UDC 517.9

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A GENERAL CLASS OF EVOLUTIONARY EQUATIONS

ЗАГАЛЬНИЙ КЛАС ЕВОЛЮЦІЙНИХ РІВНЯНЬ

Using observable quantities and state variable of a dynamical process, a general evolutionary equation is defined which unifies classical ordinary differential equations, partial differential equations, and hereditary systems of retarded and neutral type. Specific illustrations are given using transmission lines nearest neighbor coupled at the boundary and the theory of heat transfer in solids. Some spectral theory for linearization of the equations also is discussed.

За допомогою спостережуваних величин та змінної стану динамічного процесу визначено загальне еволюційне рівняння, що узагальнює класичні звичайні диференціальні рівняння, диференціальні рівняння з частинними похідними та спадкові системи із запізненням і системи нейтрального типу. Наведено специфічні ілюстрації з використанням ліній трансмісії із зчепленням „найближчих сусідів" на межі та теорії теплопереносу у твердих тілах. Розглянуто також певну спектральну теорію для лінеаризації рівнянь.

1. Introduction. Motivated by the fact that a dynamical system may evolve through an observable quantity rather than the state of the system, a general class of evolutionary equations is defined. This class includes standard ordinary and partial differential equations as well as functional differential equations of retarded and neutral type. In this way, the theory serves as a unification of these classical problems.

Included in this general formulation is a general theory for the evolution of temperature in a solid material. In the general case, temperature is transmitted as waves with a finite speed of propagation. Special cases include a theory of delayed diffusion.

We describe also in some detail a lattice on a circle where each point on the lattice is a transmission line for current and voltage whose dynamics is governed by a linear hyperbolic equation on $[0,1]$ with dynamic boundary conditons given by the circuitry on the line. The systems are coupled to their nearest neighbor at the end point 1 through resistors. A limiting process letting the distance between the lattice points approach zero leads to an interesting set of partial differential equations on $[0,1] \times S^1$ with a hyperbolic equation on $[0,1]$ and a parabolic equation on S^1. These equations have not been analyzed in detail. However, we can show that the voltage at 1 satisfies a partial neutral functional differential equation. We analyze some properties of these equations including synchronization and the behavior of solutins near periodic orbits.

There are other applications which involve partial differential equations on lattices for which the dynamics on each lattice point is governed by a partial differential equation on a bounded domain Ω. These systems could be coupled to neighbors through interaction on some subset of the boundary $\partial\Omega$ of Ω. If certain limiting processes are justified, one can obtain a partial differential equation on Ω together with another partial differential equation on another domain Ω_1 (determined by the nature of the lattice). We do not discuss this in the text, but mention it only to suggest that there are very interesting problems associated with such equations.

When the abstract evolutionary equation is linear, one arrives at an interesting spectral problem. We give some special results for the evolution of temprature and the partial neutral functional differential equation mentioned above.

In the discussion below of the dynamical system generated by an evolutionary equation, we will often enquire about the possibility of the dynamical system being a conditional α-contraction. Dynamical systems which are conditional α-contractions play an important role in the development of a qualitative theory; for example, the existence of a compact global attractor, behavior near equilibria or periodic orbits, etc. (see, for example, [1]).

For the benefit of the reader not familiar with this concept, we recall the definition. If X is a Banach space, the Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set $B \subset X$ is defined as

$$\alpha(B) = \inf \big\{ d\colon B \ \text{ a finite cover of diameter } < d \big\}.$$

A bounded set B has compact closure if and only if $\alpha(B) = 0$.

If $S\colon X \to X$ is a continuous map, then S is said to be a *conditional α-contraction* if there is a $k \in [0, 1)$ such that, for any bounded set B for which SB is bounded, $\alpha(SB) \le k\alpha(B)$. The map is conditionally compact if it is a conditional α-contraction with $k = 0$. We remark that a linear bounded map on X is a conditional α-contraction if and only if the radius of the essential spectrum is < 1.

A dynamical system $T(t)\colon X \to X$, $t \ge 0$, is a *conditional α-contraction* if there is a t_0 such that $T(t_0)$ is a conditional α-contraction.

This paper originated from a lecture at a conference at the University of São Paulo, São Carlos, Brazil in February, 2006 celebrating the life and work of Dan Henry who died on May 4, 2002 at the young age of 57.

2. A class of evolutionary equations. In this section, we describe a general class of evolutionary equations and show by examples how it unifies the discussion of many different types of equations that have been considered in the literature.

Let Y be a Banach space which we refer to as the *observable space* and let X be a Banach space which is called the *state space*. If $D\colon \mathcal{D}(D) \subset X \to Y$ and $F\colon \mathcal{D}(F) \subset \subset X \to Y$ are given functions, we define an *abstract evolutionary equation* for a function $u(t) \in X$ as

$$\partial_t (Du(t)) = F(u(t)). \tag{2.1}$$

In this abstract form, one cannot hope to have much of a general theory and the class must be restricted. We consider a special case of which we refer to as *quasilinear*: that is, the equation

$$\partial_t Du(t) = Lu(t) + G(u(t)), \tag{2.2}$$

where $D\colon \mathcal{D}(D) \subset X \to Y$, $L\colon \mathcal{D}(L) \subset X \to Y$, are linear operators and $G\colon X \to X$ (the domain of G is X).

The first problem is to define and obtain the existence of a solution of (2.2). To make sure that the linear equation,

$$\partial_t Du(t) = Lu(t)), \tag{2.3}$$

has a solution, we make the following hypothesis:

$\mathbf{H}_1$) Equation (2.3) defines a C^0-semigroup on X. Denote this semigroup by $e^{A_{D,L}t}$, where $A_{D,L}$ is the infinitesimal generator.

To obtain an evolutionary equation on X, we make the following hypotheses:

$\mathbf{H}_2$) There is a bounded linear operator $M : Y \to X$ such that (2.2) is equivalent to the equation on X:

$$\partial_t u = A_{D,L} u + MG(u). \tag{2.4}$$

Definition 2.1. *A mild solution of* (2.4) *with initial value* $u_0 \in X$ *at* $t = 0$ *is a solution of the integral equation* (*of course, assuming that it makes sense*)

$$u(t) = e^{A_{D,L}t} u_0 + \int_0^t e^{A_{D,L}(t-s)} MG(u(s))\, ds. \tag{2.5}$$

Assuming that solutions of (2.5) exist for each $u_0 \in X$ and $t \geq 0$ and the solution $u(t, u_0)$ depends continuously on (t, u_0), we let $T_{D,L,G}(t)$, $t \geq 0$, be the dynamical system on X defined by $T_{D,L,G}(t)u_0 = u(t, u_0)$.

One of the first objectives in the theory of dynamical systems is understand basic properties of $T_{D,L,G}(t)$. For example, if it is asymptotically smooth or, more particularly, a conditional α-contraction, then we can make use of many known results from dynamical systems concerning, for example, the existence of compact global attractors, local behavior near hyperbolic equilibria and periodic orbits, etc. We will mention later some of the problems involved in obtaining such a characterization of $T_{D,L,G}(t)$. Before doing this, we give several illustrations of classical equations that fall into this general class of equations.

3. Examples.

***Example* 3.1.** For $X = Y$, $D = I$, and $\mathcal{D}(F) = X$, equation (2.1) is the standard type of evolutionary equation, which includes ordinary differential equation (ODE) in $X = \mathbb{R}^n$ as well as the case where X has infinite dimension provided that F is a smooth function.

***Example* 3.2.** If $X = Y$, $D = I$, A is the infinitesimal generator of a C^0-semigroup on X and $G\colon X \to X$, this is a quasilinear equation which includes many types of partial differential equations.

***Example* 3.3** (RFDE). Fix $r \geq 0$ and let $X = C = C\big([-r, 0], \mathbb{R}^n\big)$ and $Y = \mathbb{R}^n$. For any observable continuous function $z\colon [-r, \alpha) \to \mathbb{R}^n$, let $z_t \in C$ be the state of z defined by $z_t(\theta) = z(t + \theta)$, $\theta \in [-r, 0]$. For any continuous function $F : C \to \mathbb{R}^n$, we define a *retarded functional differential equation* (RFDE) by the relation

$$\partial_t z(t) = F(z_t). \tag{3.1}$$

This is a special case of (2.2) with with $D\varphi = \varphi(0)$ for $\varphi \in C$.

This type of equation has been discussed extensively (see, for example, [2, 3] and the references therein).

For any $\varphi \in C$, there is a solution $z(t, \varphi)$ of (3.1) through φ at $t = 0$ defined on a maximal interval $[-r, \alpha)$, $\alpha > 0$. If the solution exists globally in time, then the mapping $T(t)\colon C \to C$, $t \geq 0$, defined by

$$T(t)\varphi = z_t(\cdot, \varphi), \quad t \geq 0, \tag{3.2}$$

is a dynamical system on the state space C.

For $t \geq r$, the solution $z(t)$ of (3.1) is continuously differentiable. If $F\colon C \to \mathbb{R}^n$ takes bounded sets to bounded sets, the the Arzela – Ascoli theorem implies that $T(t)$ is a conditionally compact operator for $t \geq r$.

Let $C_0 = \big\{\varphi \in C\colon \varphi(0) = 0\big\}$. The equation $\partial_t z(t) = 0$ defines a C^0-semigroup $S(t)$ on C_0. For any $\psi \in C$, $\psi - \psi(0) \in C_0$ and $S(t)\big[\psi - \psi(0)\big] = 0$ for $t \geq r$ and, for any $\beta > 0$, there is a constant K such that $\big|S(t)\big[\psi - \psi(0)\big]\big| \leq Ke^{-\beta t}|\psi|$ for $t \geq 0$ and all $\psi \in C$. Since the solution of (3.1) with initial data $\varphi \in C$ is given by

$$z(t) = \varphi(0) + \int_0^t F(z_s)\,ds, \quad t \geq 0,$$

$$z(t) = \varphi(t), \quad t \in [-r, 0],$$

it is easy to see that

$$T(t) = S(t)\big[\varphi - \varphi(0)\big] + U(t)\varphi, \tag{3.3}$$

where $U(t)$ is conditionally compact for $t \geq 0$. Therefore, $T(t)$ is a conditional α-contraction for all $t > 0$. From this fact, it has been possible to obtain a qualitative theory for RFDE which is similar to the theory of ODE.

Is it possible to write (3.1) as an evolutionary equation in C? We consider (3.1) as a perturbation of the linear equation $\partial_t z(t) = 0$ on C. We have noted that this equation defines a C^0-semigroup $S(t)$ on C_0. The infinitesimal generator $\mathcal{A}$ of $S(t)$ is easily seen to satisfy the following:

$$\mathcal{D}(A) = \Big\{\varphi \in C^1\big([-r, 0], \mathbb{R}^n\big) : \dot{\varphi}(0) = 0\Big\}, \qquad A\varphi = \dot{\varphi}, \quad \varphi \in \mathcal{D}(A). \tag{3.4}$$

This makes it apparent that it is not possible to work in the space C and obtain an abstract evolutionary equation for the state variable z_t of (3.1) since each φ in the domain of A must satisfy the boundary condition $\dot{\varphi}(0) = 0$.

As motivation for further discussion, observe that a solution $z(t)$ of (3.1) is continuously differentiable on the interval $[-r, \infty)$ if and only if $z_t \in \mathcal{D}(A)$ for all $t \geq 0$. If this is the case, then, formally,

$$\partial_t z_t = Az_t + X_0 F(z_t), \tag{3.5}$$

where

$$X_0(\theta) = \begin{cases} 0 & \text{if} \quad \theta \in [-r, 0), \\ I & \text{if} \quad \theta = 0, \end{cases} \tag{3.6}$$

I is the $n \times n$ identity matrix.

Even though this if formal, it was used extensively in the early development of the qualitative theory of (3.1) by considering a natural integral equation from (3.5) which is valid for each $\theta \in [-r, 0]$ (see [2–4] for further discussion and references).

To obtain an abstract evolutionary equation for the state variable of (3.1), we need to consider a larger space which permits the rows of the matrix function X_0 to be in the space. We choose the space to be

$$\tilde{X} \equiv \mathbb{R}^n \times C$$

and extend the definition of the infinitesimal generator A to an operator $\tilde{A}$ defined by

$$\mathcal{D}(\tilde{A}) = \big\{\varphi \in C^1([-r, 0], \mathbb{R}^n)\big\}, \qquad \tilde{A}\varphi = \dot{\varphi} - X_0\dot{\varphi}(0). \tag{3.7}$$

With this notation, we can now show that the RFDE on $\tilde{X}$ is equivalent to an abstract evolutionary

$$\partial_t z_t = \tilde{A} z_t + X_0 F(z_t). \tag{3.8}$$

This is a special case of (2.4) with M equal to X_0.

If we replace (3.1) by

$$\partial_t z(t) = L z_t + G(x_t), \tag{3.9}$$

where $L\colon C \to \mathbb{R}^n$ is a bounded linear operator, then the semigroup defined by the linear equation

$$\partial_t z(t) = L z_t$$

has the infinitesimal generator A the same as above except that it is required that $\dot{\varphi}(0) = L\varphi$ if $\varphi \in \mathcal{D}(A)$. If we make the same transformation as in (3.7) replacing $-X_0\dot{\varphi}(0)$ by $X_0\big[L\varphi - \dot{\varphi}(0)\big]$, then we obtain the same evolutionary equation with the new $\tilde{A}$.

The transformation (3.7) was first used by Chow and Mallet – Paret (1977) in the study of the integral averaging and Hopf bifurcation. Generalizations of normal form theory also follow from (3.8) (see [3] and the references therein). For more information about RFDE, see these same references.

Abstract evolutionary equations have been associated to (3.1) using deeper concepts in functional analysis (see [6]).

***Example* 3.4** (NFDE). With the notation as in Example 3.3, a *quasilinear neutral functional differential equation* (NFDE) is defined as

$$\partial_t D z_t = F(z_t), \tag{3.10}$$

where $D\colon C \to \mathbb{R}^n$ is a continuous linear operator which is nonatomic at zero or, equivalently,

$$D\varphi = \varphi(0) - \int_{-r}^{0} \big[d\eta(\theta)\big]\varphi(\theta), \tag{3.11}$$

where η is an $n \times n$ matrix function of bounded variation which is nonatomic at zero.

It is not too difficult to show that there is a solution of (3.10) with initial function $\varphi \in C$ at $t = 0$, defined on a maximal interval $[-r, \alpha_\varphi)$, $\alpha_\varphi > 0$ (see, for example, [2]).

If the solution through φ is defined for all $t \geq 0$ and, if we define $T(t)$ as in (3.2), then T is a dynamical system on the state space C.

Let $T_D(t)$ be the semigroup defined by the linear functional equation

$$D w_t = 0, \quad w_t \in C_D = \{\varphi \in C\colon D\varphi = 0\}. \tag{3.12}$$

It is known (see, for example, [2]) that the semigroup $T(t)$ defined by (3.10) can be represented as

$$T(t) = T_D(t)\Psi + U(t), \quad t \geq 0, \tag{3.13}$$

where $U(t)$ is conditionally compact for $t \geq 0$ and

$$\Psi = I - \Phi D, \qquad \Phi = (\varphi_1, \ldots, \varphi_n), \quad \varphi_j \in C, \quad 1 \leq j \leq n, \qquad D\Phi = I. \tag{3.14}$$

It follows that $T(t)$ is a conditional α-contraction if

$$r\big(\sigma_{\rm ess}(T_D(1))\big) < 1. \tag{3.15}$$

If $D\varphi = \varphi(0) + \sum_{j=1}^{N} \beta_k \varphi(-r_k)$, then condition (3.15) is equivalent to saying that the zero solution of (3.12) is exponentially stable.

Using the same ideas as in the case of RFDE, one can obtain an abstract evolutionary equation of the form (3.8) in $\tilde{X}$ for the solutions of (3.10). The matrix function X_0 is the same as before and the corresponding operator $\tilde{A}$ has the same domain with $\tilde{A}\varphi = = \dot{\varphi} - X_0 D\dot{\varphi}$.

In contrast to RFDE, some solutions of (3.10) may not become differentiable for any $t > 0$ if $D\varphi \neq \varphi(0)$. On the other hand, the development of a qualitative theory for NFDE satisfying (3.15) which is similar to the one for ODE relies primarily on the fact that $T_D(t)$ is a conditional α-contraction. Many results are known but the theory is not nearly as complete as for RFDE (see, for example, [2, 3]).

***Example* 3.5** (More general quasilinear RFDE). With the notation as above, if Y is the observable Banach space and $X = C\big([-r,0],Y\big)$ and $A\colon \mathcal{D}(A) \subset Y \to Y$ and $f: Y \to Y$ is continuous, then then one can define a *quasilinear RFDE* on X as

$$\partial_t z(t) = Az(t) + f(z_t). \tag{3.16}$$

If A is the generator of a C^0-semigroup e^{At}, then a mild solution of (3.16) with initial value f at $t = 0$ is a solution of the equation

$$z(t) = e^{At}\varphi(0) + \int_0^t e^{A(t-s)} f(z_s)\, ds, \quad t \geq 0,$$

$$z_0 = \varphi \in X.$$

Assuming that each solution $z(t,\varphi)$ exists for all $t \geq -r$ and is continuous in (t,φ), then $T_{A,f}(t) \equiv z_t(\cdot,\varphi)\colon X \to X$ is a dynamical system on X. If $T_{A,f}(t)$ is a conditional α-contraction, there is the possibility of developing a qualitative theory similar to ODE. It is known that $T_{A,f}(t)$ is a conditional α-contraction for the situation in which $A = \Delta_{BC}$ on a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary conditions BC and Y is an appropriate space of functions on Ω. On can find a detailed discussion in [7].

One can also obtain a conditional α-contraction in a linearly damped hyperbolic equation on a bounded domain Ω provided that f satisfies some growth conditions.

***Example* 3.6** (more general quasilinear NFDE). Let Y be an observable Banach space, $X = C\big([-r,0],Y\big)$. If $D_j\colon C\big([-r,0],Y\big) \to Y$, $j = 1,2$, are bounded linear operators, $A\colon \mathcal{D}(A) \subset Y \to Y$ is a linear operator and $f\colon Y \to Y$ is continuous, then a *quasilinear NFDE* on X is

$$\partial_t D_1 z_t = AD_2 z_t + f(z_t). \tag{3.17}$$

The simplest case to consider is when

$$D_j\varphi = \varphi(0) - \int_{-r}^{0} \big[d\eta_j(\theta)\big]\varphi(\theta), \quad j = 1,2, \tag{3.18}$$

where each η_j is nonatomic at zero. Such an assumption on the η_j makes easier the verification of the existence of a solution of the initial value problem.

We have mentioned in Section 2 a procedure for obtaining the existence of solutions. Each situation will require spectial consideration of the linear equation

$$\partial_t D_1 z_t = A D_2 z_t, \tag{3.19}$$

and conditions on D_1 and D_2 which imply that the solutions generate a C^0-semigroup $S_{D_1,D_2}(t)$ on X.

Furthermore, to obtain a qualitative theory, the spectrum of $T_{D_1,D_2}(1)$ will play an important role. We will discuss some aspects of this problem later, but now we now discuss in some detail two examples illustrating the importance of considering equations as general as (3.18). The first one invovles transmission lines on a circle with resistive nearest neighbor coupling and leads to (3.18) with $D_1 = D_2$. The second example deals with a theory of heat conduction in a solid where D_1 is in general different from D_2.

4. Transmission lines on a scalar domain with boundary coupling. ***4.1. Lossless transmission line and a NFDE.*** The current i and voltage v in this system can be described by the *telegraph equation*

$$L\partial_t i = -\partial_x v, \qquad C\partial_t v = -\partial_x i, \quad 0 < x < 1, \quad t > 0, \tag{4.1}$$

with the boundary conditions expressing the circuitry at the end points of the line given by

$$E - v(0,t) - Ri(0,t) = 0, \qquad C_1\partial_t v(1,t) = i(1,t) - g(v(1,t)). \tag{4.2}$$

It has been known for a long time that the undamped wave equation in one space dimension with nonlinear boundary conditions can be reduced to a NFDE of the type considered above (see, for example, [8 – 12]). We give two ways of doing this since the manner of reduction is not unique and distinct equations are obtained. However, the qualitative dynamics of the two types of equations are the same.

Define the constants

$$s = (LC)^{-1/2}, \qquad z = \left(\frac{L}{C}\right)^{1/2}, \qquad K = \frac{z-R}{z+R}, \qquad \alpha = \frac{2E}{z+R}.$$

The general solution of the partial differential equation (PDE) is given by

$$v(x,t) = \varphi(x - st) + \psi(x + st),$$

$$i(x,t) = \frac{1}{z}\big[\varphi(x - st) - \psi(x + st)\big]$$

or

$$2\varphi(x - st) = v(x,t) + zi(x,t),$$

$$2\psi(x + st) = v(x,t) - zi(x,t).$$

This implies that

$$2\varphi(-st) = v\left(1, t + \frac{1}{s}\right) + zi\left(1, t + \frac{1}{s}\right),$$

$$2\psi(-st) = v\left(1, t - \frac{1}{s}\right) + zi\left(1, t - \frac{1}{s}\right).$$

Using these expressions in the general solution and using the first boundary condition at $t - 1/s$, we obtain

$$i(1,t) - Ki\left(1, t - \frac{2}{s}\right) = \alpha - \frac{1}{z}v(1,t) - \frac{K}{z}v\left(1, t - \frac{2}{s}\right).$$

Inserting the second boundary condition and letting $u(t) = v(1,t)$, we obtain the equation

$$\dot{u}(t) - K\dot{u}\left(t - \frac{2}{s}\right) = f\left(u(t),\ u\left(t - \frac{2}{s}\right)\right),$$

where, if $\delta = 2/s$,

$$C_1 f\big(u(t), u(t-\delta)\big) = \alpha - \frac{1}{z}u(t) - \frac{K}{z}u(t-\delta) - g\big(u(t)\big) + g\big(u(t-\delta)\big).$$

If generalized solutions are considered in the original equation (4.1), then the function u would not have a derivative and we can only expect that the difference $u(t) - Ku(t-\delta)$ is differentiable and we obtain the NFDE

$$\partial_t\big[u(t) - Ku(t-\delta)\big] = f\big(u(t), u(t-\delta)\big). \tag{4.3}$$

If we let $D\varphi = \varphi(0) - K\varphi(-\delta)$, then we remark that this equation can be written in the equivalent form

$$\partial_t Du_t = \alpha - \frac{2}{z}u(t) + \frac{1}{z}Du_t - D(g \circ u)_t,$$

where $(g \circ u)(t) = g\big(u(t)\big)$.

The equation $Du_t = 0$ in the space $C_D = \big\{\varphi \in C\big([-\delta, 0], \mathbb{R}\big) \colon D\varphi = 0\big\}$ defines a C^0-semigroup. We say that D *is exponentially stable* if the zero solution of this equation is exponentially stabe. The operator D is exponentially stable if and only if $K < 1$. This will be the case if there is nonzero resistance in the line.

Let us give another NFDE which will describe qualitatively the dynamics of (4.1) in the case where $K < 1$; that is, D is exponentially stable. Let p be the unique constant solution of the equation $Dp_t = Ez/(z+R)$; that is, $p = zE/2R$. Using the first boundary condition at $t - 1/s$ and the general solution of (4.1), we obtain

$$\varphi(1 - st) = -\frac{z}{z+R}E - K\psi(st - 1).$$

If $w(t) = \psi(1 + st) - p$, then evaluation in the general solution gives

$$v(1,t) = w(t) - Kw(t-\delta), \qquad i(1,t) = \frac{1}{z}w(t) - \frac{K}{z}w(t-\delta) + q,$$

where $zq = -(1+K)p + (z/(z+R))E$. Using the second boundary condition, we obtain the equation

$$\partial_t Dw_t = q - (\delta + K)w(t) + \left(\frac{\delta}{2} + 1\right)Dw_t - g(w(t)). \tag{4.4}$$

In (4.4), the nonlinear function g appears only with the argument $w(t)$, whereas in (4.3) it also occurs with the argument $u(t-\delta)$. This can sometimes be useful in trying to make estimates on the magnitude of solutions (see, for example, [12]).

If $K < 1$, then the operator D is exponentially stable and the semigroup generated by either (4.3) or (4.4) is a conditional α-contraction.

4.2. Ring of lossless transmission lines with resistive coupling. Following Wu and Xia [13], we consider a ring of N mutually coupled lossless transmission lines interconnected by a common resistor R_0 at the right end of the line. For $1 \le k \le N$, the system of PDE is

$$L\partial_t i_k = -\partial_x v_k, \qquad C\partial_t v_k = -\partial_x i_k, \qquad 0 < x < 1, \quad t > 0, \tag{4.5}$$

with the boundary conditions

$$\begin{gathered} E - v_k(0,t) - Ri_k(0,t) = 0, \\ C_1\partial_t v_k(1,t) = i_k(1,t) - g(v_k(1,t)) - \frac{1}{R_0}(v_{k+1} - 2v_k + v_{k-1})(1,t). \end{gathered} \tag{4.6}$$

If we make the above reduction to NFDE, we have the system

$$\partial_t Du_{k,t} = f\big(u_k(t), u_k(t-\delta)\big) + \frac{1}{R_0C_1}D(u_{k+1,t} - 2u_{kt} + u_{k-1,t}), \tag{4.7}$$

where $Du_{kt} = u_k(t) - Ku_k(t-\delta)$ and $k = 1, 2, \ldots, N$.

Wu and Xia [13] discussed the existence of periodic solutions for (4.6) with $K < 1$; that is, D is exponentially stable. If D is exponentially stable, then the semigroup defined on $C\big([-\delta, 0], \mathbb{R}^N\big)$ is a conditional α-contraction.

4.3. Transmission lines on a circle. The terms on the right-hand side of (4.6) suggest an approximation to the Laplacian operator on S^1. Following Hale [14], we suppose that h is the spacing between the transmission lines and that there is a constant d such that $(R_0)^{-1} = dC_1/h^2$. If we let s represent distance on S^1 and take the limit as $h \to 0$, then we obtain the following interesting partial differential equation for $i(x,s,t)$, $v(x,s,t)$,

$$\begin{gathered} L\partial_t i(x,s,t) = -\partial_x v(x,s,t), \\ C\partial_t v(x,s,t) = -\partial_x i(x,s,t), \qquad 0 < x < 1, \quad s \in S^1, \quad t > 0, \end{gathered} \tag{4.8}$$

with the boundary conditions

$$\begin{gathered} E - v(0,s,t) - Ri(0,s,t) = 0, \\ C_1\partial_t v(1,s,t) = i(1,s,t) - g(v(1,s,t)) - d\partial_s^2 v(1,s,t), \quad s \in S^1. \end{gathered} \tag{4.9}$$

As for the discrete version, we remark that the original dynamics on the line could contain terms not involving the derivatives.

4.4. A partial NFDE. If we make the above reduction to a NFDE, we obtain the following partial NFDE for $u(t,s)$,

$$\partial_t Du_t(\cdot, s) = d\partial_s^2 Du_t(\cdot, s) + f(u(t,s), u(t-\delta, s) \tag{4.10}$$

with $s \in S^1$.

Hale [15] discussed the existence and uniqueness of solutions of equations more general than (4.10) in the space $X = C\big([-\delta, 0], H^1(S^1)\big)$; namely,

$$\partial_t Du_t = d\partial_s^2 Du_t + H(u_t) \tag{4.11}$$

with $s \in S^1$, $u = u(s,t)$ and $u_t(\theta, s) = u(t+\theta, s)$, $s \in S^1$. The function $H\colon X \to \mathbb{R}^n$ is Lipschitzian and $D\colon X \to \mathbb{R}^n$ is a continuous linear functional which is atomic at zero which without loss of generality implies that we can assume

$$D\varphi = \varphi(0) - \int_{-\delta}^{0} d\eta(\theta)\varphi(\theta) \tag{4.12}$$

where η is an $n \times n$ matrix of bounded variation with no atom at zero.

A mild solution of (4.11) is defined to be a solution of the equation

$$Du_t = e^{dA_s t} D\varphi + \int_0^t e^{dA_s(t-\tau)} H(u_\tau)d\tau, \tag{4.13}$$

where $A_s = d\partial_s^2$ on its domain in S^1.

Let $T_{D,H}(t)$, $t \geq 0$, be the dynamical system on X generated by the solutions of (4.11). Also, let $T_{D,0}(t)$, $t \geq 0$, be the dynamical system generated by the equation

$$\partial_t Du_t = d\partial_s^2 Du_t. \tag{4.14}$$

If

$$r(\sigma_{\text{ess}}(T_{D,0}(1))) < 1, \tag{4.15}$$

then one can show that $T_{D,H}(t)$ is a conditional α-contraction. We will show lat that this if true if the solution of the functional equation $Dw_t = 0$ on $C_D = \{\varphi \in C([-\delta, 0], \mathbb{R}),\ D\varphi = 0\}$ is exponentially stable. This implies that the contribution from the partial derivatives in x is a compact perturbation of the functional equation.

4.5. Spectral properties of $T_{D,0}(t)$. If A_D is the infinitesimal generator of $T_{D,0}(t)$, then it is not difficult to show that

$$\mathcal{D}(A_D) = \{\varphi \in X\colon \varphi \in C^1,\ D\dot\varphi = d\partial_s^2 D\varphi\},$$
$$A_D\varphi = \dot\varphi. \tag{4.16}$$

The operator A_D has compact resolvent. We use eigenfunction expansions to determine the spectrum $\sigma(A_D)$.

If (μ_k, e_k), $k = 1, 2, \ldots$, is a complete set of eigenpairs of $-d\partial_s^2$ on S^1, then $\mu_k \to \infty$ as $k \to \infty$. Let $X^{(k)}$ be the span of the eigenfunction e_k in X. If σ_k denotes the spectrum of $A_D\big|X^{(k)}$, then

$$\sigma(A_D) = \cup_{k\geq 1}\sigma_k. \tag{4.17}$$

A point $\lambda \in \sigma_k$ if and only if there is a nonzero function $\varphi \in X^{(k)}$ such that $A_D\varphi = \lambda\varphi$. For any nonzero $\varphi \in X^{(k)}$, there is a nonzero $w \in C([-\delta, 0], \mathbb{R})$ such that $\varphi = we_k$. From (4.16), we see that $\dot\varphi = \lambda\varphi$, and $D\dot\varphi = -\mu_k D\varphi$, Therefore, $\dot w = \lambda w$, $D\dot w = -\mu_k Dw$. As a consequence, we see that

$$\sigma_k = \{\lambda \in \mathbb{C}\colon \lambda De^{\lambda\cdot} = -\mu_k De^{\lambda\cdot}\},$$
$$\sigma(\mathcal{A}_D) = \cup_{k\geq 1}\sigma_k.$$

That part of the spectrum of $\mathcal{A}_d$ corresponding to a fixed k coincides with the eigenvalues of the NFDE

$$\partial_t Dw_t = \mu_k Dw_t. \tag{4.18}$$

The spectrum of $T_{D,0}(1)|X^{(k)}$ can be shown to satisfy

$$\sigma\big(T_{D,0}(1)\big|X^{(k)}\big) = \operatorname{Cl} e^{\sigma_k}.$$

We have noted earlier that the essential spectrum of $T_{D,0}(1)\big|X^{(k)}$ associated with (4.18) coincides with the essential spectrum of $S_D(1)$ where $S_D(t)$, $t \geq 0$, is the semigroup on $C\big([-\delta, 0], \mathbb{R}\big)$ generated by the functional equation $Dw_t = 0$ on the space C_D. We also note that $\lambda = -\mu_k$ is an element of σ_k. Since $\mu_k \to \infty$ as $k \to \infty$, this implies that

$$r\big(\sigma_{\text{ess}}(T_{D,0}(1))\big) = r\big(\sigma_{\text{ess}}(S_D(1))\big).$$

Therefore, $T_{D,0}(t)$ is a conditional α-contraction if the zero solution of the functional equation $Dw_t = 0$ in C_D is exponentially stable. As a consequence, $T_{D,H}(t)$ is a conditional α-contraction if the same property holds.

We summarize this in the following statement.

Proposition 4.1. *The semigroup $T_{D,H}(t)$ on X defined by* (4.11) *is a conditional α-contraction if and only if the zero solution of $Dw_t = 0$ on C_D is exponentially stable.*

4.6. Synchronization for the partial NFDE. In Proposition 4.1, we have observed that $T_{D,H}(t)$ is a conditional α-contraction if D is exponentially stable. From [1], this implies that there is the compact global attractor $\mathcal{A}_{D,H}$ if the dynamical system $T_{D,H}(t)$ is point dissipative and, for any bounded set $B \subset X$, there is a $t_0 = t_0(B)$ such that $\gamma^+(T_{D,H}(t_0)B)$ is bounded. We recall that $\mathcal{A}_{D,H}$ is the *compact global attractor* if it is compact, $T_{D,H}(t)\mathcal{A}_{D,H} = \mathcal{A}_{D,H}$ for all t, and for any bounded set B in X,

$$\lim_{t\to\infty} \operatorname{dist}_X(T_{D,H}(t)B, \mathcal{A}_{D,H}) = 0.$$

In this section, w discuss some properties of the compact global attractor as a function of the diffusion coefficient d in (4.11). Therefore, we fix D, H and denote the dynamical system by $T_{d^{-1}}$ and the corresponding compact global attractor by $\mathcal{A}_{d^{-1}}$. We make the following hypotheses:

H$_3$) The NFDE $\dfrac{d}{dt}Dz_t = H(z_t)$ has the compact global attractor $\mathcal{A}_0$ in $C\big([-\delta, 0], \mathbb{R}\big)$.

H$_4$) There is a $d_1 > 0$ such that the family of compact sets $\{\mathcal{A}_{d^{-1}}, d \geq d_1\} \cup \mathcal{A}_0$ is bounded in X.

We say that the system (4.11) *is synchronized* if each element of $\mathcal{A}_{d^{-1}}$ is independent of the spatial variable $x \in S^1$; that is, $\mathcal{A}_{d^{-1}} = \mathcal{A}_0$.

The following result is proved in [14].

Theorem 4.1. *If hypotheses* **H**$_3$) *and* **H**$_4$) *are satisfied, then there is a $d_2 \geq d_1$ such that, for each $d \geq d_2$, system* (4.11) *is synchronized.*

We outline the proof. Let $X = X_0 \oplus X_1$, where X_0 consists of functions which are independent of the spatial variable and X_1 consists of all functions in X which are orthogonal to the constant functions. If we let $w_t = w_t^1 + w_t^2$, $w_t^j \in X_j$, then one obtains the equations

$$\frac{\partial}{\partial t}D(q)w_t^1 = H(w_t^1) + \pi^{-1}\int\limits_0^\pi \Big[H\big(w_t^1 + w_t^2(\cdot,x)\big) - H(w_t^1)\Big]dx,$$

$$\frac{\partial}{\partial t}D(q)w_t^2 = d\frac{\partial^2}{\partial s^2}D(q)w_t^2 +$$

$$+H(w_t^1 + w_t^2) - \pi^{-1}\int\limits_0^\pi \Big[H\big(w_t^1 + w_t^2(\cdot,x)\big)\Big]dx \equiv$$

$$\equiv d\frac{\partial^2}{\partial s^2}D(q)w_t^2 + F(w_t^1, w_t^2).$$

We first observe that solutions on the attractors must satisfy some special properties. It is not difficult to see from (4.13) that, for every d, any solution on $\mathcal{A}_{d^{-1}}$ must satisfy

$$Dw_t^2 = \int\limits_{-\infty}^t e^{dA_s(t-\tau)}F(w_\tau^1, w_\tau^2)d\tau \qquad (4.19)$$

for all $t \in \mathbb{R}$.

Since $F(\varphi_0, 0) = 0$ for each $\varphi_0 \in X$, there is a constant k_0 such that

$$\big|F(\varphi_0,\varphi_1)\big|_Y \le k_0|\varphi_1| + k_1 \quad \forall\varphi_0 + \varphi_1 \in \{\mathcal{A}_{d^{-1}}, d \ge d_1\} \cup \mathcal{A}_0. \qquad (4.20)$$

From (4.19), (4.20), the fact that

$$\|e^{dA_s t}\|_{\mathcal{L}(X_1,X_1)} \le e^{-dt}$$

for $t \ge 0$ and w_t is bounded for $t \in \mathbb{R}$, one easily shows that $w_t^2 = 0$ for all $t \in \mathbb{R}$ provided that d is sufficiently large.

4.7. Synchronization in the hyperbolic PDE with parabolic PDE boundary conditions. Theorem 4.1 allows one to obtain a type of synchronization for system (4.8), (4.9). Suppose that system (4.8), (4.9) has the compact global attractor $\tilde{\mathcal{A}}_{d^{-1}}$. we say that (4.8), (4.9) *is synchronized* if each element of the compact global attractor is independent of $y \in S^1$. Using the relationships between the solutions of (4.8), (4.9) and (4.11), one can prove the following result.

Theorem 4.2. *Under the hypotheses of Theorem* 4.1, *the solutions of* (4.8), (4.9) *are synchronizeed for* $d \ge d_2$.

It would be interesting to give a proof of Theorem 4.2 directly on the equations (4.8), (4.9) without using the partial NFDE. The method employed probably could be used to discuss other PDE with interactions through the boundary.

It also is possible to consider equations (4.11) for a nonlinearity $H(s, u_t)$ which depends upon s. In this case, one can show that $\mathrm{dist}_X(\mathcal{A}_{d^{-1}}, \mathcal{A}_0) \to 0$ as $d \to \infty$, where $\mathcal{A}_0$ is the attractor for the "averaged" NFDE

$$\frac{\partial}{\partial t}Dy_t = \bar{H}(y_t), \quad y_t \in C\big([-\delta, 0], \mathbb{R}\big),$$

$$\bar{H}(\varphi) = \frac{1}{\pi}\int\limits_0^\pi H(s,\varphi)\,ds.$$

The interested reader can obtain much more information about synchronization of ODE and PDE by consulting the references in [16].

4.8. Behavior near and perturbation of periodic orbits. Consider an abstract evolutionary equation, suppose that u_0 is a T_0-periodic solution and let $\Gamma_0 = \{u_0(s)\colon s \in \in [0, T_0)\}$. It is desirable to define a large class of such equations for which we can obatin a theory similar to the ODE case describing the behavior of the solutions near Γ_0 and the effects of autonomous as well as nonautonomous perturbations. For the ODE case, a first step in the development of the theory is to define a rotating coordinate system around the periodic orbit using the "angle" s (the parameter describing the orbit Γ_0) and an element in a transversal to Γ_0 at $u_0(s)$. One can then apply the theory of invariant manifolds of Bogoliubov and Mitropolsky [17] to treat both the autonomous and nonautonomous case. For parabolic PDE, Henry [18] has shown that one can obtain the same type of results by using similar methods. Of course, there are many more technical obstacles that must be overcome. His methods should be applicable to many other types of PDE. For RFDE, Stokes [19] has given partial results in the same spirit. His results are not as general as for the ODE because the differential equation for the angle coordinate involved delays in the angle. For the partial NFDE that are similar to the ones arising in transmission lines, Hale [15] discussed some elementary properties for a hyperbolically stable periodic orbit under autonomous perturbations without using coordinate systems. For NFDE on $\mathbb{R}^n$, Hale and Weedermann [20], have given a coordinate system around a periodic orbit for which the derivative of the angle variable does not involve the delays. In this way, the spirit of Bogoliubov and Mitropolsky [17] can be followed with modifications of the techniques of Henry [18]. The construction of this coordinate system will be described later and it should be applicable to the partial NFDE above as well as more general ones.

If the perturbations are autonomous, why not use the standard Poincaré transversal map and study the neighborhood of a fixed point of the map? This is a very common approach in ODE in $\mathbb{R}^n$. We recall definition of the Poincaré map π. One chooses a transversal Σ at $u_0(0)$ so that, for any $v \in \Sigma$, there is a $t_0(v) > 0$ such that the solution $u(t)$, $u(0) = v$, satisfies $u(t_0(v)) \in \Sigma$. One then defines $\pi v = u(t_0(v))$. Using the differentiability properties of π on the transversal, one can discuss the local behavior near the fixed point $u_0(0)$. For general evolutionary equations (or even NFDE in $\mathbb{R}^n$), the map π is not differentiabe and other methods must be employed. In many important situations, one can prove that u_0 is continuously differentiable and this is sufficient to obtain the rotating coordinate system and then consider the method of integral manifolds mentioned above.

For convenience, let $C = C\big([-\delta, 0], \mathbb{R}^n\big)$. Consider now a NFDE (3.10) on $\mathbb{R}^n$ with D exponentially stable. Suppose that $p(t)$ is a T_0-periodic solution and let $\Gamma = \{p_t, t \in \in [0, T_0)\}$. It is known that $p(t)$ is a C^1-function (actually as smooth in t as F) and, therefore, Γ is a C^1-manifold (see [21] for a proof as well as references to previous work on smoothness of Γ).

The linear variational equation about $p(t)$ is given by

$$\frac{d}{dt} D y_t = L(t) y_t, \tag{4.21}$$

where $L(t) = F'(p_t)\colon C \to \mathbb{R}^n$ is a bounded linear operator which is continuous in t and T_0-periodic.

Let $T(t,s)\varphi = y_t(\cdot, s, \varphi)$, where $y(t,s,\varphi)$ is the solution of (4.21) with initial value φ at $t = s$. Define $U(s) = T(t+s, s)$, $s \in \mathbb{R}$. The point spectrum $\sigma_P(U(s))$ is independent of s. An element $\mu \neq 0$, $\mu \in \sigma_P(U(0))$, is called a *Floquet multiplier* of (4.21). Since $\dot{p}$ is a T_0-periodic solution of (4.21), $\mu = 1$ is a Floquet multiplier. The orbit Γ is *nondegenerate* if 1 is a simple Floquet multiplier. The orbit Γ is *hyperbolic* if $(\sigma_P(U(0)) \setminus \{1\}) \cap S^1 = \varnothing$, where S^1 is the unit circle in the complex plane with center zero and radius 1.

Even though it is not necessary, we assume for simplicity that Γ is nondegenerate. There is a closed subspace $Q(s)$, $s \in \mathbb{R}$, T_0-periodic, such that for every $t \geq s$,

1) $C = [\dot{p}_0] \oplus Q(s)$,
2) $T(t,s)Q(s) \subset Q(s)$,
3) $\sigma((U(s)|Q(s)) = \sigma(U(s)) \setminus \{1\}$,
4) $Q(s)$ is homeomorphic to $Q(t)$.

One can give an explicit representation of the decomposition (1) using the formal adjoint equation of (4.21) and the classical bilinear form $\langle \cdot, \cdot \rangle$ associated with (4.21) and the adjoint equation. In fact, there is T_0-periodic solution q of the adjoint equation such that $\langle q_s, \dot{p}_s \rangle = 1$ and $Q(s)$ is the set of $\varphi \in C$ for which $\langle q_s, \varphi \rangle = 0$.

One can easily show that there is a neighborhood V of Γ such that, for any $\varphi \in V$, there is a unique $s = s(\varphi)$ and a unique $\psi = \psi(\varphi) \in Q(s)$ such that $\varphi = p(s) + \psi$. One can now use this representation to change coordinates for solutions $u(t)$ of (3.10) in V as $u_t = p(s(t)) + z_t$ with $z_t \in Q(s(t))$. This is essentially the same as the one used by Stokes [19]. As remarked earlier, the differential equation for $s(t)$ involves delays in s. Hale and Weedermann [20] avoid this difficulty in the following way.

Let $M \subset C$ be a linear closed subspace of C of codimension 1 (one could choose $M = Q(0)$, for example). From the decomposition of C by 1) above, for any $s \in [0, T_0)$, there is a bounded linear isomorphism $L_s : M \to Q(s)$ such that, for any $w \in M$,

$$w = \langle q_s, w \rangle \dot{p}_s + L_s w.$$

It is not difficult to observe that the following result is valid.

Proposition 4.2 (A local coordinate system around Γ)**.** *Suppose that the codimension* 1 *subspace* M *of* C *and the operator* L_s *are defined as above. Then there exist a neighborhood* V *of* Γ *such that, for any* $\varphi \in V$, *there exists a unique pair* $(s, w) \in$ $\in [0, T_0) \times M_\epsilon$, $M_\epsilon = \{w \in M \colon |w| < \epsilon\}$, *such that*

$$\varphi = p_s + L_s w. \tag{4.22}$$

Suppose now that $u(t)$ is a solution of (4.21) with initial value in the neighborhood V of Proposition 4.2. As long as u_t remains in Γ, relation (4.22) implies that

$$u_t = p_{s(t)} + L_{s(t)} w(t).$$

It is possible to obtain the differential equation for $s(t)$ and $w(t)$ and see that these equations depend only upon $s(t)$ and no delays in $s(t)$. The introduction of the subspace M transferred all of the dependence upon the past history to the function $w(t) \in M$ (see [20] for details and applications).

This same type of transformation should be applicable to many other types of equations including the partial NFDE above.

5. Heat conduction in a solid. The remarks given below about the physical derivation of the linear theory of heat conduction in a solid is taken from the review of Joseph and Preziosi [22, 23] which also contains extensive references on the history of the problem. As we will see, the equations are a special case of our general equations in Section 2. The objective is to obtain a physical model for rigid heat conductors that can propagate waves. The methods come from continuum mechanics and thermodynamics.

We begin with a discussion of the simplest situation. Denote by θ the *temperature*, q the *heat flux*, τ the *relaxation time* and $k = k_1 + k_2$ the *thermal conductivity*, where k_1 is the *effective thermal conductivity* and k_2 is the *elastic conductivity*. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The objective is to determine an evolutionary equation which will serve to determine the evolution of temperature $\theta(t, x)$, $x \in \Omega$ and θ satisfies specified conditions on the boundary $\partial\Omega$ of Ω.

If e is the *internal energy*, then it is assumed that

$$\partial_t e = -\operatorname{div} q. \tag{5.1}$$

For a solid, it is reasonable to suppose that small changes in e are proportional to small changes in temperature; that is, there is a constant $\gamma > 0$ such that

$$\partial_t e = \gamma \partial_t \theta. \tag{5.2}$$

From (5.1) and (5.2),

$$\gamma \partial_t \theta = -\operatorname{div} q. \tag{5.3}$$

The equation governing the evolution of the temperature is obtained by specifying the manner in which the heat flux depends upon the temperature.

The simplest situation is Fourier's law:

$$q = -k\nabla\theta, \tag{5.4}$$

where $k > 0$ is a constant. Relations (5.3) and (5.4) yield the classical heat equation

$$\partial_t \theta = \frac{k}{\gamma}\Delta\theta. \tag{5.5}$$

This equation has its drawbacks due to infinite speed of propagation.

Cattaneo's law specifies that

$$\tau\partial_t q + q = -k\nabla\theta \tag{5.6}$$

for the relaxation constant τ.

From (5.3) and (5.6), we obtain the linearly damped wave equation

$$\tau\gamma\partial_t^2\theta + \gamma\partial_t\theta - k\Delta\theta = 0. \tag{5.7}$$

For given boundary conditions, this equation will generate a C^0-semigroup $S_\tau(t)$, $t \geq 0$, on a Banach space X and the radius, $r(\sigma_{\text{ess}}(T_\tau(1)))$, of the essential spectrum of $T_\tau(1)$ is less than one. Also, there is a finite speed of propagation of temperature.

One can arrive at (5.7) in the following way by specifying that the heat flux is determined with a delay time τ (*delayed diffusion*):

$$q(t+\tau, x) = -k\nabla\theta(t, x). \tag{5.8}$$

Relations (5.8) and (5.3) imply that

$$\partial_t \theta(t+\tau, x) = -k\Delta\theta(t, x). \tag{5.9}$$

Formally, one obtains (5.7) from (5.9) by relacing $\theta(t+\tau, x)$ by $\theta(t, x) + \tau\theta(t, x)$. This formal procedure has been used by several authors. Unfortunately, the behavior of the solutions of (5.7) and (5.9) are completely different. To see this, we write (5.9) in the equivalent form of an equation with the delay in the diffusion term:

$$\gamma\partial_t \theta(t, x) = -k\Delta\theta(t-\tau, x). \tag{5.10}$$

Let us analyze the behavior of some of the eigenvalues of (5.10). If (μ_k, e_k) is a complete set of eigenpairs of $-\Delta$ with the boundary conditions, then we may order the μ_k so that $\mu_k \to \infty$ as $k \to \infty$. For any fixed k, there is a solution $e^{\lambda t}\epsilon_k$ of (4.10) if and only if $\lambda = (k/\gamma)e^{-\lambda\tau}\mu_k$ or, equivalenty, if $\lambda = \mu_k\zeta$, then

$$\zeta = \frac{k}{\gamma} e^{-\zeta\mu_k\tau}. \tag{5.11}$$

Fix $\tau > 0$. There is a k_0 such that $\mu_k > 0$ for all $k \geq k_0$. For each such k, let $\zeta_k = \zeta^*(\mu_k\tau)$ be the unique real solution of (5.11). Then $\zeta_k > 0$ and $\zeta_k \to 0$ as $k \to \infty$. As a consequence, there are infinitely many positive eigenvalues of (5.10) which accumulate at zero and, in particular, $r(\sigma_{\text{ess}}(T_\tau(t))) = 1$ for all $t \geq 0$. The behavior of solutions is in stark contrast to the damped hyperbolic equation (5.7).

These remarks indicate that neither of the above models are appropriate for heat conduction if it is required that there is a finite speed of propagation.

The difficulty arose in the specification of the manner in which the heat flux q depends upon θ. Let us assume that q depends upon the past history through an expression of the form

$$q(t, x) = \int_{-\infty}^{0} \big[d\eta(s)\big]\theta(t+s, x), \tag{5.12}$$

where η is a function of bounded variation. Relations (5.3) and (5.12) imply that

$$\partial_t \theta(t, x) = \frac{k}{\gamma}\Delta \int_{-\infty}^{0} \big[d\eta(s)\big]\theta(t+s, x). \tag{5.13}$$

The internal energy should also depend upon the history of the temperature. If we assume that

$$e(t, x) = \gamma \int_{-\infty}^{0} \big[d\mu(s)\big]\theta(t+s, x), \tag{5.14}$$

where μ is of bounded variation, then the equation for the conduction of heat is given by

$$\partial_t \int\limits_{-\infty}^{0} d\mu(s)\theta(t+s,x) = \frac{k}{\gamma}\Delta \int\limits_{-\infty}^{0} d\eta(s)\theta(t+s,x), \tag{5.15}$$

which is of the form (3.19).

From our formal discussion of a delayed heat equation, conditions must be imposed on η and μ in order for this equation to define a dynamical system for which it is possible to have the radius of the essential spectrum be < 1. To attain this goal we assume that each of these functions have an atom at zero. In this case, (5.12) and (5.14) can be written as

$$\begin{aligned} e(x,t) &= b + \gamma\theta(x,t) + \int\limits_{-\infty}^{0} \big[dE(s)\big]\theta(x,t+s), \\ q(x,t) &= -k_2\theta(x,t) - \int\limits_{-\infty}^{0} \big[dQ(s)\big]\nabla\theta(x,t+s), \end{aligned} \tag{5.16}$$

and (5.15) as

$$\gamma\partial_t\theta(x,t) + \int\limits_{\infty}^{0} \big[dE(s)\big]\theta(x,t+s) = k_1\Delta\theta(x,t) + \Delta\int\limits_{\infty}^{0} \big[dQ(s)\big]\nabla\theta(x,t+s), \tag{5.17}$$

where E and Q are of bounded variation with no atom at 0.

This equation was introduced by Nunziato [24]. For $k_1 = 0$, Gurtin and Pipkin [25] introduced the equation as a model. Integrals of this type for $E = 0$ are used in the Boltzman theory of linear viscoelasticity to express the present value of stress in terms of past values of strain.

If there is a $\delta > 0$ such that the functions $E(s)$ and $Q(s)$ are constant for $s \le -\delta$, then (5.17) involves only finite delays. If this is not the case, then all of the past history is required to determine a solution of (5.17). At the present time, there is no general theory available for the case of infinite delay. For RFDE of retarded type in $\mathbb{R}^n$ with infinite delay, there is an extensive theory in a Banach space X satisfying certain properties and also conditions on the space X which will ensure that the corresponding semigroup is a conditional α-contraction (see [26]). Equation (5.17) will exhibit finite speed of propagation for most kernels E, Q. The proof of this fact requires the discussion of some spectral theory in an appropriate Banach space for which (4.8) defines a semigroup. We briefly discuss this problem in the next section for the case of finite delays.

6. Spectrum of linear equations. In this section, we consider the equation

$$\partial_t D_1 z_t = A D_2 z_t \tag{6.1}$$

on an observable space Y and the state space $X = C\big([-r,0],Y\big)$, where $A\colon \mathcal{D}(A) \subset \subset Y \to Y$ is the generator of a C^0-semigroup on Y and D_1, D_2 are bounded linear operators from X to Y.

We also assume that (6.1) generates a C^0-semigroup $T_{D_1,D_2}(t)$, $t \ge 0$, on the state space X and denote by $\mathcal{A}_{D_1,D_2}$ the infinitesimal generator.

If $\varphi \in \mathcal{D}(\mathcal{A}_{D_1,D_2})$, then

$$\mathcal{A}_{D_1,D_2}\varphi = \lim_{t\to 0^+} \frac{1}{t}\big[T_{D_1,D_2}(t)\varphi - \varphi\big].$$

If $-r \le \theta < 0$, then

$$\lim_{t\to 0^+} \frac{1}{t}\big[T_{D_1,D_2}(t)(\theta)\varphi - \varphi(\theta)\big] =$$

$$= \lim_{t\to 0^+} \frac{1}{t}\big[\varphi(t+\theta) - \varphi(\theta)\big] = \dot\varphi(\theta) = (\mathcal{A}_{D_1,D_2}\varphi)(\theta).$$

Since $\varphi \in \mathcal{D}(\mathcal{A}_{D_1,D_2}) \subset X$, it follows that

$$\mathcal{D}(\mathcal{A}_{D_1,D_2}) \subset \big\{\varphi \in X \colon \varphi \in C^1\big\}.$$

If $\varphi \in \mathcal{D}(\mathcal{A}_{D_1,D_2})$, then

$$\partial_t T_{D_1,D_2}(t)\varphi = \mathcal{A}_{D_1,D_2} T_{D_1,D_2}(t)\varphi$$

for all $t \ge 0$. Therefore,

$$\partial_t D_1 T_{D_1,D_2}(t)\varphi = D_1 \partial_t T_{D_1,D_2}(t)\varphi = D_1 \mathcal{A}_{D_1,D_2}(t) T_{D_1,D_2}(t)\varphi.$$

Also,

$$\partial_t T_{D_1,D_2}(t)\varphi = A D_2 T_{D_1,D_2}(t)\varphi.$$

Letting $t \to 0+$, we see that $D_1\dot\varphi = AD_2\varphi$ and we have shown that

$$\mathcal{D}\mathcal{A}_{D_1,D_2} = \big\{\varphi \in X \colon \varphi \in C^1, D_1\dot\varphi = AD_2\varphi\big\},$$

$$\mathcal{A}_{D_1,D_2}\varphi = \dot\varphi.$$

The operator $\mathcal{A}_{D_1,D_2}$ has compact resolvent and the spectrum consists of only point spectrum which is given by the set

$$\sigma(\mathcal{A}_{D_1,D_2}) = \Big\{\lambda \in \mathbb{C} \colon \exists \varphi \in X, \quad \varphi \ne 0 \colon \lambda D_1\varphi = AD_2\varphi\Big\},$$

where A is the operator in (6.1).

We make the following hypothesis:

$$\sigma(T_{D_1,D_2}(1)) = \ \mathrm{Cl}\, e^{\sigma(\mathcal{A}_{D_1,D_2})} \quad \text{plus possibly} \quad \{0\}.$$

From hypothesis (bfH), it is very important to determine the spectrum of $\mathcal{A}_{D_1,D_2}$. We now discuss in some detail a special case of (6.1). However, it will be clear that much of the analysis is valid in a more general context.

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $A = \Delta$, the Laplacian, with elements in the domain of Δ satisfying homogeneous Neumann boundary conditions. Let $Y = H^1(\Omega)$ be the observable space and let $X = C\big([-r,0], Y\big)$. Consider the equation

$$\partial_t D_1 z_t = \Delta D_2 z_t \tag{6.2}$$

We assume that D_1 considered as a map from $C\big([-r], \mathbb{R}\big)$ is atomic at zero.

Assume that (6.2) defines a C^0-semigroup on X. Let $(-\mu_k, \varphi_k)$, $k = 1, 2, \ldots$, be a complete set of eigenpairs of Δ, $\mu_1 < \mu_2 \le \mu_3 \le \ldots, \mu_k \le \ldots$, $\mu_k \to \infty$ as $k \to \infty$.

Let $X^{(k)} = [\varphi_k]$, the span of φ_k and let $\mathcal{A}^{(k)}_{D_1,D_2} = \mathcal{A}_{D_1,D_2}|X^{(k)}$. Each subspace $X^{(k)}$ is invariant under the solutions of (6.2). If $z_t \in X^{(k)}$, then there is a $y_t \in C([-r,0], \mathbb{R})$ such that $z_t = y_t\varphi_k$ for all $t \geq 0$ and $y(t)$ satisfies the NFDE

$$\partial_t D_1 y_t = -\mu_k D_2 y_t. \tag{6.3}$$

The spectrum σ_k of the generator $\mathcal{A}_{D_1,D_2}$ of the semigroup defined by this equation is given by

$$\sigma_k = \big\{\lambda \in \mathbb{C}\colon \lambda D_1 e^{\lambda\cdot} = -\mu_k D_2 e^{\lambda\cdot}\big\}.$$

It also is clear that

$$\sigma(\mathcal{A}_{D_1,D_2}) = \cup_{k\geq 1}\, \sigma_k.$$

If $T^{(k)}_{D_1,D_2}(t) = T_{D_1,D_2}(t)X^{(k)}$ and $S_{D_1}(t)$ is the C^0-semigroup on $C_0 \equiv \big\{\varphi \in C([-r,0],\mathbb{R})\colon \varphi(0) = 0\big\}$ generated by the functional equation $D_1 w_t = 0$, then we know that

$$r\Big(\sigma_{\rm ess}\big(T^{(k)}_{D_1,D_2}(1)\big)\Big) = r\big(\sigma_{\rm ess}(S_{D_1}(1)\big) \quad \forall k \geq 1.$$

In Section 4, we have discussed the case in which $D_1 = D_2$ and observed that

$$r(\sigma_{\rm ess}(T^{(k)}_{D_1,D_1}(1)) = \{0\} \cup r(\sigma_{\rm ess}(S_{D_1}(1)).$$

In the general case, there are several possibilities for the behavior of the spectrum. It is instructive to consider some simple examples.

Let us consider the case where $D_1\varphi = \varphi(0)$; that is, the equation

$$\partial_t z(t) = \Delta D_2 z_t. \tag{6.4}$$

The spectrum of the generator on $X^{(k)}$ is given by the solutions of the equation

$$\frac{\lambda}{\mu_k} = -D_2 e^{\lambda\cdot}, \quad k = 1, 2, \ldots. \tag{6.5}$$

Suppose further that

$$D_2\varphi = \varphi(0) + \beta\varphi(-r), \quad \varphi \in C\big([-r,0],\mathbb{R}\big). \tag{6.6}$$

The behavior of the spectrum of the generator of (6.5) is well known (see, for example, [2]). Using this information, we can make the following remarks. For D_2 as in (6.6), if $|\beta| > 1$, there is a k_0 such that, for $k \geq k_0$, there are elements of σ_k with real parts > 0. This implies that there is no way to obtain exponential decay of the “delayed diffusion equation” (6.4), (6.6).

If D_2 satisfies (6.6) and if we let $\zeta = \lambda/\mu_k$ in (6.5), then

$$\zeta = -1 - \beta e^{-\mu_k r\zeta}. \tag{6.7}$$

This is the characteristic equation for the RFDE

$$\partial_t z(t) = -z(t) - \beta z(t - \mu_k r). \tag{6.8}$$

Thus, in order to obtain the exponential stability of all solutions of (6.4), (6.6), we must have a $\delta > 0$ such that, for every k, all solutions of (6.7) satisfy $\mathrm{Re}\,\zeta \leq -\delta$. To have this property, it is necessary and sufficient that $|\beta| < 1$ (see, for example, [2]). This is the same as saying that the solutions of the RFDE

$$\partial_t z(t) = -z(t) - \beta z(t-\gamma) \tag{6.9}$$

is *exponentially stable independent of the delay* γ. More precisely, there must be positive constants K, α such that for any $\gamma \in [o, \infty)$, each solution of (6.9) satisfies $|z_t| \leq$ $\leq Ke^{-\alpha t}|z_0|$.

In the same way, if D_2 is a general difference operator,

$$D_2\varphi = -\varphi(0) - \sum_{j=1}^{M} \beta_j \varphi(-r_j), \tag{6.10}$$

then it is not difficult to see that each solution of the equation (6.4), (6.10) approaches zero exponentially if and only if the zero solution of the RFDE

$$\partial_t z(t) = -\varphi(0) - \sum_{j=1}^{M} \beta_j z(t - \alpha r_j), \tag{6.11}$$

is exponentially stable independent of α.

It can be shown that a sufficient condition for this to be true is that $\sum_{j=1}^{M} |\beta_j| <$ < 1; that is, the solutions of (6.11) is exponentially stable independently of the delays $r_1, r_2, \ldots, r_k$.

Consider now equation (6.3) with

$$D_1\varphi = \varphi(0) - \alpha\varphi(-1), \qquad D_2\varphi = \varphi(0) + \beta\varphi(-1). \tag{6.12}$$

In this case, for every k, we must solve the equation

$$\frac{\lambda + \mu}{\lambda\alpha - \mu_k \beta} = e^{-\lambda}$$

and determine conditions on α, β so that each solution of (6.3), (6.12) approaches zero exponentially. If $0 < |\alpha| < 1$, then there is a constant R such that $|\mathrm{Re}\lambda| < R$ for all k. Therefore, there is finite propagation speed. Furthermore, if $\mu_1 > 0$ and $|\beta| < 1$, then the zero solution of (6.3), (6.12) is exponentially stable uniformly in k.

These examples suggest the following conjecture.

Conjecture. Suppose that each D_j is a difference operator with delays $0 < r_1 <$ $< r_2 < \ldots < r_n$. The radius of the essential spectrum of $T_{D_1,D_2}(1)$ is < 1 if the zero solution of each of the equations $D_1 w_t = 0$ and $D_2 w_t = 0$ is exponentially stable independently of the delays.

The results mentioned above depend very strongly on the fact that z in (6.3) is a scalar. The situation for $z \in \mathbb{R}^n$ is much more complicated. In the vector situation, there are known results, there are some results on the exponential stability of difference equations independent of delays (see [2]). These results apply to some special equations.

The spectral properties of these equations certainly needs to be discussed in more detail.

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Received 30.11.2006