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On Samuelson Submanifolds in Four Space

The object of study in this talk is a general class of submanifolds of $\mathbf{R}^4$. The motivation for this work was the derivation of the following equation which answered a question posed by the distinguished economist P. A. Samuelson.

1. The Holy Grail Equation

The Holy Grail Equation has the following form:

$$
-b^3\frac{\partial^2 a}{\partial y^2} - a^3\frac{\partial^2 b}{\partial y^2} + 2\left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial x}\right)^2 - 2b^2\left(\left(\frac{\partial a}{\partial y}\right)^2 + \frac{\partial^2 a}{\partial x \partial y}\right)
$$
$$
+ a^2\left(b\frac{\partial^2 b}{\partial y^2} - 2\left(\left(\frac{\partial b}{\partial y}\right)^2 + \frac{\partial^2 b}{\partial x \partial y}\right)\right)
$$
$$
+ a\left(b^2\frac{\partial^2 a}{\partial y^2} + \frac{\partial b}{\partial y}\left(3\frac{\partial a}{\partial y} - 4\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} + 2b\left(2\frac{\partial a}{\partial y}\frac{\partial b}{\partial y} + \frac{\partial^2 a}{\partial x \partial y} + \frac{\partial^2 b}{\partial x \partial y}\right)\right)\right)
$$
$$
+ \left(\frac{\partial^2 a}{\partial x^2} - \frac{\partial^2 b}{\partial x^2}\right) + b\left(\frac{\partial b}{\partial y}\frac{\partial a}{\partial x} + \frac{\partial a}{\partial y}\right)\left(-4\frac{\partial a}{\partial y} + 3\frac{\partial b}{\partial x} - \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 b}{\partial x^2}\right) = 0.
$$

We will discuss the origins of this equation below.

We turn to our central definition. For reasons which will soon be apparent, we write (x, y, u, v) for the coordinates of a typical point of $\mathbb{R}^4$. Consider such a point. If it lies on a given two dimensional

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submanifold M then by the implicit function theorem, indeed by the definition of a submanifold, there is a neighbourhood of U of (x, y, u, v) so that $U \cap M$ is the graph of a smooth function. There are 6 possibilities:

  I. u and v as functions of x and y,
  II. x and y as functions of u and v,
  III. u and x as functions of v and y,

and the 3 analogous cases: u and y as functions of v and x, v and x as functions of u and y resp. u and x as functions of v and y.

We say that M is a Samuelson submanifold if, in case I, $u = f(x, y)$, $v = g(x, y)$, where f and g satisfy the equations

$$\left(\frac{g_2}{J} \frac{\partial}{\partial x} + \frac{g_1}{J} \frac{\partial}{\partial y} \right) \left(-\frac{f_2}{J} \frac{\partial J}{\partial x} + \frac{f_1}{J} \frac{\partial J}{\partial y} \right) =$$
$$\left(\frac{g_2}{J} \frac{\partial J}{\partial x} + \frac{g_1}{J} \frac{\partial J}{\partial y} \right) \left(-\frac{f_2}{J} \frac{\partial J}{\partial x} + \frac{f_1}{J} \frac{\partial J}{\partial y} \right).$$

Here J is the Jacobi-determinant $f_1 g_2 - f_2 g_1$ – subscripts denote partial derivatives.

In case II we demand that

$$J^{-1} J_{uv}^{-1} = J_u J_v$$

where $J^{-1} = f_1 g_2 - f_2 g_1$ and $x = f(u, v)$, $y = g(u, v)$.

In case III with $u = f(v, x)$, $y = g(v, x)$, the required equation is

$$J \frac{1}{f_2} \frac{\partial}{\partial y} \left(\frac{\partial J}{\partial v} - \frac{f_1}{f_2} \frac{\partial J}{\partial y} \right) = \frac{1}{f_2} \frac{\partial J}{\partial y} \left(\frac{\partial J}{\partial v} - \frac{f_1}{f_2} \frac{\partial J}{\partial y} \right)$$

where $J = -f_1 / g_2$.

Thus definition is motivated by the concept of a Samuelson configuration which was introduced in [9] following a suggestion of Professor Samuelson. In view of further applications it seems to be advantageous to introduce the more symmetric version above. The relation between the two concepts is clarified below.

2. The Connection with Webs

For the general concept of webs, we refer to the classical text [6]. A more modern version is in [1]. In this paper we will only be concerned with 1-webs in 2-space.

The relation between the above notion and the subject of webs lies in the following simple observation.

Suppose we have such a manifold. Then this, generically, describes a two web in $\mathbb{R}^2$ as follows: we project the manifold successively onto the (x, y, u) and (x, y, v) spaces and then consider the intersections with planes parallel to the (x, y) plane.

At this point, we describe briefly various ways to specify webs in the plane:

(1) As the level curves $u(x, y) = c$ of a smooth real valued function. (*Example*: if $u(x, y) = x^2 + y^2$ we get the family of concentric circles centered at the origin.)

(2) As indexed families $x = c_1(\lambda, t)$, $y = c_2(\lambda, t)$ of parametrized curves. (*Example*: $x = \lambda \cos t$, $y = \lambda \sin t$ describes the same family.)

(3) As the flow lines of a vector field $\mathbf{F} = (F_1, F_2)$ i.e. the solution curves of the system

$$\frac{dx}{dt} = F_1(x(t), y(t))$$
$$\frac{dy}{dt} = F_2(x(t), y(t)).$$

(*Example*: $\mathbf{F} = (-y, x)$ again produces the family of concentric circles.)

(4) As the family of solutions of the differential equation $\frac{dy}{dx} = a(x, y)$ for a given smooth functions on the plane ($a(x, y)$ is the gradient of the curve through (x, y)). This can be regarded as a special case of 3 where $\mathbf{F}$ is the directional field $(1, a(x, y))$, (the concentric circles correspond to $a(x, y) = -\frac{x}{y}$).

A key point in our discussion is the fact that these associations are not uniquely determined. Thus if we are given two functions u and v, then their level curves form a 2-web in $\mathbb{R}^2$. However these webs do not uniquely determine u and v. This is the motivation for the following definition.

Let u be a smooth real-valued function on the plane. Then a *recalibration* of u is a function of the form $U = \phi \circ V$ where ϕ is a diffeomorphism between open intervals of $\mathbb{R}$. More formally, U is a member of the orbit of u under the group of diffeomorphisms of the line (more accurately, the pseudo-group of diffeomorphisms of the above type).

We now turn to our main result which gives various characterizations of Samuelson submanifolds and which we will formulate in the spirit of classical differential geometry (i.e. without detailing what may occur in non-generic cases). Hence the quotation marks around the word equivalent in our formulation.

Let be M a 2-manifold in $\mathbb{R}^4$, $\mathcal{W}$ the 2-web it induces on $\mathbb{R}^2$ (i.e. the configuration of two webs).

Then following concepts are "equivalent":

(1) *M is a Samuelson submanifold.*
(2) *$\mathcal{W}$ is an S-configuration [9].*
(3) *The Jacobi-determinant $J = u_x v_y - u_y v_x$ splits multiplicatively as a function of u and v.*
(4) *The gradient fields $a(x,y)$ and $b(x,y)$ of the webs satisfy the Holy Grail equation.*
(5) *The derivative of the differential form*

$$\operatorname{div} \mathbf{F} \cdot \omega_1 - \omega_1([\mathbf{F}, \mathbf{G}])\omega_2$$

vanishes. Here ω_1 and ω_2 are the differential forms dual to the bases generated by the fields $\mathbf{F}, \mathbf{G}$. In coordinates,

$$\omega_1 = \frac{1}{F_1 G_2 - G_2 F_1}(G_2, -G_1)$$

and

$$\omega_2 = \frac{1}{F_1 G_2 - G_2 F_1}(-F_2, F_1).$$

(6) *There are two vector fields* **F** *and* **G** *with*

$$\operatorname{div} \mathbf{F} = \operatorname{div} \mathbf{G} = [\mathbf{F}, \mathbf{G}] = 0$$

so that the webs consist of the flow lines of **F** *and* **G** *resp.*

(7) *There are recalibrations* U *and* V *of* u *and* v *for which the Jacobi-determinant is* 1.

In the formulation of the next conditions we remark that $\operatorname{Diff}(\mathbb{R})^2$ acts on $\mathbb{R}^4(\mathbb{R}^5)$ and so on the family of submanifolds of $\mathbb{R}^4(\mathbb{R}^5)$ via the formula

$$\begin{aligned} (x, y, u, v) &\mapsto (x, y, \phi(u), \psi(v)) \\ (x, y, u, v, E) &\mapsto (x, y, \phi(u), \phi(\psi), E). \end{aligned}$$

(8) M *is equivalent to a Lagrangian submanifold of* $\mathbb{R}^4$ *with the canonical symplectic structure* $x \wedge y - u \wedge v$, *under this group action.*

(9) *The manifold is equivalent to the projection of a Legendrian submanifold of* $\mathbb{R}^5$ *onto* $\mathbb{R}^4$, *when we regard* $\mathbb{R}^5$ *as a contact manifold with the differential form* $E \pm x\, dv \pm u\, dy$. *(The four possible choices of sign depend on which of the four cases in III above holds)* [4].

We remark that in [17] Tabachnikov has introduced a connection on 2-webs in the symplectic space $(\mathbb{R}^2, x \wedge y)$ where he states that any of the above conditions is equivalent to the vanishing of this connection.

We mention briefly connections with themes in economics, cartography, thermodynamics, elementary geometry.

3. Economics

The first use of the mathematical theory of webs in economics was by Debreu, [11, 12]. Debreu had earlier obtained conditions for a preference ordering to be representable by a numerical function, Debreu [10]. Once such a function exists, it was natural for

economists to ask when it would be additively separable. Debreu showed that this question was equivalent to requiring that a 3-web in the plane given by the level curves of the function, the verticals, and the horizontals be equivalent to the trivial 3-web. This in turn required the satisfaction of the Blasche/Thomsen/Bol hexagon condition, Thomsen [18], Blaschke [5], see also Vind [19] and Wakker [20].

Here we will use web theory in a quite different way. As noted in the introduction, the original motivation for the derivation of the Holy Grail equation was to provide a test for the presence of maximization processes in economics. Such processes, Samuelson [16], are mathematically equivalent to the processes of classical thermodynamics. In this section we will show how web theory can also be used to provide a test for maximizing behavior. We focus on the economic situation, but, suitably reinterpreted, the underlying ideas are equally applicable to classical thermodynamics.

We use the same framework as Samuelson [16]. We consider a firm which hires 2 inputs, say $\mathbf{L}_1$ and $\mathbf{L}_2$, at competitive prices $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively. The firm sells its output as a monopolist into a downward sloping demand function. In principle we can experimentally construct two families of demand functions. The first, steeper, family, $\mathbf{L}_1(\mathbf{W}_1, \overline{\mathbf{L}_2})$ gives the demand for $\mathbf{L}_1$ at input price $\mathbf{W}_1$ when $\mathbf{L}_2$, the quantity of the other input, is held fixed. The second, flatter family $\mathbf{L}_1(\mathbf{W}_1, \overline{\mathbf{W}_2})$, gives the demand when the price of the second input is held fixed, see Samuelson [16].

As discussed earlier, view this data as a 2-web given by the projection of a manifold in $\mathbb{R}^4$ onto 3-dimensional $\mathbf{W}_1\mathbf{L}_1\mathbf{W}_2$ and $\mathbf{W}_1\mathbf{L}_1\mathbf{L}_2$ space respectively, the resulting projected manifold being then intersected by planes parallel to the $\mathbf{W}_1\mathbf{L}_1$ plane.

A key question in economics is the identification of the conditions on this 2-web necessary for the firm to be acting so as to maximize profits. Samuelson proposed the following simple test.

Samuelson Maximization Test: A firm whose input demand data is given by a 2-web cannot be a profit maximizer unless $ab =$

cd, where a, b, c, d are areas of quadrilaterals bounded by "threads" of the web.

As Tabachnikov has shown, Tabachnikov [17], when the Samuelson area condition is satisfied, the 2-web is trivial under an area preserving transformation. Thus we can calibrate the leaves of the web in such a way that there is unit area between the threads labeled x and $x + 1$ and y and $y + 1$, for each respective member of the family. In classical thermodynamics this calibration corresponds to the passage from empirical temperature and entropy to absolute temperature and entropy. In economics this recalibration is not possible, so the relevant test for maximization is whether or not the already calibrated 2-web, when trivialized to the horizontal/vertical web already satisfies the equal area condition.

Tabachnikov [17] gives a further characterization of the profit maximizing condition. If we place the standard area (symplectic) form $\mathbf{dL}_1 \wedge \mathbf{dW}_1$ on $\mathbb{R}^2$, the 2-web of factor demands is a Lagrangian 2-web since all curves on the symplectic plane are Lagrangian sub-manifolds. Tabachnikov's theorem 0.1 now applies directly and a symplectic, torsion free, connection (the Hess connection [15], can be associated with the web. As Tabachnikov shows, when the Samuelson test is satisfied with equality, this connection is flat. This provides an alternative characterization of profit maximizing behavior.

Finally we note that if we impose the standard symplectic form

$$\mathbf{dL}_1 \wedge \mathbf{dW}_1 + \mathbf{dL}_2 \wedge \mathbf{dW}_2$$

on $\mathbb{R}^4$, the 2-dimensional sub-manifold is a Lagrangian sub-manifold of $\mathbb{R}^4$. This means that the mapping from $\mathbf{L}_1, \mathbf{W}_1$ to $\mathbf{L}_2, \mathbf{W}_2$ is area preserving and orientation reversing as we have just seen. Since maximizing economic processes take place on Lagrangian sub-manifolds, economics, too, succumbs to Weinstein's Lagrangian creed. That everything is a Lagrangian submanifold [21].

4. CARTOGRAPHY

(The Central Problem of Mathematical Cartography.)

Our starting points are the two basic map projections:

- the Lambert projection which is an equal area projection,
- the Mercator projection which is a conformal projection (see, for example, Brown [7]).

All equal area (conformal) projections arise from the composition of these projections with area preserving maps (conformal maps) of the plane. (We remark that it follows from our main result that a 2 web is the family of meridians resp. parallels of an equal-area projection iff they form an S-configuration.)

The central problems of cartography were to characterize all equal area (conformal) projections whose meridians and parallels are (generalized) circles (cf. [7]).

The conformal case was solved by Lagrange, the equal area case by Gravé, a mathematician at the University of Kiev. In order to get the flavour of Gravé's result, consider the equation

$$x = \delta \cos \frac{u + \epsilon}{k} - \sin \frac{u + \epsilon}{k} \sqrt{2kv + e},$$
$$y = \delta \sin \frac{u + \epsilon}{k} + \cos \frac{u + \epsilon}{k} \sqrt{2kv + e}$$

(δ, ϵ, k, l are parameters), which is taken from article [14] with his solution to the second problem. Then the mapping from (u, v) to (x, y) described by this formula is an equal area mapping of the plane with the required property. Gravé's result consists of a list of analogous formulae which includes all possible equal-area projections with the above property.

We note that it follows easily from the main result that certain special S-configurations can be generated by groups in either of the following ways:

A) if we are given 2 one-parameter groups of area-preserving transformations $A(t), B(t)$ so that A and B commute, then their orbits form an S-configuration;

B) if $A(t)$ is as above and Γ is a generic curve, then the webs formed by the orbits of A resp. the images of Γ under A form an S-configuration.

It is natural to express the result of Gravé's in terms of such groups. Thus we require one-parametric groups whose orbits are generalized circles. There are three natural candidates:

- the translation group.
- the rotation group.
- the dilatation group.

We can then use these three groups together with an additional generalized circle to generate S-configurations as in B) and hence equal area projections. It turns out that all of Gravé's configurations arise in this manner and this provides a much more accessible version of his characterization.

5. Thermodynamics

Suppose we are given two functions u and v (empirical temperature and empirical entropy). Then the basic facts of thermodynamics are contained in the statement: u and v can be recalibrated to satisfy the Maxwell relations, if and only if their level curves (the "isotherms" and "adiabatics") form an S-configuration. These recalibrations, which are essentially unique, are absolute temperature and absolute entropy.

We remark that this application was our motivation for seeking a more symmetric version of Samuelson's area condition. The equations of state for a thermodynamic substance involve two relationships between the four quantities p, V, T, S (pressure, volume, (empirical) temperature, (empirical) entropy) but there is no necessity (empirical or theoretical) which dictates the choice of dependent and independent variables.

Examples:

- Ideal gas: $u = xy$, $v = xy^\gamma$,

- Van der Waals gas: $u = (x + \frac{1}{y^2})(y - 1)$, $v = (x + \frac{1}{y^2})(y - 1)^\gamma$,
- Feynman gas: $u = xy$, $v = xy^{\gamma(xy)}$,
- $u = (x + \frac{1}{y^2})(y - 1)$, $v = (x + \frac{1}{y^2})(y - 1)^{\gamma((x + \frac{1}{y^2})(y - 1))}$.

All of these satisfy the Holy Grail theorem and so have unique recalibrations for which they satisfy the Maxwell relationships. The resulting computations provide insights into the nature of real gases.

6. GEOMETRY

We close with the remark that a number of elegant geometrical configurations which we call Apostol-Tabachnikov configurations are examples of S-configurations and this gives a simple and elegant approach to some of their properties. These are generated by a single curve and a web of tangential curves, one at each point. The second web is obtained by moving the initial curve along the latter in a suitable manner.

(1) *Apostol configurations.* In this case we move a curve along its tangents. Analytically, thus is given by the equations

$$x(u, v) = c_1(u) + v \frac{(\dot{c}_1(u), \dot{c}_2(u))}{\sqrt{\dot{c}_1^2(u) + \dot{c}_2^2(u)}},$$

$$y(u, v) = c_2(u) + v \frac{(\dot{c}_1(u), \dot{c}_2(u))}{\sqrt{\dot{c}_1^2(u) + \dot{c}_2^2(u)}}.$$

(2) *Tait-Kneser configurations.* Here the second foliation consists of the family of osculating circles:

$$(x(u, v), y(u, v)) =$$

$$(c_1(u), c_2(u)) + \begin{bmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{bmatrix} \begin{bmatrix} c_1(u) - C_1(u) \\ c_2(u) - C_2(u) \end{bmatrix}.$$

($C(u)$ is the center of curvature).

(3) *Tait-Kneser-Tabachnikov configurations* generated by the
     curve $y = f(x)$.

$$x(u, v) = u + v,$$

$$y(u, v) = \sum_{r=0}^{n} \frac{v^r}{r!} f^{(r)}(u).$$

(For these configurations, see references [2], [3] and [13].) In
each case it is a routine matter to compute the Jacobi-determinant
and to verify that it splits multiplicatively in the required manner.

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