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ONE EXAMPLE OF A RANDOM CHANGE OF TIME THAT TRANSFORMS A GENERALIZED DIFFUSION PROCESS INTO AN ORDINARY ONE

We propose a random change of time for a class of generalized diffusion processes such that the corresponding stochastic differential equation (with generalized coefficients) is transformed into an ordinary one (its coefficients are some non-generalized functions). It turns out that the latter stochastic differential equation has no property of the (weak) uniqueness of a solution.

Introduction

Let S denote a closed bounded surface in a d-dimensional Euclidean space $\mathbb{R}^d$ such that it divides the space $\mathbb{R}^d$ into two open parts: interior D_i and exterior D_e, so that $\mathbb{R}^d = D_i \cup D_e \cup S$; we put $D = D_i \cup D_e$. The surface S is assumed to be smooth enough (see below for the precise assumptions), so that there exists a normal vector $\nu(x)$ to S at any point $x \in S$; it will be considered as a unit vector directed into D_e.

Suppose that an operator-valued function $b(x), x \in \mathbb{R}^d$, is given such that for any $x \in \mathbb{R}^d$, the operator $b(x)$ is a symmetric positively definite linear operator on $\mathbb{R}^d$. The positive square root of $b(x)$ will be denoted by $b(x)^{1/2}$. For $x \in S$, the vector $N(x) = b(x)\nu(x)$ is called the co-normal vector to S at the point x.

A continuous function $q(x), x \in S$, with its values being in the interval $[-1, 1]$ of a real line will be supposed to be given.

Under some assumptions on the function $(b(x))_{x\in\mathbb{R}^d}$ (see below), a continuous Markov process $(x_0(t))_{t\geq 0}$ in $\mathbb{R}^d$ exists such that its trajectories satisfy the stochastic differential equation (see [4], Ch. 3)

$$dx_0(t) = q(x_0(t))\delta_S(x_0(t))N(x_0(t))dt + b(x_0(t))^{1/2}dw(t), \tag{1}$$

where $(w(t))_{t\geq 0}$ is a standard Wiener process in $\mathbb{R}^d$ and $(\delta_S(x))_{x\in\mathbb{R}^d}$ is a generalized function on $\mathbb{R}^d$ that acts on a test function $(\varphi(x))_{x\in\mathbb{R}^d}$ according to the rule

$$\langle \delta_S, \varphi\rangle = \int_S \varphi(x)d\sigma \tag{2}$$

(the integral in this equality is a surface integral).

Let a continuous bounded function $r(\cdot) : S \to (0, +\infty)$ be now given. For $t \geq 0$, we put

$$x(t) = x_0(\zeta_t),$$

where

$$\zeta_t = \inf\left\{s :\ s + \int_0^s r(x_0(\tau))\delta_S(x_0(\tau))d\tau \geq t\right\}, \tag{3}$$

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the functional

$$\eta_t = \int_0^t r(x_0(\tau))\delta_S(x_0(\tau))d\tau,\ t \geq 0, \tag{4}$$

of the process $(x_0(t))_{t\geq 0}$ is well defined as an additive homogeneous continuous functional (see [4], Ch.3). As is known (see [2], Theorem 10.11), the process $(x(t))_{t\geq 0}$ is a continuous Markov process in $\mathbb{R}^d$ as a result of the random change of time for the process $(x_0(t))_{t\geq 0}$.

The aim of this article is to show that the trajectories of the process $(x(t))_{t\geq 0}$ satisfy the stochastic differential equation

$$dx(t) = \frac{q(x(t))}{r(x(t))} \mathbb{I}_S(x(t))N(x(t))dt + \mathbb{I}_D(x(t))b(x(t))^{1/2}dw(t), \tag{5}$$

where $\mathbb{I}_\Gamma(x),\ x \in \mathbb{R}^d$, stands for the indicator function of a set $\Gamma \subset \mathbb{R}^d$.

As a consequence of this result, we obtain the fact that Eq. (5) has no property of th (weak) uniqueness of a solution, since the representation of a given function $A(x), x \in S$, in the form $A(x) = \frac{q(x)}{r(x)}$ does not determine uniquely any pair of functions $q(\cdot)$ and $r(\cdot)$.

The situation of $b(x) \equiv I$ (I is an identical operator in $\mathbb{R}^d$) and S being a hyperplane in $\mathbb{R}^d$ was considered in [1]. The reader can find there some discussion about the place of the result in the modern context of the theory of stochastic differential equations. The main point that draws a distinction between the situation of this article and the one of [1] is that the transition probability density of the process $(x_0(t))_{t\geq 0}$ in [1] is given by an explicit formula, whereas here, on the contrary, we have only some equations for the corresponding density. This makes the computation of the local characteristics of the process $(x(t))_{t\geq 0}$ more complicated.

Nevertheless, as in [1], there is a very simple and intuitively evident way to explain why, under the time changing (3), the coefficients of Eq. (1) are transformed into the ones of Eq. (5). Namely, according to some very general result (see [2], Ch. 10), in order to obtain the diffusion coefficients of the proccess $(x(t))_{t\geq 0}$, we have to divide the ones of the proccess $(x_0(t))_{t\geq 0}$ by the function $1 + r(x)\delta_S(x),\ x \in \mathbb{R}^d$. So, for the coefficient of dt, we get

$$\frac{q(x)\delta_S(x)N(x)}{1 + r(x)\delta_S(x)} = \frac{q(x)}{r(x)} \mathbb{I}_S(x)N(x),\ x \in \mathbb{R}^d, \tag{6}$$

and, for the one of $dw(t)$, we have

$$\frac{b(x)^{1/2}}{(1 + r(x)\delta_S(x))^{1/2}} = \mathbb{I}_D(x)b(x)^{1/2},\ x \in \mathbb{R}^d. \tag{7}$$

Of course, formulae (6) and (7) cannot serve as a proof of our main result. Some arguments for proving it can be found below in Section 2. Section 1 is devoted to a brief description of the process $(x_0(t))_{t\geq 0}$ constructed in [4]. Moreover, some new properties of this process (as compared with [4]) are established in Section 1. These properties are then used in the considerations of Section 2.

1. Some properties of the process $(x_0(t))_{t\geq 0}$

1.1. A brief description of the process. Fix an orthonormal basis in $\mathbb{R}^d$ and denote, by x^j for $j = 1, 2, \dots, d$, the coordinates of a vector $x \in \mathbb{R}^d$ and, by $b_{ij}(x)$ for $i = 1, 2, \dots, d$ and $j = 1, 2, \dots, d$, the elements of the matrix of the operator $b(x)$ in that basis. We assume that the functions $b_{ij}(\cdot)$ for all $i = 1, 2, \dots, d$ and $j = 1, 2, \dots, d$ are bounded and Hölder continuous, that is, the inequality

$$|b_{ij}(x) - b_{ij}(y)| \leq C|x - y|^\alpha, \tag{8}$$

holds true for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with some constants $C > 0$ and $\alpha \in (0, 1]$. In addition, the condition of uniform non-degeneracy

$$\inf_{x\in\mathbb{R}^d} \min_{\theta\in\mathbb{R}^d:|\theta|=1} (b(x)\theta, \theta) > 0 \tag{9}$$

will be assumed to be held. Under these assumptions, there exists the fundamental solution of the equation (see [3], Ch.I)

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{d} b_{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j}.$$

Denote it by $g_0(t, x, y), t > 0, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$. This function is the transition probability density of a stochastic process $(\xi_0(t))_{t\geq 0}$ that is the solution to the following stochastic differential equation:

$$d\xi_0(t) = b(\xi_0(t))^{1/2} dw(t).$$

This means that the equality

$$E_x\varphi(\xi_0(t)) = \int_{\mathbb{R}^d} g_0(t, x, y)\varphi(y)dy$$

is valid for all $t > 0, x \in \mathbb{R}^d$, and $\varphi \in \mathbb{B}$, where $\mathbb{B}$ stands for the Banach space of all real-valued bounded measurable functions $\varphi(x), x \in \mathbb{R}^d$, with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d} |\varphi(x)|$ (the subspace of this space consisting of all continuous functions is itself a Banach space denoted by $\mathbb{C}$).

As for the surface S, we will assume the following conditions to be fulfilled (see [4], Ch.3, § 3):

(i) S belongs to the class $H^{1+\varkappa}$ for some $\varkappa \in (0, 1)$;

(ii) each point of S possesses the property of sphericity from the side of both the interior and exterior domains.

Let a continuous function $q(\cdot) : S \to [-1, 1]$ be now given. For $\varphi \in \mathbb{B}$, we consider the integral equation

$$V(t, x, \varphi) = \int_{\mathbb{R}^d} \frac{\partial g_0(t, x, z)}{\partial N(x)} \varphi(z)dz + \int_0^t d\tau \int_S \frac{\partial g_0(\tau, x, z)}{\partial N(x)} V(t - \tau, z, \varphi)q(z)d\sigma_z \tag{10}$$

in the domain $t > 0$ and $x \in S$. As proved in [4] (see Ch.3, § 3), this equation has the unique solution that is continuous in the arguments $t > 0$ and $x \in S$ and satisfies the inequality

$$|V(t, x, \varphi)| \leq K_T\|\varphi\|t^{-1/2} \tag{11}$$

in any domain of the form $(0, T] \times S$ with some positive constant K_T.

For $t > 0, x \in \mathbb{R}^d$, and $\varphi \in \mathbb{B}$, we put

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g_0(t, x, z)\varphi(z)dz + \int_0^t d\tau \int_S g_0(\tau, x, z)V(t - \tau, z, \varphi)q(z)d\sigma_z. \tag{12}$$

It turns out that a continuous Markov process $(x_0(t))_{t\geq 0}$ in $\mathbb{R}^d$ exists such that

$$E_x\varphi(x_0(t)) = u(t, x, \varphi) \tag{13}$$

for all $t > 0, x \in \mathbb{R}^d$, and $\varphi \in \mathbb{B}$. Moreover, the trajectories of this process satisfy the stochastic differential equation (1). All these assertions are proved in [4] (see Ch.3, §§ 3–4).

1.2. Transition probability density of the process. We now show that the process $(x_0(t))_{t\geq 0}$ possesses a transition probability density. We denote it by $G_0(t,x,y)$, $t>0$, $x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$. This means that the function $u(t,x,\varphi)$ given by (12) can be written in the form

$$u(t,x,\varphi)=\int_{\mathbb{R}^d} G_0(t,x,y)\varphi(y)dy \tag{14}$$

for all $t>0,\ x\in\mathbb{R}^d$, and $\varphi\in\mathbb{B}$.

For an arbitrary $y\in\mathbb{R}^d$, we formally put $\varphi(z)=\delta_y(z), z\in\mathbb{R}^d$, in (12) and (13), where $\delta_y(\cdot)$ is Dirac's δ-function concentrated at the point y (this means that $\langle\delta_y,\psi\rangle=\psi(y)$ for any test function ψ on $\mathbb{R}^d$). We then arrive at the relation

$$G_0(t,x,y)=g_0(t,x,y)+\int_0^t d\tau\int_S g_0(\tau,x,z)V(t-\tau,z,y)q(z)d\sigma_z, \tag{15}$$

where the notation $V(t,x,y)$ for $V(t,x,\delta_y), t>0, x\in S$, and $y\in\mathbb{R}^d$, is used. Equation (10) now implies the integral equation

$$V(t,x,y)=\frac{\partial g_0(t,x,y)}{\partial N(x)}+\int_0^t d\tau\int_S \frac{\partial g_0(\tau,x,z)}{\partial N(x)}V(t-\tau,z,y)q(z)d\sigma_z \tag{16}$$

that must be held for $t>0, x\in S$, and $y\in\mathbb{R}^d$.

We will prove that representation (15), (16) for the function G_0 holds indeed true. Moreover, one more representation formula for G_0 will be given. Namely, we show that

$$G_0(t,x,y)=g_0(t,x,y)+\int_0^t d\tau\int_S \widetilde{V}(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial N(z)}q(z)d\sigma_z \tag{17}$$

for $t>0,\ x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$, where $\widetilde{V}$ is the solution to the integral equation

$$\widetilde{V}(t,x,y)=g_0(t,x,y)+\int_0^t d\tau\int_S \widetilde{V}(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial N(z)}q(z)d\sigma_z, \tag{18}$$

where $t>0,\ x\in\mathbb{R}^d$, and $y\in S$.

To prove these statements, consider the function $\frac{\partial g_0(t,x,y)}{\partial N(x)}$ that is defined for $t>0,\ x\in S$, and $y\in\mathbb{R}^d$. Its restriction on the set $(0,+\infty)\times S\times S$ will be denoted by $Q(t,x,y)$. We now put $Q^{(1)}=Q$ and, for $k\geq 1$,

$$Q^{(k+1)}(t,x,y)=\int_0^t d\tau\int_S Q^{(k)}(\tau,x,z)Q(t-\tau,z,y)q(z)d\sigma_z=$$
$$=\int_0^t d\tau\int_S Q(\tau,x,z)Q^{(k)}(t-\tau,z,y)q(z)d\sigma_z,$$

where $t>0,\ x\in S$, and $y\in S$.

Proposition 1. *The series*

$$R(t,x,y)=\sum_{k=1}^{\infty}Q^{(k)}(t,x,y)$$

is convergent uniformly in $x\in S$ and $y\in S$ and locally uniformly in $t>0$. The kernel R is continuous in the arguments $t>0,\ x\in S,\ y\in S,\ x\neq y$ and satisfies the inequality

$$|R(t,x,y)|\leq\frac{\widetilde{K}_T}{t^{\beta_1}|y-x|^{\beta_2}} \tag{19}$$

in any domain of the form $(0,T]\times S\times S$ *with some constants* $\beta_1\in(0,1)$ *and* $\beta_2\in(0,d-1)$. *In addition, this kernel is the solution to each one of the following pair of integral equations* $(t>0, x\in S, y\in S)$

$$R(t,x,y)=Q(t,x,y)+\int_0^t d\tau\int_S R(\tau,x,z)Q(t-\tau,z,y)q(z)d\sigma_z,$$

(20)

$$R(t,x,y)=Q(t,x,y)+\int_0^t d\tau\int_S Q(\tau,x,z)R(t-\tau,z,y)q(z)d\sigma_z.$$

Finally, each equation in (20) has no more than one solution satisfying estimate (19).

Proof. Denote, by δ, the minimal one of the numbers α from (8) and $\varkappa$ mentioned in (i) above. It is clear that δ is a positive number. As proved in [3] (see Ch. V, § 2), for any $\beta\in\big(1-\frac{\delta}{2},1\big)$, the kernel Q satisfies the inequality

$$|Q(t,x,y)|\le\frac{\text{const}}{t^\beta|y-x|^{d+1-\delta-2\beta}}$$

in any domain of the form $(0,T]\times S\times S$. We now put $\rho=d+1-\delta-2\beta$, $\gamma=2\beta+\delta-2$ and $\sigma=1-\beta$; then $\gamma>0$ and $\sigma>0$. Making use of Lemma 2 from [3] (see Ch. V, § 2), one can obtain the estimate

$$|Q^{(2)}(t,x,y)|\le\frac{\text{const}}{t^{\beta-\sigma}|y-x|^{\rho-\gamma}}$$

valid for $(t,x,y)\in(0,T]\times S\times S$. In the same way, we get the estimate

$$|Q^{(3)}(t,x,y)|\le\frac{\text{const}}{t^{\beta-2\sigma}|y-x|^{\rho-2\gamma}}$$

valid for $(t,x,y)\in(0,T]\times S\times S$. Therefore, an integer k_0 exists such that $|Q^{(k_0)}(t,x,y)|\le\text{const}$ in any domain of the form $(0,T]\times S\times S$ for $T<+\infty$. By induction on k, we then obtain the inequality

$$|Q^{(k_0+n)}(t,x,y)|\le C_T^n\frac{t^{n-\beta}}{(1-\beta)(2-\beta)\dots(n-\beta)}$$

fulfilled for $(t,x,y)\in[0,T]\times S\times S$ and $n=1,2,\dots$ with some constant $C_T<\infty$ for $T<\infty$. This inequality guarantees the convergence of the series mentioned above. The remaining assertions of Proposition 1 are elementary.

Corollary. *The solution to Eq. (16) can be given by*

$$V(t,x,y)=\frac{\partial g_0(t,x,y)}{\partial N(x)}+\int_0^t d\tau\int_S R(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial N(z)}q(z)d\sigma_z \tag{21}$$

for $t>0, x\in S$, *and* $y\in\mathbb{R}^d$, *and the solution to Eq. (18) can be written as*

$$\widetilde{V}(t,x,y)=g_0(t,x,y)+\int_0^t d\tau\int_S g_0(\tau,x,z)R(t-\tau,z,y)q(z)d\sigma_z \tag{22}$$

for $t>0, x\in\mathbb{R}^d$, *and* $y\in S$.

It is now not difficult to verify that the result of substituting (21) into (15) coincides with that of substituting (22) into (17). So, the right-hand sides of (15) and (17) define the same function G_0. The fact that it is the transition probability density of the process $(x_0(t))_{t\ge0}$ follows now from the comparison of (10) and (12) with (16) and (15). We have thus proved that there exists the transition probability density for the process $(x_0(t))_{t\ge0}$ and it can be represented by (15) and (17).

Remark 1. Applying the well-known theorem on the jump of the co-normal derivative of a single-layer potential (see, for example, [4], Ch. 3, § 3) to (15), we get the relation

$$\frac{\partial G_0(t,x\pm,y)}{\partial N(x)} = (1\mp q(x))V(t,x,y) \tag{23}$$

valid for $t>0, x\in S$ and $y\in\mathbb{R}^d$, where $\frac{\partial G_0(t,x+,y)}{\partial N(x)}$ means the limit of $\frac{\partial G_0(t,z,y)}{\partial N(x)}$ as $z\to x$ in such a non-tangent way that $(z,\nu(x))>0$; the value of $\frac{\partial G_0(t,x-,y)}{\partial N(x)}$ is determined in the same way, but $(z,\nu(x))<0$ must be held this time. By analogy, (17) and (18) yeild the relation

$$G_0(t,x,y\pm) = (1\pm q(y))\widetilde{V}(t,x,y) \tag{24}$$

fulfilled for $t>0, x\in\mathbb{R}^d$, and $y\in S$.

Remark 2. Formulae (20) – (24) and the theorem mentioned in Remark 1 imply the relations

a) $R(t,x,y) = \frac{1}{2}[V(t,x,y+)+V(t,x,y-)] =$

$$= \frac{1}{2}\left[\frac{\partial\widetilde{V}(t,x+,y)}{\partial N(x)} + \frac{\partial\widetilde{V}(t,x-,y)}{\partial N(x)}\right];$$

b) $R(s+t,x,y) = \int\limits_{\mathbb{R}^d} V(s,x,z)\widetilde{V}(t,z,y)dz$

valid for $s>0, t>0, x\in S$, and $y\in S$.

Remark 3. The function δ_S in formulae (1) and (4) possesses the property of symmetry in the sense

$$\langle\delta_S,\psi\rangle = \frac{1}{2}\int_S[\psi(x+)+\psi(x-)]d\sigma \tag{25}$$

for any function ψ having, at the points of S, the non-tangent limits from the sides of D_e and D_i such that they are integrable. Notice that (25) is a natural generalization of (2).

This statement follows from the definition of the functional $(\eta_t)_{t\geq 0}$ (see [4], Ch. 3, § 4), according to which

$$\eta_t = \underset{h\to 0}{\mathrm{l.i.m.}}\int_0^t h^{-1}f_h(x_0(\tau))d\tau,\ \ t\geq 0,$$

where

$$f_t(x) = \int_0^t d\tau\int_S\widetilde{V}(\tau,x,y)r(y)d\sigma_y,\ \ t\geq 0,\ \ x\in\mathbb{R}^d.$$

One can easily verify that

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}^d} f_h(x)\psi(x)dx = \frac{1}{2}\int_S[\psi(x+)+\psi(x-)]r(x)d\sigma$$

for any test function ψ having the same properties as those in (25).

Remark 4. Equation (1) is analytically equivalent to the two relations (see [4], Ch. 3, § 4)

$$\int_{\mathbb{R}^d}(y,\theta)G_0(t,x,y)dy = (x,\theta) + \int_0^t d\tau\int_S(N(y),\theta)q(y)\widetilde{V}(\tau,x,y)d\sigma_y, \tag{26}$$

$$\int_{\mathbb{R}^d}(y,\theta)^2 G_0(t,x,y)dy = (x,\theta)^2 + \int_0^t d\tau \int_{\mathbb{R}^d} G_0(\tau,x,y)(b(y)\theta,\theta)dy+ \tag{27}$$
$$+2\int_0^t d\tau \int_S (N(z),\theta)(z,\theta)\widetilde{V}(\tau,x,z)q(z)d\sigma_z$$

valid for $t>0, x\in\mathbb{R}^d$, and $\theta\in\mathbb{R}^d$. Indeed, equality (26) means that the process

$$\widetilde{\xi}_0(t) = x_0(t) - x_0(0) - \int_0^t q(x_0(\tau))\delta_S(x_0(\tau))N(x_0(\tau))d\tau,\ t\geq 0,$$

is a martingale. Equality (27) implies that its square characteristic is given by (see [4], Ch. 3, § 4)

$$\langle\widetilde{\xi}_0\rangle_t = \int_0^t b(x_0(\tau))d\tau,\ t\geq 0.$$

1.3. The Feynman–Kac formula. For $\lambda\geq 0$, we define a function G_λ of the arguments $t>0, x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$ by the relation

$$E_x(\varphi(x_0(t))\exp\{-\lambda\eta_t\}) = \int_{\mathbb{R}^d}\varphi(y)G_\lambda(t,x,y)dy \tag{28}$$

that must be fulfilled for all $t>0,\ x\in\mathbb{R}^d$, and $\varphi\in\mathbb{B}$.

Proposition 2. *The function G_λ is determined uniquely by (28) and it satisfies each one of the following pair of equations (in the domain $t>0, x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$):*

$$G_\lambda(t,x,y) = G_0(t,x,y) - \lambda\int_0^t d\tau\int_S \widetilde{V}(\tau,x,z)G_\lambda(t-\tau,z,y)r(z)d\sigma_z, \tag{29}$$

$$G_\lambda(t,x,y) = G_0(t,x,y) - \lambda\int_0^t d\tau\int_S G_\lambda(\tau,x,z)G_0(t-\tau,z,y)r(z)d\sigma_z. \tag{30}$$

Proof of this assertion is quite similar to that given in [5] for the case of $b(x)\equiv I$ and S being a hyperplane in $\mathbb{R}^d$. We omit it here.

Notice that G_λ as a function of the third argument has a jump at the points of S (at those of them, for which the function q does not vanish). Namely, we have

$$G_\lambda(t,x,y\pm) = G_0(t,x,y\pm) - \lambda\int_0^t d\tau\int_S G_\lambda(\tau,x,z)G_0(t-\tau,z,y\pm)r(z)d\sigma_z$$

from (30) for $y\in S$. In this equality (and in (30) too), we have

$$G_\lambda(\tau,x,z) = \frac{1}{2}[G_\lambda(\tau,x,z+) + G_\lambda(\tau,x,z-)],\ z\in S,$$

because of the property of symmetry of δ_S. So, for $t>0, x\in\mathbb{R}^d$, and $y\in S$, Eqs. (29) and (30) can be rewritten in the following way:

$$G_\lambda(t,x,y) = \widetilde{V}(t,x,y) - \lambda\int_0^t d\tau\int_S \widetilde{V}(\tau,x,z)G_\lambda(t-\tau,z,y)r(z)d\sigma_z, \tag{31}$$

$$G_\lambda(t,x,y) = \widetilde{V}(t,x,y) - \lambda\int_0^t d\tau\int_S G_\lambda(\tau,x,z)\widetilde{V}(t-\tau,z,y)r(z)d\sigma_z. \tag{32}$$

We now introduce the kernels

$$\mathcal{G}_\lambda(x,y) = \int_0^\infty e^{-\lambda t}G_\lambda(t,x,y)dt,\ \lambda>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d;$$

$$\widetilde{G}_0(\lambda,x,y)=\int_0^\infty e^{-\lambda t}G_0(t,x,y)dt,\ \ \lambda>0,\ \ x\in\mathbb{R}^d,\ \ y\in\mathbb{R}^d;$$
$$\widetilde{\mathcal{V}}_\lambda(x,y)=\int_0^\infty e^{-\lambda t}\widetilde{V}(t,x,y)dt,\ \ \lambda>0,\ \ x\in\mathbb{R}^d,\ \ y\in S.$$

Then equalities (29) and (30) yield the relations

$$\mathcal{G}_\lambda(x,y)=\widetilde{G}_0(\lambda,x,y)-\lambda\int_S\widetilde{\mathcal{V}}_\lambda(x,z)\mathcal{G}_\lambda(z,y)r(z)d\sigma_z, \tag{33}$$

$$\mathcal{G}_\lambda(x,y)=\widetilde{G}_0(\lambda,x,y)-\lambda\int_S\mathcal{G}_\lambda(x,z)\widetilde{G}_0(\lambda,z,y)r(z)d\sigma_z \tag{34}$$

valid for $\lambda>0, x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$. If $y\in S$, we have

$$\mathcal{G}_\lambda(x,y)=\widetilde{\mathcal{V}}_\lambda(x,y)-\lambda\int_S\mathcal{G}_\lambda(x,z)\widetilde{\mathcal{V}}_\lambda(z,y)r(z)d\sigma_z, \tag{35}$$

$$\mathcal{G}_\lambda(x,y)=\widetilde{\mathcal{V}}_\lambda(x,y)-\lambda\int_S\widetilde{\mathcal{V}}_\lambda(x,z)\mathcal{G}_\lambda(z,y)r(z)d\sigma_z. \tag{36}$$

2. A martingale characterization of the process $(x(t))_{t\geq 0}$

2.1. The main formula. Supposing that $\varphi\in\mathbb{C}$ and taking into account (3), we obtain

$$E_x\int_0^\infty e^{-\lambda t}\varphi(x(t))dt=E_x\int_0^\infty e^{-\lambda t}\varphi(x_0(\zeta_t))dt=$$
$$=E_x\int_0^\infty e^{-\lambda t-\lambda\eta_t}\varphi(x_0(t))dt+E_x\int_0^\infty e^{-\lambda t-\lambda\eta_t}\varphi(x_0(t))r(x_0(t))\delta_S(x_0(t))dt=$$
$$=\int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,y)\varphi(y)dy+\int_S\mathcal{G}_\lambda(x,y)r(y)\varphi(y)d\sigma_y.$$

By the standard reason, this equality can be extended on all functions $\varphi\in\mathbb{B}$. We have just proved the following statement.

Proposition 3. *The formula*

$$E_x\int_0^\infty e^{-\lambda t}\varphi(x(t))dt=\int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,y)\varphi(y)dy+ \\ +\int_S\mathcal{G}_\lambda(x,y)r(y)\varphi(y)d\sigma_y \tag{37}$$

holds true for all $\lambda>0, x\in\mathbb{R}^d$, *and* $\varphi\in\mathbb{B}$.

It is clear that this formula remains to be true for some unbounded measurable functions φ.

2.2. Calculating the local characteristics of the process. Equality (26) can be rewritten as

$$\int_{\mathbb{R}^d}(y,\theta)\widetilde{G}_0(\lambda,x,y)dy=\frac{1}{\lambda}(x,\theta)+\frac{1}{\lambda}\int_S(N(y),\theta)q(y)\widetilde{\mathcal{V}}_\lambda(x,y)d\sigma_y,$$

where $\lambda>0, x\in\mathbb{R}^d$, and $\theta\in\mathbb{R}^d$. Using this equality and (34), we get

$$\int_{\mathbb{R}^d}(y,\theta)\mathcal{G}_\lambda(x,y)dy=\frac{1}{\lambda}(x,\theta)+\frac{1}{\lambda}\int_S\widetilde{\mathcal{V}}_\lambda(x,y)(N(y),\theta)q(y)d\sigma_y-$$
$$-\int_S\mathcal{G}_\lambda(x,z)(z,\theta)r(z)d\sigma_z-$$

$$-\int_S \mathcal{G}_\lambda(x,z)r(z)\Big[\int_S (N(y),\theta)q(y)\widetilde{\mathcal{V}}_\lambda(z,y)d\sigma_y\Big]d\sigma_z.$$

This equality and (35) together with (37) yield

$$E_x\int_0^\infty e^{-\lambda t}(x(t),\theta)dt = \int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,y)(y,\theta)dy + \int_S \mathcal{G}_\lambda(x,y)r(y)(y,\theta)d\sigma_y =$$

$$= \frac{1}{\lambda}(x,\theta) + \frac{1}{\lambda}\int_S \widetilde{\mathcal{V}}_\lambda(x,y)(N(y),\theta)q(y)d\sigma_y -$$

$$-\frac{1}{\lambda}\int_S (N(y),\theta)q(y)[\widetilde{\mathcal{V}}_\lambda(x,y)-\mathcal{G}_\lambda(x,y)]d\sigma_y =$$

$$= \frac{1}{\lambda}(x,\theta) + \frac{1}{\lambda}\int_S (N(y),\theta)q(y)\mathcal{G}_\lambda(x,y)d\sigma_y =$$

$$= \frac{1}{\lambda}(x,\theta) + \frac{1}{\lambda}\int_S (N(y),\theta)\frac{q(y)}{r(y)}\mathcal{G}_\lambda(x,y)r(y)d\sigma_y =$$

$$= \frac{1}{\lambda}(x,\theta) + E_x\int_0^\infty e^{-\lambda t}dt\int_0^t (N(x(\tau)),\theta)\frac{q(x(\tau))}{r(x(\tau))}\,\mathrm{I\!I}_S(x(\tau))d\tau.$$

This means that the process

$$\xi(t) = x(t) - x(0) - \int_0^t \frac{q(x(\tau))}{r(x(\tau))}\,\mathrm{I\!I}_S(x(\tau))N(x(\tau))d\tau,\ \ t\ge 0,$$

is a martingale. It remains to prove that the equality

$$E_x(\xi(t),\theta)^2 = E_x\int_0^t \mathrm{I\!I}_D(x(\tau))(b(x(\tau))\theta,\theta)d\tau \tag{38}$$

holds true for all $t>0, x\in\mathbb{R}^d$, and $\theta\in\mathbb{R}^d$.

From (27) and (34), we get

$$\int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,y)(y,\theta)^2dy = \frac{1}{\lambda}(x,\theta)^2 + \frac{1}{\lambda}\int_{\mathbb{R}^d}\widetilde{G}_0(\lambda,x,y)(b(y)\theta,\theta)dy +$$

$$+\frac{2}{\lambda}\int_S (z,\theta)(N(z),\theta)\widetilde{\mathcal{V}}_\lambda(x,z)q(z)d\sigma_z -$$

$$-\lambda\int_S \mathcal{G}_\lambda(x,z)r(z)\Big[\frac{1}{\lambda}(z,\theta)^2 + \frac{1}{\lambda}\int_{\mathbb{R}^d}\widetilde{G}_0(\lambda,z,y)(b(y)\theta,\theta)dy +$$

$$+\frac{2}{\lambda}\int_S \widetilde{\mathcal{V}}_\lambda(z,y)(y,\theta)(N(y),\theta)q(y)d\sigma_y\Big]d\sigma_z.$$

Making use of this relation, (34), and (35), we obtain

$$\int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,y)(y,\theta)^2dy + \int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,z)r(z)(z,\theta)^2d\sigma_z =$$

$$= \frac{1}{\lambda}(x,\theta)^2 + \frac{2}{\lambda}\int_S \mathcal{G}_\lambda(x,y)(y,\theta)(N(y),\theta)q(y)d\sigma_y + \frac{1}{\lambda}\int_{\mathbb{R}^d}\mathcal{G}_\lambda(x,y)(b(y)\theta,\theta)dy. \tag{39}$$

We now taking into account the relation

$$E_x(\xi(t),\theta)^2 = E_x(x(t),\theta)^2 - (x,\theta)^2 -$$

$$-2E_x\int_0^t (x(\tau),\theta)(N(x(\tau)),\theta)\frac{q(x(\tau))}{r(x(\tau))}\,\mathrm{I\!I}_S(x(\tau))d\tau$$

that is a simple consequence of the definition of the process $(\xi(t))_{t\geq 0.}$ This equality and (39) together with the main formula allow us to arrive at the equality

$$E_x \int_0^\infty e^{-\lambda t}(\xi(t),\theta)^2 dt = \frac{1}{\lambda}\int_{\mathbb{R}^d} \mathcal{G}_\lambda(x,y)(b(y)\theta,\theta)dy =$$
$$= \frac{1}{\lambda}\int_{\mathbb{R}^d} \mathcal{G}_\lambda(x,y)(b(y)\theta,\theta)\mathbb{I}_D(y)dy,$$

and it can be rewritten as

$$E_x \int_0^\infty e^{-\lambda t}(\xi(t),\theta)^2 dt = E_x \int_0^\infty e^{-\lambda t} dt \int_0^t (b(x(\tau))\theta,\theta)\mathbb{I}_D(x(\tau))d\tau.$$

This relation is equivalent to (38), and the proof of our main result has been completed.

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