Theory of Stochastic Processes
Vol. 14(30), no.3-4, 2008, pp.104-113

OLEKSANDER PONOMARENKO AND YURIY PERUN

MULTIVARIATE RANDOM FIELDS ON SOME HOMOGENEOUS SPACES

The generalized continuous random fields of second order with values in arbitrary complex normed space X in the case when their arguments belong to homogeneous space with compact transformation group G are considered. Such fields are harmonizable in some sense. The spectral representations of homogeneous random fields in X and G-invariant positive definite operator-valued kernels are obtained. The special case of random fields with values in complex Hilbert space and random fields on three-dimensional spheres are also studied.

1. Introduction

The central role in the theory of second-order random functions (processes and fields) plays different representations of these functions in the form of stochastic integrals and series.

Such representations for complex-valued and multivariate finite-dimensional homogeneous random fields over different homogeneous spaces were studied by A.M.Yaglom [1]. The theory of integral representations of multivariate generalized random functions of second order with values in linear topological spaces was studied in [2].

In this paper we consider the representations of generalized random fields of second order with values in complex normed and Hilbert spaces defined over compact groups and compact homogeneous spaces by mean of random series. The main attention is devoted to representations of homogeneous second order random fields and invariant operator-valued positive definite functions.

Let X be a complex normed space and X^* be its topological dual space endowed by strong topology. Denote by $L_2(\Omega)$ Hilbert space of all complex

2000 *Mathematics Subject Classifications.* Primary 60G10, 60G57.

Key words and phrases. Generalized random fields of second order in normed spaces, homogeneous space with compact transformation group, unitary representations, spectral representations, harmonizable field, invariant positive definite operator-valued kernels.

second order random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ (inner product in $L_2(\Omega)$ is equal to covariance of r.v. from $L_2(\Omega)$) endowed by strong topology.

The generalized random element Ξ on Ω in X is continuous linear operator $\Xi : X^* \to L_2(\Omega)$. Every such element is generated by some usual random element ξ on Ω with values in some extension of the space X (see [3]). The space of generalized random elements in X will denoted by $\mathcal{L}(X^*, L_2(\Omega))$ and endowed by strong operator topology.

Let $\overline{\mathcal{L}}(X^*, X^{**})$ be a space of all antilinear (or conjugate linear) continuous operators from X^* into the X^{**} (second dual space for X) endowed by weak operator topology. Define the expectation $\mathbb{E}\Xi \in X^{**}$ of random element $\Xi \in \mathcal{L}(X^*, L_2(\Omega))$ by the equality

$$(\mathbb{E}\Xi)(x^*) = \mathbb{E}(\Xi x^*), \ x^* \in X^* \tag{1}$$

and covariance operator $[\Xi, \Phi] \in \overline{\mathcal{L}}(X^*, X^{**})$ of elements $\Xi, \Phi \in \mathcal{L}(X^*, L_2(\Omega))$ by the equality

$$\mathbb{E}(\Xi x^*)(\overline{\Phi y^*}) = ([\Xi, \Phi] y^*)(x^*), x^*, y^* \in X^* \tag{2}$$

Note that operator $[\Xi, \Xi]$ is nonnegative in $\overline{\mathcal{L}}(X^*, X^{**})$, i.e. for all $x^* \in X^* : \ ([\Xi, \Xi]x^*)(x^*) \geq 0$.

Let T be a topological Hausdorff space. The generalized random field of second order $\Xi_t \ t \in T$ in X is mapping of T into $\mathcal{L}(X^*, L_2(\Omega))$. In further we assume that this mapping is continuous. Denote by $R(t,s)$ the covariance function of random field $\Xi_t, t \in T$,

$$R(t,s) = [\Xi_t, \Xi_s], t, s \in T \tag{3}$$

Then $R(t,s)$ is continuous $\overline{\mathcal{L}}(X^*, X^{**})$-valued kernel on T.

Note that the class of covariance functions of the set of all second order random fields in X coinsides with the set of all continuous positive definite $\overline{\mathcal{L}}(X^*, X^{**})$-valued operator kernels on T, i.e. for all $n \in \mathbb{N}$, elements $x_i^* \in X^*, t_i \in T, i = 1, ..., n$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (R(t_i, t_j) x_j^*)(x_i^*) \geq 0 \tag{4}$$

(see [4]). This fact give to us the opportunity to obtain spectral representations of some classes of invariant operator positive definite functions on compact groups and compact homogeneous spaces by mean of probabilistic methods.

2. Random fields on compact groups

The simplest type of compact homogeneous space is the group space $G = \{g\}$ consisting of the elements g of some compact group G. There are two different families of continuous transformations of G namely left and right shifts

$$V_g^l : h \to gh \ \ and \ \ V_g^r : h \to hg. \tag{5}$$

Let $\Xi_g, g \in G$ be a second order continuous random field in X over G. The field Ξ_g will be called left (right) homogeneous if its mean function $\mathbb{E}\Xi_g$ and covariance function $R(g,h) = [\Xi_g, \Xi_h], g, h \in G$ are invariant with respect to all left shifts V_g^l (right shifts V_g^r). Due to transitivity of transformation group $V_g^l(V_g^r),\ g \in G$ we always have that $\mathbb{E}\Xi_g = m = const$ and that for all $s \in G$ for left homogeneous field

$$[\Xi_g, \Xi_h] = [\Xi_{sg}, \Xi_{sh}] = R(h^{-1}g), g, h \in G \tag{6}$$

and for right homogeneous field

$$[\Xi_g, \Xi_h] = [\Xi_{gs}, \Xi_{hs}] = R(gh^{-1}), g, h \in G. \tag{7}$$

For two-way homogeneous field conditions (6) and (7) must be satisfied simultaneously, so that the covariance function R such field must be a constant for a class of conjugate elements of G:

$$R(h) = R(ghg^{-1}), g, h \in G. \tag{8}$$

In the further we assume that for homogeneous field Ξ_g the mean $\mathbb{E}\Xi_g$ is zero.

According to the theory of unitary representations of compact group G (see[5],[6]) there exits the system U_g of not more than a countable nonequivalent finite dimensional unitary irreducible representations $U_g = \{U_g^{(\lambda)}, \lambda = 1, ...\}$, where $U_g^{(\lambda)}$ are homomorphisms of the group G into group of unitary matrices of finite order d_λ, namely

$$U_g^{(\lambda)} = \|u_{ij}^{(\lambda)}(g)\|_{i,j=1}^{d_\lambda}, d_\lambda < \infty, \lambda = 1, 2, ...$$

$$U_{gh}^{(\lambda)} = U_g^{(\lambda)} U_h^{(\lambda)}, U_{g^{-1}}^{(\lambda)} = [U_g^{(\lambda)}]^{-1} = [U_g^{(\lambda)}]^*. \tag{9}$$

The matrix elements $u_{ij}^{(\lambda)}(g)$ of these representations satisfy the following orthogonality relations

$$\int_G u_{ij}^{(\lambda)}(g)\overline{u_{kl}^{(\mu)}}(g)dg = \delta_{\lambda\mu}\delta_{ik}\delta_{jl}d_\lambda^{-1}, \tag{10}$$

where $\delta_{\lambda\mu}$ is a Kroneker symbol and dg is unique normed invariant measure on G, $\int_G dg = 1$.

Theorem 2.1 *Let $\Xi_g, g \in G$ be a generalized continuous random field of second order in normed space X on compact group G. Then the field Ξ_g is harmonizable with respect to the system $U_g = \{U_g^{(\lambda)}\}$ of unitary representations of G (9) in sense that Ξ_g admits expansion by the random series of the form*

$$\Xi_g = \sum_{\lambda} \sum_{i,j=1}^{d_\lambda} u_{ij}^{(\lambda)}(g) \Phi_{ji}^{(\lambda)}, \tag{11}$$

where $u_{ij}^{(\lambda)}(g)$ are the matrix elements of representations $\{U_g^{(\lambda)}\}$ and $\Phi_{ji}^{(\lambda)} \in \mathcal{L}(X^*, L_2(\Omega))$,

$$\Phi_{ji}^{(\lambda)} = d_\lambda \int_G \overline{u_{ij}^{(\lambda)}(g)} \Xi_g dg, \tag{12}$$

where integral in (12) is understood in sense of strong topology in $\mathcal{L}(X^*, L_2(\Omega))$.

The statement of this theorem is simple consequence of the fact that the set of all matrix elements $u_{ij}^{(\lambda)}(g)$ for $1 \leq i, j \leq d_\lambda, \lambda = 1, 2, ...$ of system $\{U_g^{(\lambda)}\}$ form a complete orthogonal system in the Hilbert space $L_2(G)$ of complex functions over G whose square of modulus are integrable with respect to dg.

Theorem 2.2 I. *The continuous random field $\Xi_g, g \in G$ of second order in X is left homogeneous if and only if it admits the spectral representation of the form (11) with random elements $\Phi_{ji}^{(\lambda)}$ which satisfy the conditions*

$$[\Phi_{ji}^{(\lambda)}, \Phi_{lk}^{(\mu)}] = \delta_{\lambda\mu} \delta_{ik} F_{jl}^{(\lambda)}, \tag{13}$$

where

$$F_{jl}^{(\lambda)} = \int_G \overline{u_{jl}^{(\lambda)}(g)} R(g) dg \in \overline{\mathcal{L}}(X^*, X^{**}), \tag{14}$$

matrices $F^{(\lambda)} = \|F_{ij}^{(\lambda)}\|_{i,j=1}^{d_\lambda}$ are positive definite in the sense of similar to (4), and such that the series

$$\sum_{\lambda} \sum_{j} F_{jj}^{(\lambda)} = \sum_{\lambda} T_r(F^{(\lambda)}) \tag{15}$$

is convergent in $\overline{\mathcal{L}}(X^*, X^{**})$.

II. *The covariance function $R(g)$ of Ξ_g can be represented in the form*

$$R(g) = \sum_{\lambda} \sum_{j,l} u_{lj}^{(\lambda)}(g) F_{jl}^{(\lambda)}. \tag{16}$$

Conversely, any $\overline{\mathcal{L}}(X^*, X^{**})$-valued function of the form (16), where $F^{(\lambda)} = \|F_{jl}^{(\lambda)}\|$ are positive definite operator matrices satisfying the condition of convergence (15) is a covariance function of some left homogeneous

random field over G in X.
III. The foregoing situation with spectral representation (11) is quite analogous for right homogeneous random field Ξ_g in X, the only change being that condition (13) is replaced by condition

$$[\Phi_{ji}^{(\lambda)}, \Phi_{lk}^{(\mu)}] = \delta_{\lambda\mu}\delta_{jl}F_{ik}^{(\lambda)}. \tag{17}$$

Proof. The necessity of condition (13) in I. From the equalities (6), (12), (10) and invariance of measure dg it easily follows that

$$[\Phi_{ji}^{(\lambda)}, \Phi_{lk}^{(\mu)}] = \delta_{\lambda\mu}\delta_{ik}d_\lambda \int_G \overline{u_{lj}(g)}R(g)dg = \delta_{\lambda\mu}\delta_{ik}F_{jl}^{(\lambda)}. \tag{18}$$

Substituting (11) into (6) we can verify equality (16). The operator matrices $F^{(\lambda)}$ are obviously all positive definite. Because $u_{ik}^{(\lambda)}(e) = \delta_{ik}$, where e is unit of G, it follows that convergence of series (15) is condition for convergence of right side of (11) and (16).

The sufficiency of condition (12) and II. If $F^{(\lambda)}, \lambda = 1, 2, ...$ are arbitrary positive definite $(d_\lambda \times d_\lambda)$ operator $\overline{\mathcal{L}}(X^*, X^{**})$ - valued matrices, then one can always select $\Phi_{jl}^{(\lambda)} \in \mathcal{L}(X^*, L_2(\Omega))$ such that (13) satisfied. Under condition of convergence of (15) the series in right side of (11) converges and defines a random field Ξ_g in X for which $[\Xi_{hg}, \Xi_h]$ is given by the formula (16), i.e. Ξ_g is left homogeneous.

The proof of the part III of the theorem is analogous to the proof of parts I and II.

Corollary 2.3. *Continuous $\overline{\mathcal{L}}(X^*, X^{**})$ - valued operator function $R(g)$ is left positive definite, i.e. for all $g_i \in G, x_i^* \in X^*, i = 1, ..., n, n \in \mathbb{N}$*

$$\sum_{i,j=1}^{n} [R(g_j^{-1}g_i)x_j^*](x_i^*) \geq 0,$$

if and only if it admits representation of the form (16).

This result follows from characterization of the class of covariance functions of random fields over G in X by condition (4) and theorem 2.2. This is an operator version of Bochner theorem for complex positive definite function over G [7].

Theorem 2.4. *The continuous random field $\Xi_g, g \in G$ in X is two-way homogeneous if and only if its admits the spectral representation (11) with random elements $\Phi_{jl}^{(\lambda)} \in \mathcal{L}(X^*, L_2(\Omega))$ satisfying the condition*

$$[\Phi_{ji}^{(\lambda)}, \Phi_{lk}^{(\mu)}] = \delta_{\lambda\mu}\delta_{jl}\delta_{ik}F^{(\lambda)}, \tag{19}$$

where $F^{(\lambda)}$ is nonnegative operators in $\overline{\mathcal{L}}(X^*, X^{**})$ such that series $\sum_\lambda d_\lambda F^{(\lambda)}$ is convergent in $\overline{\mathcal{L}}(X^*, X^{**})$.

The covariance function $R(g)$ of two-way homogeneous random field $\Xi_g, g \in G$ in X is represented in the form

$$R(g) = \sum_{\lambda} \chi^{(\lambda)}(g) F^{(\lambda)}, \tag{20}$$

where $\chi^{(\lambda)} = T_r(U_g^{(\lambda)})$ are characters of the group G.

This theorem is consequence of part I, II and III of theorem 2.2.

Corollary 2.5. *The expansion (20) is general characterization of $\overline{\mathcal{L}}(X^*, X^{**})$ valued operator continuous positive definite functions over G which is invariant with respect to two-way shifts. This expansion is operator version of corresponding result of Bochner for complex functions [7].*

3. Random fields on homogeneous spaces

Let $Q = \{q\}$ be a homogeneous space with transitive transformation group $G = \{g\}$. It is well known that Q can be identified with the set of left cosets G/K, where K is a stationary subgroup of G, which leaves invariant some point $q_0 \in Q$, and for $q = hK, h \in G$ the actoin $gq = ghK$. The topology of G induced naturally a topology in Q [6]. Functions in Q are continuous if and only if the corresponding functions assuming a constant value over elements from G/K are continuous on G.

Second order random field $\Xi_q, q \in Q$ in X is a continuous mapping of Q into $\mathcal{L}(X^*, L_2(\Omega))$. The field Ξ_q is called homogeneous if for all $g \in G, p, q \in Q$

$$\mathbb{E}\Xi_{gq} = \mathbb{E}\Xi_q = const, \quad [\Xi_{gq}, \Xi_{gp}] = [\Xi_q, \Xi_p] = R(q,p). \tag{21}$$

In the following we suppose that $\mathbb{E}\Xi_q = 0$.

It is obvious that class of homogeneous random fields on Q coincides with the class of homogeneous random fields on G which are constant over all left cosets modulo K.

Now assume that the group G is compact. In order to obtain spectral representation for Ξ_q we must use general theory of spherical functions (or harmonics)on Q (see [8], [9]). Let us consider the complete system $U_g^{(\lambda)}$ of unitary continuous nonequivalent representation of G (9) and choose in the space of irreducible representations a basis such that to obtain the irreducible representation of K. In order that a matrix element $u_{ij}^{(\lambda)}(g)$ be a constant on all elements of G/K we must have

$$u_{ij}^{(\lambda)}(gk) = \sum_m u_{im}^{(\lambda)}(g) u_{mj}^{(\lambda)}(k) = u_{ij}^{(\lambda)}(g), g \in G, k \in K. \tag{22}$$

It means that the equalities

$$u_{mj}^{(\lambda)}(k) = \delta_{mj}, m = 1, ..., d_\lambda, k \in K$$

take place. So elements $u_{ij}^{(\lambda)}(g)$which are constant on cosets of G/K fill out the column of $U_g^{(\lambda)}$ corresponding to the identity representation of K. If representation $U_g^{(\lambda)}$ contains r_λ times the identity representation of K, suppose that in basis $e_1, ...e_{d_\lambda}$ these identity representations correspond to the first r_λ basis vectors, $U_k^{(\lambda)}e_j = e_j, k \in K, j = 1, ..., r_\lambda$. In this case the functions of q

$$\psi_{ij}^{(\lambda)}(q) = u_{ij}^{(\lambda)}(g), i = 1, ..., d_\lambda, j = 1, ..., r_\lambda, \lambda = 1, 2, ... \tag{23}$$

are called spherical functions on Q while the functions

$$\psi_{ij}^{(\lambda)}(q) = u_{ij}^{(\lambda)}(g), i = 1, ..., r_\lambda, j = 1, ..., r_\lambda, \lambda = 1, 2, ... \tag{24}$$

are called zonal spherical functions. The functions (24)are constant on all spheres with center at q_0, i.e. sets of points $kq, k \in K, q \in Q$ (set $kq, k \in K$ is a sphere with center $q_0(= K)$ and passing through the point q). The zonal function $\psi_{ij}^{(\lambda)}(q)$ depends only on the invariants of the ordered pair of points q and q_0 which remain unaltered under all transformations $g \in G$, i.e. on the composite distance from q to q_0:

$$\psi_{ij}^{(\lambda)}(q) = \psi_{ij}^{(\lambda)}(q, q_0) = \psi_{ij}^{(\lambda)}(gq, gq_0), g \in G, i, j = 1, ..., r_\lambda, \lambda = 1, 2, ... \tag{25}$$

Theorem 3.1. *The continuous random field $\Xi_q, q \in Q$ of second order in X on compact homogeneous space $Q = G/K$ is harmonizable in the sense that it admits the representation*

$$\Xi_q = \sum_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{r_\lambda} \psi_{ij}^{(\lambda)}(q)\Phi_{ji}^{(\lambda)}, \tag{26}$$

where $\psi_{ij}^{(\lambda)}$ are spherical harmonics (23)on Q and $\Phi_{ji}^{(\lambda)} \in \mathcal{L}(X^*, L_2(\Omega))$ have the form

$$\Phi_{ij}^{(\lambda)} = (\int_Q |\psi_{ij}^{(\lambda)}(q)|^2 dq)^{-1} \int_Q \overline{\psi_{ij}^{(\lambda)}(q)}\Xi_q dq, \tag{27}$$

where dq is G-invariant measure on Q.

The statement of the theorem is consequence of the fact that according to general theory of spherical harmonics the functions (23) represents a complete orthogonal system in the Hilbert space $L_2(Q)$ of complex functions $\varphi(q), q \in Q$ such that $|\varphi(q)|^2$ is integrable with respect to G-invariant measure dq on Q. So only the harmonics (23) enter into the expansion of the function $\varphi(g)$ which is constant over all cosets from G/K.

Theorem 3.2 *The continuous homogeneous random field $\Xi_q, q \in Q$ on compact homogeneous space $Q = G/K$ in X admits the spectral representation (26) if and only if random elements $\Phi_{ji}^{(\lambda)} \in \mathcal{L}(X^*, L_2(\Omega))$ satisfy the relations*

$$[\Phi_{ji}^{(\lambda)}, \Phi_{lk}^{(\mu)}] = \delta_{\lambda\mu}\delta_{ik}F_{jl}^{(\lambda)}, \tag{28}$$

where operators $F_{jl}^{(\lambda)} \in \overline{\mathcal{L}}(X^*, X^{**})$

The covariance function $R(q,p) = [\Xi_q, \Xi_p]$ *of such field* Ξ_q *can be represented in the form*

$$R(q,p) = \sum_{\lambda} \sum_{j,l=1}^{r_\lambda} \psi_{lj}^{(\lambda)}(q,p) F_{jl}^{(\lambda)} \tag{29}$$

where $\psi_{lj}^{(\lambda)}(q,p)$ *are the function (25). Conversely, any* $\overline{\mathcal{L}}(X^*, X^{**})$*-valued function of the form (29), where* $\|F_{jl}^{(\lambda)}\|$ *are positive definite matrices in* $\overline{\mathcal{L}}(X^*, X^{**})$ *such that series (29) converges is a covariance function of some homogeneous field* $\Xi_q, q \in Q$.

This theorem is consequence of foregoing theory of spherical harmonics in view of theorems 2.2 and 3.1.

Corollary 3.3. *The representation (29) gives general form of all continuous* G*-invariant positive definite operator* $\overline{\mathcal{L}}(X^*, X^{**})$*-valued kernels on* $Q = G/K$*. This is operator version of related result of Bochner [7] for complex-valued functions.*

4. Some special cases

The special case of generalized random field in complex normed space X is such field in complex Hilbert space H with inner product $(\cdot|\cdot)$ and strong topology. In this case it is more appropriate to give the definition of generalized random element of second order Ξ in H as continuous linear mapping from H into $L_2(\Omega), \Xi \in \mathcal{L}(H, L_2(\Omega))$, the definition of expectation $\mathbb{E}\Xi$ as vector of H, for which

$$\mathbb{E}(\Xi x) = (x|\mathbb{E}\Xi), x \in H, \tag{30}$$

and the definition of covariance operator $[\Xi, \Phi]$ of elements $\Xi,\ \Phi \in \mathcal{L}(H, L_2(\Omega))$ as linear continuous mapping of H into H which satisfy the relation

$$([\Xi, \Phi]x|y) = (\mathbb{E}(\Xi x)(\overline{\Phi y}), x, y \in H. \tag{31}$$

So covariance $[\Xi, \Phi]$ belongs to the space of bounded linear operator $\mathcal{L}(H, H) = \mathcal{L}(H)$ in space H. Every generalized random elements $\Xi \in \mathcal{L}(H, L_2(\Omega))$ is generated by some usual random element in some quasinuclear extension of H (see [3]).

In the further we assume that the space $\mathcal{L}(H, L_2(\Omega))$ is endowed by strong operator topology and the space $\mathcal{L}(H)$ is endowed by weak operator topology.

The second order generalized random field $\Xi_t, t \in T$ over some topological space T is a continuous mapping of T into $\mathcal{L}(H, L_2(\Omega))$. The class of covariance functions of such fields, $R(t,s) = [\Xi_t, \Xi_s], t, s \in T$ coinsides with the class of continuous $\mathcal{L}(H)$-valued operator positive definite kernels, i.e. for all $t_i \in T, x_i \in H, i = 1, ..., n, n \in \mathbb{N}$

$$\sum_{i,j=1}^{n} (R(t_i, t_j)x_i|x_j) \geq 0. \tag{32}$$

The all results section 2 and 3 of this paper are valid for corresponding generalized random fields over G and G/K in Hilbert space H and positive definite invariant $\mathcal{L}(H)$-valued operator kernels with using instead of space $\mathcal{L}(X^*, L_2(\Omega))$ of generalized random element in X the space $\mathcal{L}(H, L_2(\Omega))$ of such elements in H and instead of covariance operators from $\overline{\mathcal{L}}(X^*, X^{**})$ the covariance operators from $\mathcal{L}(H)$.

In the case, when Q is sphere S_2 in three-dimensional Euclidean space $\mathbb{R}^3$ with transformation group G as the group of all rotations g around center of sphere 0, the homogeneous random field $\Xi_{\theta,\varphi}, (\theta, \varphi) \in S_2$ in X admits the expansion in the form of series

$$\Xi_{\theta,\varphi} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi)\Phi_m^l \tag{33}$$

in accordance of theorem 3.2, where $Y_l^m(\theta, \varphi)$ are spherical harmonics and Φ_m^l are random elements from $\mathcal{L}(X^*, L_2(\Omega))$ with covariances

$$[\Phi_m^l, \Phi_j^k] = \delta_{mj}\delta_{lk}F_m, \tag{34}$$

where F_m are nonnegative operators from $\overline{\mathcal{L}}(X^*, X^{**})$.

It follows from the equalities (33) and (34) and the addition theorem of associated Legendre functions that corresponding covariance function R of the field $\Xi_{\theta,\varphi}$ depends only on angular distance $\theta_{1,2}$ between the points $(\theta_1, \varphi_1), (\theta_2, \varphi_2) \in S_2$, i.e.

$$R(\theta_{1,2}) = [\Xi_{(\theta_1,\varphi_1)}, \Xi_{(\theta_2,\varphi_2)}]$$

and has representation of the form

$$R(\theta_{1,2}) = \sum_{l=0}^{m} P_l(cos\theta_{1,2})B_l, \tag{35}$$

where

$$P_l(cos\theta_{1,2}) = \frac{2}{2l+1} \sum_{m=-l}^{l} Y_l^m(\theta_1\varphi_1)\overline{Y_l^m(\theta_2\varphi_2)}, B_l = \frac{2l+1}{2}F_l.$$

Conversely, for every Φ_l^m and F_l satisfying (34) and such that the series (35) converges the field (33) is homogeneous and the function (35) is a covariance function of a homogeneous random field over S_2.

Note that representations (33) and (35)are multivariate analogue of results of Obukhov [10] (see also [11]) for complex-valued fields on S_2. Expansion (35) gives general form of G-invariant positive definite $\overline{\mathcal{L}}(X^*, X^{**})$-valued kernel on S_2. This is a operator version of result of Shoenberg [12] for complex-valued case.

References

1. Yaglom A.M., *Second-order homogeneous random fields.* Proc. Fourth Berkeley Symp. Math.Stat. and Probab., 2 (1961), p.593-622.
2. Ponomarenko O.I., *Integral representation of random functions with values in locally convex spaces.* Theory Probab. and Math. Stat., N46, (1992), p.132–141.
3. Ponomarenko O.I., *Random linear functionals of Second Order I.* Theory Probab. and Math. Stat., N54, (1997), p.145–154.
4. Ponomarenko O.I., *Infinite-dimensional random fields on semigroups.* Theory Probab. and Math. Stat., 30, (1985), p.153–158.
5. Pontrjagin L., *Topological groups.* Princeton Univ. Press,(1939).
6. Weil A., *L'integration dans les groupes Topologiquies et ses Applications.* Paris, Harmann, (1940).
7. Bochner S., *Hilbert distance and positive definite functions.* Ann. of Math. 42 (1941), p.647-656.
8. Cartan E., *Sur la détermination dun système orthgonal complet dans un espace de Riemann symmétrique elos.* Rend. Cire. Mat.Palermo, 53 (1929), p. 217-252.
9. Weil H., *Harmonics on homogeneous manifolds.* Ann. of Math. 35(1943), p.486-494.
10. Obukhov A.M., *Statistically homogeneous random fields on a sphere.* Uspehi Mat. Nauk. 2, (1947), p.196-198.
11. Ogura H., *Representations of the random fields on a sphere.* Mem. Fac.Eng. Kyoto Univ. 52, N2 (1990), p.81–105.
12. Shoenberg I.J., *Positive definite functions on spheres.* Duck. Math. J. 9 (1942), p.96–108.

Department of Probability and Mathematical Statistics, Kiev National Tarasa Shevchenko University, Kiev, Ukraine
E-mail: probability@mechmat.univ.kiev.ua

Department of Auditing, National Bank of Ukraine, Kiev, Ukraine
E-mail: perun@bank.gov.ua