Theory of Stochastic Processes
Vol. 14 (30), no. 3-4, 2008, pp.129-164

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NONLINEARLY PERTURBED STOCHASTIC PROCESSES

This paper is a survey of results presented in the recent book [25][1]. This book is devoted to studies of quasi-stationary phenomena in nonlinearly perturbed stochastic systems. New methods of asymptotic analysis for nonlinearly perturbed stochastic processes based on new types of asymptotic expansions for perturbed renewal equation and recurrence algorithms for construction of asymptotic expansions for Markov type processes with absorption are presented. Asymptotic expansions are given in mixed ergodic (for processes) and large deviation theorems (for absorption times) for nonlinearly perturbed regenerative processes, semi-Markov processes, and Markov chains. Applications to analysis of quasi-stationary phenomena in nonlinearly perturbed queueing systems, population dynamics and epidemic models, and risk processes are presented. The book also contains an extended bibliography of works in the area.

1. Introduction

The book mentioned above presents new methods of asymptotic analysis of nonlinearly perturbed stochastic processes and systems with random lifetimes.

Usually the behaviour of a stochastic system can be described in terms of some Markov type stochastic process $\eta^{(\varepsilon)}(t)$ and its lifetime defined to be

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2000 *Mathematics Subject Classifications*: Primary: 60F05, 60F10; 60K05, 60K15; Secondary: 60J22, 60J27, 60K20, 60K25, 65C40.

Key words: nonlinear perturbation, quasi-stationary phenomenon, pseudo-stationary phenomenon, stochastic system, renewal equation, asymptotic expansion, ergodic theorem, limit theorem, large deviation, regenerative process, regenerative stopping time, semi-Markov process, Markov chain, absorption time, queueing system, population dynamics, epidemic model, lifetime, risk process, ruin probability, Cramér-Lundberg approximation, diffusion approximation.

[1]Gyllenberg, M., Silvestrov, D.S. (2008). *Quasi-Stationary Phenomena in Nonlinearly Perturbed Stochastic Systems*, De Gruyter Expositions in Mathematics **44**, Walter de Gruyter, Berlin, XII + 579 pp.

the time $\mu^{(\varepsilon)}$ at which the process $\eta^{(\varepsilon)}(t)$ hits a special absorption subset of the phase space of this process for the first time. The object of interest is the joint distribution of the process $\eta^{(\varepsilon)}(t)$ subject to a condition of non-absorption of the process up to a moment t, i.e., the probabilities $\mathsf{P}\{\eta^{(\varepsilon)}(t) \in A, \mu^{(\varepsilon)} > t\}$, and the asymptotic behaviour of these probabilities for $t \to \infty$.

A typical situation is when the process $\eta^{(\varepsilon)}(t)$ and the absorption time $\mu^{(\varepsilon)}$ depend on a small parameter $\varepsilon \geq 0$ in the sense that some of their local "transition" characteristics depend on the parameter ε. The parameter ε is involved in the model in such a way that the corresponding local characteristics are continuous at the point $\varepsilon = 0$, if regarded as functions of ε. These continuity conditions permit to consider the process $\eta^{(\varepsilon)}(t)$, for $\varepsilon > 0$, as a perturbed version of the process $\eta^{(0)}(t)$.

In models with perturbations, it is natural to study the asymptotic behaviour of the probabilities $\mathsf{P}\{\eta^{(\varepsilon)}(t) \in A, \mu^{(\varepsilon)} > t\}$ when the time $t \to \infty$ and the perturbation parameter $\varepsilon \to 0$ simultaneously. Without loss of generality it can be assumed that the time $t = t^{(\varepsilon)}$ is a function of the parameter ε such that $t^{(\varepsilon)} \to 0$ as $\varepsilon \to 0$.

The corresponding studies obviously relate to two classical asymptotic problems.

The first problem is connected with limit theorems for lifetimes of stochastic systems, i.e., propositions about convergence of probabilities $\mathsf{P}\{\mu^{(\varepsilon)} > t^{(\varepsilon)}\}$ to some non-zero limits as $\varepsilon \to 0$, as well as with large deviation theorems for lifetimes, i.e., propositions about asymptotic behaviour of these probabilities in the case where they tend to zero as $\varepsilon \to 0$.

The second problem is connected with a mathematical description of quasi-stationary phenomena for the corresponding processes. These phenomena describe the behaviour of stochastic systems with random lifetimes. The core of the quasi-stationary phenomenon is that one can observe something that resembles a stationary behaviour of the system before the lifetime goes to the end, i.e., stabilisation of conditional probabilities $\mathsf{P}\{\eta^{(\varepsilon)}(t^{(\varepsilon)}) \in A/\mu^{(\varepsilon)} > t^{(\varepsilon)}\}$ as $\varepsilon \to 0$.

The principal novelty of results presented in book [25] is that the models with nonlinear perturbations are studied. Local transition characteristics that were mentioned above are usually some scalar or vector moment functionals $p^{(\varepsilon)}$ of local transition probabilities for the corresponding processes. By a nonlinear perturbation we mean that these characteristics are nonlinear functions of the perturbation parameter ε and that the assumptions made imply that the characteristics can be expanded in an asymptotic power series with respect to ε up to and including some order k,

$$p^{(\varepsilon)} = p^{(0)} + p[1]\varepsilon + \cdots + p[k]\varepsilon^k + o(\varepsilon^k). \tag{1}$$

The case $k = 1$ corresponds to models with usual linear perturbations while the cases $k > 1$ correspond to models with nonlinear perturbations.

It turns out that the relation between the velocities with which ε tends to zero and the time $t = t^{(\varepsilon)}$ tends to infinity has a delicate influence on the quasi-stationary asymptotics. The balance between the rate of the perturbation and the rate of growth of time can be characterized by the following condition which is assumed to hold for some $1 \leq r \leq k$:

$$\varepsilon^r t^{(\varepsilon)} \to \lambda_r < \infty \quad \text{as} \quad \varepsilon \to 0. \tag{2}$$

The main results are represented by so-called mixed ergodic and limit/large deviation theorems given in a form of exponential expansions for the probabilities $\mathsf{P}\{\eta^{(\varepsilon)}(t^{(\varepsilon)}) \in A, \mu^{(\varepsilon)} > t^{(\varepsilon)}\}$. Under the mentioned above assumptions on the perturbations and some additional natural conditions of Cramér type, the following exponential asymptotic relations are obtained:

$$\frac{\mathsf{P}\{\eta^{(\varepsilon)}(t^{(\varepsilon)}) \in A, \mu^{(\varepsilon)} > t^{(\varepsilon)}\}}{\exp\{-(\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})t^{(\varepsilon)}\}} \to \pi^{(0)}(A)e^{-\lambda_r a_r} \quad \text{as} \quad \varepsilon \to 0. \tag{3}$$

With these relations, explicit algorithms for calculating the coefficients $\rho^{(0)}$ and $a_1, \ldots, a_k$, as functions of the coefficients in the expansions of the local transition characteristics that appear in the initial nonlinear perturbation conditions of type (1), are given. It is a non-trivial problem due to the nonlinear character of the perturbations involved.

The asymptotic behaviour of $\mathsf{P}\{\eta^{(\varepsilon)}(t) = j, \mu^{(\varepsilon)} > t\}$ is very much different in the two alternative cases: **(a)** $\rho^{(0)} = 0$ and **(b)** $\rho^{(0)} > 0$. In the first case, the absorption times $\mu^{(\varepsilon)}$ are stochastically unbounded random variables, that is, they tend to ∞ in probability as $\varepsilon \to 0$. In the second case, the absorption times $\mu_0^{(\varepsilon)}$ are stochastically bounded as $\varepsilon \to 0$.

In the literature, the asymptotics related to the second model is known as *quasi-stationary* asymptotics. To distinguish between the asymptotics (3) in the first and the second cases, the term *pseudo-stationary* was coined for the first one.

Another class of asymptotic expansions systematically studied in the book concerns the so-called quasi-stationary distributions. Under some natural moment, communication and aperiodicity conditions imposed on the local transition characteristics there exists a so-called quasi-stationary distribution for the process $\eta^{(\varepsilon)}(t)$, which is given by the formula,

$$\pi^{(\varepsilon)}(A) = \lim_{t \to \infty} \mathsf{P}\{\eta^{(\varepsilon)}(t) \in A \;/\; \mu^{(\varepsilon)} > t\}. \tag{4}$$

Asymptotics (3) let one also obtain asymptotic expansions for the quasi-stationary distributions of the nonlinearly perturbed processes $\eta^{(\varepsilon)}(t)$,

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + g_1(A)\varepsilon + \cdots + g_k(A)\varepsilon^k + o(\varepsilon^k). \tag{5}$$

As in the case of asymptotics (3), an explicit recurrence algorithm for calculating the coefficients $g_1(A), \ldots, g_k(A)$, as functions of the coefficients in the expansions for local transition characteristics of the processes $\eta^{(\varepsilon)}(t)$ in the corresponding initial perturbation conditions, is given.

The classes of processes for which this program is realised include nonlinearly perturbed regenerative processes, semi-Markov processes, and continuous time Markov chains with absorption.

The approach is based on advanced techniques, developed in the book, of nonlinearly perturbed renewal equations.

Applications to the analysis of quasi-stationary phenomena in models of nonlinearly perturbed stochastic systems considered in the book pertain to models of highly reliable queueing systems, M/G queueing systems with quick service, stochastic systems of birth-death type, including epidemic and population dynamics models, metapopulation dynamic models, and perturbed risk processes.

As was mentioned above, quasi-stationary phenomena were a subject of intensive studies during several decades. An extensive survey of the literature and comments can be found in the bibliographical remarks given in book [25].

The results related to the asymptotics given in (3) and (5) and known in the literature mainly cover the case $k = 1$, which corresponds to the model of linearly perturbed processes. Some known results also relate to the case where the perturbation condition (1) has the special form of relation **(c)** $p^{(\varepsilon)} = p^{(0)} + p[1]\varepsilon$. In context of nonlinearly perturbed models, this is a particular case of nonlinear perturbation condition (1) which holds for every $k \geq 1$ with all higher order terms $p[2], p[3], \ldots \equiv 0$. It should be noted that the asymptotic expansions given in (3) and (5) obviously cover this case but not vise versa. Indeed, the asymptotic expansions obtained for the models with perturbation condition of the form **(c)** do not give any information about contribution of the terms $p[2], p[3], \ldots$ in nonlinear asymptotic expansions given in (3) and (5) for the models where these higher order terms take non-zero values.

The book [25] contains an extended introduction, where the main problems, methods, and algorithms that constitute the content of the book are presented in informal form. In Chapters 1 and 2, results which deal with a generalisation of the classical renewal theorem to a model of the perturbed renewal equation are presented. These results are interesting by their own and, as we think, can find various applications beyond the areas mentioned in the book. In Chapters 3, 4, and 5 asymptotics of the types (3) and (5) for nonlinearly perturbed regenerative processes, semi-Markov processes, and continuous time Markov chains with absorption are studied. Chapters 6 and 7 are devoted to applications of the theoretical results to studies of quasi-stationary phenomena for various nonlinearly perturbed models of stochastic systems. In Chapter 6, quasi-stationary phenomena are studied for highly reliable queueing systems, M/G queueing systems with quick service, stochastic systems of birth-death type, including epidemic and population dynamics models, and metapopulation dynamic models; Chapter 7 deals with perturbed risk processes. The last Chapter 8 contains three supplements. The first one gives some basic operation formulas for scalar and matrix asymptotic expansions. In the second supplement some new prospective directions for future research in the are discussed. In the last supplement, bibliographical remarks to the bibliography that includes more than 1000 references are given.

2. Nonlinearly perturbed renewal equation

2.1. Renewal theorem for perturbed renewal equation. Let us consider the family of renewal equations,

$$x^{(\varepsilon)}(t) = q^{(\varepsilon)}(t) + \int_0^t x^{(\varepsilon)}(t-s)F^{(\varepsilon)}(ds),\ t \geq 0, \tag{6}$$

where, for every $\varepsilon \geq 0$, we have the following: (a) $q^{(\varepsilon)}(t)$ is a real-valued function on $[0, \infty)$ that is Borel measurable and locally bounded, i.e., bounded on every finite interval, and (b) $F^{(\varepsilon)}(s)$ is a distribution function on $[0, \infty)$ which is not concentrated at 0 but can be improper, i.e. $F^{(\varepsilon)}(\infty) \leq 1$.

As well known, there exists the unique Borel measurable and bounded on every finite interval solution $x^{(\varepsilon)}(t)$ of equation (6).

In the model of perturbed renewal, the forcing function $q^{(\varepsilon)}(t)$ and distribution $F^{(\varepsilon)}(s)$ depend on some perturbation parameter $\varepsilon \geq 0$ and converge in some sense to $q^{(0)}(t)$ and $F^{(0)}(s)$ as $\varepsilon \to 0$.

The fundamental fact of the renewal theory connected with this equation is the renewal theorem given in its final form by Feller (1966). This theorem describes the asymptotic behavior of solution in the form of asymptotic relation $x^{(0)}(t) \to x^{(0)}(\infty)$ as $t \to \infty$ for non-perturbed renewal equation.

The renewal theorem is a very powerful tool for proving ergodic theorems for regenerative stochastic processes. This class of processes is very broad. It includes Markov processes with discrete phase space. Moreover, Markov processes with a general phase space can be included, under some minor conditions, in a model of regenerative processes with the help of the procedure of artificial regeneration.

Applying the renewal theorem to ergodic theorems for regenerative type processes is based on the well known fact that the distribution of a regenerative process at a moment t satisfies a renewal equation. This makes it possible to apply the renewal theorem and to describe the asymptotic behaviour of the distribution of the regenerative process as $t \to \infty$.

Theorems that generalise the classical renewal theorem to a model of the perturbed renewal equation was proved in papers Silvestrov (1976, 1978, 1979). These results are presented in Chapter 1 of book [25].

As usual the symbol $F^{(\varepsilon)}(\cdot) \Rightarrow F^{(0)}(\cdot)$ as $\varepsilon \to 0$ means weak convergence of the distribution functions that is, the pointwise convergence in each point of continuity of the limiting distribution function.

Further, the following notations are used,

$$f^{(\varepsilon)} = 1 - F^{(\varepsilon)}(\infty), \quad m_r^{(\varepsilon)} = \int_0^\infty s^r F^{(\varepsilon)}(ds), \quad r \geq 1.$$

We assume that the functions $q^{(\varepsilon)}(t)$ and the distributions $F^{(\varepsilon)}(s)$ satisfy the following continuity conditions at the point $\varepsilon = 0$, if regarded as functions of ε:

$\mathbf{D_1}$: $F^{(\varepsilon)}(\cdot) \Rightarrow F^{(0)}(\cdot)$ as $\varepsilon \to 0$, where $F^{(0)}(s)$ is a proper non-arithmetic distribution function;

$\mathbf{M_1}$: $m_1^{(\varepsilon)} \to m_1^{(0)} < \infty$ as $\varepsilon \to 0$;

and

$\mathbf{F_1}$: **(a)** $\lim_{u\to 0} \overline{\lim}_{0\leq\varepsilon\to 0} \sup_{|v|\leq u} |q^{(\varepsilon)}(t+v) - q^{(0)}(t)| = 0$ almost everywhere with respect to the Lebesgue measure on $[0, \infty)$;

(b) $\overline{\lim}_{0\leq\varepsilon\to 0} \sup_{0\leq t\leq T} |q^{(\varepsilon)}(t)| < \infty$ for every $T \geq 0$;

(c) $\lim_{T\to\infty} \overline{\lim}_{0\leq\varepsilon\to 0} h \sum_{r\geq T/h} \sup_{rh\leq t\leq (r+1)h} |q^{(\varepsilon)}(t)| = 0$ for some $h > 0$.

It is easy to show that, under conditions **A** and $\mathbf{M_1}$, the following relation holds,

$$f^{(\varepsilon)} \to f^{(0)} = 0 \quad \text{as} \quad \varepsilon \to 0. \tag{7}$$

Let also assume the following condition that balances the rate at which time $t^{(\varepsilon)}$ approaches infinity, and the convergence rate of the defect $f^{(\varepsilon)}$ to zero as $\varepsilon \to 0$:

B: $0 \leq t^{(\varepsilon)} \to \infty$ and $f^{(\varepsilon)} \to 0$ as $\varepsilon \to 0$ in such a way that $f^{(\varepsilon)} t^{(\varepsilon)} \to \lambda$, where $0 \leq \lambda \leq \infty$.

The starting point for the research studies presented in book [25] is the following theorem (Silvestrov, 1976, 1978, 1979).

Theorem 1. *Let conditions* $\mathbf{D_1}$, $\mathbf{M_1}$, $\mathbf{F_1}$, *and* **B** *hold. Then,*

$$x_\varepsilon(t_\varepsilon) \to e^{-\lambda/m_1^{(0)}} \frac{\int_0^\infty q^{(0)}(s)ds}{m_1^{(0)}} \quad \text{as} \quad \varepsilon \to 0. \tag{8}$$

Remark 1. It is worth to note that this theorem reduces to the classical renewal theorem in the case of non-perturbed renewal equation, i.e., where the forcing functions $q^{(\varepsilon)}(t) \equiv q^{(0)}(t)$ and distribution functions $F^{(\varepsilon)}(s) \equiv$

$F^{(0)}(s)$ do not depend on ε. In particular, condition $\mathbf{D_1}$ reduces to the assumption that $F^{(0)}(s)$ is a proper non-arithmetic distribution function; $\mathbf{M_1}$ to the assumption that the expectation $m_1^{(0)}$ is finite; and $\mathbf{F_1}$ to the assumption that the function $q^{(0)}(t)$ is directly Riemann integrable on $[0, \infty)$.

In this case, the defect $f^{(\varepsilon)} \equiv 0$ and the balancing condition $\mathbf{B}$ holds for any $t^{(\varepsilon)} \to \infty$ as $\varepsilon \to 0$ with the parameter $\lambda = 0$.

Note that condition $\mathbf{D_1}$ does not require and does not provide that the pre-limit ($\varepsilon > 0$) distribution functions $F^{(\varepsilon)}(s)$ are non-arithmetic.

Also, condition $\mathbf{F_1}$ does not provide that the pre-limit ($\varepsilon > 0$) forcing functions $q^{(\varepsilon)}(t)$ are directly Riemann integrable on $[0, \infty)$. However, this condition does imply that the limit forcing functions $q^{(0)}(t)$ has this property.

In general case, the balancing condition $\mathbf{B}$ restrict the rate of growth for time $t^{(\varepsilon)}$. This restriction becomes unnecessary if an additional Cramér type condition is imposed on the distributions $F_\varepsilon(s)$.

In this case, one can also weaken condition $\mathbf{D_1}$ and accept also the possibility for the limit distribution be improper:

$\mathbf{D_2}$: **(a)** $F^{(\varepsilon)}(\cdot) \Rightarrow F^{(0)}(\cdot)$ as $\varepsilon \to 0$, where $F^{(0)}(t)$ is a non-arithmetic distribution function (possibly improper);

(b) $f^{(\varepsilon)} \to f^{(0)} \in [0, 1)$ as $\varepsilon \to 0$.

The Cramér type condition mentioned above takes the following form:

$\mathbf{C_1}$: There exists $\delta > 0$ such that:

(a) $\overline{\lim}_{0 \leq \varepsilon \to 0} \int_0^\infty e^{\delta s} F^{(\varepsilon)}(ds) < \infty$;

(b) $\int_0^\infty e^{\delta s} F^{(0)}(ds) > 1$.

Let us introduce the moment generation function,

$$\phi^{(\varepsilon)}(\rho) = \int_0^\infty e^{\rho s} F^{(\varepsilon)}(ds), \ \rho \geq 0.$$

Consider the following characteristic equation,

$$\phi^{(\varepsilon)}(\rho) = 1. \tag{9}$$

Under condition $\mathbf{D_2}$ and $\mathbf{C_1}$, there exists $\varepsilon_1 > 0$ such that $\phi^{(\varepsilon)}(\delta) \in (1, \infty)$, and, therefore, equation (9) has a unique non-negative root $\rho^{(\varepsilon)}$ and $\rho^{(\varepsilon)} \leq \delta$, for every $\varepsilon \leq \varepsilon_1$. Also,

$$\rho^{(\varepsilon)} \to \rho^{(0)} \text{ as } \varepsilon \to 0. \tag{10}$$

Note also that **(a)** $\rho^{(0)} = 0$ if and only if $f^{(0)} = 0$ and **(b)** $\rho^{(0)} > 0$ if and only if $f^{(0)} > 0$.

In this case, condition $\mathbf{F_1}$ takes the following modified form:

$\mathbf{F_2}$: **(a)** $\lim_{u\to 0}\overline{\lim}_{0\le\varepsilon\to 0}\sup_{|v|\le u}|q^{(\varepsilon)}(t+v)-q^{(0)}(t)| = 0$ almost everywhere with respect to the Lebesgue measure on $[0,\infty)$;

(b) $\overline{\lim}_{0\le\varepsilon\to 0}\sup_{0\le t\le T}|q^{(\varepsilon)}(t)| < \infty$ for every $T \ge 0$;

(c) $\lim_{T\to\infty}\overline{\lim}_{0\le\varepsilon\to 0} h\sum_{r\ge T/h}\sup_{rh\le t\le(r+1)h} e^{\gamma t}|q^{(\varepsilon)}(t)| = 0$ for some $h > 0$ and $\gamma > \rho^{(0)}$.

Let us denote,

$$\tilde{x}^{(\varepsilon)}(\infty) = \frac{\int_0^\infty e^{\rho^{(\varepsilon)}s}q^{(\varepsilon)}(s)m(ds)}{\int_0^\infty s e^{\rho^{(\varepsilon)}s}F^{(\varepsilon)}(ds)},$$

where $m(ds)$ is the Lebesgue measure on a real line.

Conditions $\mathbf{D_2}$, $\mathbf{C_1}$, and $\mathbf{F_2}$ imply, due to relation $\rho^{(\varepsilon)} \to \rho^{(0)}$ as $\varepsilon \to 0$, that there exists $0 < \varepsilon_2 \le \varepsilon_1$ such that $\rho^{(\varepsilon)} < \gamma$ and $\int_0^\infty e^{\rho^{(\varepsilon)}s}q^{(\varepsilon)}(s)m(ds) < \infty$ for $\varepsilon \le \varepsilon_2$. Thus, the functional $\tilde{x}^{(\varepsilon)}(\infty)$ is well defined for $\varepsilon \le \varepsilon_2$.

The following theorem was also proved in Silvestrov (1976, 1978, 1979).

Theorem 2. *Let conditions* $\mathbf{D_2}$*,* $\mathbf{C_1}$*, and* $\mathbf{F_2}$ *hold. Then,*

$$\frac{x^{(\varepsilon)}(t^{(\varepsilon)})}{e^{-\rho^{(\varepsilon)}t^{(\varepsilon)}}} \to \tilde{x}^{(0)}(\infty) \ \text{ as } \ \varepsilon \to 0. \tag{11}$$

The asymptotic relation (11) given in Theorem 2 should be compared with the asymptotic relation (8) given in Theorem 1, in the case where $\rho^{(0)} = 0$.

Indeed, relation (8) can be re-written in the form given in (11), with coefficients $\rho^{(\varepsilon)} = f^{(\varepsilon)}/m_1^{(\varepsilon)}$. The Cramér type condition $\mathbf{C_1}$ makes it possible to use in (11) an alternative coefficients $\rho^{(\varepsilon)}$ defined as the solution of the characteristic equation (9). The latter coefficients provide better fitting of the corresponding exponential approximation for solution of renewal equation. That is why the asymptotic relation (11) does not restrict the rate of growth for time $t^{(\varepsilon)}$ while the asymptotic relation (8) does impose such restriction.

Remark 2. It is worth to note that this theorem reduces to the variant of renewal theorem for improper renewal equation in the case of non-perturbed renewal equation, also given by Feller (1966). Condition $\mathbf{D_2}$ reduces to the assumption that $F^{(0)}(s)$ is a non-arithmetic distribution function with defect $f^{(0)} \in [0,1)$; $\mathbf{C_1}$ to the assumption that the exponential moment $\phi^{(0)}(\delta) \in (1,\infty)$; and $\mathbf{F_2}$ to the assumption that the function $e^{\gamma t}q^{(0)}(t)$ is directly Riemann integrable on $[0,\infty)$ for some $\gamma > \rho^{(0)}$.

The results formulated in Theorems 1 and 2 created the base for further research studies in the area. For example, Shurenkov (1980a, 1980b) generalised the results of these theorems to the case of matrix renewal equation using possibility of imbedding the matrix model to the scalar model considered in Theorems 1 and 2.

2.2. Exponential expansions in renewal theorem for nonlinearly perturbed renewal equation. A new improvement was achieved in the paper Silvestrov (1995) and then in the papers Gyllenberg and Silvestrov (1998, 1999a, 2000a). Under natural additional perturbation conditions, which assume that the defect $f^{(\varepsilon)}$ and the corresponding moments of the distribution $F^{(\varepsilon)}(s)$ can be expanded in power series with respect to ε up to and including an order k, explicit expansions for the corresponding characteristic roots were given, and the corresponding exponential expansions were obtained for solutions of nonlinearly perturbed renewal equations. In Silvestrov (1995), the case with asymptotically proper distributions $F^{(\varepsilon)}(s)$ was considered, while, in Gyllenberg and Silvestrov (1998, 1999a, 2000a), the case with asymptotically improper distributions $F^{(\varepsilon)}(s)$ was investigated. These results are presented in Chapter 2 of book [25].

Let us introduce the mixed power-exponential moment generating functions,

$$\phi^{(\varepsilon)}(\rho, r) = \int_0^\infty s^r e^{\rho s} F^{(\varepsilon)}(ds), \ \rho \geq 0, \ r = 0, 1, \dots .$$

Note that, by the definition, $\phi^{(\varepsilon)}(\rho, 0) = \phi^{(\varepsilon)}(\rho)$. Under condition $\mathbf{D_2}$ and $\mathbf{C_1}$, for any $0 < \delta' < \delta$, there exists $0 < \varepsilon_3 < \varepsilon_2$ such that $\phi^{(\varepsilon)}(\delta', r) < \infty$ for $r = 0, 1, \dots$ and $\varepsilon \leq \varepsilon_3$. Also, $\phi^{(\varepsilon)}(\rho, r) \to \phi^{(0)}(\rho, r)$ as $\varepsilon \to 0$ for $r = 0, 1, \dots$ and $\rho \leq \delta'$. Let δ' is chosen such that $\phi^{(0)}(\delta') \in (1, \infty)$. In this case, the characteristic root $\rho^{(0)} < \delta'$, and, also, there exists $0 < \varepsilon_4 < \varepsilon_3$ such that the characteristic roots $\rho^{(\varepsilon)} < \delta'$ for $\varepsilon \leq \varepsilon_4$.

The basic role plays the following nonlinear perturbation condition:

$\mathbf{P_1^{(k)}}$: $\phi^{(\varepsilon)}(\rho^{(0)}, r) = \phi^{(0)}(\rho^{(0)}, r) + b_{1,r}\varepsilon + \cdots + b_{k-r,r}\varepsilon^{k-r} + o(\varepsilon^{k-r})$ for $r = 0, \dots, k$, where $|b_{n,r}| < \infty, n = 1, \dots, k-r, \ r = 0, \dots, k$.

It is convenient to define $b_{0,r} = \phi^{(0)}(\rho^{(0)}, r), r = 0, 1, \dots$. From the definition of $\rho^{(0)}$ it is clear that $b_{0,0} = \phi^{(0)}(\rho^{(0)}, 0) = 1$.

It should be noted that, in the case $f^{(0)} = 0$, where characteristic root $\rho^{(0)} = 0$, the perturbation condition $\mathbf{P_1^{(k)}}$ involves usual power moments of distributions $F^{(\varepsilon)}(s)$. While, in the case $f^{(0)} > 0$, where characteristic root $\rho^{(0)} > 0$, the perturbation condition involve mixed power-exponential moments of distributions $F^{(\varepsilon)}(s)$.

Let also formulate the following condition that balances the rate at which time $t^{(\varepsilon)}$ approaches infinity, and the convergence rate of perturbation in different asymptotic zones, for $1 \leq r \leq k$:

$\mathbf{B^{(r)}}$: $0 \leq t^{(\varepsilon)} \to \infty$ in such a way that $\varepsilon^r t^{(\varepsilon)} \to \lambda_r$, where $0 \leq \lambda_r < \infty$.

The following theorem given in Silvestrov (1995) and Gyllenberg and Silvestrov (1998, 1999a, 2000a).

Theorem 3. *Let conditions* $\mathbf{D_2}$, $\mathbf{C_1}$, *and* $\mathbf{P_1^{(k)}}$ *hold. Then,*

(i) *The root* $\rho^{(\varepsilon)}$ *of equation* (9) *has the asymptotic expansion*

$$\rho^{(\varepsilon)} = \rho^{(0)} + a_1\varepsilon + \cdots + a_k\varepsilon^k + o(\varepsilon^k), \tag{12}$$

where the coefficients a_n *are given by the recurrence formulas* $a_1 = -b_{1,0}/b_{0,1}$ *and, in general, for* $n = 1, \dots, k$,

$$\begin{aligned} a_n = & - b_{0,1}^{-1}(b_{n,0} + \sum_{q=1}^{n-1} b_{n-q,1}a_q \\ & + \sum_{2\le m\le n}\sum_{q=m}^{n} b_{n-q,m} \cdot \sum_{n_1,\dots,n_{q-1}\in D_{m,q}} \prod_{p=1}^{q-1} a_p^{n_p}/n_p!), \end{aligned} \tag{13}$$

where $D_{m,q}$, *for every* $2 \le m \le q < \infty$, *is the set of all nonnegative, integer solutions of the system*

$$n_1 + \cdots + n_{q-1} = m, \quad n_1 + \cdots + (q-1)n_{q-1} = q. \tag{14}$$

(ii) *If* $b_{l,0} = 0, l = 1, \dots, r$, *for some* $1 \le r \le k$, *then* $a_1, \dots, a_r = 0$. *If* $b_{l,0} = 0, l = 1, \dots, r-1$ *but* $b_{r,0} < 0$, *for some* $1 \le r \le k$, *then* $a_1, \dots, a_{r-1} = 0$ *but* $a_r > 0$.

(iii) *If, additionally, conditions* $\mathbf{B^{(r)}}$, *for some* $1 \le r \le k$, *and* $\mathbf{F_2}$ *hold, then the following asymptotic relation holds*:

$$\begin{aligned} & \frac{x^{(\varepsilon)}(t^{(\varepsilon)})}{\exp\{-(\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})t^{(\varepsilon)}\}} \\ & \qquad \to e^{-\lambda_r a_r}\tilde{x}^{(0)}(\infty) \ \text{ as } \ \varepsilon \to 0. \end{aligned} \tag{15}$$

The asymptotic relation (15) given in Theorem 3 should be compared with the asymptotic relation (11) given in Theorem 2.

The asymptotic relation (11) looks nicely but has actually a serious drawback. Indeed, the exponential normalisation with the coefficient $\rho^{(\varepsilon)}$ is not so effective because of this coefficient is given us only as the root of the nonlinear equation (9), for every $\varepsilon \ge 0$.

Relation (15) essentially improve the asymptotic relation (11) replacing this simple convergence relation by the corresponding asymptotic expansion. The exponential normalisation with the coefficient $\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1}$ involves the root $\rho^{(0)}$. To find it one should solve only one nonlinear equation (9), for the case $\varepsilon = 0$. As far as the coefficients $a_1, \dots, a_{r-1}$ are concerned, they are given in the explicit algebraic recurrence form.

Moreover, the root $\rho^{(0)} = 0$ in the most interesting case, where $f^{(0)} = 0$, i.e., the limit renewal equation is proper. Here, the non-linear step connected with finding of the root of equation (9) can be omitted.

If there exist $0 < \varepsilon' \leq e_5$ such that the conditions listed in Remark 2 holds for the distribution function $F^{(\varepsilon)}(s)$ and the forcing function $q^{(\varepsilon)}(t)$ for every $\varepsilon \leq \varepsilon'$, then according Theorem 2, the following asymptotic relation holds for every $\varepsilon \leq \varepsilon'$,

$$\frac{x^{(\varepsilon)}(t)}{e^{-\rho^{(\varepsilon)}t}} \to \tilde{x}^{(\varepsilon)}(\infty) \ \text{ as } \ t \to \infty. \tag{16}$$

Let us now define for the mixed power-exponential moment functionals for the forcing functions,

$$\omega^{(\varepsilon)}(\rho, r) = \int_0^\infty s^r e^{\rho s} q^{(\varepsilon)}(s) m(ds), \ \rho \geq 0, \ r = 0, 1, \ldots.$$

Under conditions $\mathbf{C_1}$ and $\mathbf{F_2}$, for any $0 < \gamma' < \gamma$, there exists $0 < \varepsilon_6 < \varepsilon_5$ such that $\bar{\omega}^{(\varepsilon)}(\gamma', r) = \int_0^\infty s^r e^{\gamma' s} |q^{(\varepsilon)}(s)| m(ds) < \infty$ for $r = 0, 1, \ldots$ and $\varepsilon \leq \varepsilon_6$. Also, $\omega^{(\varepsilon)}(\rho, r) \to \omega^{(0)}(\rho, r)$ as $\varepsilon \to 0$ for $r = 0, 1, \ldots$ and $\rho \leq \gamma'$. Let γ' is chosen such that $\rho^{(0)} < \gamma'$. In this case, there exists $0 < \varepsilon_7 < \varepsilon_6$ such that the characteristic roots $\rho^{(\varepsilon)} < \gamma'$ for $\varepsilon \leq \varepsilon_7$.

Note that the renewal limit $\tilde{x}^{(\varepsilon)}(\infty)$ is well defined for $\varepsilon \leq \varepsilon_7$ even without the assumptions made above in order to provide asymptotic relation (16) and, moreover,

$$\tilde{x}^{(\varepsilon)}(\infty) = \frac{\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0)}{\phi^{(\varepsilon)}(\rho^{(\varepsilon)}, 1)}. \tag{17}$$

Let us now formulate a perturbation condition for mixed power-exponential moment functionals for the forcing functions:

$\mathbf{P_2^{(k)}}$: $\omega^{(\varepsilon)}(\rho^{(0)}, r) = \omega^{(0)}(\rho^{(0)}, r) + c_{1,r}\varepsilon + \cdots + c_{k-r,r}\varepsilon^{k-r} + o(\varepsilon^{k-r})$ for $r = 0, \ldots, k$, where $|c_{n,r}| < \infty, n = 1, \ldots, k - r, r = 0, \ldots, k$.

It is convenient to set $c_{0,r} = \omega^{(0)}(\rho^{(0)}, r), r = 0, 1, \ldots$.

The following theorem supplements Theorem 3.

Theorem 4. *Let conditions* $\mathbf{D_2}$, $\mathbf{C_1}$, $\mathbf{F_2}$, $\mathbf{P_1^{(k+1)}}$, *and* $\mathbf{P_2^{(k)}}$ *hold. Then the functional* $\tilde{x}^{(\varepsilon)}(\infty)$ *has the following asymptotic expansions*:

$$\begin{aligned} \tilde{x}^{(\varepsilon)}(\infty) &= \frac{\omega^{(0)}(\rho^{(0)}, 0) + f'_1\varepsilon + \cdots + f'_k\varepsilon^k + o(\varepsilon^k)}{\phi^{(0)}(\rho^{(0)}, 1) + f''_1\varepsilon + \cdots + f''_k\varepsilon^k + o(\varepsilon^k)} \\ &= \tilde{x}^{(0)}(\infty) + f_1\varepsilon + \cdots + f_k\varepsilon^k + o(\varepsilon^k), \end{aligned} \tag{18}$$

where the coefficients f_n', f_n'' *are given by the formulas* $f_0' = \omega^{(0)}(\rho^{(0)}, 0) = c_{0,0}, f_1' = c_{1,0} + c_{0,1}a_1,\ f_0'' = \phi^{(0)}(\rho^{(0)}, 1) = b_{0,1}, f_1'' = b_{1,1} + b_{0,2}a_1,$ *and in general for* $n = 0, \dots, k$,

$$f_n' = c_{n,0} + \sum_{q=1}^{n} c_{n-q,1}a_q + \sum_{2 \le m \le n} \sum_{q=m}^{n} c_{n-q,m} \cdot \sum_{n_1,\dots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} a_p^{n_p}/n_p!, \tag{19}$$

and

$$f_n'' = b_{n,1} + \sum_{q=1}^{n} b_{n-q,2}a_q + \sum_{2 \le m \le n} \sum_{q=m}^{n} b_{n-q,m+1} \cdot \sum_{n_1,\dots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} a_p^{n_p}/n_p!, \tag{20}$$

and the coefficients f_n *are given by the recurrence formulas* $f_0 = \tilde{x}^{(0)}(\infty) = f_0'/f_0''$ *and in general for* $n = 0 \dots, k$,

$$f_n = (f_n' - \sum_{q=0}^{n-1} f_{n-q}'' f_q)/f_0''. \tag{21}$$

It should be noted that one should require the perturbation condition $\mathbf{P_1^{(k+1)}}$ stronger than $\mathbf{P_1^{(k)}}$ in Theorem 4. This is because of the former condition is needed to get the corresponding expansion for $\phi^{(\varepsilon)}(\rho^{(\varepsilon)}, 1)$ in an asymptotic power series with respect to ε up to and including the order k.

Chapter 2 of book [25] also contains asymptotic results based on more general perturbation conditions.

It worth to mention that discrete time analogues of some of the results presented above are given in papers by Englund and Silvestrov (1997), Englund (2000, 2001), and Silvestrov (2000b). Also, exponential asymptotic expansions for renewal equation with non-polynomial perturbations are studied in papers by Englund and Silvestrov (1997), Englund (2001), and Ni, Silvestrov, and Malyarenko (2008).

3. Nonlinearly perturbed stochastic processes

3.1. Nonlinearly perturbed regenerative processes. Method of asymptotic analysis of nonlinearly perturbed renewal equation can be directly used in studies of quasi- and pseudo-stationary asymptotics for nonlinearly perturbed regenerative processes. The corresponding mixed ergodic and limit theorems and mixed ergodic and large deviation theorems for nonlinearly perturbed regenerative processes are given in a form of exponential asymptotics for joint distributions of positions of regenerative processes and regenerative stopping times. These distributions satisfy some renewal equations, and the corresponding theorems may be obtained by applying the above methods of asymptotic analysis for perturbed renewal equations. The corresponding results are presented in Chapter 3 of book [25].

Let $\xi^{(\varepsilon)}(t), t \geq 0$ be a regenerative process with a measurable phase space X and regeneration times $\tau_n^{(\varepsilon)}, n = 1, 2, \ldots$, and $\mu^{(\varepsilon)}$ be a regenerative stopping time that regenerates jointly with the process $\xi^{(\varepsilon)}(t)$, at times $\tau_n^{(\varepsilon)}$.

Both the regenerative process $\xi^{(\varepsilon)}(t)$ and the regenerative stopping time $\mu^{(\varepsilon)}$ are assumed to depend on a small perturbation parameter $\varepsilon \geq 0$. The processes $\xi^{(\varepsilon)}(t)$, for $\varepsilon > 0$ are considered as a perturbation of the process $\xi^{(0)}(t)$, and therefore we assume some weak continuity conditions for certain characteristic quantities of these processes regarded as functions of ε at point $\varepsilon = 0$.

As far as the regenerative stopping times are concerned, we consider two cases. The first one is a pseudo-stationary case, where the random variables $\mu^{(\varepsilon)}$ are stochastically unbounded, i.e., $\mu^{(\varepsilon)}$ tend to ∞ in probability as $\varepsilon \to 0$. The second one is the quasi-stationary case, where the random variables $\mu^{(\varepsilon)}$ are stochastically bounded as $\varepsilon \to 0$.

The object of studies is the probabilities $P^{(\varepsilon)}(t, A) = \mathsf{P}\{\xi^{(\varepsilon)}(t) \in A, \mu^{(\varepsilon)} > t\}$. These probabilities satisfy the following renewal equation,

$$P^{(\varepsilon)}(t, A) = q^{(\varepsilon)}(t, A) + \int_0^\infty P^{(\varepsilon)}(t - s, A) F^{(\varepsilon)}(ds),\ t \geq 0, \tag{22}$$

where the forcing function $q^{(\varepsilon)}(t, A) = \mathsf{P}\{\xi^{(\varepsilon)}(t) \in A, \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > t\}$ and distribution function $F^{(\varepsilon)}(s) = \mathsf{P}\{\tau_1^{(\varepsilon)} \leq s, \mu^{(\varepsilon)} \geq \tau_1^{(\varepsilon)}\}$.

Note that the distribution $F^{(\varepsilon)}(s)$ has the defect $f^{(\varepsilon)} = \mathsf{P}\{\mu^{(\varepsilon)} < \tau_1^{(\varepsilon)}\}$.

In this case, the mixed power-exponential moment generating function $\phi^{(\varepsilon)}(\rho, r) = \mathsf{E}(\tau_1^{(\varepsilon)})^r e^{\rho\tau_1^{(\varepsilon)}} \chi(\mu^{(\varepsilon)} \geq \tau_1^{(\varepsilon)})$ and the characteristic equation (9) takes the form $\phi^{(\varepsilon)}(\rho, 1) = 1$. The corresponding perturbation condition assumes that function $\phi^{(\varepsilon)}(\rho, r)$ (taken in point $\rho^{(0)}$ which is the root of the limit characteristic equation) can be expanded in a power series with respect to ε up to and including the order $k - r$ for every $r = 1, \ldots, k$.

The direct application of Theorems 1–4 to the renewal equation (22) yields the exponential asymptotic expansions of type (3) for the probabilities $P^{(\varepsilon)}(t, A)$ as well as the corresponding asymptotic expansions for the renewal limits $\tilde{P}^{(\varepsilon)}(\infty, A) = \lim_{t\to\infty} e^{\rho^{(\varepsilon)}t} P^{(\varepsilon)}(t, A)$ and then the asymptotic expansions of type (5) for the quasi-stationary distributions $\pi^{(\varepsilon)}(A) = \tilde{P}^{(\varepsilon)}(\infty, A)/\tilde{P}^{(\varepsilon)}(\infty, X)$. Finally, these asymptotic results is expanded to the model of nonlinearly perturbed regenerative processes with transition period.

3.2. Perturbed semi-Markov processes. The asymptotic results obtained in Chapters 1–3 play the key role in further studies. They let one make a very detailed analysis of pseudo- and quasi-stationary phenomena for perturbed semi-Markov processes with a finite set of states and absorption. It can be done by using the fact that a semi-Markov process can be considered as a regenerative process with regeneration times which are subsequent return moments to any fixed state $j \neq 0$. The first hitting time to the absorption state 0 is, in this case, the regenerative stopping time. The asymptotic results mentioned above are obtained by applying the corresponding results for regenerative processes given in Chapter 3. These results are presented in Chapters 4 and 5 of book [25].

A semi-Markov process $\eta^{(\varepsilon)}(t), t \geq 0$, with a phase space $X = \{0, \dots, N\}$ and transition probabilities $Q^{(\varepsilon)}_{ij}(u)$ is the object of studies. The semi-Markov process $\eta^{(\varepsilon)}(t)$ is assumed to depend on a perturbation parameter $\varepsilon \geq 0$. The processes $\eta^{(\varepsilon)}(t)$, for $\varepsilon > 0$, are considered as perturbations of the process $\eta^{(0)}(t)$, and, therefore, some weak continuity conditions are imposed on transition characteristics. Namely, the assumptions are made that moment functionals of transition probabilities are continuous, if regarded as functions of ε, at the point $\varepsilon = 0$.

Let us denote by $\mu^{(\varepsilon)}_j$ is the first hitting time (as result of a jump) for the process $\eta^{(\varepsilon)}(t)$ to a state $j \in X$. It is also assumed that 0 is an absorption state. The first hitting time $\mu^{(\varepsilon)}_0$ to the state 0 is an absorption time. In applications, the absorption times $\mu^{(\varepsilon)}_0$ are often interpreted as transition times for different stochastic systems described by semi-Markov processes; these are occupation times or waiting times in queueing systems, lifetimes in reliability models, extinction times in population dynamic models, etc.

The object of studies is probabilities $P^{(\varepsilon)}_{ij}(t) = \mathsf{P}_i\{\eta^{(\varepsilon)}(t) = j, \mu^{(\varepsilon)}_0 > t\}$.

Not only the generic case, where the limiting semi-Markov process has one communication class of recurrent-without absorption states, is considered in details, but also the case, where the limiting semi-Markov process has one communication class of recurrent-without absorption states and, additionally, the class of non-recurrent-without absorption states. The latter model covers a significant part of applications.

Semi-Markov processes possess a regeneration property at moments of hitting to a fixed state. This let one write down the following renewal type equations for the above probabilities,

$$P^{(\varepsilon)}_{ij}(t) = q^{(\varepsilon)}_{ij}(t) + \int_0^\infty P^{(\varepsilon)}_{jj}(t - s)\, {}_0G^{(\varepsilon)}_{ij}(ds), \ t \geq 0, \ i, j \neq 0, \tag{23}$$

where the forcing function $q^{(\varepsilon)}_{ij}(t) = \mathsf{P}\{\eta^{(\varepsilon)}(t) = j, \mu^{(\varepsilon)}_j \wedge \mu^{(\varepsilon)}_0 > t\}$ and distribution function ${}_0G^{(\varepsilon)}_{ij}(s) = \mathsf{P}_i\{\mu^{(\varepsilon)}_j \leq s, \mu^{(\varepsilon)}_j \leq \mu^{(\varepsilon)}_0\}$.

In fact, the relations above supply a renewal equation for every $j \neq 0$

if to choose $i = j$ and a transition renewal type relation in the case where $i \neq 0, j$.

In Chapter 4, asymptotic results for probabilities $P_{ij}^{(\varepsilon)}(t)$, which follows from the basic Theorems 1–2 and are based on the weak perturbation assumption about weak convergence of transition probabilities $Q_{ij}^{(\varepsilon)}(u)$ and minimal moment conditions involving only first moments of these transition probabilities, are given as well as asymptotic relations involving Cramér type conditions imposed on mixed power-exponential moments of these transition probabilities.

There exist analogs of these results in the literature, for example, those obtained in papers by Alimov and Shurenkov (1990a, 1990b), Shurenkov and Degtyar (1994), and Eleĭko and Shurenkov (1995).

However, Chapter 4 also contains a number of new asymptotic solidarity propositions for perturbed semi-Markov processes that clarify the role of initial state in the corresponding asymptotic relations as well as solidarity properties of cyclic absorption probabilities, and moment generating functions for hitting times for perturbed semi-Markov processes. They show that the corresponding conditions for convergence formulated in terms of cyclic characteristics of return times, and the form of the corresponding asymptotic relations are invariant with respect to the choice of the recurrent-without-absorption state used to construct the regeneration cycles. Also, a number of auxiliary propositions dealing with convergence of hitting probabilities, distributions, expectations, and exponential moments of hitting times is formulated and proved.

In this case, the distribution function ${}_0G_{jj}^{(\varepsilon)}(t)$ of the return-without-absorption time in a state $j \neq 0$ generates the renewal equation. The corresponding characteristic equation takes the form $\int_0^\infty e^{\rho s}\, {}_0G_{jj}^{(\varepsilon)}(ds) = 1$. In particular, it is shown that the characteristic root $\rho^{(\varepsilon)}$ of this equation does not depend on the choice of a recurrent-without-absorption state $j \neq 0$.

3.3. Nonlinearly perturbed semi-Markov processes. The asymptotic results presented in Chapter 4 play a preparatory role. They can be improved to the form of exponential expansions for joint distributions of positions of semi-Markov processes and absorption times. Such expansions were obtained in papers Gyllenberg and Silvestrov (1998, 1999a, 2000a), Silvestrov (2000b, 2007a). They are presented in more advanced form in Chapter 5 of book [25]. Also, asymptotic expansions for quasi-stationary distributions of nonlinearly perturbed semi-Markov processes are given in this chapter.

We consider the same model of perturbed semi-Markov processes as in Chapter 4. Cramér type conditions of type $\mathbf{C_1}$ and nonlinear perturbation conditions are imposed on the transition probabilities of the semi-Markov processes $Q_{ij}^{(\varepsilon)}(u)$. These conditions assume that mixed power-exponential moment generation functions $p_{ij}^{(\varepsilon)}[\rho, r] = \int_0^\infty s^r e^{\rho s} Q_{ij}^{(\varepsilon)}(ds), i \neq 0, j \in X$

(taken in point $\rho^{(0)}$ which is the root of the corresponding characteristic equation for the limit case $\varepsilon = 0$) can be expanded in a power series with respect to ε up to and including the order $k - r$ for every $r = 1, \dots, k$.

As above, the relationship between the rate with which ε tends to zero and the time t tends to infinity has a delicate influence upon the results. The balance between the rate of perturbation and the rate of growth of time is characterised by the following asymptotic relation $\varepsilon^r t^{(\varepsilon)} \to \lambda_r < \infty$ as $\varepsilon \to 0$ that is assumed to hold for some $1 \leq r \leq k$.

With the above assumptions, some natural additional non-periodicity conditions for the transition probabilities $Q_{ij}^{(\varepsilon)}(u)$, the following asymptotic relations are obtained,

$$\frac{\mathsf{P}_i\{\eta^{(\varepsilon)}(t^{(\varepsilon)}) = j, \mu_0^{(\varepsilon)} > t^{(\varepsilon)}\}}{e^{-(\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})t^{(\varepsilon)}}} \to \tilde{\tilde{\pi}}_{ij}^{(0)}(\rho^{(0)}) e^{-\lambda_r a_r} \quad \text{as} \quad \varepsilon \to 0, \ i, j \neq 0. \tag{24}$$

There are to alternatives in this relation, where the characteristic root **(a)** $\rho^{(0)} = 0$ and **(b)** $\rho^{(0)} > 0$.

The first alternative **(a)** holds if an absorption in 0 is impossible for the limiting process $\eta^{(0)}(t)$. In such a case, the absorption times $\mu_0^{(\varepsilon)}$ tend in probability to ∞ as $\varepsilon \to 0$. The asymptotic relation (24) describes pseudo-stationary phenomena for perturbed semi-Markov processes.

The second alternative **(b)** holds if an absorption in 0 is possible for the limiting process $\eta^{(0)}(t)$. In this case, the absorption times $\mu_0^{(\varepsilon)}$ are stochastically bounded as $\varepsilon \to 0$. The asymptotic relation (24) describes, in this case, quasi-stationary phenomena for perturbed semi-Markov processes.

Relations (24) let us get a new type of so-called mixed ergodic (for the process $\eta^{(\varepsilon)}(t)$) and large deviation (for the lifetime $\mu_0^{(\varepsilon)}$) theorems.

Let us, for simplicity, restrict the presentation by the basic case, where the set of non-absorbing states $X_1 = \{1, \dots, N\}$ is one communicative class of recurrent-without absorption states for the limit semi-Markov process.

Let us consider the pseudo-stationary case, where **(a)** $\rho^{(0)} = 0$. In this case, the limiting coefficients $\tilde{\tilde{\pi}}_{ij}^{(0)}(0) = \tilde{\pi}_j^{(0)}$, where $\tilde{\pi}_j^{(0)}$ are the stationary probabilities of the limiting process $\eta^{(0)}(t)$.

If $k = 1$, then the only case $r = 1$ is possible for the balancing condition.

In this case, (24) is equivalent to the following asymptotic relation $\mathsf{P}_i\{\eta^{(\varepsilon)}(t^{(\varepsilon)}) = j, \varepsilon\mu_0^{(\varepsilon)} > \varepsilon t^{(\varepsilon)}\} \to \tilde{\pi}_j^{(0)}(0) e^{-\lambda_1 a_1}$ as $\varepsilon \to 0, i, j \neq 0$. It shows that the position of the semi-Markov process $\eta(t^{(\varepsilon)})$ and the normalised absorption time $\varepsilon\mu_0^{(\varepsilon)}$ are asymptotically independent and have, in the limit, a stationary distribution and an exponential distribution, respectively. This can be interpreted as a mixed ergodic theorem (for the regenerative processes) and a limit theorem (for regenerative stopping times).

If $k = 2$, then two cases, $r = 1$ and $r = 2$, are possible for the balancing condition.

The case $r = 1$ was already commented and interpreted above. In this case, relation (24) can be given in the equivalent alternative form, $P_{ij}(t^{(\varepsilon)})/\tilde{\pi}_j^{(0)} e^{-a_1 \varepsilon t^{(\varepsilon)}} \to 1$ as $\varepsilon \to 0$. It shows that probability $P_{ij}(t^{(\varepsilon)})$ can be approximated by the exponential type mixed tail probability $\tilde{\pi}_j^{(0)} e^{-a_1 \varepsilon t^{(\varepsilon)}}$, with the zero asymptotic relative error, in every asymptotic time zone which is determined by the relation $\varepsilon t^{(\varepsilon)} \to \lambda_1$ as $\varepsilon \to 0$, where $0 \leq \lambda_1 < \infty$.

In the case $r = 2$, the asymptotic relation (24) reduces to the asymptotic relation $P_{ij}(t^{(\varepsilon)})/\tilde{\pi}_j^{(0)} e^{-a_1 \varepsilon t^{(\varepsilon)}} \to e^{-a_2 \lambda_2}$ as $\varepsilon \to 0$. It shows that probability $P_{ij}(t^{(\varepsilon)})$ can be approximated by the the exponential type mixed tail probability $\tilde{\pi}_j^{(0)} e^{-a_1 \varepsilon t^{(\varepsilon)}}$ as $\varepsilon \to 0$, with the asymptotic relative error $1 - e^{-a_2 \lambda_2}$, in every asymptotic time zone which is determined by the relation $\varepsilon^2 t^{(\varepsilon)} \to \lambda_2$ as $\varepsilon \to 0$, where $0 \leq \lambda_2 < \infty$.

If $\lambda_2 = 0$, then $\varepsilon t^{(\varepsilon)} = o(\varepsilon^{-1})$, and the asymptotic relative error is 0. Note that this case also covers the situation where $\varepsilon t^{(\varepsilon)}$ is bounded, which corresponds to the asymptotic relation (24) with $r = 1$. This is already an extension of this asymptotic relation since it is possible that $\varepsilon t^{(\varepsilon)} \to \infty$.

If $\lambda_2 > 0$, then $\varepsilon t^{(\varepsilon)} = O(\varepsilon^{-1})$, and the asymptotic relative error is $1 - e^{-\lambda_2 a_2}$. It differs from 0. Therefore, $o(\varepsilon^{-1})$ is an asymptotic bound for the large deviation zone with zero asymptotic relative error in the above approximation.

To get the approximation with zero asymptotic relative error in the asymptotic time zone which are determined by the relation $\varepsilon^2 t^{(\varepsilon)} \to \lambda_2$ one should approximate the mixed tail probabilities $P_{ij}(t^{(\varepsilon)})$ by the exponential type mixed tail probabilities $\tilde{\pi}_j^{(0)} e^{-(a_1 \varepsilon + a_2 \varepsilon^2) t^{(\varepsilon)}} \sim \tilde{\pi}_j^{(0)} e^{-a_1 \varepsilon t^{(\varepsilon)} - a_2 \lambda_2}$, i.e., to introduce the corresponding corrections for the parameters in the exponents.

The comments above let one interpret relation (24) in the case $r = 2$ as a new type of mixed ergodic and large deviation theorem for the nonlinearly perturbed process $\eta^{(\varepsilon)}(t)$ and the lifetime $\mu_0^{(\varepsilon)}$.

A similar interpretation can be made for the asymptotic relation (24) if $k > 2$. As above, relation (24) can also be interpreted as a mixed ergodic and limit theorem if $r = 1$. On the other hand, this relation is naturally to consider as a mixed ergodic and large deviation theorem for the corresponding asymptotic time large deviation zones if $1 < r \leq k$.

A similar interpretation remains also in the quasi-stationary case, where **(b)** $\rho^{(0)} > 0$.

As was mentioned above, in the pseudo-stationary case **(a)**, where $\rho^{(0)} = 0$, the limiting coefficients $\tilde{\tilde{\pi}}_{ij}^{(0)}(0)$ do not depend of initial states i and coincides with the corresponding stationary probabilities $\tilde{\pi}_j^{(0)}$ for the limiting process $\eta^{(0)}(t)$. The situation is not so simple in the quasi-stationary case **(b)**. In this case, under some natural conditions, there exists a so-called quasi-stationary distribution for the semi-Markov process $\eta^{(\varepsilon)}(t)$ given by the formulas $\pi_j^{(\varepsilon)}(\rho^{(\varepsilon)}) = \tilde{\tilde{\pi}}_{ij}^{(\varepsilon)}(\rho^{(\varepsilon)}) / \sum_{r \neq 0} \tilde{\tilde{\pi}}_{ir}^{(\varepsilon)}(\rho^{(\varepsilon)})$, $j \neq 0$. In this case, the coefficients $\tilde{\tilde{\pi}}_{ij}^{(\varepsilon)}(\rho^{(\varepsilon)})$, which are defined as in (24) but for a fixed $\varepsilon \geq 0$, may depend on the initial state $i \neq 0$. But, the quasi-stationary probabilities $\pi_j^{(\varepsilon)}(\rho^{(\varepsilon)}), j \neq 0$, do not depend on the choice of the state $i \neq 0$ in the formulas above.

In Chapter 5, the asymptotic expansions for the quasi-stationary distributions for nonlinearly perturbed semi-Markov processes with absorption also are given,

$$\pi_j^{(\varepsilon)}(\rho^{(\varepsilon)}) = \pi_j^{(0)}(\rho^{(0)}) + g_j[1]\varepsilon + \cdots + g_j[k]\varepsilon^k + o(\varepsilon^k), \ j \neq 0. \tag{25}$$

Both asymptotic expansions (24) and (25) are provided by the explicit algorithms for calculating the coefficients in these expansions.

Mixed large deviation and ergodic theorems for nonlinearly perturbed semi-Markov processes presented above are obtained by application exponential expansions in mixed ergodic and large deviation theorems developed in Chapter 3 for nonlinearly perturbed regenerative processes. The fact that any semi-Markov processes is a regenerative process with the return times to some fixed state being regeneration times can be employed. The time of absorption (the first time of hitting the absorption state 0) is a regenerative stopping time. A semi-Markov process is characterized by prescribing its transition probabilities. It is natural to formulate the perturbation conditions in terms of these transition probabilities. However, the conditions and expansions formulated for regenerative processes are specified in terms of expansions for the moments of regeneration times. Therefore, the corresponding asymptotic expansions for absorption probabilities and the moments of return and hitting times for perturbed semi-Markov processes must be developed.

Thus, as the first step, asymptotic expansions for hitting probabilities, power and mixed power-exponential moments of hitting times are constructed using a procedure that is based on recursive systems of linear equations for hitting probabilities and moments of hitting times. These moments satisfy recurrence systems of linear equations with the same perturbed coefficient matrix and the free terms connected by special recurrence systems of relations. In these relations, the free terms for the moments of a given order are given as polynomial functions of moments of lower orders. This permits to build an effective recurrence algorithm for constructing the corresponding asymptotic expansions. Each sub-step in this recurrence algorithm is of a matrix but linear type, where the solution of the system of linear equations with nonlinearly perturbed coefficients and free terms should be expanded in asymptotic series. These expansions are also provided with a detailed analysis of their pivotal properties. These results have their own values and possible applications beyond the problems studied in the book.

As soon as asymptotic expansions for moments of return-without-absorption times are constructed, the second nonlinear but scalar step of construction asymptotic expansions expansions for the characteristic root $\rho^{(\varepsilon)}$ can be realised.

The separation of two steps described above, the first one matrix and recurrence but linear and the second one nonlinear but scalar, significantly simplify the whole algorithm.

Asymptotic expansion for the characteristic root $\rho^{(\varepsilon)}$ let one combine it with asymptotic relations for probabilities $\mathsf{P}_i\{\eta^{(\varepsilon)}(t^{(\varepsilon)}) = j, \mu_0^{(\varepsilon)} > t^{(\varepsilon)}\}$ given in Chapter 4 and obtain the exponential expansion (24).

The asymptotic expansion (25) for the quasi-stationary distributions $\pi_j^{(\varepsilon)}(\rho^{(\varepsilon)})$ requires two more steps, which are needed for constructing asymptotic expansions at the point $\rho^{(\varepsilon)}$ for the corresponding moment generation functions, giving expressions for quasi-stationary probabilities in the quotient form, and for transforming the corresponding asymptotic quotient expressions to the form of power asymptotic expansions.

As a result, one get an effective algorithm for a construction of the asymptotic expansions given in relations (24) and (25). It seems, that the method used for obtaining the expansions mentioned above has its own value and great potential for future studies.

As a particular but important example, the model of nonlinearly perturbed continuous time Markov chains with absorption is also considered. In this case, it is more natural to formulate the perturbation conditions in terms of generators of the perturbed Markov chains. Here, an additional step in the algorithms is needed, since the initial perturbation conditions for generators must be expressed in terms of the moments for the corresponding semi-Markov transition probabilities. Then, the basic algorithms obtained for nonlinearly perturbed semi-Markov processes can be applied.

Chapters 1–5 present a theory that can be applied in studies of pseudo- and quasi-stationary phenomena in nonlinearly perturbed stochastic systems.

4. Applications

4.1. Nonlinearly perturbed stochastic systems. Chapter 6 of book [25] deals with applications of the results obtained in Chapters 1–5 to an analysis of pseudo- and quasi-stationary phenomena in nonlinearly perturbed stochastic systems. This chapter is partly based on the results of the papers Gyllenberg and Silvestrov (1994, 1999a, 2000a).

Examples of stochastic systems under consideration are queueing systems, epidemic, and population dynamics models with finite lifetimes. In queueing systems, the lifetime is usually the time at which some kind of a fatal failure occurs in the system. In epidemic models, the time of extinction of the epidemic in the population plays the role of the lifetime, while in population dynamics models, the lifetime is usually the extinction time for the corresponding population.

Several classical models being the subject of long term research studies were selected. These models serve nowadays mainly as platforms for demonstration of new methods and innovation results. Our goal also is to show what kind of new types results related to quasi-stationary asymptotics can be obtained for such models with nonlinearly perturbed parameters.

All types of asymptotic results studied in Chapters 1–5 are given. They include the following: (i) mixed ergodic theorems (for the state of the system) and limit theorems (for the lifetimes) that describe transition phenomena; (ii) mixed ergodic and large deviation theorems that describe pseudo- and quasi-stationary phenomena; (iii) exponential expansions in mixed ergodic and large deviation theorems; (iv) theorems on convergence of quasi-stationary distributions; and (v) asymptotic expansions for quasi-stationary distributions. In all the examples, we try to specify and to describe, in a more explicit form, conditions and algorithms for calculating the limit expressions, the characteristic roots, and the coefficients in the corresponding asymptotic expansions.

As the first example, a M/M queueing system with highly reliable main servers is considered. The simplest variant of such system with nonlinearly perturbed parameters was the subject of discussion in the first three subsections above. This queueing system is our first choice because of its function can be described by some nonlinearly perturbed continuous time Markov chains with absorption. Here, all conditions take a very explicit and clear form.

Also a M/G queueing system with quick service and a bounded queue buffer is considered. In this case, the perturbed stochastic processes, which describe the dynamics of the queue in the system, belong to the class of so-called stochastic processes with semi-Markov modulation. These processes admit a construction of imbedded semi-Markov processes and are more general than semi-Markov processes. This example was chosen because it shows in which way the main results obtained in the book can be applied to stochastic processes more general than semi-Markov processes, in particular, to stochastic processes with semi-Markov modulation.

The next example is based on classical semi-Markov and Markov birth-and-death type processes. Some classical models of queueing systems, epidemic or population dynamic models can be described with the use of such processes. We show in which way nonlinear perturbation conditions should be used and what form will take advanced quasi- and pseudo-stationary asymptotics developed in Chapters 1–5.

Finally, an example of nonlinearly perturbed metapopulation model is considered. This example is interesting since it brings, for the first time, the discussion on advanced quasi- and pseudo-stationary asymptotics in this actual area of research in mathematical biology.

4.2. Nonlinearly perturbed risk processes. The classical risk processes are still the object of intensive research studies as show, for example, references given in the bibliography. Of course, the purpose of these studies is not any more to derive formulas relevant for field applications. These studies intend to illustrate new methods and types of results that can later be expanded to more complex models. The same approach was used by us when choosing this model. The aim was to show that the innovative methods of analysis for nonlinearly perturbed processes developed in the book can yield new results for this classical models.

Chapter 7 of book [25] contains results that extend the classical Cramér-Lundberg and diffusion approximations for the ruin probabilities to a model of nonlinearly perturbed risk processes. Both approximations are presented in a unified way using the techniques of perturbed renewal equations developed in Chapters 1 and 2. This chapter is partly based on the results of the papers Gyllenberg and Silvestrov (1999b, 2000b) and Silvestrov (2000a, 2007b).

The main new element in the results presented in Chapter 7 is a high order exponential asymptotic expansion in these approximations for nonlinearly perturbed risk processes. Correction terms are obtained for the Cramér-Lundberg and diffusion type approximations, which provide the right asymptotic behaviour of relative errors in the perturbed model. We study the dependence of these correction terms on the relations between the rate of perturbation and the rate of growth of the initial capital.

Also various variants of the diffusion type approximation, including the asymptotics for increments and derivatives of the ruin probabilities are given.

Finally, we give asymptotic expansions in the Cramér-Lundberg and diffusion type approximations for distribution of the capital surplus prior and at ruin for nonlinearly perturbed risk processes.

It seems to us that results presented in Chapters 6 and 7 illustrate well a potential of asymptotic methods developed in the book.

The works by Englund (1999a, 1999b, 2001) and Ni, Silvestrov, and Malyarenko (2008) may also be mentioned. They also deal with applications of methods based on perturbed renewal to asymptotic analysis of nonlinearly perturbed queuing systems and nonlinearly perturbed risk processes.

5. New directions for the research

The last Chapter 8 of book [25] contains three supplements. The first one gives some basic arithmetic operation formulas for scalar and matrix asymptotic expansions.

The book contains also extended and carefully gathered bibliography that has more than 1000 references to works in the areas related to the subject of the book. The third supplement in Chapter 8 contains the corresponding brief bibliographical remarks.

In the second supplement, some new directions in the research concerned pseudo- and quasi-stationary phenomena for perturbed stochastic systems that relate to the theory developed in this book are discussed and commented on. There is a hope that this discussion will be especially useful for young researchers and stimulate their interest to research studies in these areas.

5.1. Nonlinearly perturbed stochastic systems with discrete time. There is no doubt that all results on continuous time stochastic processes and systems presented in the book should have their analogues for discrete time stochastic processes and systems. It should be noted that discrete time models are interesting by themselves and have important applications.

Some results concerned asymptotic expansions in mixed ergodic and large deviation theorems for nonlinearly perturbed regenerative processes with discrete time can be found in Englund and Silvestrov (1997), Englund (2000, 2001), and in Silvestrov (2000b) for discrete time Markov chains with absorption. However, these results give discrete time analogues just for a small part of the results from the theory developed in the present book for continuous time processes.

A development of a similar complete theory for nonlinearly perturbed stochastic processes and system with discrete time requires an additional comprehensive research.

5.2. Asymptotic expansions based on non-polynomial systems of infinitesimals. Asymptotic expansions in mixed ergodic and limit/large deviation theorems and asymptotic expansions for quasi-stationary distributions can also be obtained for nonlinearly perturbed stochastic processes and systems where the expansions in the initial perturbation conditions are based on different systems of infinitesimals.

In this book, all expansions are based on the integer powers $\varphi_n(\varepsilon) = \varepsilon^n, n = 0, 1, \ldots$, for the simplest infinitesimal $\varphi_1(\varepsilon) = \varepsilon$. This is a very natural system of infinitesimals that uses Taylor type expansions for nonlinearly perturbed characteristics of the corresponding processes and systems.

There is a conjecture that analogous asymptotic expansions can also be constructed for a model where the corresponding expansions include products of integer powers $\varphi_{\bar{n}}(\varepsilon) = \prod_{i=1}^{m}(\varphi_i(\varepsilon))^{n_i}, \bar{n} = (n_1, \ldots, n_m), n_1, \ldots, n_m = 0, 1, \ldots$, $m = 1, 2, \ldots$, of infinitesimals taken from a finite or countable base set $\{\varphi_i(\varepsilon), i = 1, 2, \ldots\}$. The only condition should be imposed on these infinitesimals that any two infinitesimals of the product form given above, $\varphi'_{\bar{n}'}(\varepsilon)$ and $\varphi''_{\bar{n}''}(\varepsilon)$, should be asymptotically comparable, i.e., there should be a constant $0 \leq c \leq \infty$ (determined by these infinitesimals) such that $\varphi'_{\bar{n}'}(\varepsilon)/\varphi''_{\bar{n}'}(\varepsilon) \to c$ as $\varepsilon \to 0$. The difference in the corresponding expansions should mainly be caused by new forms in arithmetics for asymptotic expansions, i.e., in formulas for coefficients in sums, products, quotients, as well as in the power, exponential, and other functions in the asymptotic expansions. Such expansions permit to obtain a more dense net of infinitesimals in the corresponding expansions and also to get the expansions in models with corresponding types of perturbations.

Examples of models with non-polynomial perturbations can be found in Englund and Silvestrov (1997) and Englund (1999a, 1999b, 2000, 2001) and Ni, Silvestrov, and Malyarenko (2008). In the first five works, the model of nonlinear perturbations with the base set of the infinitesimals $\{\varepsilon, e^{-a/\varepsilon}\}$ is considered. The last paper deals with the model of nonlinear perturbations with the base set of the infinitesimals $\{\varepsilon, \varepsilon^{\omega}\}$, where $\omega > 1$ is some irrational number, is considered.

5.3. Asymptotic expansions in mixed ergodic and large deviation theorems for semi-Markov type processes with countable and general phase spaces. The asymptotic results obtained in Chapters 4 and 5 of this book relate to nonlinearly perturbed Markov chains and semi-Markov processes with finite phase spaces. These processes possess a regeneration property at return moments in a fixed state. This makes it possible to use the asymptotic results for nonlinearly perturbed regenerative processes presented in Chapter 3.

Markov chains and semi-Markov processes with countable phase spaces possess similar regeneration properties. Moreover, the method of artificial regeneration developed in the works of Kovalenko (1977), Nummelin (1978), Athreya and Ney (1978), permits to construct regeneration moments for Markov and semi-Markov processes with a general phase space. In this way, the results concerning the asymptotic analysis of pseudo- and quasi-stationary phenomena for nonlinearly perturbed regenerative processes can be applied to nonlinearly perturbed Markov chains and semi-Markov processes with countable and general phase spaces.

There exists a large number of works devoted to ergodic theorems, limit theorems for random functionals of hitting time types, as well as large deviation theorems for such functionals in non-mixed or mixed forms (with ergodic theorems). These works relate to Markov and semi-Markov type processes with finite, countable, and general phase spaces. The results presented in Chapter 4 are related to this direction. The corresponding references are listed and commented on in the bibliographical remarks of book [25].

It should be noted that the results given in Chapter 4 play only a preparatory role with respect to the results related to exponential asymptotic expansions in mixed ergodic and large deviation theorem, and asymptotic expansions for quasi-stationary distributions given in Chapter 5. These theorems also require many auxiliary results that are important by themselves. Such results, for example, include asymptotic cyclic solidarity properties for the corresponding processes and asymptotic expansions for absorption and hitting probabilities, power and mixed power-exponential moments for the first hitting times and related functionals, and others. Most of these results still have no analogues for Markov and semi-Markov type processes with countable and general phase spaces, and they may constitute a new direction for the future research. As usual, operator analogues for matrix techniques used in the models with finite phase spaces should be employed for models with countable and general phase spaces, which significantly complicates the problem under consideration.

The corresponding references are given in the bibliographical remarks of book [25]. Here, we would like only to mention some originating papers related research studies of quasi-stationary phenomena Vere-Jones (1962), Kingman (1963), Darroch and Seneta (1965), and some books related to these problems, Seneta (1981), Stewart and Sun Ji Guang (1990), Meyn and Tweedie (1993), Kijima (1997), and Stewart (1998, 2001).

5.4. Mixed ergodic and large deviation theorems for semi-Markov type processes with asymptotically uncoupled phase space. Results presented in Chapters 4 and 5 are related to a basic model of nonlinearly perturbed semi-Markov processes with one limit class of recurrent-without-absorption states. It would be very interesting to extend the theory to models in which the set of non-absorption states for perturbed processes asymptotically splits into several classes of recurrent-without-absorption states which do not communicate with each others.

It should be noted that most of the results for models with an asymptotically uncoupled phase space are related to weak convergence limit theorems for distributions of hitting times for pseudo-stationary models mixed with the corresponding ergodic theorems and related theorems on asymptotic aggregation of Markov and semi-Markov type processes. Also a few results concerned asymptotic expansions for linearly perturbed Markov and semi-Markov type processes are known.

At the same time, a study of large deviation theorems for hitting times, as well as exponential asymptotic expansions in mixed ergodic and large deviation theorems for hitting times and asymptotic expansions for quasi-stationary distributions in models with asymptotically uncoupled sets of recurrent-without-absorption states for linearly and all the more nonlinearly perturbed and Markov and semi-Markov type processes, have not been conducted before. Such theorems certainly constitute an additional prospective direction for the future research.

In principle, the method based on applying asymptotic results to perturbed regenerative processes developed in Chapter 3 can be used in combination with the method of recurrence asymptotic analysis for power and mixed power-exponential moments of hitting times developed in Chapter

5. However, one should expect that pseudo-absorption effects caused by asymptotic vanishing of communication between different recurrent-without-absorption classes of states may complicate the asymptotic analysis for power and mixed power-exponential moments of hitting times for models with asymptotically uncoupled phase space.

The corresponding references are in the bibliographical remarks of book [25]. Here, we would like only to mention some originating papers in this area that are Dobrushin (1953), Meshalkin (1958), Simon and Ando (1961), Hanen (1963), and Korolyuk (1969) as well as the latest books in the area, Stewart and Sun Ji Guang (1990), Stewart (1998, 2001), Yin and Zhang (1998), Koroliuk and Limnios (2005), and Anisimov (2008).

5.5. Mixed ergodic and large deviation theorems for stochastic processes with semi-Markov modulation (switchings). Methods of the asymptotic analysis of pseudo- and quasi-stationary phenomena in nonlinearly perturbed Markov chains and semi-Markov processes also can be applied to more general processes of Markov and semi-Markov types. In order to be able to apply the methods of asymptotic analysis developed for perturbed renewal equations, such processes should possess appropriate regeneration properties. In particular, these are so-called stochastic processes with semi-Markov modulation (switchings), which possess imbedded semi-Markov processes. In the Markov case, these processes are also known as Markov processes with discrete components. The books of Gikhman and Skorokhod (1975), Silvestrov (1980), Koroliuk and Limnios (2005), and Anisimov (2008) contain descriptions of these classes of stochastic processes.

In Chapter 6 of book [25], a detailed description of a typical example of M/G queueing systems with quick service and a bounded queue buffer, which can be described with the use of processes with semi-Markov modulation, is given. This example shows in which way the methods developed in the book can be applied to the asymptotic analysis of pseudo- and quasi-stationary phenomena in nonlinearly perturbed processes with semi-Markov modulation. The main point in the corresponding asymptotic analysis is to express the perturbation and other conditions in terms of local characteristics related to the individual semi-Markov regeneration cycles.

There are no doubts that the main results of the theory presented in this book can be extended to this class of processes that are more general than regenerative and semi-Markov processes.

5.6. Double asymptotic expansions in limit and large deviation theorems for lifetimes in nonlinearly perturbed stochastic processes and systems. The expansions obtained in Chapter 5, applied to marginal distributions of absorption times, have the following form $\mathsf{P}\{\varepsilon\mu^{(\varepsilon)} > t/\varepsilon^{r-1}\}/\pi e^{-(\rho_0+a_1\varepsilon+\cdots+a_r\varepsilon^r)t/\varepsilon^r} = 1 + o(1)$ for $r = 1, \ldots, k$. These expansions describe the behaviour of relative errors in large deviation zones of different orders, when approximating the distribution of absorption times with exponential type distributions that have specially fitted parameters.

Our conjecture is that it should also be possible to expand the residual term $o(1)$ in the asymptotic relation given above. This hope is based on the results given in Korolyuk, Penev and Turbin (1972, 1981), Piliščuk and Turbin (1973), and Abadov (1984), Silvestrov and Abadov (1984, 1991, 1993), where such expansions have been obtained in pseudo-stationary model ($\rho_0 = 0$), respectively, for the case $r = 1$ in the first three papers and for the cases $r = 1$ and $r = 2$ in the last four papers. Note that the case $r = 1$ corresponds to asymptotic expansions in usual weak limit theorem while the case $r = 2$ corresponds asymptotic expansions large deviation theorems for the zone of large deviation of the order $O(\varepsilon^{-1})$. In the cases $r > 2$, the question about such "double" expansions is an open problem. These remarks can be also related to exponential expansions in mixed ergodic and large deviation theorems.

5.7. Explicit upper bounds for the remainder terms in asymptotic expansions for nonlinearly perturbed stochastic systems. The problem of finding explicit upper bounds for remainder terms in exponential asymptotic expansions in mixed ergodic and limit/large deviation theorems for nonlinearly perturbed stochastic processes and systems is of a special importance. For example, such explicit upper bounds for the remainder terms are obtained in the asymptotic expansions given in the papers Silvestrov and Abadov (1991, 1993) mentioned above.

Of course, the investigation of the rates of convergence in limit theorems for lifetime functionals for regenerative processes and Markov- and semi-Markov type processes (which correspond to the problem of finding explicit upper bounds for the remainder terms in zero-order expansions) continue to be actual.

Some explicit upper bounds for the remainder terms in asymptotic expansions of stationary and quasi-stationary distributions are known for linearly perturbed Markov- and semi-Markov processes. Similar problems related to explicit upper bounds for the remainder terms in asymptotic expansions of stationary and quasi-stationary distributions for nonlinearly perturbed stochastic processes and systems have not been conducted so far.

The corresponding references are given in the bibliographical remarks of book [25]. Here, we would like only to mention some papers and books, which contain explicit estimates in stability problems, rates of approximation for the perturbation of stationary distributions and related characteristics for Markov chains and semi-Markov type processes. These are, Solov'ev (1983), Kalashnikov and Rachev (1988), Meyn and Tweedie (1993), Kartashov (1996), and Kalashnikov (1997).

5.8. Quasi-stationary phenomena in nonlinearly perturbed queueing and reliability systems. Asymptotic analysis of pseudo- and quasi-stationary phenomena in nonlinearly perturbed queueing systems and net-works of M/M, M/G, and G/M types, which can be described with the use of Markov chains, semi-Markov processes, and stochastic processes with semi-Markov modulation, is an unlimited area for applications of the theory developed in the book. Examples of such applications are given in Chapter 6 of book [25]. These examples show that, in such applications, the main problems translate to technical and sometimes non-trivial and interesting calculations connected with a re-formulation of the perturbation conditions and rewriting formulas for the coefficients of the asymptotic expansions and quasi-stationary distributions in terms of local characteristics used to define the corresponding queueing systems.

It would be interesting to develop an asymptotic analysis of pseudo- and quasi-stationary phenomena for nonlinearly perturbed queueing systems with a more general structure of input flows of customers and service processes, e.g., systems with input flows of group income of customers and group service, different models of vacations, systems with parameters in input flows and service processes depending on a queue, systems with modulated input flows and switching service regimes, systems with several kinds of customers, systems with several types of servers, with and without reservation, etc.

As well known, queueing systems of G/G type can also be described with the use of regenerative processes. Here the moments when the queue becomes empty play, as a rule, the role of regeneration times. Thus, we can use the methods of asymptotic analysis of pseudo- and quasi-stationary phenomena for perturbed regenerative processes. However, the analysis of regeneration cycles and a reduction of the conditions and formulas that are based on the global characteristics connected with the regeneration cycles to, correspondingly, conditions and formulas that would be based on local characteristics used to define the corresponding systems is expected to be much more difficult for such systems.

The corresponding references are given in the bibliographical remarks of book [25]. Here, we would like only to mention some surveys and latest books related to asymptotic problems for queueing systems; these are Asmussen (1987, 2003), Kalashnikov and Rachev (1988), Kalashnikov (1994), Kovalenko (1994), Kovalenko, Kuznetsov, and Pegg (1997), Borovkov (1998), Limnios and Oprişan (2001), Whitt (2002), and Anisimov (2008).

5.9. Quasi-stationary phenomena in nonlinearly perturbed models of biological type systems. Another unlimited area for applications of the asymptotic results dealing with pseudo- and quasi-stationary phenomena in stochastic systems are nonlinearly perturbed epidemic and population dynamics models which can be described with the use of Markov chains, semi-Markov processes with finite phase spaces. Examples of such applications are given in Chapter 6 of book [25]. Similarly to the examples from the queueing theory, the main problems are to reformulate the perturbation conditions and to express the formulas for the coefficients in the asymptotic expansions and quasi-stationary distributions in terms of the local characteristic used to define the corresponding epidemic or population dynamics models.

Nonlinearly perturbed epidemic and population dynamics models with different types, e.g., sex, age, health status, etc., of individuals in the population, forms of interactions between individuals, environmental modulation, complex spatial structure, etc., make interesting objects for asymptotic analysis of pseudo- and quasi-stationary phenomena.

The related references are given and commented in the bibliographical remarks of book [25].

5.10. Asymptotics of probabilities and other characteristics connected with ruins for perturbed risk type processes. Chapter 7 of the book is devoted to studies of quasi-stationary phenomena for nonlinearly perturbed risk processes. Asymptotic results for ruin probabilities, densities of non-ruin probabilities, and distributions of the capital surplus prior to and at the time of a ruin are given. However, there are many other functional characteristics connected with risk processes that satisfy renewal equations. In such cases, analogous asymptotic results can be obtained with the use of asymptotic methods developed for perturbed renewal equations in Chapters 1 and 2. More general risk type processes may also supply models with functional characteristics satisfying renewal equations. For example, such are risk processes with investment components described by Brownian motions, more general Lévy type risk processes, risk processes with modulated claim flows, etc.

We give corresponding references in the bibliographical remarks. Here, we would like only to mention some surveys and latest books related to asymptotic problems for risk processes; these are Embrechts, Klüppelberg, and Mikosch (1997), Rolski, Schmidli, Schmidt, and Teugels (1999), Asmussen (2000), and Bening and Korolev (2002).

5.11. Markov and semi-Markov processes describing dynamics of credit ratings. Finally, we would like also to mention a possible prospective area of financial applications for asymptotic methods developed in the book. The methods of asymptotic analysis of pseudo- and quasi-stationary characteristics may be also applied to studies of default type distributions for Markov and semi-Markov processes with absorption which become common to use for description of dynamics of credit ratings.

5.12. Numerical studies connected with asymptotic expansions describing quasi-stationary phenomena in nonlinearly perturbed stochastic systems. Numerical and simulation studies make a natural supplement to analytical results, especially such as asymptotic expansions for functional characteristics of perturbed stochastic processes and systems. For example, asymptotic results presented in Chapter 7 of book [25] yield formulas for approximations of ruin probabilities based on higher order moments of the claim distributions. These approximations are asymptotically optimal in the sense that the corresponding relative errors tend to zero. It would be interesting to compare them with other known approximations. This, however, would require to carry out additional comprehensive analytical, numerical, and simulation studies that are beyond the frame of the present book. There is a hope that these experimental studies will be realised in a future for this model as well for other perturbed models of stochastic processes and systems.

5.13. Asymptotics for perturbed equations of renewal type. The program for studies of pseudo- and stationary phenomena in nonlinearly perturbed stochastic processes and systems realised in the book is based on the method of the asymptotic analysis for perturbed renewal equations presented in Chapters 1 and 2. There are no doubts that the main asymptotic results for the perturbed renewal equations presented in these chapters should hold for more general equations of renewal type. In many cases, such results can be achieved by a suitable reduction to the case of classical renewal equation treated in the book. For example, Shurenkov (1980a, 1980b, 1980c) gave such a generalisation of some results presented in Chapter 1 to the case of a matrix renewal equation. Chapter 4 contains a more through and extended presentation of matrix version of asymptotic results for perturbed renewal equation given in Chapter 1 for the model case of perturbed semi-Markov processes with absorption. Chapter 5 essentially improves these asymptotic results to the much more advanced form of the corresponding asymptotic expansions. In fact, this chapter contains a matrix generalisation of the results of Chapter 2 to the model case of nonlinearly perturbed semi-Markov processes with absorption.

There exist other versions of renewal type equations, for example, the so-called renewal equations with waits, operator renewal type equations, renewal equations on a whole line etc. It would be interesting to extend the theory presented in the present book to such equations.

There are also other areas, beyond ergodic, pseudo- and quasi-stationary problems, where the renewal equation and the corresponding asymptotic results play an essential role. For example, these are asymptotic problems for moment functionals appearing in the theory of branching processes and many others. We hope that the book will stimulate future works directed to new applications of the theory presented in this book.

5.14. Bibliography. The bibliography of book [25] contains more than 1000 references to works in related areas, dealing with ergodic and quasi-ergodic theorems, stability theorems, limit and large deviation theorems for lifetime-type functionals and asymptotic aggregation theorems for regenerative, Markov, and semi-Markov type processes, as well as applications of such theorems to queueing systems, models of population dynamics, epidemic models, and other stochastic systems.

This bibliography also contains a representative sample of references to works, where alternative methods and results on asymptotic expansions for characteristic of perturbed stochastic processes. Here, we would like only to point the books already mentioned above and some additional relevant books related to perturbation problems for stochastic processes. These are Korolyuk and Turbin (1976, 1978), Wentzell and Freidlin (1979), Seneta (1981, 2006), Asmussen (1987, 2000, 2003), Kalashnikov and Rachev (1988), Stewart and Sun Ji Guang (1990), Ho and Cao (1991), Meyn and Tweedie (1993), Kalashnikov (1994), Kartashov (1996), Embrechts, Klüppelberg, and Mikosch (1997), Borovkov (1998), Stewart (1998, 2001), Yin and Zhang (1998), Korolyuk V.S. and Korolyuk, V.V. (1999), Latouche and Ramaswami (1999), Whitt (2002), Silvestrov (2004), Koroliuk and Limnios (2005), Anisimov (2008), and Gyllenberg and Silvestrov (2008).

6. Conclusion

Quasi-stationary phenomena and related problems are a subject of intensive studies during several decades. However, the development of theory of quasi-stationary phenomena is still far from its completion. The part of the theory related to conditions of existence of quasi-stationary distributions is comparatively well developed while computational aspects of the theory are underdeveloped. The content of the book [25] is concentrated in this area. The book presents new effective methods for asymptotic analysis of pseudo- and quasi-stationary phenomena for nonlinearly perturbed stochastic processes and systems. Moreover, the results presented in the book unite, for the first time, research studies of pseudo- and quasi-stationary phenomena in the frame of one theory. Methods of asymptotic analysis for nonlinearly perturbed stochastic processes and systems developed in the book have their own values and possible applications beyond the problems studied in the book.

The results presented in the book will be interesting to specialists, who work in such areas of the theory of stochastic processes as ergodic, limit, and large deviation theorems, analytical and computational methods for Markov chains, regenerative, Markov, semi-Markov, risk and other classes of stochastic processes, renewal theory, and their queueing, reliability, population dynamics, and other applications. There is a hope that the book will also attract attention of those researchers, who are interested in new analytical methods of analysis for nonlinearly perturbed stochastic processes and systems, especially those who like serious analytical work.

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