} \tag{24}$$

and the vorticity vector ω must satisfy

$$\frac{\mathcal{D}\omega}{\mathcal{D}t} = S\omega + \nu\nabla^2\omega\,, \tag{25}$$

where S is the strain matrix whose elements are

$$S_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

In the following Lemma $\boldsymbol{v}(x, y, t) = (u_1, u_2)$, and $\mathcal{P}(x, y, t)$ is a two-dimensional pressure variable which is related to the full pressure P in (31). The material derivative is now

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla \tag{26}$$

Lemma 1. *(see Gibbon* et al. *1999) Consider the velocity field* $\boldsymbol{u} = (\boldsymbol{v},\, z\gamma + \phi)$*; then* $\boldsymbol{v}$, ω_3, ϕ *and* γ *satisfy*

$$\frac{\mathcal{D}\boldsymbol{v}}{\mathcal{D}t} + \nabla\mathcal{P} = \nu\nabla^2\boldsymbol{v} \qquad\qquad \frac{\mathcal{D}\omega_3}{\mathcal{D}t} = \gamma\omega_3 + \nu\nabla^2\omega_3\,, \tag{27}$$

$$\frac{\mathcal{D}\phi}{\mathcal{D}t} = -\gamma\phi + \nu\nabla^2\phi\,, \tag{28}$$

$$\frac{\mathcal{D}\gamma}{\mathcal{D}t} + \gamma^2 + P_{zz}(t) = \nu\nabla^2\gamma\,. \tag{29}$$

The velocity field $\boldsymbol{v}$ *satisfies the continuity condition* $\div\,\boldsymbol{v} = -\gamma$ *and the second partial* z*-derivative of the pressure* P_{zz} *is constrained to be spatially uniform.*

Remark 1. *While (27) looks like a two-dimensional Navier-Stokes flow, the continuity condition implies that the two-dimensional divergence div*$\boldsymbol{v} \neq 0$*; thus an element of three-dimensionality remains.*

Proof. The evolution of the third velocity component $u_3 = \gamma z + \phi$ is given by

$$(30) \quad -P_z = \frac{\mathrm{D}u_3}{\mathrm{D}t} - \nu\nabla^2 u_3 = = z\left(\frac{\mathcal{D}\gamma}{\mathcal{D}t} + \gamma^2 - \nu\nabla^2\gamma\right) + \left(\frac{\mathcal{D}\phi}{\mathcal{D}t} + \gamma\phi - \nu\nabla^2\phi\right)$$

which, on integration with respect to z, gives

$$(31) \quad -P(x,y,z,t) = \tfrac{1}{2}z^2\left(\frac{\mathcal{D}\gamma}{\mathcal{D}t} + \gamma^2 - \nu\nabla^2\gamma\right) + + z\left(\frac{\mathcal{D}\phi}{\mathcal{D}t} + \gamma\phi - \nu\nabla^2\phi\right) - \mathcal{P}(x,y,t).$$

It is in this way that $\mathcal{P}(x,y,t)$ is related to $P(x,y,z,t)$. However, from the first two components of the Navier-Stokes equations, we know that ∇P must be independent of z. For this to be true the coefficients of z and z^2 in (31) must necessarily satisfy

$$(32) \quad \frac{\mathcal{D}\phi}{\mathcal{D}t} + \gamma\phi - \nu\nabla^2\phi = c_1(t), \qquad \frac{\mathcal{D}\gamma}{\mathcal{D}t} + \gamma^2 - \nu\nabla^2\gamma = c_2(t).$$

$c_1(t)$ is an acceleration of the co-ordinate frame which can be taken as zero without loss of generality. Equation (31) shows that $c_2(t) = -P_{zz}(t)$ which restricts P_{zz} to being spatially uniform. To find the evolution of ω_3 we consider the strain matrix $S = \{S_{ij}\}$

$$(33) \quad S = \begin{pmatrix} u_{1,x} & \frac{1}{2}(u_{1,y} + u_{2,x}) & \frac{1}{2}(z\gamma_x + \phi_x) \\ \frac{1}{2}(u_{1,y} + u_{2,x}) & u_{2,y} & \frac{1}{2}(z\gamma_y + \phi_y) \\ \frac{1}{2}(z\gamma_x + \phi_x) & \frac{1}{2}(z\gamma_y + \phi_y) & \gamma \end{pmatrix}.$$

Working out the vorticity field ω from (23) it is easily seen that $(S\omega)_3 = \gamma\omega_3$. Thus (25) shows that ω_3 decouples from ϕ to give the equation for ω_3 in (27). □

Now let us consider the class of Burgers' velocity fields given in (21) with a stream function $\psi(x,y,t)$. The strain rate variable

γ is taken as a function of time only. The continuity condition is now automatically satisfied. The material derivative is given by

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} - \tfrac{1}{2}\gamma(t)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + J_{x,y}(\psi,\,\cdot). \tag{34}$$

New co-ordinates can be taken (Lundgren's transformation (Lundgren (1982)))

$$s(t) = \exp\left(\int_0^t \gamma(t')\,\mathrm{d}t'\right) \tag{35}$$

$$\tilde{x} = s^{1/2}x, \qquad \tilde{y} = s^{1/2}y, \qquad \tilde{t} = \int_0^t s(t')\,\mathrm{d}t', \tag{36}$$

which re-scale ω_3 and ϕ into new variables

$$\tilde{\omega}_3(\tilde{x},\tilde{y},\tilde{t}) = s^{-1}\omega_3(x,y,t), \qquad \tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = s\,\phi(x,y,t). \tag{37}$$

The material derivative is

$$\frac{\mathcal{D}}{\mathcal{D}\tilde{t}} = \frac{\partial}{\partial\tilde{t}} + \tilde{\boldsymbol{v}}\cdot\tilde{\nabla} \tag{38}$$

where $\psi(x,y,t) = \tilde{\psi}(\tilde{x},\tilde{y},\tilde{t})$, $\tilde{\boldsymbol{v}} = \left(\tilde{\psi}_{\tilde{y}}\,, -\tilde{\psi}_{\tilde{x}}\right)$ and $\tilde{\nabla} = \hat{\boldsymbol{\imath}}\,\partial_{\tilde{x}} + \hat{\boldsymbol{\jmath}}\,\partial_{\tilde{y}}$. The relation between $\boldsymbol{v} = (u_1,\, u_2)$ and $\tilde{\boldsymbol{v}}$ is given by

$$u_1 = -\tfrac{1}{2}\gamma(t)x + s^{1/2}\tilde{v}_1, \qquad u_2 = -\tfrac{1}{2}\gamma(t)y + s^{1/2}\tilde{v}_2 \tag{39}$$

and the relation between the two material derivatives in combination with the respective Laplacians is

$$\frac{\mathcal{D}}{\mathcal{D}t} - \nu\nabla^2 = s\left(\frac{\mathcal{D}}{\mathcal{D}\tilde{t}} - \nu\tilde{\nabla}^2\right). \tag{40}$$

Introducing a new pressure variable $\tilde{P}$ as

$$\tilde{P} = s^{-1}\left[\mathcal{P} - \tfrac{1}{4}(x^2+y^2)\left(\dot{\gamma} - \tfrac{1}{2}\gamma^2\right)\right] \tag{41}$$

our results can be summarized in our second Lemma:

Lemma 2. *The re-scaled velocity field $\tilde{\boldsymbol{v}}$ satisfies the two-dimensional re-scaled Navier-Stokes equations ($\div \tilde{\boldsymbol{v}} = 0$)*

$$\frac{\mathcal{D}\tilde{\boldsymbol{v}}}{\mathcal{D}\tilde{t}} + \tilde{\nabla}\tilde{P} = \nu\tilde{\nabla}^2\tilde{\boldsymbol{v}}\,. \tag{42}$$

The vorticity $\tilde{\omega}_3(\tilde{x},\tilde{y},\tilde{t}) = -\tilde{\nabla}^2\,\tilde{\psi}$ and the passive scalar $\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t})$ satisfy

$$\frac{\mathcal{D}\tilde{\omega}_3}{\mathcal{D}\tilde{t}} = \nu\tilde{\nabla}^2\omega_3, \qquad\qquad \frac{\mathcal{D}\tilde{\phi}}{\mathcal{D}\tilde{t}} = \nu\tilde{\nabla}^2\phi\,. \tag{43}$$

Proof. From (35) we note the useful result that $\mathrm{D}s/\mathrm{D}t = \gamma s$. Using (39) we write

$$\frac{\mathrm{D}u_1}{\mathrm{D}t} - \nu\nabla^2 u_1 = -\tfrac{1}{2}x\left(\dot{\gamma} - \tfrac{1}{2}\gamma^2\right) + s^{3/2}\left(\frac{\mathcal{D}\tilde{v}_1}{\mathcal{D}\tilde{t}} - \nu\nabla^2\tilde{v}_1\right)\,, \tag{44}$$

$$\frac{\mathrm{D}u_2}{\mathrm{D}t} - \nu\nabla^2 u_2 = -\tfrac{1}{2}y\left(\dot{\gamma} - \tfrac{1}{2}\gamma^2\right) - s^{3/2}\left(\frac{\mathcal{D}\tilde{v}_2}{\mathcal{D}\tilde{t}} - \nu\nabla^2\tilde{v}_2\right)\,. \tag{45}$$

Next we appeal to the definition of the pressure $\tilde{P}$ in (41) to give the velocity pressure relation in (42). The results for $\tilde{\phi}$ and $\tilde{\omega}_3$ follow immediately. □

The proof of Theorem 1 is now ready to be completed. To obtain the full three-dimensional Laplacian of the pressure $\nabla_3^2 P$ we use (41) and (29) and write

$$\begin{aligned} -\nabla_3^2 P &= \tfrac{3}{2}\gamma^2 + s^2\left[\frac{\partial}{\partial\tilde{x}}\left(\frac{\mathcal{D}\tilde{v}_1}{\mathcal{D}\tilde{t}} - \nu\nabla^2\tilde{v}_1\right) + \frac{\partial}{\partial\tilde{y}}\left(\frac{\mathcal{D}\tilde{v}_2}{\mathcal{D}\tilde{t}} - \nu\nabla^2\tilde{v}_2\right)\right] \\ &= \tfrac{3}{2}\gamma^2 - s^2\tilde{\nabla}^2\tilde{P}\,. \end{aligned} \tag{46}$$

Thus if $\nabla_3^2 P$ satisfies the condition in Theorem 1 then the corresponding Kähler positivity condition (15) on the Laplacian for two-dimensional flow is satisfied. □

Lundgren's mapping breaks down under one condition: while the strain $\gamma(t)$ can take either sign, if it is forever negative or for long intervals, the domain $t \in [0, \infty]$ maps on to a finite section of the $\tilde{t}$-axis. For example, if

$$\gamma = -\gamma_0 = \text{const} < 0$$

then $s = \exp(-\gamma_0 t)$ and $\tilde{t} = \gamma_0^{-1}\,[1 - \exp(-\gamma_0 t)]$. Hence $t \in [0, \infty]$ maps onto $\tilde{t} \in [0, \gamma_0^{-1}]$.

5. Summary

We have shown how Kähler geometry arises in the Navier-Stokes equations of incompressible hydrodynamics, via a Monge–Ampère equation associated with (4). Although it is certainly not the case that all two-dimensional flows will satisfy the condition for the Kähler structure to exist, the situation looks much more promising for two-and-a-half-dimensional flows, of which Burgers vortex is one example.

Issues relating to the existence and interpretation of Kähler structures, the integrability conditions, and related matters involving contact and symplectic structures, were discussed by McIntyre & Roulstone (2002) in connection with various Monge–Ampère equations arising in geophysical fluid dynamics. The semi-geostrophic equations of meteorology, which are a particularly useful model for studying the formation of fronts, were the starting point in McIntyre & Roulstone *op. cit.* for an investigation into the role of novel coordinate systems, similar to those we have found here in (19) and (49). In semi-geostrophic theories, such coordinates facilitate significant simplifications of difficult nonlinear problems, and they are associated with canonical Hamiltonian formulations of these systems. Issues relating to contact and symplectic geometry may also be relevant to the results presented in this paper, and this suggests one direction for further study.

A further variation on this theme revolves around the addition of rotation to the system, which, as we pointed out in the Introduction, has important meteorological applications. Euler's equations of motion are

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + f(\boldsymbol{k} \times \boldsymbol{v}) + \nabla P = 0\,, \tag{47}$$

where $\frac{1}{2}f$ is the angular frequency of the rotation. If we examine these equations in two dimensions, with constant rotation, then taking the divergence of (47) gives

$$\nabla^2 P = -2(\psi_{xy}^2 - \psi_{xx}\psi_{yy}) + f\nabla^2\psi. \tag{48}$$

This equation, which is commonly referred to as the Charney balance condition in the geophysical fluid dynamics literature, is an elliptic Monge–Ampère equation for ψ if $\nabla^2 P + f^2/2 > 0$. The associated complex structure is integrable when $\nabla^2 P$ is a constant (cf. (10)), and in this case we can introduce new complex coordinates

$$\tilde{X} = ax + i(fy + 2q), \;\; \tilde{Y} = ay - i(fx + 2p), \tag{49}$$

with $a = (2\nabla^2 P + f^2)^{1/2}$. Once again, (48) together with (17) are equivalent to

$$\omega_{\tilde{X}\tilde{Y}} \equiv \mathrm{d}\tilde{X} \wedge \mathrm{d}\tilde{Y}, \;\; \triangle_{\omega_{\tilde{X}\tilde{Y}}} = 0.$$

If the pressure is zero, or harmonic, then (48) is suggestive of a special Lagrangian structure. A special Lagrangian structure has also been noted in the work of Roubtsov & Roulstone (2001), but its role in that context is obscure (see McIntyre & Roulstone (2002) equation (13.27) *et seq.*).

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