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ALGEBRAS OF UNBOUNDED OPERATORS OVER THE RING OF MEASURABLE FUNCTIONS AND THEIR DERIVATIONS AND AUTOMORPHISMS

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Abstract. In the present paper derivations and $*$-automorphisms of algebras of unbounded operators over the ring of measurable functions are investigated and it is shown that all L^0-linear derivations and L^0-linear $*$-automorphisms are inner. Moreover, it is proved that each L^0-linear automorphism of the algebra of all linear operators on a *bo*-dense submodule of a Kaplansky-Hilbert module over the ring of measurable functions is spatial.

0. Introduction

The theory of derivations and automorphisms of operator algebras is an important branch of the theory of operator algebras and mathematical physics. The present paper is devoted to the study of derivations and automorphisms of the algebras of unbounded operators over the ring of measurable functions. Derivations on the algebras of bounded operators are rather well-investigated [1]. A certain method of investigation of derivations was suggested in [3], where it was proved that any derivation of a standard algebra of bounded operators on a normed space is inner and any automorphism of such algebra is spatial.

A survey of results and open problems in the theory of derivations on unbounded operators algebras were given in [2]. Later the existence of non-inner derivations on the algebra $L(M)$ of measurable operators affiliated with an abelian von Neumann algebra M was established in [4]. Recently it was proved [5] that in the algebra of (equivalence classes of) measurable complex functions on a locally separable measure space there exist non trivial derivations and non-extendable automorphisms which are not identical.

Derivations and automorphisms of special classes of unbounded operator algebras (so-called O^*-algebras) were considered in [6], in particular it was proved that all derivations and all $*$-automorphisms of the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D})$ are inner and every automorphism of the algebra $\mathcal{L}(\mathcal{D})$ is spatial. In the present paper we study derivations and automorphisms of standard algebras of unbounded L^0-linear operators and obtain L^0-valued versions of the above results from [6].

It should be noted that L^0-valued analogues of some classic results become very useful in solving problems of classical operator algebras. For example, in [7] the theory of Kaplansky-Hilbert modules over L^0 has been applied for the investigation of derivations on algebras of τ-measurable operators affiliated with a type I von Neumann algebra and faithful normal semi-finite trace τ.

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The Section 1 contains preliminaries from the theory of Kaplansky-Hilbert modules over L^0. In Section 2 we develop the theory of unbounded L^0-linear operators on Kaplansky-Hilbert modules over L^0 and introduce and study notions such as O-modules, O^*-modules, O-algebras, O^*-algebras for the L^0-valued case. Further we show that every L^0-linear derivation of the algebra $\mathcal{L}^+(\mathcal{D})$ is inner and each automorphism of the algebra $\mathcal{L}(\mathcal{D})$ is spatial. We also consider $*$-isomorphisms of O^*-algebras over the ring of measurable functions and prove that every L^0-linear $*$-isomorphism between O^*-algebras is spatial and each L^0-linear $*$-automorphism of the algebra $\mathcal{L}^+(\mathcal{D})$ is inner.

1. Kaplansky-Hilbert modules over the ring of measurable functions

Let (Ω, Σ, μ) be a space with a complete finite measure, and let $L^0 = L^0(\Omega)$ be the algebra of all measurable complex-valued functions on (Ω, Σ, μ) (functions equal almost everywhere are identified).

Consider a vector space X over the field $\mathbb{C}$ of complex numbers. A map $\|\cdot\| : X \longrightarrow L^0$ is called an L^0-valued norm on X, if for any $\varphi, \psi \in X$, $\lambda \in \mathbb{C}$ the following conditions are fulfilled:
1) $\|\varphi\| \geq 0$;
2) $\|\varphi\| = 0 \Longleftrightarrow \varphi = 0$;
3) $\|\lambda\varphi\| = |\lambda|\|\varphi\|$;
4) $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$.

The pair $(X, \|\cdot\|)$ is said to be a *lattice-normed* space (shortly, LNS) over L^0. An LNS X is called d-decomposable, if for any $\varphi \in X$ and for each decomposition $\|\varphi\| = e_1 + e_2$ into the sum of disjoint elements there exist $\varphi_1, \varphi_2 \in X$ such that $\varphi = \varphi_1 + \varphi_2$ and $\|\varphi_1\| = e_1, \|\varphi_2\| = e_2$. A d-decomposable norm is also called a *Kantorovich norm.* A net $(\varphi_\alpha)_{\alpha\in A}$ of element from X is called (bo)-*convergent* to $\varphi \in X$, if the net $(\|\varphi_\alpha - \varphi\|)_{\alpha\in A}$ (o)-converges to zero in L^0 (recall that (o)-convergence of a net from L^0 is equivalent to its convergent almost everywhere). A *Banach-Kantorovich space* (further, BKS) over L^0 is a (bo)-complete d-decomposable LNS over L^0.

Any BKS X over L^0 is a module over L^0, i.e. for any $\lambda \in L^0$ and $\varphi \in X$ the element $\lambda\varphi \in X$ is determined and $\|\lambda\varphi\| = |\lambda|\|\varphi\|$ (see [8, 9]).

A module E over L^0 is said to be *finite-generated,* if there exist $\varphi_1, \varphi_2, \ldots, \varphi_n$ in E such that every $\varphi \in E$ can be decomposed as $\varphi = \alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n$ where $\alpha_i \in L^0, i = \overline{1, n}$. The elements $\varphi_1, \varphi_2, \ldots, \varphi_n$ are called *generators* of the module E. A minimal number of generators of a finite-generated module E is denoted by $d(E)$. A module E over L^0 is called σ-*finite-generated,* if there exists a partition $(\pi_n)_{n\in\mathbb{N}}$ of the unit in ∇ (∇ is the Boolean algebra of all idempotents in L^0) such that each $\pi_n E$ is finite-generated. A finite-generated module E over L^0 is called *homogeneous of type* n, if $n = d(\pi E)$ for every nonzero $\pi \in \nabla$.

Elements $\varphi_1, \varphi_2, \ldots, \varphi_n \in E$ are called ∇-*linear independent*, if for every $\pi \in \nabla$ and any $\alpha_1, \alpha_2, \ldots, \alpha_n \in L^0$ the equality $\pi \sum\limits_{k=1}^{n} \alpha_k\varphi_k = 0$ implies $\pi\alpha_1 = \pi\alpha_2 = \cdots = \pi\alpha_n = 0$ (see [7]).

If E is module over L^0 which is a homogeneous of type n then there exists a basis $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ in E, consisting of ∇-linear independent elements, i.e. each element $\varphi \in E$ can be uniquely represented in the form $\varphi = \alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n$, $\alpha_i \in L^0, i = \overline{1, n}$ (see [10], Proposition 6).

Let X and Y be BKS over L^0. An operator $a : X \to Y$ is L^0-*linear* if $a(\alpha\varphi + \beta\psi) = \alpha a(\varphi) + \beta a(\psi)$ for all $\alpha, \beta \in L^0, \varphi, \psi \in X$. The set of all L^0-*linear* operators is denoted by $\mathcal{L}(X, Y)$. An operator $a \in \mathcal{L}(X, Y)$ is called L^0-*bounded*, if there exists $c \in L^0$ such that $\|a(\varphi)\| \leq c\|\varphi\|$ for all $\varphi \in X$. For an L^0-bounded operator a we put $\|a\| = \sup\{\|a(\varphi)\| : \|\varphi\| \leq \mathbf{1}\}$. An L^0-linear operator $a : X \to Y$ is said to be *finite-generated* (respectively,

σ-finite-generated, homogeneous of type n), if $a(X) = \{a(\varphi) : \varphi \in X\}$ is a finite-generated (respectively, σ-finite-generated, homogeneous of type n) submodule in Y.

It is clear that each L^0-linear σ-finite-generated operator $a : X \to Y$ can be represented as $a = \sum\limits_{n=1}^{\infty} \pi_n a_n$, where $(\pi_n)_{n\in\mathbb{N}}$ is a partition of the unit ∇, and a_n are homogeneous operators of finite type. Moreover if a is a finite-generated operator then (π_n) is a finite partition of unit.

Let $a : X \to Y$ be a homogeneous of type n L^0-linear operator and let $\{\psi_1, \dots, \psi_n\}$ be a basis in $a(X)$. Denote by X^* the space of all L^0-bounded L^0-linear functionals from X into L^0. Then there exists a system $\{f_1, \dots, f_n\} \subset Y^*$ such that $f_i(\psi_j) = \delta_{ij}\mathbf{1}$, where δ_{ij} is Kroenecker symbol (see [10], Proposition 2). We define $g_i \in X^*, i = \overline{1,n}$ as follows

$$g_i(\varphi) = f_i(a(\varphi)), \quad \varphi \in X.$$

It is clear that

$$a(\varphi) = \sum_{k=1}^{n} g_k(\varphi)\psi_k, \quad \varphi \in X.$$

This formula gives the general form of L^0-bounded L^0-linear operators from X into Y which are homogeneous of type $n (n \in \mathbb{N})$.

If X and Y coincide then $\mathcal{L}(X)$ is used for $\mathcal{L}(X, X)$.

An algebra $\mathcal{U} \subset \mathcal{L}(X)$ over L^0 is said to be *standard* if $\mathcal{F}(X) \subset \mathcal{U}$, where $\mathcal{F}(X)$ is the algebra of all finite-generated L^0-linear operators from $\mathcal{L}(X)$. The following algebras over L^0 are examples of standard algebras: the algebra $\mathcal{F}(X)$; the algebra $\mathcal{F}_\sigma(X)$ of all σ-finite-generated L^0-linear operators from $\mathcal{L}(X)$; the algebra $\mathcal{K}(X)$ of all L^0-linear cyclically compact operators from $\mathcal{L}(X)$; the whole algebra $\mathcal{L}(X)$.

Let $\mathcal{A}$ be a module over L^0. A map $\langle\cdot,\cdot\rangle : \mathcal{A} \times \mathcal{A} \to L^0$ is called an *L^0-valued inner product*, if for all $\varphi, \psi, \eta \in \mathcal{A}$, $\lambda \in L^0$ the following conditions are fulfilled:
1) $\langle\varphi,\varphi\rangle \geq 0$;
2) $\langle\varphi,\varphi\rangle = 0 \Leftrightarrow \varphi = 0$;
3) $\langle\varphi,\psi\rangle = \overline{\langle\psi,\varphi\rangle}$;
4) $\langle\lambda\varphi,\psi\rangle = \lambda\langle\varphi,\psi\rangle$;
5) $\langle\varphi+\psi,\eta\rangle = \langle\varphi,\eta\rangle + \langle\psi,\eta\rangle$.

If $\langle\cdot,\cdot\rangle : \mathcal{A} \times \mathcal{A} \to L^0$ is an L^0-valued inner product then the following formula

$$\|\varphi\| = \sqrt{\langle\varphi,\varphi\rangle}$$

determines an *L^0-valued norm* on $\mathcal{A}$. A pair $\langle\mathcal{A}, \langle\cdot,\cdot\rangle\rangle$ is called *Kaplansky-Hilbert module* over L^0 or *L^0-Hilbert space* if $(\mathcal{A}, \|\cdot\|)$ is BKS over L^0 (see [8, 9]).

Let X be a Kaplansky-Hilbert module over L^0, and $X_0 \subset X$. Note that X_0 is a *bo-closed submodule* of the Kaplansky-Hilbert module X if and only if X_0 is a submodule in the usual sense, i.e. X_0 is a set containing all sums of the form *bo*-$\sum\limits_{\alpha\in A} \pi_\alpha\varphi_\alpha$, where $(\varphi_\alpha)_{\alpha\in A}$ is any bounded family in X_0 and $(\pi_\alpha)_{\alpha\in A}$ is a partition of the unit in ∇, and it is also closed with respect to the norm of the module X.

Let I be an index set. For every $i \in I$ consider a Kaplansky-Hilbert module X_i over L^0. Put $X_I = \{\varphi \in \prod\limits_{i\in I} X_i : (o)\text{-}\sum\limits_{i\in I} \|\varphi_i\|_i^2 \in L^0\}$. Considered with the pointwise operations, X_I forms a module over L^0. The inner product $\langle\cdot,\cdot\rangle : X_I \times X_I \to L^0$ is defined as follows:

$$\langle\varphi,\psi\rangle = (o)\text{-}\sum_{i\in I}\langle\varphi_i,\psi_i\rangle_i,$$

where $\varphi, \psi \in X_I$ and $\langle\cdot,\cdot\rangle_i : X_i \times X_i \to L^0$ is the inner product in the corresponding X_i. Then $\|\varphi\| = \sqrt{\langle\varphi,\varphi\rangle}$ gives an L^0-valued norm on X_I, and it clear that $\|\varphi\| = ((o)\text{-}\sum\limits_{i\in I}\langle\varphi_i,\varphi_i\rangle_i)^{1/2}$. Besides X_I equipped with this structure forms a Kaplansky-Hilbert module over L^0. We say that X_I is the direct sum of the family $(X_i)_{i\in I}$ and denote it by $\bigoplus_{i\in I} X_i$.

Let X_1, X_2 be Kaplansky-Hilbert modules over L^0, and let a be an operator from X_1 into X_2. The domain of the operator a is denoted by $\mathcal{D}(a)$. The set of all pairs $(\varphi, a\varphi)$, $\varphi \in \mathcal{D}(a)$, in the direct sum $X_1 \oplus X_2$, is called the *graph* of the operator a. The graph of the operator a is denoted by $G(a)$. Thus

$$G(a) = \{(\varphi, a\varphi) : \ \varphi \in D(a)\}.$$

It is clear that two operators a and b coincide if and only if $G(a) = G(b)$. The set $S \subset X_1 \oplus X_2$ is the graph of an appropriate operator if and only if the relations $(\varphi, \psi) \in S$, $(\varphi, \psi') \in S$ imply $\psi = \psi'$. An operator $a : X_1 \to X_2$ is L^0-linear if and only if $G(a)$ is a submodule of $X_1 \oplus X_2$. An operator $a : X_1 \to X_2$ is called *bo-closed* if its graph $G(a)$ *bo*-closed in $X_1 \oplus X_2$.

If an operator a is not *bo*-closed then by the definition its graph $G(a)$ is not *bo*-closed in $X_1 \oplus X_2$. If the *bo*-closure $\overline{G(a)}$ of the set $G(a)$ in $X_1 \oplus X_2$ is the graph of some operator, then this operator is denoted by $\widetilde{a}$ and it is called the *bo-closure* of a. In this case the operator a is said to be *bo-closable* operator.

Note that $\widetilde{a}$ is the least *bo*-closed extension of the operator a. The set $\overline{G(a)}$, which is the graph of the operator $\widetilde{a} : X_1 \to X_2$, consists of elements of the form $(\varphi, a\varphi)$, $\varphi \in \mathcal{D}(a)$ and their *bo*-limits.

For a Kaplansky-Hilbert module X over L^0 an L^0-valued version of the Riesz theorem is also true , i.e. for every L^0-bounded L^0-linear functional $f : X \to L^0$ there exists a vector $\psi \in X$ such that $f(\varphi) = \langle \varphi, \psi\rangle$ for all $\varphi \in X$ (see [9]).

Let $a : X \to Y$ be an L^0-linear operator. An *adjoint* operator to a is an operator $a^* : Y \to X$, satisfying the condition $\langle a\varphi, \psi\rangle = \langle \varphi, a^*\psi\rangle$ for all $\varphi \in X$ and $\psi \in Y$.

Let $\varphi, \ \psi \in X$. We define an L^0-linear operator $\varphi \otimes \psi$ on X by the rule

$$(\varphi \otimes \psi)\eta = \langle \eta, \psi\rangle\varphi.$$

An element $\lambda \in L^0$ is called *strictly positive* (denoted by $\lambda \gg 0$) if $\lambda(\omega) > 0$ for almost every $\omega \in \Omega$. If $\|\varphi\| \gg 0$, $\|\psi\| \gg 0$, then the operator $\varphi\otimes\psi$ is homogeneous of type one. Moreover, $\varphi \otimes \psi$ is a projection if and only if $\psi = \varphi$ and $\|\varphi\| = \mathbf{1}$.

2. Derivations and automorphisms of O^*-algebras over L^0

Let X be a Kaplansky-Hilbert module over L^0, and let $\mathcal{D} \subset X$ be a dense domain. By $I_{\mathcal{D}}$ we denote the identity map on $\mathcal{D}$.

Definition 1. A set of *bo*-closable L^0-linear operators with the domain $\mathcal{D}$ and containing $I_{\mathcal{D}}$ is said to be an *O-family* over L^0. In this case $\mathcal{D}$ is called the *domain* of this family.

If $\mathcal{A}$ is an O-family over L^0 then the domain of this family will be denoted by $\mathcal{D}(\mathcal{A})$. If $a \in \mathcal{A}$ then according to the definition we have $\mathcal{D}(\mathcal{A}) = \mathcal{D}(a) = \mathcal{D}$.

Definition 2. An *O-module* over L^0 is an O-family $\mathcal{A}$ over L^0 such that $\alpha a + \beta b \in \mathcal{A}$ for all $a, \ b \in \mathcal{A}$ and $\alpha, \ \beta \in L^0$.

Recall that by ab we denote the composition of the operators a and b. If a and b are operators on $\mathcal{D}$ and $b\mathcal{D} \subset \mathcal{D}$ then ab is also an operator on $\mathcal{D}$ defined by $ab\varphi = a(b\varphi)$, $\varphi \in \mathcal{D}$.

Definition 3. An *O-algebra* over L^0 is an O-module $\mathcal{A}$ over L^0 such that $b\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ and $ab \in \mathcal{A}$ for all $a, \ b \in \mathcal{A}$.

It is easy to see that every O-algebra over L^0 with the operations of addition, multiplication by elements of L^0 and the product defined as the composition of operators, is an algebra over L^0. Note also that $I_{\mathcal{D}}$ is the unit of this algebra.

Definition 4. An O^*-*family* over L^0 on $\mathcal{D}$ is a set $\mathcal{A}$ of L^0-linear operators with the domain $\mathcal{D}$ such that $I_{\mathcal{D}} \in \mathcal{A}$, $\mathcal{D} \subset \mathcal{D}(a^*)$, and $a^+ \in \mathcal{A}$ for all $a \in \mathcal{A}$, where $a^+ = a^*|\mathcal{D}$.

Let $\mathcal{A}$ be an O^*-family over L^0 on $\mathcal{D}$. Then $\mathcal{A}$ is an O-family over L^0 on $\mathcal{D}$. Indeed, each operator $a \in \mathcal{A}$ is *bo*-closable because $\mathcal{D} \subset \mathcal{D}(a^*)$ and $\mathcal{D}$ is dense in X.

If $a \in \mathcal{A}$ then

$$\langle a\varphi, \psi\rangle = \langle \varphi, a^+\psi\rangle \quad \text{for all} \quad \varphi, \psi \in \mathcal{D} \tag{1}$$

and hence $a = (a^+)^+$. From the above we obtain, in particular, that $a \to a^+$ is a bijective map of $\mathcal{A}$ onto itself.

Definition 5. An O^*-*module* over L^0 is an O-module over L^0 which is an O^*-family over L^0.

If $\mathcal{A}$ is an O^*-module over L^0 on $\mathcal{D}$ then the map $a \to a^+$, $a \in \mathcal{A}$, is an involution on $\mathcal{A}$.

Definition 6. An O^*-*algebra* over L^0 is an O-algebra over L^0 which is an O^*-family over L^0.

Let $\mathcal{L}^+(\mathcal{D})$ denote the set of all L^0-linear operators a on a Kaplansky-Hilbert module X over L^0 which satisfy $a\mathcal{D} \subset \mathcal{D}$, $\mathcal{D} \subset \mathcal{D}(a^*)$ and $a^*\mathcal{D} \subset \mathcal{D}$.

Theorem 1. *$\mathcal{L}^+(\mathcal{D})$ is the largest O^*-algebra over L^0 with the domain $\mathcal{D}$.*

Proof. At first we check that $\mathcal{L}^+(\mathcal{D})$ is an O^*-family over L^0. Let $a \in \mathcal{L}^+(\mathcal{D})$. We have $a^+\mathcal{D} = a^*\mathcal{D} \subset \mathcal{D}$, $(a^+)^* = (a^*|\mathcal{D})^* \supset a^{**} \supset a$, and hence $(a^+)^*\mathcal{D} = a\mathcal{D} \subset \mathcal{D}$, i.e. $a^+ \in \mathcal{L}^+(\mathcal{D})$, as it was asserted.

Now let us show that $\mathcal{L}^+(\mathcal{D})$ is an O-algebra over L^0. Let $a, b \in \mathcal{L}^+(\mathcal{D})$. It is easy to see that $\lambda a \in \mathcal{L}^+(\mathcal{D})$ for all $\lambda \in L^0$. From $\mathcal{D} \subset \mathcal{D}(a^*) \cap \mathcal{D}(b^*) \subset \mathcal{D}((a+b)^*)$ and $(a+b)^*\mathcal{D} = (a^* + b^*)\mathcal{D}$ it follows that $(a+b) \in \mathcal{L}^+(\mathcal{D})$.

We shall show that $ab \in \mathcal{L}^+(\mathcal{D})$. Let $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{D}$. According to (1) we have $\langle ab\varphi, \psi\rangle = \langle b\varphi, a^+\psi\rangle$. By virtue of $a^+\mathcal{D} \subset \mathcal{D}$, applying again (1), we obtain $\langle ab\varphi, \psi\rangle = \langle \varphi, b^+a^+\psi\rangle$. Besides, $b^+a^+ \subset (ab)^*$ and $b^+a^+ = (b^*|\mathcal{D})(a^*|\mathcal{D}) = (b^*a^*)|\mathcal{D} = (ab)^*|\mathcal{D} = (ab)^+$. These imply that $\mathcal{D} \subset \mathcal{D}((ab)^*)$, $(ab)^*\mathcal{D} = b^+a^+\mathcal{D} \subset \mathcal{D}$. Thus, $ab \in \mathcal{L}^+(\mathcal{D})$.

From the above it is clear that $\mathcal{L}^+(\mathcal{D})$ is an O^*-algebra over L^0.

Now let $\mathcal{A}$ be an arbitrary O^*-algebra over L^0 with the domain $\mathcal{D}$ and let $a \in \mathcal{A}$. According to the definition 3 we have $a\mathcal{D} \subset \mathcal{D}$ since $\mathcal{A}$ is an O-algebra. The definition 4 yields that $a^+ \in \mathcal{A}$ since $\mathcal{A}$ is an O^*-algebra. Hence, $a^*\mathcal{D} = a^+\mathcal{D} \subset \mathcal{D}$. This means that $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$. Theorem 1 is proved. □

Let X be a Kaplansky-Hilbert module over L^0, and let $\mathcal{D} \subset X$ be a *bo*-dense submodule. By the symbol $\mathcal{L}(\mathcal{D})$ we denote the algebra of all L^0-linear operators $a : \mathcal{D} \to \mathcal{D}$. Let $\mathcal{U}$ be a standard algebra in $\mathcal{L}(\mathcal{D})$. Recall that a linear operator $\delta : \mathcal{U} \to \mathcal{L}(\mathcal{D})$ is said to be a *derivation*, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{U}$. If for a derivation $\delta : \mathcal{U} \to \mathcal{L}(\mathcal{D})$ there exists an element $x \in \mathcal{U}$ such that $\delta(a) = xa - ax$ for all $a \in \mathcal{U}$ then δ is called an *inner derivation.*

Further in theorems 2 and 3 we suppose that there exists a vector e in the *bo*-dense submodule $\mathcal{D}$ of the Kaplansky-Hilbert module X over L^0 such that $\|e\| = \mathbf{1}$, where $\mathbf{1}$ is the unit in L^0.

Theorem 2. *Let $\delta : \mathcal{U} \to \mathcal{L}(\mathcal{D})$ be an L^0-linear derivation of a standard algebra $\mathcal{U}$. Then there exists $x \in \mathcal{L}(\mathcal{D})$ such that*

$$\delta(a) = xa - ax$$

for all $a \in \mathcal{U}$.

Proof. At first consider the case $\mathcal{U} = \mathcal{F}(\mathcal{D})$, where $\mathcal{F}(\mathcal{D})$ is the algebra of finite-generated operators $a : \mathcal{D} \to \mathcal{D}$.

Fix a vector $e \in \mathcal{D}$ with $\|e\| = \mathbf{1}$ and a functional $f : \mathcal{D} \to L^0$ such that $f(e) = \mathbf{1}$. Define a projection $p \in \mathcal{F}(\mathcal{D})$ by

$$p(\varphi) = f(\varphi)e, \quad \varphi \in \mathcal{D}.$$

Since $p^2 = p$ then $\delta(p) = p\delta(p) + \delta(p)p$ and therefore $p\delta(p)p = 0$. Put $\psi = p\delta(p) - \delta(p)p$. Then $p\psi - \psi p = p\delta(p) + \delta(p)p = \delta(p)$.

Putting $\delta'(a) = \delta(a) - (a\psi - \psi a)$ we get $\delta'(p) = 0$. Thus, one may assume that $\delta(p) = 0$. Then we have

$$\delta(ap) = a\delta(p) + \delta(a)p = \delta(a)p. \tag{2}$$

Consider a vector $\varphi \in \mathcal{D}$ and an operator $a \in \mathcal{F}(\mathcal{D})$ such that $a(e) = \varphi$. Define an operator $x : \mathcal{D} \to \mathcal{D}$ by the formula

$$x(\varphi) = \delta(a)e.$$

The operator x is defined correctly. Indeed, let $\varphi \in \mathcal{D}$ be a vector and let $a_1,\ a_2 \in \mathcal{F}(\mathcal{D})$ be operators such that $a_1(e) = a_2(e) = \varphi$. For each $\eta \in \mathcal{D}$ we have $(a_ip)\eta = f(\eta)a_i(e)$, $i = 1,\ 2$, i.e. $a_1p = a_2p$. Therefore by virtue of (2) it follows that $\delta(a_1)(e) = (\delta(a_1)p)(e) = \delta(a_1p)(e) = \delta(a_2p)(e) = (\delta(a_2)p)(e) = \delta(a_2)(e)$, i.e. $\delta(a_1) = \delta(a_2)$.

It easy to see that the operator x is L^0-linear.

Let $\varphi \in \mathcal{D}$ and $a \in \mathcal{F}(\mathcal{D})$. Then $(xap)\varphi = x(a(p(\varphi))) = x(f(\varphi)a(e)) = f(\varphi)x(a(e)) = f(\varphi)\delta(a)(e) = \delta(a)p(\varphi) = \delta(ap)\varphi$. Thus, $xap = \delta(a)p$ for all $a \in \mathcal{F}(\mathcal{D})$. Therefore for $b \in \mathcal{F}(\mathcal{D})$ we have $xabp = \delta(ab)p = a\delta(b)p + \delta(a)bp = axbp + \delta(a)bp$, i.e.

$$\delta(a)bp = xabp - axbp. \tag{3}$$

Now for an arbitrary $\varphi \in \mathcal{D}$ take $b \in \mathcal{F}(\mathcal{D})$ such that $b(e) = \varphi$. Then $(bp)(e) = \varphi$. Hence from (3) we obtain $\delta(a) = xa - ax$ for all $a \in \mathcal{F}(\mathcal{D})$.

Let now $\mathcal{U} \subset \mathcal{L}(\mathcal{D})$ be an arbitrary standard algebra and take $b \in \mathcal{U}$. Then $ba \in \mathcal{F}(\mathcal{D})$ for all $a \in \mathcal{F}(\mathcal{D})$. Therefore

$$\delta(ba) = xba - bax. \tag{4}$$

On the other hand according to the definition of derivation we have

$$\delta(ba) = \delta(b)a + b\delta(a) = \delta(b)a + b(xa - ax). \tag{5}$$

From (4) and (5) we obtain $\delta(b)a = xba - bxa = (xb - bx)a$.

Now for an arbitrary $\varphi \in \mathcal{D}$ take $a \in \mathcal{F}(\mathcal{D})$ such that $a(\varphi) = \varphi$. Then $\delta(b)(\varphi) = \delta(b)(a(\varphi)) = (\delta(b)a)(\varphi) = ((xb-bx)a)(\varphi) = (xb-bx)(a(\varphi)) = (xb-bx)(\varphi)$, i.e. $\delta(b)(\varphi) = (xb - bx)(\varphi)$ for any $\varphi \in \mathcal{D}$. This means that $\delta(b) = xb - bx$ for all $b \in \mathcal{U}$. Theorem 2 is proved. □

Replacing $\mathcal{F}(\mathcal{D})$ by $\mathcal{F}^+(\mathcal{D}) := \mathcal{F}(\mathcal{D}) \cap \mathcal{L}^+(\mathcal{D})$ and $\mathcal{L}(\mathcal{D})$ by $\mathcal{L}^+(\mathcal{D})$, we get

Corollary 1. *Let $\delta : \mathcal{U} \to \mathcal{L}^+(\mathcal{D})$ be an L^0-linear derivation of the algebra $\mathcal{U} \supset \mathcal{F}^+(\mathcal{D})$, where $\mathcal{D}$ is a bo-dense submodule of a Kaplansky-Hilbert module X with a vector $e \in \mathcal{D}$ with $\|e\| = \mathbf{1}$. Then there exists $x \in \mathcal{L}^+(\mathcal{D})$ such that*

$$\delta(a) = xa - ax$$

for all $a \in \mathcal{U}$. In particular each L^0-linear derivation of the algebra $\mathcal{L}^+(\mathcal{D})$ over L^0 is inner.

Recall that a bijective linear operator $\alpha : \mathcal{L}(\mathcal{D}) \to \mathcal{L}(\mathcal{D})$ is called *automorphism* if $\alpha(ab) = \alpha(a)\alpha(b)$ for all $a,\ b \in \mathcal{L}(\mathcal{D})$.

Theorem 3. *Let $\alpha : \mathcal{F}(\mathcal{D}) \to \mathcal{F}(\mathcal{D})$ be an L^0-linear automorphism of the algebra $\mathcal{F}(\mathcal{D})$. Then there exists $x \in \mathcal{L}(\mathcal{D})$ such that $x^{-1} \in \mathcal{L}(\mathcal{D})$ and*

$$\alpha(a) = xax^{-1}$$

for all $a \in \mathcal{F}(\mathcal{D})$.

Proof. Let $e \in \mathcal{D}$ be a vector with $\|e\| = \mathbf{1}$ and let $f : \mathcal{D} \to L^0$ be an L^0-linear functional such that $\|e\| = \mathbf{1},\ f(e) = \mathbf{1}$. We define a projection $p \in \mathcal{F}(\mathcal{D})$ as follows

$$p(\varphi) = f(\varphi)e, \quad \varphi \in \mathcal{D}.$$

Then obviously $p(e) = e$. Moreover the projection $\alpha(p)$ is homogeneous of type one because α is an L^0-linear automorphism. Now take $e_1 \in \mathcal{D}$ such that $\|e_1\| = \mathbf{1},\ \alpha(p)(e_1) = e_1$.

We define an operator $x : \mathcal{D} \to \mathcal{D}$ as follows: for any $\varphi \in \mathcal{D}$ take an operator $a \in \mathcal{F}(\mathcal{D})$ such that $a(e) = \varphi$ and put

$$x(\varphi) = \alpha(a)(e_1), \quad \varphi \in \mathcal{D}.$$

Let $\varphi \in \mathcal{D}$ and take $a_1, a_2 \in \mathcal{F}(\mathcal{D})$ such that $a_1(e) = a_2(e) = \varphi$. For each $\psi \in \mathcal{D}$ we have $(a_ip)(\psi) = f(\psi)a_i(e)$, $i = 1, 2$, i.e. $a_1p = a_2p$. Therefore $\alpha(a_1)(e_1) = \alpha(a_1)\alpha(p)(e_1) = \alpha(a_1p)(e_1) = \alpha(a_2p)(e_1) = \alpha(a_2)\alpha(p)(e_1) = \alpha(a_2)(e_1)$. This means that x is defined correctly.

Obviously x is L^0-linear.

Now we shall show that x is a bijection. Let $\varphi_1,\ \varphi_2 \in \mathcal{D}$ such that $\varphi_1 \neq \varphi_2$. Choose $a_1,\ a_2 \in \mathcal{F}(\mathcal{D})$ such that $a_i(e) = \varphi_i,\ i = 1,\ 2$. Then $a_1 \neq a_2$, and hence $a_1p \neq a_2p$. Since a_ip, $i = 1, 2$, are one-generated operators and α is an automorphism then $\alpha(a_1)(e_1) = \alpha(a_1)\alpha(p)(e_1) = \alpha(a_1p)(e_1) \neq \alpha(a_2p)(e_1) = \alpha(a_2)\alpha(p)(e_1) = \alpha(a_2)(e_1)$. Hence, $x(\varphi_1) \neq x(\varphi_2)$. Now take $\psi \in \mathcal{D}$, and $a \in \mathcal{F}(\mathcal{D})$ such that $a(e_1) = \psi$. Put $b = \alpha^{-1}(a)$. Then for $\varphi = b(e)$ one has $x(\varphi) = \alpha(b)(e_1) = \alpha(\alpha^{-1}(a))(e_1) = a(e_1) = \psi$, i.e. $x(\varphi) = \psi$.

Let $\varphi \in \mathcal{D}$ and $a \in \mathcal{F}(\mathcal{D})$. Take $b \in \mathcal{F}(\mathcal{D})$ such that $b(e) = \varphi$. Then $(xa)(\varphi) = x(a(\varphi)) = x(ab(e)) = \alpha(ab)(e_1) = \alpha(a)\alpha(b)(e_1) = \alpha(a)x(\varphi)$. Thus, $xa = \alpha(a)x$, i.e. $\alpha(a) = xax^{-1}$ for all $a \in \mathcal{F}(\mathcal{D})$. Theorem 3 is proved. □

Corollary 2. *For each L^0-linear automorphism of a standard algebra $\mathcal{U}$ there exists $x \in \mathcal{L}(\mathcal{D})$ such that $x^{-1} \in \mathcal{L}(\mathcal{D})$ and*

$$\alpha(a) = xax^{-1}$$

for all $a \in \mathcal{U}$. In particular, each L^0-linear automorphism of the algebra $\mathcal{L}(\mathcal{D})$ is spatial.

Let $\mathcal{D}_1$, resp. $\mathcal{D}_2$ be (bo)-dense submodules in the Kaplansky-Hilbert modules X_1, resp. X_2 over L^0, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be $*$-subalgebras respectively in the O^*-algebras $\mathcal{L}^+(\mathcal{D}_1)$ and $\mathcal{L}^+(\mathcal{D}_2)$ over L^0.

Definition 7. An L^0-linear $*$-isomorphism $\pi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is said to be *spatial* if there exists an isometry $U : X_1 \xrightarrow{\text{on}} X_2$ such that

(i) $U\mathcal{D}_1 = \mathcal{D}_2$,

(ii) $\pi(a)\varphi = UaU^{-1}\varphi$ for all $\varphi \in \mathcal{D}_2,\ a \in \mathcal{A}_1$.

Then we say that π is *implemented* by the operator U.

An L^0-linear $*$-automorphism of an algebra $\mathcal{A}$ is called *inner*, if it is spatial and it may be implemented by a unitary operator U on a Kaplansky-Hilbert module X over L^0 such that $U|\mathcal{D} \in \mathcal{A}$, where $\mathcal{D}$ is a (bo)-dense submodule of X.

Let A be a module over L^0 and a $*$-algebra over L^0. The set of all projections in A is denoted by $I_{sa}(A)$. If $p_1, p_2 \in I_{sa}(A)$ then we write $p_1 \leq\ p_2$ if and only if $p_1p_2 = p_1$. The relation $\leq$ is a reflexive, antisymmetric and transitive relation in $I_{sa}(A)$. If there exists an L^0-linear $*$-isomorphism π from the algebra A onto a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$, and if p is a projection in A then $\widetilde{\pi(p)}$ is also a projection in $\mathcal{L}^+(\mathcal{D})$, i.e. $\widetilde{\pi(p)} \in \mathcal{L}^+(\mathcal{D})$ and $\widetilde{\pi(p)}^2 = \widetilde{\pi(p)}$. Obviously the relation $p_1 \leq p_2$ is equivalent to the usual relation $\widetilde{\pi(p_1)} \leq \widetilde{\pi(p_2)}$ between the projections $\widetilde{\pi(p_1)}$ and $\widetilde{\pi(p_2)}$. Let $H_1(A)$ denote the set of all homogeneous of type one projections of the algebra A. For $p_1, p_2 \in H_1(A)$ we shall write $p_1 \approx p_2$, if $p_1Ap_2 \neq \{0\}$. Further on, the elements of the set $H_1(A)$ will be called projections of rank one.

Let $\mathcal{D}_i$ be a (bo)-dense submodule of a Kaplansky-Hilbert module X_i over L^0 such that there exists $\varphi_i \in \mathcal{D}_i$, $\|\varphi_i\| = \mathbf{1}$, $i \in I$. By $\mathcal{D}_I$ we denote a (bo)-dense submodule of the Kaplansky-Hilbert module X_I over L^0, consisting of all vectors $(\varphi_i) := (\varphi_i)_{i\in I}$, which have only finitely many nonzero coordinates $\varphi_i \in \mathcal{D}_i$.

Note that every element $(a_i) := (a_i)_{i\in I}$ of the product $\prod\limits_{i\in I} \mathcal{L}^+(\mathcal{D}_i)$ is an operator on $\mathcal{D}_I$ which acts according to the formula:

$$(a_i)(\varphi_i) = (a_i\varphi_i), \quad (\varphi_i) \in \mathcal{D}_I.$$

The set of all such operators forms an O^*-algebra with the domain $\mathcal{D}_I$. This algebra is denoted by $\mathcal{L}^+(\mathcal{D}_i : i \in I)$

Lemma 1. *Let $\mathcal{A}$ be a $*$-subalgebra of the algebra $\mathcal{L}^+(\mathcal{D}_I)$ over L^0 and let $M(\mathcal{A})$ be the set of all projections $p \in H_1(\mathcal{A})$, for which the generators of the images $p\mathcal{D}_I$ have a unique nonzero coordinate. Then:*

(i) The set $M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$ consists of the projections of the form $\varphi_i \otimes \varphi_i$, where $\varphi_i \in \mathcal{D}_i$, $\|\varphi_i\| = \mathbf{1}$, $i \in I$. If $\varphi_i \otimes \varphi_i$ and $\psi_j \otimes \psi_j$ are two such operators then $\varphi_i \otimes \varphi_i \approx \psi_j \otimes \psi_j$ if and only if $i = j$.

(ii) $M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$ if and only if $\mathcal{A} \subseteq \mathcal{L}^+(\mathcal{D}_i : i \in I)$ and $\mathcal{F}(\mathcal{D}_i) \subseteq \mathcal{A}$ for all $i \in I$.

(iii) If $M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$ then on the set $M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$ the relation "$\approx$" corresponding to the $*$-algebra $\mathcal{A}$ coincides with the relation "$\approx$" corresponding to the $*$-algebra $\mathcal{L}^+(\mathcal{D}_i : i \in I)$.

(iv) The set $H_1(\mathcal{L}^+(\mathcal{D}_i : i \in I))$ of projections of rank one consists of all L^0-linear projections of the form (o)-$\sum\limits_{i\in I} \pi_i(\varphi_i \otimes \varphi_i)$, where $\varphi_i \in \mathcal{D}_i$, $\|\varphi_i\| = \mathbf{1}$, and $(\pi_i)_{i\in I}$ is a partition of the unit in ∇.

Proof. (i) From the definition it follows that the operators of the form $\varphi_i \otimes \varphi_i$, $\varphi_i \in \mathcal{D}_i$, $\|\varphi_i\| = \mathbf{1}$, $i \in I$, are projections of rank one.

Let $\varphi_i \otimes \varphi_i, \psi_j \otimes \psi_j \in M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$. If $i \neq j$ then $\varphi_i \otimes \varphi_i M(\mathcal{L}^+(\mathcal{D}_i : i \in I))\psi_j \otimes \psi_j = \{0\}$. This implies that $\varphi_i \otimes \varphi_i \approx \psi_j \otimes \psi_j$ if and only if $i = j$.

ii) Suppose that $M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I)$. At first we shall prove that $\mathcal{A} \subseteq \mathcal{L}^+(\mathcal{D}_i : i \in I)$. Fix $i \in I$. If we prove that $a\varphi \in \mathcal{D}_i$ for some $\varphi \in \mathcal{D}_i$ then by virtue of the linearity of the operator $a \in \mathcal{A}$ we have $a\psi \in \mathcal{D}_i$ for any $\psi \in \mathcal{D}_i$. Therefore without loss of generality we may suppose that $\|\varphi\| = \mathbf{1}$ and $a\varphi \neq 0$. Then $\varphi \otimes \varphi \in \mathcal{A}$ and hence $a + \varphi \otimes \varphi \in \mathcal{A}$. Apply the operator $a + \varphi \otimes \varphi$ to the element $\varphi \in \mathcal{D}_i :\quad (a + \varphi \otimes \varphi)(\varphi) = a(\varphi) + \varphi$. This implies that $a\varphi \in \mathcal{D}_i$.

Now let us show that $\mathcal{F}(\mathcal{D}_i) \subset \mathcal{A}$. For this it is enough to prove that $\varphi \otimes \psi \in \mathcal{A}$ for all unit elements $\varphi,\ \psi \in \mathcal{D}_i$ since each finite-generated operator from $\mathcal{F}(\mathcal{D}_i)$ may be represented as a linear combination of operators of rank one. Let $\varphi, \psi \in \mathcal{D}_i$ and $\|\varphi\| = \|\psi\| = \mathbf{1}$. By virtue of (i) we have $\varphi \otimes \varphi, \psi \otimes \psi \in M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$. From this it follows that the operators $\varphi \otimes \varphi,\ \psi \otimes \psi$ belong to $\mathcal{A}$ and hence

$$(\psi \otimes \psi)(\varphi \otimes \varphi) = \langle \psi, \varphi \rangle (\varphi \otimes \psi) \in \mathcal{A},$$

i.e. $(\varphi \otimes \psi) \in \mathcal{A}$.

The inverse statement is obvious.

(iii) If $M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I))$ then from (ii) it follows that $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D}_i : i \in I)$. Therefore according to (i) it is sufficient to show that $\psi \otimes \psi \mathcal{A} \varphi \otimes \varphi \neq \{0\}$ for all unit elements $\varphi, \psi \in \mathcal{D}_i$. Consider $\xi \otimes \xi \in \mathcal{A}$, where the vector $\xi \in \mathcal{D}_i$ is defined by the formula

$$\xi = \begin{cases} \frac{1}{\sqrt{2}}(\varphi + \psi), & \text{if } \langle \varphi, \ \psi \rangle = 0, \\ \varphi, & \text{in other cases.} \end{cases}$$

Then we have

$$(\psi \otimes \psi)(\xi \otimes \xi)(\varphi \otimes \varphi) = \langle \varphi, \xi \rangle \langle \psi, \xi \rangle \varphi \otimes \psi \neq 0,$$

i.e. $\psi \otimes \psi \mathcal{A} \varphi \otimes \varphi \neq \{0\}$.

(iv) Let $a = (a_i)_{i \in I} \in \mathcal{L}^+(\mathcal{D}_i : i \in I)$ be a projection of rank one. Then a_i is a projection in $\mathcal{L}^+(\mathcal{D}_i)$ for all $i \in I$. Since a is a projection of rank one there exist a partition $(\pi_i)_{i \in I}$ of the unit in ∇ and a vector $\varphi_i \in \mathcal{D}_i$, $\|\varphi_i\| = \mathbf{1}$, such that $a_i = \pi_i(\varphi_i \otimes \varphi_i)$. From this we have $a = (o)\text{-}\sum_{i \in I} \pi_i(\varphi_i \otimes \varphi_i)$. Lemma 1 is proved. □

Theorem 4. *Let $\mathcal{D}_i$ and $\mathcal{D}_j$ be (bo)-dense submodules of Kaplansky-Hilbert modules X_i $(i \in I)$ and X_j $(j \in J)$ over L^0, respectively, such that for each $i \in I$ and $j \in J$ there exist $e_i \in \mathcal{D}_i$ and $f_j \in \mathcal{D}_j$ with $\|e_i\| = \mathbf{1}$ and $\|f_j\| = \mathbf{1}$. Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-subalgebras of the algebras $\mathcal{L}^+(\mathcal{D}_I)$ and $\mathcal{L}^+(\mathcal{D}_J)$ over L^0, respectively, satisfying the following conditions*

$$M(\mathcal{A}) = M(\mathcal{L}^+(\mathcal{D}_i : i \in I)),$$
$$M(\mathcal{B}) = M(\mathcal{L}^+(\mathcal{D}_j : j \in J)).$$

Suppose that there exists an L^0-linear $*$-isomorphism π, mapping $\mathcal{A}$ onto $\mathcal{B}$. Then π is a spatial L^0-linear $*$-isomorphism. Moreover, there exist a partition (π_α) of the unit in ∇, bijective maps $\chi_\alpha : I \to J$ and surjective isometries $U_\alpha : X_I \to X_J$ such that $U = \sum_\alpha \pi_\alpha U_\alpha$ implements π and $U_\alpha(\pi_\alpha \mathcal{D}_i) = \pi_\alpha \mathcal{D}_{\chi_\alpha(i)}$ for all $i \in I$.

Proof. Since π is a $*$-isomorphism, it preserves the relation $\approx$ and $\pi(M(\mathcal{A})) \subset H_1(\mathcal{B})$. Hence

$$\pi(M(\mathcal{L}^+(\mathcal{D}_i : i \in I))) \subset H_1(\mathcal{L}^+(\mathcal{D}_j : j \in J)). \tag{6}$$

From (6) we have $\pi(\varphi_i \otimes \varphi_i) \in H_1(\mathcal{L}^+(\mathcal{D}_j : j \in J))$. By virtue of lemma 1, $\pi(\varphi_i \otimes \varphi_i)$ has the form $(o)\text{-}\sum_{j \in J} \pi_{ij}(\psi_{ij} \otimes \psi_{ij})$, where $(\pi_{ij})_{j \in J}$ is a partition of the unit in ∇ such that $(\pi_{ij})_{i \in I}$ is also a partition of the unit in ∇.

Since π is a $*$-isomorphism the cardinalities of the sets I and J are equal. Let $S(I, J)$ be the set of all bijections from I onto J. For each $\alpha \in S(I, J)$ put $\chi_\alpha(i) = \alpha(i)$ and $\pi_\alpha = \bigwedge_{i \in I} \pi_{i\chi_\alpha(i)}$. Then $\pi_\alpha \pi_{\alpha'} = 0$ at $\alpha \neq \alpha'$ and $\bigvee_\alpha \pi_\alpha = \mathbf{1}$. Indeed, if $\alpha \neq \alpha'$ then there exists $i_0 \in I$ such that $\alpha(i_0) \neq \alpha'(i_o)$. Then $\pi_{i_0\chi_\alpha(i_0)} \pi_{i_0\chi_{\alpha'}(i_0)} = 0$. From this it follows that $\pi_\alpha \pi_{\alpha'} = 0$ at $\alpha \neq \alpha'$. Further, $\bigvee_\alpha \pi_\alpha = \bigvee_\alpha (\bigwedge_{i \in I} \pi_{i\chi_\alpha(i)}) = \bigwedge_{i \in I} (\bigvee_{\alpha \pi_{i\chi_\alpha(i)}}) = \mathbf{1}$.

Suppose that $\varphi_i \in \mathcal{D}_i$, $\psi_{\chi_\alpha(i)} \in \mathcal{D}_j$, are unit elements such that $\pi(\pi_\alpha(\varphi_i \otimes \varphi_i)) = \pi_\alpha(\psi_{\chi_\alpha(i)} \otimes \psi_{\chi_\alpha(i)})$. We shall prove that

$$\|\pi_\alpha x \varphi_i\| = \|\pi_\alpha \pi(x) \psi_{\chi_\alpha(i)}\| \tag{7}$$

for any $x \in \mathcal{A}$. From the lemma 1 it follows that $x\varphi_i \in \mathcal{D}_i$ and hence $\pi_\alpha x \varphi_i \in \mathcal{D}_i$, $\pi_\alpha(x\varphi_i \otimes x\varphi_i) \in \mathcal{A}$. One has

$$\begin{aligned} \pi(\pi_\alpha(x\varphi \otimes x\varphi)) &= \pi_\alpha \pi(x(\varphi \otimes \varphi)x^+) \\ &= \pi_\alpha \pi(x)\pi(\varphi \otimes \varphi)\pi(x)^+ = \pi(x)\pi(\pi_\alpha(\varphi \otimes \varphi))\pi(x)^+ \\ &= \pi_\alpha(\pi(x)\psi_{\chi_\alpha(i)} \otimes \pi(x)\psi_{\chi_\alpha(i)}). \end{aligned} \tag{8}$$

If $\pi(x)\psi = 0$ then (7) is true. If $\pi(x)\psi_{\chi_\alpha(i)} \neq 0$ then

$$(\pi(\pi_\alpha(x\varphi_i \otimes x\varphi_i))^2 = \pi(\pi_\alpha(x\varphi \otimes \ x\varphi)^2) = \pi_\alpha\pi\|x\varphi_i\|^2(x\varphi_i \otimes \ x\varphi_i) \\ = \|\pi_\alpha x\varphi\|^2(\pi(x)\psi_{\chi_\alpha(i)} \otimes \pi(x)\psi_{\chi_\alpha(i)}). \tag{9}$$

On the other hand according to (8) we have

$$(\pi(x\varphi_i \otimes x\varphi_i))^2 = \|\pi_\alpha\pi(x)\psi_{\chi_\alpha(i)}\|^2(\pi(x)\psi_{\chi_\alpha(i)} \otimes \pi(x)\psi_{\chi_\alpha(i)}). \tag{10}$$

From the equalities (9) and (10) we obtain (7). If $i \in I$ then from (7) it follows that the equality

$$U_{\alpha i}(\pi_\alpha x\varphi_i) = \pi_\alpha\pi(x)\psi_{\chi_\alpha(i)}, \quad x \in \mathcal{A},$$

defines a unique norm preserving L^0-linear surjective map

$$U_{\alpha i} \colon \pi_\alpha\mathcal{A}\varphi_i \to \pi_\alpha\pi(\mathcal{A})\psi_{\chi_\alpha(i)} \equiv \pi_\alpha\mathcal{B}\psi_{\chi_\alpha(i)}.$$

By virtue of lemma 1 the inclusions $\mathcal{F}(\mathcal{D}_i) \subseteq \mathcal{A}|\mathcal{D}_i \subseteq \mathcal{L}^+(\mathcal{D}_i)$ are true. From this it follows that $\pi_\alpha\mathcal{A}\varphi_i = \pi_\alpha\mathcal{D}_i$. Similarly, $\pi_\alpha\mathcal{B}\psi_{\chi_\alpha(i)} = \pi_\alpha\mathcal{D}_{\chi_\alpha(i)}$. Thus, $U_\alpha(\pi_\alpha\mathcal{D}_i) = \pi_\alpha\mathcal{D}_{\chi_\alpha(i)}$, where $U_\alpha = \bigoplus\limits_{i\in I} U_{\alpha i}$. Since the index $i \in I$ is arbitrary it follows that $U_\alpha(\pi_\alpha\mathcal{D}_I) = \pi_\alpha\mathcal{D}_J$. Put $U = (o)\text{-}\sum\limits_\alpha \pi_\alpha U_\alpha$. It is clear that U is a surjective isometry from X_I onto X_J. For $a \in \mathcal{A}$ one has

$$\pi(a)(\pi_\alpha\pi(x)\psi_{\chi_\alpha(i)}) = \pi_\alpha\pi(ax)\psi_{\chi_\alpha(i)} = U_\alpha\pi_\alpha ax\varphi_i = U_\alpha aU_\alpha^{-1}(\pi_\alpha\pi(x)\psi_{\chi_\alpha(i)}),$$

i.e. $\pi(a)(\pi_\alpha\pi(x)\psi_{\chi_\alpha(i)}) = U_\alpha aU_\alpha^{-1}(\pi_\alpha\pi(x)\psi_{\chi_\alpha(i)})$ for all $x \in \mathcal{A}$, π_α and $i \in I$. The latter equality implies $\pi(a)\psi = UaU^{-1}\psi$ for all $\psi \in \mathcal{D}_J$ and $a \in \mathcal{A}$. Thus, π is spatial and it is implemented by U. Theorem 4 is proved. □

Corollary 3. *Let $\mathcal{D}_i$ and $\mathcal{D}_j$ be (bo)-dense submodules of Kaplansky-Hilbert modules X_i $(i \in I)$ and X_j $(j \in J)$ over L^0, respectively, such that for each $i \in I$ and $j \in J$ there exist $e_i \in \mathcal{D}_i$ and $f_j \in \mathcal{D}_j$ with $\|e_i\| = \mathbf{1}$ and $\|f_i\| = \mathbf{1}$. If π is an L^0-linear $*$-isomorphism from $\mathcal{L}^+(\mathcal{D}_i : i \in I)$ onto a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D}_J)$ such that $M(\pi(\mathcal{L}^+(\mathcal{D}_i : i \in I))) = M(\mathcal{L}^+(\mathcal{D}_j : j \in J))$ then π is a spatial L^0-linear $*$-isomorphism from $\mathcal{L}^+(\mathcal{D}_i : i \in I)$ onto $\mathcal{L}^+(\mathcal{D}_j : j \in J)$.*

Proof. Assume that $\mathcal{A} = \mathcal{L}^+(\mathcal{D}_i : i \in I)$ and $\mathcal{B} = \pi(\mathcal{A})$. Then according to theorem 4 π is spatial. By the properties of the isometry U listed in theorem 4 the map $a \mapsto UaU^{-1}$ is a surjection from $\mathcal{L}^+(\mathcal{D}_i : i \in I)$ onto $\mathcal{L}^+(\mathcal{D}_j : j \in J)$. The equality $\pi(a) = UaU^{-1}$ implies that $\pi(\mathcal{A}) = \mathcal{L}^+(\mathcal{D}_j : j \in J)$. Corollary 3 is proved. □

Corollary 4. *Let $\mathcal{D}$ be a bo-dense submodule of a Kaplansky-Hilbert module X over L^0 such that there exists $e \in \mathcal{D}$ with $\|e\| = \mathbf{1}$. Then each L^0-linear $*$-automorphism of the O^*-algebra $\mathcal{L}^+(\mathcal{D})$ is inner.*

Proof. Put $\mathcal{A} = \mathcal{B} = \mathcal{L}^+(\mathcal{D})$ and apply Theorem 4. Then every L^0-linear $*$-automorphism π of the algebra $\mathcal{L}^+(\mathcal{D})$ is spatial. If π is implemented by some U then $U\mathcal{D} = \mathcal{D}$ and $U^*\mathcal{D} = \mathcal{D}$. So $U|\mathcal{D} \in \mathcal{L}^+(\mathcal{D})$ and therefore by definition 7 π is inner. Corollary 4 is proved. □

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