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ON ONE STOCHASTIC OPTIMAL CONTROL PROBLEM WITH VARIABLE DELAY

The purpose of this paper is to give necessary conditions for the optimality of nonlinear stochastic control systems with variable delay and with constraint on the right end of a trajectory. The necessary optimality conditions in the form of a stochastic analogy of the maximum principle are obtained. These conditions are contained in Theorems 1 and 2.

Introduction

Stochastic differential equations with delay find many applications in authomatic control theory, in the theory of self-oscillating systems, etc., where real systems are subjected to the influence of random disturbances which cannot be ignored [1,2]. Optimal control problems for the systems described by means of such equations have been already investigated in [3]-[5]. This research is devoted to a problem of stochastic optimal control with delay both on control and state, when the cost function contains a variable delay as well.

Statement of the problem

Let (Ω, F, P) be a complete probability space with the filtration $\{F^t : t_0 \le t \le t_1\}$ generated by the Wiener process w_t and $F^t = \sigma(w_s; t_0 \le s \le t)$. $L^2_F(t_0, t_1, R^n)$ – space of predictable processes $x_t(\omega)$ such that: $E\int_{t_0}^{t_1} |x_t|^2 dt < +\infty$. Consider the following stochastic system with delay:

(1) $$dx_t = g(x_t, x_{t-h(t)}, u_t, u_{t-h_1(t)}, t)dt + \sigma(x_t, x_{t-h(t)}, t)dw_t, \quad t \in (t_0, t_1];$$

(2) $$x_t = \Phi(t), \ t \in [t_0 - h(t_0), t_0);$$

(3) $$x_{t_0} = x_0;$$

(4) $$u_t = Q(t), \ t \in [t_0 - h_1(t_0), t_0);$$

(5) $$u_t \in U_\partial \equiv \{u(\cdot,\cdot) \in L^2_F(t_0, t_1; R^m) | u(t,\cdot) \in U \subset R^m, \text{ a.s.}\}$$

where U – non-empty bounded set, $\Phi(t), Q(t)$ – piecewise continuous non-random functions, $h(t) \ge 0$ and $h_1(t) \ge 0$ – continuously differentiable non-random functions, and $\frac{dh(t)}{dt} < 1$, $\frac{dh_1(t)}{dt} < 1$.

It is required to minimize the following functional in a set of admissible controls:

(6) $$J(u) = E\left\{p(x_{t_1}) + \int_{t_0}^{t_1} l(x_t, x_{t-h(t)}, u_t, u_{t-h_1(t)}, t)dt\right\}$$

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under the condition

$$Eq(x_{t_1}) \in G \subset R^k, \tag{7}$$

where G – closed convex set in R^k.

Let assume that the following requirements are satisfied:

I. Functions l, g, and σ are continuous in all arguments.

II. When (t, u) are fixed, then l, g, σ functions are continuously differentiable with respect to (x, y) and satisfy the condition of linear growth:

$$(1 + |x| + |y|)^{-1}(|g(x, y, u, v, t)| + |g_x(x, y, u, v, t)| +$$

$$+|g_y(x, y, u, v, t)| + |\sigma(x, y, t)| + |\sigma_x(x, y, t)| + |\sigma_y(x, y, t)|) \le N$$

$$(1 + |x|)^{-1}(|l(x, y, u, v, t)| + |l_x(x, y, u, v, t)| + |l_y(x, y, u, v, t)|) \le N.$$

III. Function $p(x) : R^n \to R^1$ is continuously differentiable, and $|p(x)| + |p_x(x)| \le N(1 + |x|)$.

IV. Function $q(x) : R^m \to R^k$ is continuously differentiable, and $|q(x)| + |q_x(x)| \le N(1 + |x|)$.

First, we consider the stochastic optimal control problem (1)-(6).

Problem without constraint

We obtained the following result that is a necessary condition of optimality for problem (1)-(6):

Theorem 1. *Let conditions I-III hold, and let (x_t^0, u_t^0) be a solution of problem (1)–(6). Let there exist the random processes $(\psi_t, \beta_t) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{n\times n})$, which are the solutions of the adjoint equation*

$$\begin{cases} d\psi_t = -[H_x(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + H_y(\psi_z, x_z^0, y_z^0, u_z^0, v_z^0, z)|_{z=s(t)} s'(t)]dt + \\ +\beta_t dw_t,\ t_0 \le t < t_1 - h(t_1), \\ d\psi_t = -H_x(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + \beta_t dw_t,\ t_1 - h(t_1) \le t < t_1, \\ \psi_{t_1} = -p_x(x_{t_1}^0). \end{cases} \tag{8}$$

Then, $\forall\ \widetilde{u} \in U$ a.c., the following relations hold:

$$\begin{cases} H(\psi_t, x_t^0, y_t^0, u, v_t^0, t) - H(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + \\ +[H(\psi_z, x_z^0, y_z^0, u_z^0, u, z) - H(\psi_z, x_z^0, y_z^0, u_z^0, v_z^0, z)]|_{z=r(t)} r'(t) \le 0, \\ a.e.\ t \in [t_0, t_1 - h_1(t_1)), \\ H(\psi_t, x_t^0, y_t^0, u, v_t^0, t) - H(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) \le 0,\ \ a.e.\ t \in [t_1 - h_1(t_1), t_1]. \end{cases} \tag{9}$$

Here, $\tau = s(\tau)$ is a solution of the equation $\tau = t - h(t)$, $\tau = r(\tau)$ is a solution of equation $\tau = t - h_1(t)$, $y_t = x_{t-h(t)}$, $v_t = u_{t-h_1(t)}$, and

$$H(\psi_t, x_t, y_t, u_t, v_t, t) = \psi_t^* g(x_t, y_t, u_t, v_t, t) + \beta_t^* \sigma(x_t, y_t, t) - l(x_t, y_t, u_t, v_t, t).$$

Proof. Let $\overline{u}_1 = u_t^0 + \Delta u_t$ be some admissible control, and let $\overline{x}_1 = x_t^0 + \Delta x_t$ be the trajectory of system (1)-(5) corresponding to this control. We use the identities

$$\begin{aligned} & d\Delta x_t = [g(\overline{x}_t, \overline{y}_t, \overline{u}_t, \overline{v}_t, t) - g(x_t^0, y_t^0, u_t^0, v_t^0, t)]dt + [\sigma(\overline{x}_t, \overline{y}_t, t) - \\ & - \sigma(x_t^0, y^0, t)]dw_t = \{\Delta_{\overline{u}} g(x_t^0, y_t^0, u_t^0, v_t^0, t) + \Delta_{\overline{v}} g(x_t^0, y_t^0, u_t^0, v_t^0, t) + \\ & + g_x(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta x_t + g_y(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta y_t\}dt + \\ & + \{\sigma_x(x_t^0, y_t^0, t)\Delta x_t + \sigma_y(x_t^0, y_t^0, t)\Delta y_t\}dw_t + \eta_t^1,\quad t \in (t_0, t_1] \\ & \Delta x_t = 0,\ t \in [t_0 - h(t_0), t_0], \end{aligned} \tag{10}$$

where

$$\eta_t^1=\Big\{\int_0^1[g_x^*(x_t^0+\mu\Delta x_t,\overline{y}_t,\overline{u}_t,\overline{v}_t,t)-g_x^*(x_t^0,y_t^0,u_t^0,v_t^0,t)]\Delta x_t d\mu+ \\ +\int_0^1[g_y^*(x_t^0,y_t^0+\mu\Delta y_t,\overline{u}_t,\overline{v}_t,t)-g_y^*(x_t^0,y_t^0,\overline{u}_t,\overline{v}_t,t)]\Delta y_t d\mu\Big\}dt+ \\ +\Big\{\int_0^1[\sigma_x^*(x_t^0+\mu\Delta x_t,\overline{y}_t,t)-\sigma_x^*(x_t^0,\overline{y}_t,t)]\Delta x_t d\mu+ \\ +\int_0^1[\sigma_y^*(x_t^0,y_t^0+\mu\Delta y_t,t)-\sigma_y^*(x_t^0,y_t^0,t)]\Delta y_t d\mu\Big\}dw_t$$

and

$$d(\psi_t^*\cdot\Delta x_t)=d\psi_t^*\cdot\Delta x_t+\psi_t^*\cdot d\Delta x_t+\{\beta_t^*\sigma_x(x_t^0,y_t^0,t)\cdot\Delta x_t+\beta_t^*\sigma_y(x_t^0,y_t^0,t)\cdot\Delta y_t+ \\ +\beta_t^*\int_0^1[\sigma_x(x_t^0+\mu\Delta x_t,\overline{y},t)-\sigma_x(x_t^0,\overline{y},t)]\Delta x_t d\mu+ \\ +\beta_t^*\int_0^1[\sigma_y(x_t^0,y_t^0+\mu\Delta y_t,t)-\sigma_y(x_t^0,y_t^0,t)]\Delta y_t d\mu\}dt. \tag{11}$$

The increment of functional (6) along the admissible control looks like

$$\Delta_{\overline{u}}J(u)=E\Big\{p\Big(\overline{x}_{t_1}-p(x_{t_1}^0)+\int_{t_0}^{t_1}\Big)[l(\overline{x}_t,\overline{y}_t,\overline{u}_t,\overline{v}_t,t)-l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt\Big\}= \\ =Ep_x(x_{t_1}^0)\Delta x_{t_1}+E\int_{t_0}^{t_1}[\Delta_u l(x_t^0,y_t^0,u_t^0,v_t^0,t)+\Delta_v l(x_t^0,y_t^0,u_t^0,v_t^0,t)+ \\ +l_x(x_t^0,y_t^0,u_t^0,v_t^0,t)\Delta x_t+l_y(x_t^0,y_t^0,u_t^0,v_t^0,t)\Delta y_t]dt+\eta^2, \tag{12}$$

where

$$\eta^2=E\int_0^1[p_x^*(x_{t_1}^0+\mu\Delta x_{t_1})-p_x^*(x_{t_1}^0)]\Delta v x_{t_1}d\mu \\ +E\int_{t_0}^{t_1}\Big\{\int_0^1[l_x^*(x_{t_1}^0+\mu\Delta x_t,\overline{y}_t,\overline{u}_t,\overline{v}_t,t)-l_x^*(x_t^0,\overline{y}_t,\overline{u}_t,\overline{v}_t,t)\Delta x_t d\mu \\ +\int_0^1[l_y^*(x_t^0,y_t^0+\mu\Delta y_t,\overline{u}_t,\overline{v}_t,t)-l_y^*(x_t^0,y_t^0,\overline{u}_t,\overline{v}_t,t)]\Delta y_t d\mu\Big\}dt.$$

Taking (10) and (11) into consideration, expression (12) takes the form

$$\Delta_{\overline{u}}J(u^0)=-E\int_{t_0}^{t_1}d\psi_t^*\Delta x_t-E\int_{t_0}^{t_1}\psi_t^*\{[\Delta_{\overline{u}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)+ \\ +\Delta_{\overline{v}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)+g_x(x_t^0,y_t^0,u_t^0,v_t^0,t)\Delta x_t+ \\ +g_y(x_t^0,y_t^0,u_t^0,v_t^0,t)\Delta y_t]dt+[\sigma_x(x_t^0,y_t^0,t)\Delta x_t+\sigma_y(x_t^0,y_t^0,t)\Delta y_t]\}dw_t- \\ -E\int_{t_0}^{t_1}\beta_t^*[\sigma_x(x_t^0,y_t^0,t)\Delta x_t+\sigma_y(x_t^0,y_t^0,t)\Delta y_t]dt+E\int_{t_0}^{t_1}[\Delta_{\overline{u}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)+ \\ +\Delta_{\overline{v}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)+l_x(x_t^0,y_t^0,u_t^0,v_t^0,t)\Delta x_t+l_y(x_t^0,y_t^0,u_t^0,v_t^0,t)\Delta y_t dt]+\eta_{t_0,t_1}, \tag{13}$$

where

$$\eta_{t_0,t_1}=\eta^2+E\int_{t_0}^{t_1}\left\{\int_0^1\psi_t^*(g_x(x_t^0+\mu\Delta x_t,\overline{y}_t,u_t^0,v_t^0,t)-g_x(x_t^0,\overline{y}_t,u_t^0,v_t^0,t))\Delta x_t d\mu+\right.$$
$$\left.+\int_0^1\psi_t^*(g_y(x_t^0,y_t^0+\mu\Delta y_t,u_t^0,v_t^0,t)-g_y(x_t^0,y_t^0,u_t^0,v_t^0,t))\Delta y_t d\mu\right\}+$$
$$+E\int_{t_0}^{t_1}\left\{\int_0^1\beta_t^*(\sigma_x(x_t^0+\mu\Delta x_t,\overline{y}_t,t)-\sigma_x(x_t^0,y_t^0,t))\Delta x_t d\mu+\right.$$
$$\left.+\int_0^1\beta_t^*(\sigma_y(x_t^0,y_t^0+\mu\Delta y_t,t)-\sigma_y(x_t^0,y_t^0,t))\Delta y_t d\mu\right\}dt.$$

Using simple transformations and taking (8) into consideration, expression (13) takes the form

$$\Delta J(u^0)=-E\int_{t_0}^{t_1}[\psi_t^*\Delta_{\overline{u}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)-\Delta_{\overline{u}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt-$$
$$-E\int_{t_0}^{t_1}[\psi_t^*\Delta_{\overline{v}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)-\Delta_{\overline{v}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt+\eta_{t_0,t_1}. \tag{14}$$

Let's consider the following spike variation:

$$\Delta u_t=\Delta u_{t,\varepsilon}^{\theta}=\begin{cases}0, & t\overline{\in}[\theta,\theta+\varepsilon),\varepsilon>0,\theta\in[t_0,t_1)\\ \widetilde{u}-u_t^0, & t\in[\theta,\theta+\varepsilon),\widetilde{u}\in L^2(\Omega,F^{\theta},P;R^m).\end{cases}$$

Then (14) takes the form

$$\Delta_\theta J(u^0)=-E\int_{\theta}^{\theta+\varepsilon}[\psi_t^*\Delta_{\widetilde{u}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)+\psi_t^*\Delta_{\widetilde{v}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)-$$
$$-\Delta_{\widetilde{u}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)-\Delta_{\widetilde{v}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt+\eta_{\theta,\theta+\varepsilon}. \tag{15}$$

We will use the following lemma.

Lemma 1. *Let conditions I-III be satisfied. Then*

$$E|x_{t,\varepsilon}^{\theta}-x_t^0|^2\le N\varepsilon^2,\ \textit{if}\ \varepsilon\to 0,$$

where $x_{t,\varepsilon}^{\theta}$ *is the trajectory corresponding to the control* $u_{t,\varepsilon}^{\theta}=u_t^{\theta}+\Delta u_{t,\varepsilon}^{\theta}$.

Proof. Let's designate

$$\widetilde{x}_{t,\varepsilon}=\frac{x_{t,\varepsilon}^{\theta}-x_t^0}{\varepsilon},\ \widetilde{y}_{t,\varepsilon}=\widetilde{x}_{t-h(t),\varepsilon}=\frac{x_{t-h(t),\varepsilon}^{\theta}-x_{t-h(t)}^0}{\varepsilon}.$$

It is clear that $\forall\ t\in[t_0,\theta)$ $\widetilde{x}_{t,\varepsilon}=0$. Then, for $\forall\ t\in[\theta,\theta+\varepsilon)$,

$$d\widetilde{x}_{t,\varepsilon}=\frac{1}{\varepsilon}[g(x_t^0+\varepsilon\widetilde{x}_{t,\varepsilon},y_t^0+\varepsilon\widetilde{y}_{t,\varepsilon},\widetilde{u},v_t^0,t)-g(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt+$$
$$+\frac{1}{\varepsilon}[\sigma(x_t^0+\varepsilon\widetilde{x}_{t,\varepsilon},y_t^0+\varepsilon\widetilde{y}_{t,\varepsilon},t)-\sigma(x_t^0,y_t^0,t)]dw_t,\ t\in(\theta,\theta+\varepsilon)$$
$$\widetilde{x}_{\theta,\varepsilon}=-(g(x_\theta^0,y_\theta^0,\widetilde{u},v_\theta^0,\theta)-g(x_\theta^0,y_\theta^0,u_\theta^0,v_\theta^0,\theta)).$$

Therefore, conditions I-II and the Gronwall inequality yield

$$E|\widetilde{x}_{\theta+\varepsilon,\varepsilon}|^2 \le N\Big[E\sup_{\theta\le t\le\theta+\varepsilon}|x^\theta_{t,\varepsilon}-x^0_t|^2+E\sup_{\theta\le t\le\theta+\varepsilon}|x^0_t-x^0_\theta|^2+E\sup_{\theta\le t\le\theta+\varepsilon}|y^\theta_{t,\varepsilon}-y^0_t|^2+$$
$$+E\sup_{\theta\le t\le\theta+\varepsilon}|y^0_t-y^0_\theta|^2+\sup_{\theta\le t\le\theta+\varepsilon}E|g(x^0_t,y^0_t,\widetilde{u},v^0_t,t)-g(x^0_\theta,y^0_\theta,\widetilde{u},v^0_\theta,\theta)|^2+$$
$$+\frac{1}{\varepsilon}ER\int_\theta^{\theta+\varepsilon}|g(x^0_t,y^0_t,u^0_t,v^0_t,t)-g(x^0_\theta,y^0_\theta,u^0_\theta,v^0_\theta,\theta)|^2dt\Big].$$

Hence: $E|\widetilde{x}_{t+\varepsilon,\varepsilon}|^2\le N,\ \varepsilon\to 0,\ \forall\ t\in[\theta,\theta+\varepsilon)$. In the same way for $\forall\ t\in[\theta+\varepsilon,t_1]$, we have

$$d\widetilde{x}_{t,\varepsilon}=\frac{1}{\varepsilon}[g(x^0_t+\varepsilon\widetilde{x}_{t,\varepsilon},y^0_t+\varepsilon\widetilde{y}_{t,\varepsilon},u^0_t,\widetilde{u},t)-g(x^0_t,y^0_t,u^0_t,v^0_t,t)]dt+$$
$$+\frac{1}{\varepsilon}[\sigma(x^0_t+\varepsilon\widetilde{x}_{t,\varepsilon},y^0_t+\varepsilon\widetilde{y}_{t,\varepsilon},t)-\sigma(x^0_t,y^0_t,t)]dw_t.$$

Whence we have $E|\widetilde{x}_{t,\varepsilon}|^2\le N$, for $\forall\ t\in[\theta+\varepsilon,t_1]$, if $\varepsilon\to 0$. Thus, $\sup\limits_{t_0\le t\le t_1}E|\widetilde{x}_{t,\varepsilon}|^2\le N$. Lemma 1 is proved.

According to Lemma 1 and from expression for η_{t_0,t_1}, we obtain $\eta_{\theta,\theta+\varepsilon}=o(\varepsilon)$.

Then it follows from (15) that

$$\Delta_\theta J(u^0)=-E[\psi^*_\theta\Delta_{\widetilde{u}}g(x^0_\theta,y^0_\theta,x^0_\theta,u^0_\theta,v^0_\theta,\theta)-\Delta_{\widetilde{u}}l(x^0_\theta,y^0_\theta,x^0_\theta,u^0_\theta,v^0_\theta,\theta)+$$
$$+[\psi^*_z\Delta_{\widetilde{v}}g(x^0_z,y^0_z,x^0_z,u^0_z,v^0_z,\theta)-\Delta_{\widetilde{v}}l(x^0_z,y^0_z,x^0_z,u^0_z,v^0_z,z]|_{z=r(\theta)}r'(\theta)]\varepsilon+o(\varepsilon)\ge 0.$$

Hence, due to the sufficient smallness of ε, relation (9) is fulfilled. Theorem 1 is proved.

Problem with constraint

Using the obtained result and the variation principle of Ekeland [6], we will prove the following theorem for a stochastic optimal control problem with the endpoint constraint (7).

Theorem 2. *Let conditions I-IV hold, and let (x^0_t,u^0_t) be a solution of problem (1)–(7). Let there exist the random processes $(\psi_t,\beta_t)\in L^2_F(t_0,t_1;R^n)\times L^2_F(t_0,t_1;R^{n\times n})$ which are solutions of the adjoint system*

$$(16)\qquad \begin{cases} d\psi_t=-[H_x(\psi_t,x^0_t,y^0_t,u^0_t,v^0_t,t)+H_y(\psi_z,x^0_z,y^0_z,u^0_z,v^0_z,z)|_{z=s(t)}s'(t)]dt+\\ +\beta_t dw_t,\ t_0\le t<t_1-h(t_1),\\ d\psi_t=-H_x(\psi_t,x^0_t,y^0_t,u^0_t,v^0_t,t)dt+\beta_t dw_t,\ t_1-h(t_1)\le t<t_1,\\ \psi_{t_1}=-\lambda_0 p_x(x^0_{t_1})-\lambda_1 q_x(x^0_{t_1}),\end{cases}$$

where $(\lambda_0,\lambda_1)\in R^{k+1}$, $\lambda_0\ge 0$, λ_1 is the normal to the set G at the point $Eq(x^0_{t_1})$, and $\lambda_0^2+|\lambda_1|^2=1$. Then, $\forall\ \widetilde{u}\in U$ a.c., the following relations hold:

$$(17)\qquad \begin{cases} H(\psi_t,x^0_t,y^0_t,u,v^0_t,t)-H(\psi_t,x^0_t,y^0_t,u^0_t,v^0_t,t)+\\ +[H(\psi_z,x^0_z,y^0_z,u^0_z,u,z)-H(\psi_z,x^0_z,y^0_z,u^0_z,v^0_z,z)]|_{z=r(t)}r'(t)\le 0,\\ a.e.\ t\in[t_0,t_1-h_1(t_1)),\\ H(\psi_t,x^0_t,y^0_t,u,v^0_t,t)-H(\psi_t,x^0_t,y^0_t,u^0_t,v^0_t,t)\le 0,\ t\in[t_1-h_1(t_1),t_1],\ a.e.\end{cases}$$

Proof. For any natural j, we introduce the approximating functional

$$J_j(u) = S_j(Ep(x_{t_1}) + E\int_{t_0}^{t_1} l(x_t, y_t, u_t, v_t, t)dt, Eq(x_{t_1})) =$$

$$= \min_{(c,y)\in\varepsilon} \sqrt{\left|c - 1/j - Ep(x_{t_1}) - E\int_{t_0}^{t_1} l(x_t, y_t, u_t, v_t, t)dt\right|^2 + \|y - Eq(x_{t_1})\|^2},$$

$\mathcal{E} = \{(c, y) : c \le J^0, y \in G\}$, where J^0 is the minimal value of the functional in (1)-(7). By $V \equiv (U_\partial, d)$, we denote the space of controls obtained by means of introducing the metric

$$d(u, v) = (l \otimes P)\{(t, \omega) \in [t_0, t_1] \times \Omega : v_t \ne u_t\},$$

so that V is a complete metric space. In what follows, we need the following lemma.

Lemma 2. *We assume that conditions I-IV hold, u_t^n – a sequence of admissible controls from V, x_t^n – a sequence of the corresponding trajectories of system (1)-(3). If $d(u_t^n, u_t) \to 0, n \to \infty$, then $\lim\limits_{n\to\infty}\left\{\sup\limits_{t_0\le t\le t_1} E|x_t^n - x_t|^2\right\} = 0$, where x_t is a trajectory corresponding to an admissible control u_t.*

Proof. Let u_t^n be a sequence of admissible controls from V, and let x_t^n be a sequence of the corresponding trajectories. Then, for any $t \in (t_0; t_1]$, we have

$$|x_t^n - x_t| =$$

$$= \left|\int_{t_0}^{t} [g(x_s^n, y_s^n, u_s^n, v_s^n, s) - g(x_s, y_s, u_s, v_s, s)]ds + \int_{t_0}^{t} [\sigma(x_s^n, y_s^n, s) - \sigma(x_s, y_s, s)]dw_s\right|.$$

Let's square and take expectation of both sides of the last expression. Due to assumption II, we have

$$E|x_t^n - x_t|^2 \le$$

$$\le NE\int_{t_0}^{t} |\Delta_{u^n} g(x_s, y_s, u_s, v_s, s)|^2 ds + NE\int_{t_0}^{t} |x_t^n - x_t|^2 dt + N\int_{t_0}^{t} E|y_t^n - y_t|^2 dt.$$

Hence, condition I and the Gronwall inequality yield

$$E|x_t^n - x_t|^2 \le C\exp(C(t - t_0)),$$

where $C = NE\int_{t_0}^{t} |\Delta_{u^n} g(x_s, y_s, u_s, v_s, s)|^2 ds$. Lemma 2 is proved.

Due to continuity of the functional $J_j : V \to R^n$, according to the variation principle of Ekeland, we have that there exists a control $u_t^j : d(u_t^j, u_t^0) \le \sqrt{\varepsilon_j}$ and, $\forall\ u \in V$, the following inequality holds: $J_j(u^j) \le J_j(u) + \sqrt{\varepsilon_j} d(u^j, u),\ \varepsilon_j = \frac{1}{j}$.

This inequality means that (x_t^j, u_t^j) is a solution of the following problem:

$$\text{(18)} \qquad \begin{cases} I_j(u) = J_j(u) + \sqrt{\varepsilon_j} E\int_{t_0}^{t_1} \delta(u_t, u_t^j)dt \to \min \\ dx_t = g(x_t, y_t, u_t, v_t, t)dt + \sigma(x_t, y_t, t)dw_t,\ t \in (t_0, t_1] \\ x_t = \Phi(t),\ t \in [t_0 - h(t_0), t_0] \\ u_t = Q(t),\ t \in [t_0 - h_1(t_0), t_0] \\ u_t \in U_\partial. \end{cases}$$

The function $\delta(u, v)$ is determined in the following way: $\delta(u, v) = \begin{cases} 0, & u = v \\ 1, & u \ne v. \end{cases}$

Let (x_t^j, u_t^j) be a solution of problem (18). If there exist the random processes $\psi_t^j \in L_F^2(_0, t_1; R^n)$, $\beta_t^j \in L_F^2(t_0, t_1; R^{n\times n})$, which are solutions of the system

$$(19)\quad \begin{cases} d\psi_t^j = -[H_x(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t) + H_y(\psi_z^j, x_z^j, y_z^j, u_z^j, v_z^j, z)|_{z=s(t)} s'(t)]dt + \\ +\beta_t^j dw_t,\ t_0 \le t \le t_1 - h(t_1) \\ d\psi_t^j = -H_x(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t)dt + \beta_t^j dw_t,\ t_1 - h(t_1) \le t < t_1 \\ \psi_{t_1}^j = -\lambda_0^j p_x(x_{t_1}^j) - \lambda_1^j q_x(x_{t_1}^j), \end{cases}$$

where the non-zero $(\lambda_0^j, \lambda_1^j) \in R^{k+1}$ meet the requirement

$$(20)\quad (\lambda_0^j, \lambda_1^j) = (-c_j + 1/j + Ep(x_{t_1}^j) + E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt, -y_j + Eq(x_{t_1}^j))/I_j^0,$$

then, according to Theorem 1,

$$(21)\quad \begin{cases} H(\psi_t^j, x_t^j, y_t^j, u, v_t^j, t) - H(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t) + [H(\psi_z^j, x_z^j, y_z^j, u_z^j, u, z) - \\ -H(\psi_z^j, x_z^j, y_z^j, u_z^j, v_z^j, z)]|_{z=r(t)} r'(t) \le 0,\ \text{a.c., a.e. } t \in [t_0, t_1 - h_1(t_1)), \\ H(\psi_t^j, x_t^j, y_t^j, u, v_t^j, t) - H(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t) \le 0,\ \text{a.c, a.e. } t \in [t_1 - h_1(t_1), t_1]. \end{cases}$$

Here,

$$I_j^0 = \sqrt{\left| c_j - 1/j - Ep(x_{t_1}^j) - E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt \right|^2 + |y_j - Eq(x_{t_1}^j)|^2}.$$

Since $\|(\lambda_0^j, \lambda_1^j)\| = 1$, we can think that $(\lambda_0^j, \lambda_1^j) \to (\lambda_0, \lambda_1)$.

It is known that S_j is a convex function which is Gateaux-differentiable at a point: $(Ep(x_{t_1}^j) + E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt, Eq(x_{t_1}^j)$. Then, for all $(c, y) \in \mathcal{E}$,

$$\left(\lambda_0^j, c - \frac{1}{j} - Ep(x_{t_1}^j) - E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt\right) + (\lambda_1^j, y - Eq(x_{t_1}^j)) \le \frac{1}{j}.$$

Proceeding to the limit in the last inequality, we get that $\lambda_0 \ge 0$ and λ_1 is a normal to the set G at $Eq(x_{t_1}^0)$. Since

$$(22)\qquad \psi_{t_1}^j = -\lambda_0^j p_x(x_{t_1}^j) - \lambda_1^j q_x(x_{t_1}^j),\ \text{ we have } \psi_{t_1}^j \to \psi_{t_1} \text{ in } L_F^2(t_0, t_1; R^n).$$

Lemma 3. *Let ψ_t^j be a solution of system (19), and let ψ_t be a solution of system (16). Then*

$$E\int_{t_0}^{t_1} |\psi_t^j - \psi_t|^2 dt + E\int_{t_0}^{t_1} |\beta_t^j - \beta_t|^2 dt \to 0,\ \text{ if } d(u_t^j, u_t) \to 0,\ j \to \infty.$$

Proof. According to Ito formula $\forall\ s \in [t_1 - h(t), t_1]$,

$$\begin{aligned} & E|\psi_{t_1}^j - \psi_{t_1}|^2 - E|\psi_s^j - \psi_s|^2 = \\ & = 2E\int_s^{t_1} [\psi_t^j - \psi_t][(g_x^*(x_t^j, y_t^j, u_t^j, v_t^j, t) - g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t))\psi_t^j + \\ & + g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t)(\psi_t^j - \psi_t) + (\sigma_x^*(x_t^j, y_t^j, t) - \sigma_x^*(x_t^0, y_t^0, t) \times \\ & \times (\beta_t^j - \beta_t) - l_x(x_t^j, y_t^j, u_t^j, v_t^j, t) + l_x(x_t^0, y_t^0, u_t^0, v_t^0, t)]dt + E\int_s^{t_1} |\beta_t^j - \beta_t|^2 dt. \end{aligned}$$

Due to assumptions I-II and using simple transformations, we obtain

$$E\int_s^{t_1} |\beta_t^j - \beta_t|^2 dt + E|\psi_t^j - \psi_t|^2 dt + EN\varepsilon\int_s^{t_1} |\beta_t^j - \beta_t|^2 dt + E|\psi_{t_1}^j - \psi_{t_1}|^2.$$

Hence, according to the Gronwall inequality, we have

$$E|\psi_s^j-\psi_s|^2 \le De^{N(t_1-s)} \text{ a.e. in } [t_1-h(t),t_1], \tag{23}$$

where the constant D is determined in the following way: $D=E|\psi_{t_1}^j-\psi_{t_1}|^2$. According to Ito formula $\forall\ s\in[t_0,t_1-h(t_1))$,

$$\begin{aligned}
&E|\psi_{t_1-h(t_1)}^j-\psi_{t_1-h(t_1)}|^2-E|\psi_s^j-\psi_s|^2=2E\int_s^{t_1-h(t_1)}(\psi_t^j-\psi_t)[(g_x^*(x_t^j,y_t^j,u_t^j,v_t^j,t)-\\
&-g_x^*(x_t^0,y_t^0,u_t^0,v_t^0,t))\psi_t^j+g_x^*(x_t^0,y_t^0,u_t^0,v_t^0,t)(\psi_t^j-\psi_t)+(\sigma_x^*(x_t^j,y_t^j,t)-\\
&-\sigma_x^*(x_t^0,y_t^0,t))\beta_t^j+\sigma_x^*(x_t^0,y_t^0,t)(\beta_t^j-\beta_t)+(g_y^*(x_z^j,y_z^j,u_z^j,v_z^j,z)-\\
&-g_y^*(x_z^0,y_z^0,u_z^0,v_z^0,z))\psi_z^j s'(t)+g_y^*(x_z^0,y_z^0,u_z^0,v_z^0,z)(\psi_z^j-\psi_z)s'(t)\sigma_y^*(x_z^j,y_z^j,z)-\\
&-\sigma_y^*(x_z^0,y_z^0,z))\beta_z^j s'(t)+\sigma_y^*(x_z^0,y_z^0,z)(\beta_z^j-\beta_z)s'(t)+l_x(x_t^0,y_t^0,u_t^0,v_t^0,t)-\\
&-l_x(x_t^j,y_t^j,u_t^j,v_t^j,t)+l_y(x_t^0,y_t^0,u_t^0,v_t^0,t)-\\
&-l_y(x_t^j,y_t^j,u_t^j,v_t^j,t)]dt+E\int_s^{t_1-h(t_1)}|\beta_t^j-\beta_t|^2dt.
\end{aligned}$$

In view of assumptions I-II and expression (22), we obtain

$$\begin{aligned}
&E\int_s^{t_1-h(t_1)}|\beta_t^j-\beta_t|^2dt+E|\psi_s^j-\psi_s|^2\le\\
&\le EN\int_s^{t_1-h(t_1)}|\psi_t^j-\psi_t|^2dt+EN\varepsilon\int_s^{t_1-h(t_1)}|\psi_z^j-\psi_z|^2dt+\\
&+EN\varepsilon\int_s^{t_1-h(t_1)}|\beta_t^j-\beta_t|^2dt+E|\psi_{t_1-h(t_1)}^j-\psi_{t_1-h(t_1)}|^2.
\end{aligned}$$

Hence, using simple transformations, we have

$$\begin{aligned}
&E(1-2N\varepsilon)\int_s^{t_1-h(t_1)}|\beta_t^j-\beta_t|^2dt+E|\psi_s^j-\psi_s|^2\le E(N+N\varepsilon)\int_s^{t_1-h(t_1)}|\psi_t^j-\psi_t|^2dt+\\
&EN\varepsilon\int_{t_1-h(t_1)}^{t_1}|\psi_t^j-\psi_t|^2dt+EN\varepsilon\int_{t_1-h(t_1)}^{t_1}|\beta_t^j-\beta_t|^2dt+E|\psi_{t_1-h(t_1)}^j-\psi_{t_1-h(t_1)}|^2.
\end{aligned}$$

According to the Gronwall inequality,

$$E|\psi_s^j-\psi_s|^2\le D\exp[-(N+N\varepsilon)(t_1-h(t_1)-s)], \text{ a.e. in } [t_0,t_1-h(t_1)),$$

where

$$D=E|\psi_{t_1-h(t_1)}^j-\psi_{t_1-h(t_1)}|^2+EN\varepsilon\int_{t_1-h(t_1)}^{t_1}|\psi_t^j-\psi_t|^2dt+EN\varepsilon\int_{t_1-h(t_1)}^{t_1}|\beta_t^j-\beta_t|^2dt.$$

Due to sufficient smallness of ε and from inequality (23), we get $D\to 0$. Thus, $\psi_t^j-\psi_t$ $L_F^2(t_0,t_1;R^n)$, $\beta_t^j\to\beta_t$ $L_F^2(t_0,t_1;R^{n\times n})$. Lemma 3 is proved.

It follows from Lemma 3 and assumptions I-III that we can proceed to the limit in systems (19), (21) and get the fulfillment of (16) and (17). Theorem 2 is proved.

Corollary. *In the case where $g\equiv g(x_t,y_t,u_t,t)$ and $l\equiv l(x_t,u_t,t)$ we obtain the result proved in [4].*

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