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A CLASS OF DISTAL FUNCTIONS ON SEMITOPOLOGICAL SEMIGROUPS

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Abstract. The norm closure of the algebra generated by the set

$$\{n \mapsto \lambda^{n^k} : \lambda \in \mathbb{T} \text{ and } k \in \mathbb{N}\}$$

of functions on $(\mathbb{Z}, +)$ was studied in [11] (and was named as the Weyl algebra). In this paper, by a fruitful result of Namioka, this algebra is generalized for a general semitopological semigroup and, among other things, it is shown that the elements of the involved algebra are distal. In particular, we examine this algebra for $(\mathbb{Z}, +)$ and (more generally) for the discrete (additive) group of any countable ring. Finally, our results are treated for a bicyclic semigroup.

1. Introduction

Distal functions on topological groups were extensively studied by A. W. Knapp [7]. The norm closure of the algebra generated by the set $F = \{n \mapsto \lambda^{n^k} \colon \lambda \in \mathbb{T} \text{ and } k \in \mathbb{N}\}$ of functions on $(\mathbb{Z}, +)$ was called the Weyl algebra by E. Salehi in [11]. A. W. Knapp [7], showed that all of the elements of F are distal on $(\mathbb{Z}, +)$. Also I. Namioka [9, Theorem 3.6] proved the same result by using a very fruitful result ([9, Theorem 3.5]) which played an important role for the construction of this paper. By the above mentioned results of A. W. Knapp and I. Namioka, all elements of the Weyl algebra are distal, however it dose not exhaust all distal functions on $(\mathbb{Z}, +)$, [11, Theorem 2.14]. In this paper, we generalize the notion of Weyl algebra to an arbitrary semitopological semigroup and also we show that all elements of the involved algebra are distal. In particular, our method provides a convenient way to deduce a result of M. Filali [4] on the distality of the functions $\chi(q(t))$, where χ is a character on the discrete additive group of a (countable) ring R and $q(t)$ is a polynomial with coefficients in R.

2. Preliminaries

For the background materials and notations we follow J. F. Berglund et al. [1] as much as possible. For a semigroup S, the right translation r_t and the left translation l_s on S are defined by $r_t(s) = st = l_s(t)$, $(s, t \in S)$. The semigroup S, equipped with a topology, is said to be right topological if all of the right translations are continuous, semitopological if all of the left and right translations are continuous. If S is a right topological semigroup then the set $\Lambda(S) = \{s \in S : l_s \text{ is continuous}\}$ is called the topological centre of S.

Throughout this paper, unless otherwise stated, S is a semitopological semigroup. The space of all bounded continuous complex valued functions on S is denoted by $C(S)$. For $f \in C(S)$ and $s \in S$ the right (respectively, left) translation of f by s is the function $R_s f = f \circ r_s$ (respectively, $L_s f = f \circ l_s$).

A left translation invariant unital C^*-subalgebra F of $C(S)$ (i.e., $L_s f \in F$ for all $s \in S$ and $f \in F$) is called m-admissible if the function $s \mapsto (T_\mu f)(s) = \mu(L_s f)$ belongs to F

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for all $f \in F$ and $\mu \in S^F$(=the spectrum of F). If F is m-admissible then S^F under the multiplication $\mu\nu = \mu \circ T_\nu$ $(\mu, \nu \in S^F)$, furnished with the Gelfand topology is a compact Hausdorff right topological semigroup and it makes S^F a compactification (called the F-compactification) of S.

Some of the usual m-admissible subalgebras of $C(S)$, that are needed in the sequel, are the left multiplicatively continuous, strongly almost periodic and distal functions on S. These are denoted by $LMC(S)$, $SAP(S)$ and $D(S)$, respectively. Here and also for other emerging spaces when there is no risk of confusion, we have suppressed the letter S from the notation.

The interested reader may refer to [1] for ample information about these m-admissible subalgebras and the properties of their corresponding compactifications.

3. Main results

The idea of defining our new algebras W_k and W, in the form given below, came from a nice result of I. Namioka [9, Theorem 3.5].

Let $\Sigma = \{T_\mu : LMC(S) \to LMC(S); \mu \in S^{LMC}\}$. Let F_0 be the set of all constant functions of modulus 1. For every $k \in \mathbb{N}$ assume that we have defined F_i for $i = 1, 2, \ldots, k-1$ and define F_k by

$$F_k = \{f \in LMC : |f| = 1 \text{ and for every } \sigma \in \Sigma, \sigma(f) = f_\sigma f, \text{ for some } f_\sigma \in F_{k-1}\}.$$

It is clear from definitions that $F_k \subseteq F_{k+1}$, for all $k \in \mathbb{N} \cup \{0\}$. Let W_k and W be the norm closure of the algebras generated by F_k and $\bigcup_{k\in\mathbb{N}} F_k$ in $C(S)$, respectively; then trivially $W_k \subseteq W_{k+1} \subseteq W$. Hence, W is the uniform closure of the algebra $\bigcup_{k\in\mathbb{N}} W_k$. It is also readily verified that W is the direct limit of the family $\{W_i : \ i \in \mathbb{N}\}$ (ordered by inclusion, and with the inclusion maps as morphisms). From now on, we assume that $k \in \mathbb{N}$ is arbitrary. We leave the following simple observations without proof.

Proposition 3.1. *(i) For every $f \in F_k$ and $\sigma \in \Sigma$ the function f_σ with the property $\sigma(f) = f_\sigma f$ is unique.*

(ii) For every $f, g \in F_k$ and $\sigma \in \Sigma$, $(fg)_\sigma = f_\sigma g_\sigma$. In particular, F_k is a multiplicative subsemigroup of LMC.

(iii) F_k is conjugate closed; in other words, if $f \in F_k$ then $\overline{f} \in F_k$.

(iv) F_k contains the constant functions.

Lemma 3.2. *The set F_k is left translation invariant and it is also invariant under Σ; in other words, $L_S(F_k) \subseteq F_k$ and $\Sigma(F_k) \subseteq F_k$.*

Proof. A direct verification reveals that F_k is left translation invariant. For the invariance under Σ let $f \in F_k$ and $\sigma \in \Sigma$, the equality $\sigma(f) = f_\sigma f$ for some $f_\sigma \in F_{k-1}$ implies that $|\sigma(f)| = 1$ and so for every $\tau \in \Sigma$,

$$\tau(\sigma(f)) = \tau(f_\sigma f) = \tau(f_\sigma)\tau(f) = ((f_\sigma)_\tau f_\sigma)(f_\tau f) = ((f_\sigma)_\tau f_\tau)(f_\sigma f) = ((f_\sigma)_\tau f_\tau)\sigma(f).$$

Since $(f_\sigma)_\tau f_\tau \in F_{k-1}$ we have $\sigma(f) \in F_k$; in other words $\Sigma(F_k) \subseteq F_k$, as required. $\square$

Lemma 3.3. *All elements of F_k remain fixed under the idempotents of Σ.*

Proof. It is easily seen that the result holds for $k = 1$. Assume that $k > 1$ and that the result holds for $k-1$. Let $f \in F_k$ and let $\varepsilon \in \Sigma$ be an idempotent, then $\varepsilon(f) = f_\varepsilon f$ for some $f_\varepsilon \in F_{k-1}$. Therefore

$$f_\varepsilon f = \varepsilon(f) = \varepsilon^2(f) = \varepsilon(\varepsilon(f)) = \varepsilon(f_\varepsilon f) = \varepsilon(f_\varepsilon)\varepsilon(f) = f_\varepsilon(f_\varepsilon f) = {f_\varepsilon}^2 f,$$

hence $f_\varepsilon = 1$ and $\varepsilon(f) = f$, as claimed. $\square$

Lemma 3.4. $F_k \subseteq D$.

Proof. Let $f \in F_k$. To show that $f \in D$, using [1, Lemma 4.6.2], it is enough to show that $\varepsilon\sigma(f) = \sigma(f)$ for each $\sigma \in \Sigma$ and each idempotent ε in Σ. By Lemma 3.2, $\sigma(f) \in F_k$, so that Lemma 3.3 implies that $\varepsilon(\sigma(f)) = \sigma(f)$, as required. $\square$

Using parts (iii) and (iv) of Proposition 3.1, W_k and W are unital C^*-subalgebras of $C(S)$. The following result shows that these are indeed m-admissible subalgebras of D.

Theorem 3.5. *For every semitopological semigroup S, W_k and W are m-admissible subalgebras of $D(S)$.*

Proof. For the m-admissibility of W_k it is enough to show that it is left translation invariant and also invariant under Σ. Let $\langle F_k\rangle$ be the algebra generated by F_k. Lemma 3.2 implies that $L_S(\langle F_k\rangle) \subseteq \langle F_k\rangle$ and also $\Sigma(\langle F_k\rangle) \subseteq \langle F_k\rangle$. For every $f \in W_k$ there exists a sequence $\{f_n\} \subseteq \langle F_k\rangle$ which converges (in the norm of $C(S)$) to f. Let $\sigma \in \Sigma$ and $s \in S$, then the inequalities $\|L_s(f_n) - L_s(f)\| \leq \|f_n - f\|$ and $\|\sigma(f_n) - \sigma(f)\| \leq \|f_n - f\|$ imply that $L_s(f_n) \to L_s(f)$ and $\sigma(f_n) \to \sigma(f)$, respectively. Since for each $n \in \mathbb{N}$, $L_s(f_n)$ and $\sigma(f_n)$ lie in $\langle F_k\rangle$, we have $L_s(f) \in W_k$ and also $\sigma(f) \in W_k$. It follows that W_k is m-admissible. A similar argument may apply for the m-admissibility of W. The fact that W_k and W are contained in D follows trivially from Lemma 3.4. $\square$

The next result gives S^W in terms of the subdirect product of the family $\{S^{W_k} : k \in \mathbb{N}\}$. For a full discussion of the subdirect product of compactifications one may refer to [1, Section 3.2].

Proposition 3.6. *The compactification S^W is the subdirect product of the family $\{S^{W_k} : k \in \mathbb{N}\}$; in symbols, $S^W = \bigvee\{S^{W_k} : k \in \mathbb{N}\}$.*

Proof. The family (of homomorphisms) $\{\pi_k : S^W \to S^{W_k}; k \in \mathbb{N}\}$, where for each $\mu \in S^W$, $\pi_k(\mu) = \mu|_{W_k}$, separates the points of S^W, because for given $\mu, \nu \in S^W$ with $\mu \neq \nu$ (on W) one has $\mu \neq \nu$ of $F = \bigcup_{k\in\mathbb{N}} F_k$, hence there exists a natural number j and an element $f \in F_j$ such that $\mu(f) \neq \nu(f)$. Therefore $\mu|_{W_j} \neq \nu|_{W_j}$, that is $\pi_j(\mu) \neq \pi_j(\nu)$. Now the conclusion follows from [1, Theorem 3.2.5]. $\square$

Proposition 3.7. (i) *For every abelian semitopological semigroup S, $SAP(S) \subseteq W_k(S)$.*

(ii) *For every abelian semitopological semigroup S with a left identity, $W_1(S) = SAP(S)$.*

Proof. (i) Since S is abelian $SAP(S)$ is the closed linear span of the set of all continuous characters on S; see [1, Corollary 4.3.8]. Let f be any continuous character on S and let $\sigma \in \Sigma$. Then there exists a net $\{s_\alpha\}$ in S such that $\sigma(f) = \lim_\alpha R_{s_\alpha} f$. By passing to a subnet, if necessary, we may assume that $f(s_\alpha)$ converges to some $\lambda_\sigma \in \mathbb{T}$. Therefore for every $s \in S$, $\sigma(f)(s) = \lim_\alpha R_{s_\alpha} f(s) = \lim_\alpha f(ss_\alpha) = f(s)\lim_\alpha f(s_\alpha) = f(s)\lambda_\sigma$. Hence $\sigma(f) = \lambda_\sigma f$ or equivalently $f \in F_1$.

(ii) By part (i) it is enough to show that $W_1 \subset SAP$. Indeed we are going to show that $F_1 \subset SAP$; for this end, let $f \in F_1$ and let $s \in S$, then $R_s f = \lambda_s f$ for some λ_s in $\mathbb{T}$. Let e be a left identity of S, then for each s in S, $f(s) = R_s f(e) = \lambda_s f(e)$. Let $h = f/f(e)$, then h is a continuous character on S. But $f = f(e)h$, now using the fact that SAP is the closed linear span of continuous characters of S we have $f \in SAP$, as required. $\square$

As a consequence of the latter result we have

Corollary 3.8. *For any compact abelian topological group G, $W_k(G) = W(G) = C(G)$.*

Proof. Since for every compact topological group G, $SAP(G) = C(G)$, [1, Theorem 4.3.5], the result follows from the last proposition. $\square$

4. Examples

Example (a). Here we restrict our discussion to the discrete group $(\mathbb{Z}, +)$ and examine W and W_k for this particular case, which were studied extensively by E. Salehi in [11]. Note that although we would deal with countable discrete rings in part (b), but the proofs on $\mathbb{Z}$ are more interesting and characterizations of $F_k(\mathbb{Z})$ are more explicit. We commence with the next key lemma which characterizes F_k in $l^\infty(\mathbb{Z})$.

Lemma 4.1. *The set $F_k(\mathbb{Z},+)$ is the (multiplicative) sub-semigroup of $l^\infty(\mathbb{Z})$ generated by the set $\{n \mapsto \lambda^{n^i} : \lambda \in \mathbb{T},\ i = 0, 1, \ldots, k\}$.*

Proof. For each $k \in \mathbb{N}$, let A_k denote the multiplicative sub-semigroup of $l^\infty(\mathbb{Z})$ generated by the set $\{n \mapsto \lambda^{n^i},\ \lambda \in \mathbb{T}$ and $i = 0, 1, \ldots, k\}$. For $k = 1$ a direct verification reveals that $A_1 \subseteq F_1$; for the reverse inclusion let $f \in F_1$. Then for some $\lambda \in \mathbb{T}$, $R_1 f = \lambda f$, hence $f(1) = R_1 f(0) = \lambda f(0) = \lambda\lambda_1$, in which $\lambda_1 = f(0)$. Also $f(2) = R_1 f(1) = \lambda f(1) = \lambda^2\lambda_1$, by induction it is easily proved that for each $n \in \mathbb{N}$, $f(n) = \lambda^n\lambda_1$. Let $R_{-1}f = \beta f$, where $\beta \in \mathbb{T}$, then $f(-1) = R_{-1}f(0) = \beta f(0)$. But $f(1) = R_{-1}f(2) = \beta f(2) = \beta\lambda^2\lambda_1$, therefore $\lambda\lambda_1 = \beta\lambda^2\lambda_1$, hence $\beta = \lambda^{-1}$ and for all $n \in \mathbb{Z}$, $f(n) = \lambda^n\lambda_1$. Thus $f \in A_1$, and so $F_1 = A_1$.

Let $k \geq 2$ and assume that $A_{k-1} = F_{k-1}$. Let $n \in \mathbb{Z}$ and $\lambda \in \mathbb{T}$ and let $f \in A_k$ and assume (without loss of generality) that $f(n) = \lambda^{n^k}$, then for given $\sigma = \lim_\alpha m_\alpha \in \Sigma$ we have $\sigma(f)(n) = \lim_\alpha R_{m_\alpha} f(n) = \lim_\alpha \lambda^{(n+m_\alpha)^k} = f(n)f_\sigma(n)$, in which $f_\sigma(n) = \mu_1^{n^{k-1}}\mu_2^{n^{k-2}}\ldots\mu_{k-1}^{n}\mu_k$, where (by going through a sub-net of $\{m_\alpha\}$, if necessary) $\mu_i = \lim_\alpha \lambda^{\binom{k}{i}{m_\alpha}^i}$, for $i = 1, 2, \ldots, k$. But $f_\sigma \in A_{k-1} = F_{k-1}$, so $f \in F_k$. Hence $A_k \subseteq F_k$. Now let $f \in F_k$, we have to show that $f \in A_k$. We have $R_1 f = f_1 f$, for some $f_1 \in F_{k-1} = A_{k-1}$. Since $f_1 \in A_{k-1}$ we may assume that $f_1(n) = \lambda_1^{n^{k-1}}\lambda_2^{n^{k-2}}\ldots\lambda_{k-1}^{n}\lambda_k$, where $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{T}$. Then $f(1) = R_1 f(0) = f_1(0)f(0)$ and $f(2) = R_1 f(1) = f_1(1)f(1) = f_1(1)f_1(0)f(0)$, and by induction,

$$\begin{aligned} f(n) &= (\lambda_1^{(n-1)^{k-1}}\lambda_2^{(n-1)^{k-2}}\ldots\lambda_{k-1}^{n-1}\lambda_k)(\lambda_1^{(n-2)^{k-1}}\lambda_2^{(n-2)^{k-2}}\ldots\lambda_{k-1}^{n-2}\lambda_k) \\ &\qquad\qquad \ldots(\lambda_1\lambda_2\ldots\lambda_k)(\lambda_k)f(0) \\ &= \lambda_1^{\sum_{j=1}^{n-1} j^{k-1}}\lambda_2^{\sum_{j=1}^{n-1} j^{k-2}}\ldots\lambda_{k-1}^{\sum_{j=1}^{n-1} j}\lambda_k^n f(0). \end{aligned}$$

So for each $i = 0, 1, 2, \ldots, k-1$ the power of λ_{k-i} is a polynomial in n of degree $i+1$. Hence the power of λ_1 (which has the maximum degree) is a polynomial of degree k. It follows that $f \in A_k$ and the proof is complete by induction. $\square$

As an immediate consequence of the latter lemma we have the next theorem which characterizes our algebras for the additive group $\mathbb{Z}$.

Theorem 4.2. (i) *$W_k(\mathbb{Z},+)$ coincides with the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i} : \lambda \in \mathbb{T},\ i = 0, 1, \ldots, k\}$.*

(ii) *$W(\mathbb{Z},+)$ coincides with the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^k} : \lambda \in \mathbb{T}$, and $k \in \mathbb{N}\}$, that is, $W(\mathbb{Z},+)$ coincides with the Weyl algebra.*

Example (b). Let R be a countable discrete ring. Let χ be an arbitrary character on the additive group $(R_d,+)$, where R_d denotes R with the discrete topology. Without loss of generality, assume that R is abelian. We are going to show that for each s in R the function $f(t) = \chi(st^3)$ belongs to $F_3(R_d,+)$. To this end, let $\sigma \in \Sigma$. Thus there exists a sequence s_n in R such that for each $h \in l^\infty(R_d)$, $\sigma(h)(t) = \lim_n R_{s_n}h(t) = \lim_n h(t+s_n)$.

Thus $\sigma f(t) = \lim_n f(t+s_n) = \lim_n \chi(s(t+s_n)^3) = f(t)f_\sigma(t)$, in which $f_\sigma(t) = \lim_n \chi(ss_n^3 + 3ss_nt^2 + 3ss_n^2t)$. Since R is countable, by the diagonal process, there exists a subsequence, say s_n, of the sequence s_n such that, for all t in R, all of the limits $\lim_n \chi(ss_n^3)$, $\lim_n \chi(ss_nt^2)$ and $\lim_n \chi(ss_n^2t)$ exist. (In fact, one may first choose a subsequence of s_n, if necessary, such that $\lim_n \chi(ss_n^3)$ exist. Let $R = \{x_1, x_2, x_3, \ldots\}$. Choose a subsequence of s_n, say $s_{1,n}$ such that both limits $\lim_n \chi(ss_{1,n}x_1^2)$ and $\lim_n \chi(ss_{1,n}^2x_1)$ exist. This time, choose a subsequence of $s_{1,n}$, say $s_{2,n}$, such that both limits

$\lim_n \chi(ss_{2,n}x_2^2)$ and $\lim_n \chi(ss_{2,n}^2 x_2)$ exist. Continue this process and choose the resulting sequence $s_{n,n}$ on the diagonal, which is eventually our desired subsequence, say s_n). Hence for each $t \in R$,

$$f_\sigma(t) = \lim_n \chi(ss_n^3) \lim_n \chi^3(ss_n^2 t) \lim_n \chi^3(ss_n t^2).$$

By definition, to prove that $f \in F_3$ it is enough to show that $f_\sigma \in F_2$. To see this, let $\tau \in \Sigma$ be arbitrary. Then there exists a sequence t_m in R such that for each $h \in l^\infty(R_d)$, $\tau(h)(t) = \lim_m R_{t_m} h(t) = \lim_m h(t + t_m)$.

Thus $\tau f_\sigma(t) = \lim_m f_\sigma(t + t_m) = f_\sigma(t) f_{\sigma\tau}(t)$, where

$$f_{\sigma\tau}(t) = (f_\sigma)_\tau(t) = \lim_m \lim_n \chi^3(ss_n^2 t_m) \lim_m \lim_n \chi^3(ss_n t_m^2 + 2ss_n t_m t).$$

R is countable, hence by going through a subsequence of t_m (by using the diagonal process) one may assume that for all t in R the limits $\lim_m \lim_n \chi(ss_n t_m^2)$ and $\lim_m \lim_n \chi(ss_n t_m t)$ exist. Therefore

$$f_{\sigma\tau}(t) = \lim_m \lim_n \chi^3(ss_n^2 t_m) \lim_m \lim_n \chi^3(ss_n t_m^2) \lim_m \lim_n \chi^6(ss_n t_m t).$$

Again by definition, to prove that $f_\sigma \in F_2$ it is enough to show that $f_{\sigma\tau} \in F_1$. Let $\xi = \lim_l u_l \in \Sigma$, then it follows from the above equation that

$$\xi(f_{\sigma\tau})(t) = f_{\sigma\tau}(t) \lim_l \lim_m \lim_n \chi^6(ss_n t_m u_l).$$

That is, $\xi(f_{\sigma\tau}) = \lambda f_{\sigma\tau}$ where $\lambda = \lim_l \lim_m \lim_n \chi^6(ss_n t_m u_l) \in F_0 = \mathbb{T}$. Hence $f_{\sigma\tau} \in F_1$ and so $f_\sigma \in F_2$ and this implies that $f \in F_3$. Our claim is now established. By using the above method, one may prove that for each $k \in \mathbb{N}$ and $s \in R$ the function $t \to \chi(st^k)$ is an element of F_k.

Briefly speaking, the above argument and Lemma 3.4 imply that

Corollary 4.3. *If R is a countable discrete ring, then for each character χ on the discrete additive group of R the function $\chi(q(t))$, in which $q(t)$ is a polynomial with coefficients in R, belongs to $W(R_d, +)$ and is also a distal function.*

It should be remarked that the distality of the functions $\chi(q(t))$ was first proved by M. Filali [4] without the countability condition on R.

Example (c). (i) If S contains a right zero element, i.e. there exists $t \in S$ such that $st = t$ for all $s \in S$, then for $f \in F_1$ there exists $\lambda_t \in \mathbb{T}$ such that $R_t f = \lambda_t f$, hence for all $s \in S$, $f(t) = f(st) = R_t f(s) = \lambda_t f(s)$, that is $f = f(t)/\lambda_t \in \mathbb{T}$. Therefore $F_1 = \mathbb{T}$ and so $F_k = \mathbb{T}$ for all $k \in \mathbb{N}$. It follows that for such a semigroup S, $W_k(S) = W(S) =$ the set of constant functions.

(ii) If S is a left zero semigroup (i.e. $st = s$ for all $s, t \in S$), then for each function $f \in LMC(S)$ we have $\sigma(f) = f$ for all σ in Σ, and so if $|f| = 1$, then $f \in F_1$. Hence for all $k \in \mathbb{N}$, $W = W_k = W_1 = LMC$.

Now we examine some of the newly defined algebras on a non-trivial non-group semigroup.

Example (d). Let S be the bicyclic semigroup of [1, Example 2.10], i.e. S is a semigroup generated by elements 1, p and q, where 1 is the identity and p and q satisfy $pq = 1 \neq qp$. The relation $pq = 1$ implies that any member of S may be written uniquely in the form $q^m p^n$, where $m, n \in \mathbb{Z}^+$ and $p^0 = q^0 = 1$.

We are going to show that

$$F_1(S) = \{q^m p^n \mapsto \mu^r \nu^{1-r} : \ r = m - n \text{ and } \mu,\ \nu \in \mathbb{T}\}.$$

To see this, let $f \in F_1$, then for each $s \in S$, $R_s f = \lambda_s f$ for some $\lambda_s \in \mathbb{T}$. Hence $f(q) = \lambda_q f(1)$, $f(p) = \lambda_p f(1)$ and $f(1) = f(pq) = R_q f(p) = \lambda_q f(p)$, therefore $\lambda_p \lambda_q = 1$. It is also readily seen that $\lambda_{q^m p^n} = \lambda_q^{m-n}$. One may use induction to simply prove that for each $n \in \mathbb{Z}^+$, $f(qp^n) = f(q)(f(p)/f(1))^n$, and then use the latter to show (again by induction on m) that $f(q^m p^n) = f(p)^{n-m} f(1)^{1-(n-m)}$. But $f(p)f(q) = f(1)^2$, so

$f(q^m p^n) = f(q)^{m-n} f(1)^{1-(m-n)}$. Hence it is enough to take $\mu = f(q)$ and $\nu = f(1)$. The converse inclusion is easily verified.

To prove the next theorem, the following lemma is needed.

Lemma 4.4. *Let S be the bicyclic semigroup described above. If $f \in F_1(S)$, then $f(p)f(q) = f(1)^2$.*

Proof. Let $f \in F_1(S)$, then there exist constants λ_p and $\lambda_q \in \mathbb{T}$ such that $R_p f = \lambda_p f$ and $R_q f = \lambda_q f$. Hence $f(p)f(q) = \lambda_p \lambda_q f(1)^2$. Thus it is enough to show that $\lambda_p \lambda_q = 1$. But, $f(1) = f(pq) = R_q f(p) = \lambda_q f(p) = \lambda_q \lambda_p f(1)$, that is $\lambda_q \lambda_p = 1$, and the result follows. □

Theorem 4.5. *Let S be the bicyclic semigroup generated by $1, p$ and q, where 1 is the identity and $pq = 1 \neq qp$. Then $W_1(S)$ and $W_2(S)$ are the norm closure of the algebras generated by the sets*

$$\{q^m p^n \mapsto \mu^r \nu^{1-r} :\ r = m - n \text{ and } \mu,\ \nu \in \mathbb{T}\}$$

and

$$\{q^m p^n \mapsto \lambda^{(r^2-r)/2} \mu^{(r^2+r)/2} \nu^{1-r^2},\ r = m-n \text{ and } \lambda,\ \mu,\ \nu \in \mathbb{T}\}$$

respectively.

Proof. By what we already discussed, the result is clear for $W_1(S)$. To complete the proof, it is enough to show that

$$F_2(S) = \{q^m p^n \mapsto \lambda^{(r^2-r)/2} \mu^{(r^2+r)/2} \nu^{1-r^2},\ r = m-n \text{ and } \lambda,\ \mu,\ \nu \in \mathbb{T}\}.$$

Let A denote the right hand side of the above equation and let $f \in A$. Then there exist $\lambda, \mu, \nu \in \mathbb{T}$ such that for all $m, n \in \mathbb{Z}^+ \cup \{0\}$, $f(q^m p^n) = \lambda^{(r^2-r)/2} \mu^{(r^2+r)/2} \nu^{1-r^2}$ with $r = m - n$. By choosing suitable $m, n \in \mathbb{Z}^+ \cup \{0\}$ we derive that $\lambda = f(p)$, $\mu = f(q)$ and $\nu = f(1)$. To prove $f \in F_2(S)$ is to prove that there exist elements f_p and f_q in $F_1(S)$ such that $R_p f = f_p f$ and $R_q f = f_q f$. In fact it is enough to take

$$f_p(q^m p^n) = [f(q)^{-1} f(1)]^r [f(p) f(1)^{-1}]^{1-r}$$

and

$$f_q(q^m p^n) = [f(p) f(q)^2 f(1)^{-3}]^r [f(q) f(1)^{-1}]^{1-r},$$

where $r = m - n$. Our discussion preceding the theorem reveals that both f_p and f_q are elements of $F_1(S)$, therefore $f \in F_2(S)$.

Conversely, let $f \in F_2(S)$. To show $f \in A$ is to show that for all $m, n \in \mathbb{Z}^+ \cup \{0\}$,

$$(*) \qquad f(q^m p^n) = f(p)^{(r^2-r)/2} f(q)^{(r^2+r)/2} f(1)^{1-r^2}, \quad \text{where} \quad r = m - n.$$

Since $f \in F_2$, there exists $f_q \in F_1(S)$ such that $R_q f = f_q f$. Therefore (from the above Lemma)

$$(I) \qquad f_q(p) f_q(q) = f_q(1)^2.$$

On the other hand, $f_q(1) = f(q) f(1)^{-1}$, and also $f(1) = f(pq) = R_q f(p) = f_q(p) f(p)$, thus $f_q(p) = f(p)^{-1} f(1)$. Hence it follows from (I) that

$$(II) \qquad f_q(q) = f(p) f(q)^2 f(1)^{-3}.$$

Fix $n \in \mathbb{N}$, then by using induction on m and the fact that $f(q^{m+1} p^n) = R_q f(q^m p^n) = f_q(q^m p^n) f(q^m p^n) = f_q(q)^{m-n} f_q(1)^{1-(m-n)} f(q^m p^n)$, we deduce from (II) and $(*)$ that

$$f(q^{m+1} p^n) = f(p)^{(s^2-s)/2} f(q)^{(s^2+s)/2} f(1)^{1-s^2}, \quad \text{where} \quad s = (m+1) - n.$$

The theorem is now established by induction. □

Corollary 4.6. *Let S be the bicyclic semigroup generated by $1, p$ and q, where 1 is the identity and $pq = 1 \neq qp$. If either p or q is an idempotent, then $W(S) = W_i(S) = \mathbb{C}$, for all i.*

Proof. It is enough to show that $F_1(S) = \mathbb{T}$. To this end, let $f \in F_1(S)$, and assume that $p^2 = p$. Then there exist $\lambda_p \in \mathbb{T}$ such that $R_p f = \lambda_p f$, hence $f(p) = f(p^2) = R_p f(p) = \lambda_p f(p)$ and so $\lambda_p = 1$. That is, $R_p f = f$ and $f(p) = f(1)$. It follows from Lemma 4.4 that $f(q) = f(1)$. Now the first part of the above theorem implies that for all $m, n \in \mathbb{Z}^+ \cup \{0\}$, $f(q^m p^n) = f(q)^{m-n} f(1)^{1-(m-n)} = f(1)$. Hence $f = f(1) \in \mathbb{T}$. The proof for the case where q is an idempotent is similar. □

Remarks. (i) By the results 3.5 and 3.7, for every abelian semitopological semigroup S, W_k and also W lie between SAP and D. It would be desirable to study the structure of the (right topological abelian group) compactifications S^{W_k} and S^W. In particular, it would be more desirable if one could investigate the size of the topological centres of S^{W_k} and S^W. The latter problem is of particular interest among some authors, (the interested reader is referred to [3, 2, 5, 6, 10]). For an elegant characterization of the topological centre of the largest compactification of a locally compact group one may refer to [8] and also [10].

(ii) In [11, Theorem 2.13] and [5, Corollary 3.3.3], (by using different methods) it is proved that all elements of the Weyl algebra $W(\mathbb{Z}, +)$ are uniquely ergodic. One may seek the same result for $W(S)$, where S is an arbitrary semitopological semigroup.

(iii) It would be quite interesting to find a general formula for $F_k(S)$ in Theorem 4.5.

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