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Groups with the minimal condition for nonabelian subgroups

Для деяких дуже широких класів $\mathfrak{D}$ і $\mathfrak{B} \subset \mathfrak{D}$ груп автор доводить, що довільна (неабелева) група $G \in \mathfrak{D}$ (відповідно, $G \in \mathfrak{B}$) задовольняє умову мінімальності для (неабелевих) підгруп тоді і тільки тоді, коли вона є черніковською.

Для некоторых очень широких классов $\mathfrak{D}$ и $\mathfrak{B} \subset \mathfrak{D}$ групп автор доказывает, что произвольная (неабелева) группа $G \in \mathfrak{D}$ (соответственно, $G \in \mathfrak{B}$) удовлетворяет условию минимальности для (неабелевых) подгрупп тогда и только тогда, когда она является черниковской.

For some very wide classes $\mathfrak{D}$ and $\mathfrak{B} \subset \mathfrak{D}$ of groups, the author proves that an arbitrary (nonabelian) group $G \in \mathfrak{D}$ (respectively $G \in \mathfrak{B}$) satisfies the minimal condition for (nonabelian) subgroups iff it is Chernikov.

Recall that a group G is called Shunkov, if for any its finite subgroup K every subgroup of the quotient group $N_G(K)/K$, generated by two its conjugated elements of prime order, is finite. Recall that a group is called locally graded, if any its finitely generated subgroup $\neq 1$ contains a subgroup of finite index $\neq 1$ (S.N.Chernikov). The class of all periodic Shunkov groups is wide and contains, for instance, all binary finite and 2-groups. The class of all locally graded groups is very wide and contains, for instance, all abelian, locally finite, residually finite groups. It is easy to see that this class is local and any group having a series with locally graded factors is locally graded, and any residually (locally graded) group is locally graded. At the same time, the class of all locally graded groups contains all groups having a series with locally finite factors, all *RN*- (and so all groups of all Kurosh-Chernikov classes), locally solvable, locally hyperabelian, radical in the sense of B.I.Plotkin, residually solvable groups.

Below, as usual, $\pi(G)$ is the set of all primes p for which the group G has a p-element $\neq 1$.

Let $\mathfrak{A}$ be the class of all groups G for which the following conditions are fulfilled:

(i) If G is not torsion-free, then for any $p \in \pi(G)$ and p-element $g \neq 1$ of G and $h \in C_G(g^p)$, $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$ or $\langle g, h \rangle$ is not periodic.

(ii) If G is not periodic, then for any element g of infinite order of G and $h \in G$, $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$.

The class $\mathfrak{A}$ is very wide and contains, for instance, all periodic Shunkov groups and all locally graded groups.

Let $\mathfrak{C}$ be the class of all groups G for which (i), with "periodic $\neq 1$" instead of "not torsion-free" and deleted "or $\langle g, h \rangle$ is not periodic", is fulfilled (and (ii) is not necessary fulfilled). Then $\mathfrak{A}$ is contained in $\mathfrak{C}$, $\mathfrak{C}$ contains all nonperiodic groups and the class of all periodic $\mathfrak{A}$-groups is just the class of all periodic $\mathfrak{C}$-groups.

Let $\mathfrak{B}$ (resp. $\mathfrak{D}$) be the minimal local class of groups containing $\mathfrak{A}$ (resp. $\mathfrak{C}$) such that any group possessing a series with $\mathfrak{B}$ (resp. $\mathfrak{D}$)-factors belongs to $\mathfrak{B}$ (resp. $\mathfrak{D}$). Put $\mathfrak{B}_o = \mathfrak{A}$ (resp. $\mathfrak{D}_0 = \mathfrak{C}$) and for ordinals $\beta > 0$ by induction: if for some ordinal α, $\beta = \alpha + 1$, then $\mathfrak{B}_\beta$ (resp. $\mathfrak{D}_\beta$) be the class of all groups which have a local system of subgroups possessing a series with $\mathfrak{B}_\alpha$ (resp. $\mathfrak{D}_\alpha$)-factors, and if there is no such α, then $\mathfrak{B}_\beta = \bigcup_{\alpha<\beta} \mathfrak{B}_\alpha$ (resp. $\mathfrak{D}_\beta = \bigcup_{\alpha<\beta} \mathfrak{D}_\alpha$). It is easy to see that $\mathfrak{B}$ (resp. $\mathfrak{D}$) is the union of classes $\mathfrak{B}_\beta$ (resp. $\mathfrak{D}_\beta$). It is easy to show by induction that all $\mathfrak{B}_\beta$ and, at the same time, $\mathfrak{B}$ are closed with respect to subgroups.

The known Shunkov's [1] and S.N.Chernikov's [2] Theorems establish that a nonabelian group satisfying the minimal condition for nonabelian subgroups is Chernikov, if it is locally finite or has a series with finite factors resp. The next Theorem contains them.

Its proof below uses some results [2] and Suchkova-Shunkov Theorem [3], which asserts that a Shunkov group with the minimal condition for abelian subgroups is Chernikov.

Theorem. *A (nonabelian) group $G \in \mathfrak{D}$ (resp. $G \in \mathfrak{B}$) satisfies the minimal condition for (nonabelian) subgroups iff it is Chernikov.*

Note that Ol'shanskii's nonabelian groups, in which all proper subgroups are finite (see [4]), satisfy the minimal condition for subgroups and are non-Chernikov. Thus, in Theorem, the condition "$G \in \mathfrak{D}$" is essential. Note that Ol'shanskii's nonabelian torsion-free groups, in which all proper subgroups are cyclic (see [4]), satisfy the minimal condition for nonabelian subgroups, are Shunkov and non-Chernikov. In particular, in Theorem, the condition "$G \in \mathfrak{B}$" is essential.

Below min and min $-\overline{ab}$ are the minimal conditions for subgroups and nonabelian subgroups resp. Other notations are standard. (Remark that a group with min is periodic and an abelian group with min is Chernikov.)

Proof. *Sufficiency* is obvious.

Necessity. Let G be non-Chernikov.

(a) *Reduction to the case when G satisfies* min $-\overline{ab}$ *and also $G \in \mathfrak{A}$.* Let ζ be minimal among all ordinals α, for which $\mathfrak{B}_\alpha$

(resp. $\mathfrak{D}_\alpha$) contains a non-(Chernikov or abelian) group (resp. a non-Chernikov group) with min $-\overline{ab}$ (resp. min). We may assume that $G \in \mathfrak{B}_\zeta$ (resp. $G \in \mathfrak{D}_\zeta$). Suppose $\zeta > 0$. Clearly, for some ordinal ξ, $\zeta = \xi + 1$. So G possesses a local system of subgroups having a series with $\mathfrak{B}_\xi$ (resp. $\mathfrak{D}_\xi$)-factors. Every factor is, obviously, Chernikov or abelian (resp. Chernikov). So G is locally graded. Thus $G \in \mathfrak{B}_0$ (resp. $G \in \mathfrak{D}_0$), which is a contradiction. So $\zeta = 0$ and $G \in \mathfrak{A}$ (resp. $G \in \mathfrak{C}$). In the case of min, $G \in \mathfrak{C}$ and G is, obviously, periodic nonabelian with min $-\overline{ab}$. In particular, $G \in \mathfrak{A}$. Taking this into account, we may consider later on only the case of $G \in \mathfrak{A}$ with min $-\overline{ab}$.

Since G satisfies min $-\overline{ab}$, it contains a non-(Chernikov or abelian) subgroup L such that any its proper subgroup is Chernikov or abelian. We may assume that $G = L$. Then for every normal subgroup N of G, any proper subgroup of G/N is Chernikov or abelian.

(b) *Show that a subgroup H of G is Chernikov or abelian, if it has a subgroup K of finite index $\neq 1$ or if it is almost solvable.* Since $K \neq G$, K and so H are almost abelian. If H is almost solvable, then in view of Corollary to Theorem 1 [2], Corollary 2 [2] and Lemmas 1,2 [2], it is Chernikov or abelian.

(c) *Show that G is periodic.* Let G have elements g of infinite order. Since $G \in \mathfrak{A}$, every $\langle g, g^h \rangle$ has a proper subgroup of finite index and so is abelian (see (b)). Then for any $u \in G$, $\langle g^h : h \in G \rangle \langle u \rangle$ is non-Chernikov solvable and so is abelian (see (b)). Thus $g \in Z(G)$. Let $v \in G$ and $|\langle v \rangle| < \infty$. Then vg is of infinite order. So $v = (vg)g^{-1} \in Z(G)$. Thus G is abelian, which is a contradiction.

(d) *Show that $G/Z(G)$ is Shunkov.* Let $K/Z(G)$ be a finite subgroup of $G/Z(G)$. In view of Kalužnin's Theorem (see [5]),

$C_G(K/Z(G))/C_G(K)$ is abelian. So $N_G(K)/C_G(K)$ is almost abelian.

If $K/Z(G) \neq 1$, then $C_G(K) \neq G$ and so $C_G(K)$ is almost abelian. Thus $N_G(K)$ is almost solvable. So $N_G(K)$ is Chernikov or abelian (see (b)). Clearly, $N_G(K)/Z(G) = N_{G/Z(G)}(K/Z(G))$. Consequently, the quotient group $N_{G/Z(G)}(K/Z(G))/(K/Z(G))$ is Chernikov or abelian. Therefore any two its elements of prime order generate a finite subgroup.

Let $K/Z(G) = 1$ and $R/Z(G)$ be a subgroup of $G/Z(G)$ generated by two its conjugated element of some prime order p. Obviously, because of $Z(G)$ is periodic (see (c)), for some p-element$g \in G$ such that $g^p \in Z(G)$ and some $h \in G$, $R = \langle g, g^h \rangle Z(G)$. Clearly, $R/Z(G)$ is isomorphic to a quotient group of the group $\langle g, g^h \rangle / \langle g^p \rangle$. Since $G \in \mathfrak{A}$, $\langle g, g^h \rangle$ has a subgroup of finite index $\neq 1$. So, with regard to (b), $\langle g, g^h \rangle$ is finite. Therefore $R/Z(G)$ is finite.

(e) *Show that $G/Z(G)$ has an abelian non-Chernikov non-normal maximal subgroup $A/Z(G)$ such that*

(0.10)

$$A/Z(G) \cap (A/Z(G))^g = 1, \quad g \in (G/Z(G)) \backslash (A/Z(G)).$$

If all proper subgroups of $G/Z(G)$ are Chernikov, then it satisfies min. So because of $G/Z(G)$ is Shunkov (see (d)), by Suchkova-Shunkov Theorem [3] it is Chernikov. So G is almost solvable, which is a contradiction (see (b)). So some maximal abelian subgroup $A/Z(G)$ of $G/Z(G)$ is non-Chernikov. An arbitrary proper subgroup $H \supseteq A$ of G is non-Chernikov. So it is abelian. Therefore $H/Z(G) = A/Z(G)$ and $H = A$. Thus A is an abelian maximal subgroup of G. If A is normal in G, then $|G : A|$ is prime and G is solvable, which is a contradiction (see (b)). Consequently, for any $g \in G \backslash A$, $G = \langle A, A^g \rangle$. Then $A \cap A^g \subseteq Z(G)$. But, clearly, $Z(G) \subseteq A, A^g$. Thus $A \cap A^g = Z(G)$. Therefore (1) is valid.

(f) *Show that $A/Z(G)$ has some element a of odd prime order.* Suppose that this is not the case. Since $A/Z(G)$ is periodic (see (c)) and neither cyclic nor quasicyclic, it has some elements b and $c \neq b$ of order 2. Let $h \in (G/Z(G)) \setminus (A/Z(G))$. Then $\langle b, c^h \rangle = \langle bc^h \rangle \leftthreetimes \langle b \rangle = \langle bc^h \rangle \leftthreetimes \langle c^h \rangle$ and $|\langle bc^h \rangle| < \infty$ (see (c)). If $|\langle bc^h \rangle|$ is odd, then for some $s \in \langle bc^h \rangle$, $b = c^{hs}$. Since $A/Z(G)$ is abelian and $b, c \in A/Z(G)$, $hs \notin A/Z(G)$. But $b \in A/Z(G) \cap (A/Z(G))^{hs}$, which is a contradiction (see (1)). So $\langle bc^h \rangle$ contains some element w of order 2. But then $w \in C_{G/Z(G)}(b) \cap C_{G/Z(G)}(c^h) = A/Z(G) \cap (A/Z(G))^h$, which is a contradiction.

(g) *Final contradiction.* Let a be from (f). Since $G/Z(G)$ is Shunkov (see (d)), for any $h \in G/Z(G)$, $|\langle a, a^h \rangle| < \infty$. So with regard to (1) by Sozutov-Shunkov Theorem [6], for some normal subgroup $N/Z(G)$ of $G/Z(G)$, $G/Z(G) = (A/Z(G))(N/Z(G))$ and $A/Z(G) \cap N/Z(G) = 1$. Since $N \neq G$ and G/N is abelian, G is almost solvable, which is a contradiction (see (b)).

The following new proposition is contained in Theorem.

Proposition 1. *Let G be a nonabelian group. Assume that G is locally graded or periodic Shunkov. Then G satisfies* $\min -\overline{ab}$ *iff it is Chernikov.*

In view of Mal'cev Theorem (see Theorem 4.2 [7]), a linear group over a field is locally residually finite. Further, for a commutative and associative ring R with 1 and any finitely generated unital module M over R, $\mathrm{Aut}_R(M)$ is hyperabelian-by-residually (linear over fields) (Theorem 13.5 [7]). Consequently, $\mathrm{Aut}_R(M)$ is locally graded. Hence follows that any

$\mathbf{GL}_n(R)$ is locally graded. Therefore, in virtue of Theorem, the following proposition is valid.

Proposition 2. *A (nonabelian) group* $G \subseteq \mathrm{Aut}_R(M)$ *or* $G \subseteq \mathbf{GL}_n(R)$ *satisfies* min *(resp.* $\min -\overline{ab}$*) iff it is Chernikov.*

Finally, let $\mathfrak{C}$ be the class of all groups G for which the following conditions are fulfilled:

(i) If G is not torsion-free, then for any $p \in \pi(G)$ and p-element $g \neq 1$ of G and $h \in C_G(g^p)$, $\langle g, g^h \rangle$ possesses a $\mathfrak{B}$-homomorphic image $\neq 1$ or $\langle g, h \rangle$ is not periodic.

(ii) If G is not periodic, then for any element g of infinite order of G and $h \in G$, $\langle g, g^h \rangle$ possesses a $\mathfrak{B}$-homomorphic image $\neq 1$.

Let $\mathfrak{F}$ be the class of all groups G for which the following condition is fulfilled:

If G is periodic $\neq 1$, then for any $p \in \pi(G)$ and p-element $g \neq 1$ of G and $h \in C_G(g^p)$, $\langle g, g^h \rangle$ possesses a $\mathfrak{D}$-homomorphic image $\neq 1$.

Proposition 3. *A (nonabelian) group* $G \in \mathfrak{F}$ *(resp.* $G \in \mathfrak{C}$*) satisfies* min *(resp.* $\min -\overline{ab}$*) iff it is Chernikov.*

Proof. *Necessity.* A corresponding homomorphic image of $\langle g, g^h \rangle$ is generated by two elements. In the case when it is not abelian, by Theorem, it is finite. In the case when it is abelian, it has a subgroup of finite index $\neq 1$. Consequently, $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$. So $G \in \mathfrak{C}$ (resp. $G \in \mathfrak{A}$). Therefore by Theorem, G is Chernikov.

Sufficiency is obvious.

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