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SOME PROPERTIES OF WEIGHT FUNCTIONS IN TAUBERIAN THEOREMS. II

New representations for weight functions in Tauberian theorems are derived. The representations are given by recurrent formulae. Obtained results are used to study properties of the weight functions.

1. Introduction

The paper is the second part of the article [1] published in previous issue of the journal. Therefore, we refer to [1] for notations, problems descriptions, the bibliography, and discussions.

The paper continues investigations of weight functions in Tauberian theorems for random fields, see [1, 2, 3, 4]. New method for weight functions computation is proposed and studied.

Sections 2 and 5 give recurrent formulae which link weight functions in spaces of different dimensions. Initial functions for the recurrent formulae are investigated in Sections 3 and 4. Some weight functions properties followed from obtained representations are derived in Section 6.

In the following, $a > 0$, $r > 0$. C with various indexes are nonnegative constants which may be different in different places.

2. Recurrent formulae for weight functions

By formulae for Bessel function's derivatives (see. §3.2, [5]), we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\nu}J_{\nu}(z)\right) = z^{\nu}J_{\nu-1}(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{J_{\nu}(z)}{z^{\nu}}\right) = -\frac{J_{\nu+1}(z)}{z^{\nu}}.$$

Hence, for $n \geq 0$ integration by parts in the representation for weight function (see (1) in [1]) gives

$$f_{n,r,a}(|t|) = \frac{1}{|t|^{\frac{n}{2}-1}}\int_0^{\infty}\frac{J_{\frac{n}{2}}(r(\lambda-a))}{(r(\lambda-a))^{n/2}}\lambda^{n/2}J_{\frac{n}{2}-1}(|t|\lambda)\mathrm{d}\lambda = \frac{J_{\frac{n}{2}}(r(\lambda-a))}{|t|^{n}(r(\lambda-a))^{n/2}}\times$$

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$$(\lambda|t|)^{n/2} J_{\frac{n}{2}}(|t|\lambda)\Big|_0^\infty + \frac{r}{|t|^n}\int_0^\infty \frac{J_{\frac{n}{2}+1}(r(\lambda-a))}{(r(\lambda-a))^{n/2}}(\lambda|t|)^{n/2} J_{\frac{n}{2}}(|t|\lambda)\mathrm{d}\lambda =$$

$$-\delta_{0,n} J_0(ar) + \frac{r}{|t|^n}\int_0^\infty \frac{J_{\frac{n}{2}+1}(r(\lambda-a))}{(r(\lambda-a))^{n/2}}(\lambda|t|)^{n/2} J_{\frac{n}{2}}(|t|\lambda)\mathrm{d}\lambda\,,$$

because of Bessel function asymptotic behavior (see (2) in [1]).

Hence

$$f_{n,r,a}(|t|) = -\delta_{0,n} J_0(ar) + \frac{r^2}{|t|^{n+1}}\int_0^\infty \frac{J_{\frac{n}{2}+1}(r(\lambda-a))}{(r(\lambda-a))^{n/2+1}}(\lambda|t|)^{n/2+1} J_{\frac{n}{2}}(|t|\lambda)\mathrm{d}\lambda -$$

$$\frac{ar^2}{|t|^n}\int_0^\infty \frac{J_{\frac{n}{2}+1}(r(\lambda-a))}{(r(\lambda-a))^{n/2+1}}(\lambda|t|)^{n/2} J_{\frac{n}{2}}(|t|\lambda)\mathrm{d}\lambda =$$

$$-\delta_{0,n} J_0(ar) + r^2 f_{n+2,r,a}(|t|) - r^2 F_n(r,|t|,a),$$

where

$$F_n(r,|t|,a) := \frac{a}{|t|^n}\int_0^\infty \frac{J_{\frac{n}{2}+1}(r(\lambda-a))}{(r(\lambda-a))^{n/2+1}}(\lambda|t|)^{n/2} J_{\frac{n}{2}}(|t|\lambda)\mathrm{d}\lambda\,. \tag{1}$$

Remark. Note that due to Bessel functions behavior at infinity (see (2) in [1]) and at zero (see, for example, Poisson's integral in §2, [1]) the integral in (1) converges absolutely.

Summarizing, we have:

Lemma 1. *For $n \geq 0$ the following holds*

$$f_{n+2,r,a}(|t|) = \frac{f_{n,r,a}(|t|) + \delta_{0,n} J_0(ar)}{r^2} + F_n(r,|t|,a)\,, \tag{2}$$

where $F_n(r,|t|,a)$ is given by (1).

Due to (2) we get

Corollary. *For $n > m \geq 0$*

(i) *if both n and m are even, then*

$$f_{n,r,a}(|t|) = \frac{f_{m,r,a}(|t|) + \delta_{0,m} J_0(ar)}{r^{n-m}} + \sum_{k=\frac{m}{2}+1}^{\frac{n}{2}} \frac{F_{2k-2}(r,|t|,a)}{r^{n-2k}}\,;$$

(ii) *if both n and m are odd, then*

$$f_{n,r,a}(|t|) = \frac{f_{m,r,a}(|t|)}{r^{n-m}} + \sum_{k=\frac{m+1}{2}}^{\frac{n-1}{2}} \frac{F_{2k-1}(r,|t|,a)}{r^{n-2k-1}}\,.$$

3. Computation $f_{1,r,a}(|t|)$

We will use sine and cosine integrals defined by formulae

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(u)}{u}\,\mathrm{d}u\,, \qquad \mathrm{Ci}(z) = \gamma + \ln(z) + \int_0^z \frac{\cos(u)-1}{u}\,\mathrm{d}u\,, \tag{3}$$

where γ is Euler's constant.

Lemma 2. *$f_{1,r,a}(|t|)$ equals*

$$\frac{(\mathrm{Si}(a(|t|+r))-\mathrm{Si}(a(|t|-r)))\cos(|t|a)-(\mathrm{Ci}(a(|t|+r))-\mathrm{Ci}(a(|t|-r)))\sin(|t|a)}{\pi r}, \tag{4}$$

if $|t|>r$, and

$$\frac{(\mathrm{Si}(a(|t|+r))+\mathrm{Si}(a(r-|t|)))\cos(|t|a)-(\mathrm{Ci}(a(|t|+r))-\mathrm{Ci}(a(r-|t|)))\sin(|t|a)+\pi\cos(|t|a)}{\pi r}, \tag{5}$$

if $|t|<r$.

Proof. By §3.4, [5]

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z), \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z)\,.$$

Hence

$$f_{1,r,a}(|t|) = \int_0^\infty \frac{J_{\frac{1}{2}}(r(\lambda-a))}{\sqrt{r(\lambda-a)}}\sqrt{\lambda|t|}J_{-\frac{1}{2}}(|t|\lambda)\mathrm{d}\lambda =$$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(r(\lambda-a))\cos(|t|\lambda)}{r(\lambda-a)}\mathrm{d}\lambda\,.$$

The integral can be explicitly evaluated in terms of sine integrals Si(·) and cosine integrals Ci(·). Computations by *Mathematica* 5.0[1] give expressions (4) and (5). □

4. Computation $f_{0,r,a}(|t|)$

Lemma 3. *For $|t|\neq r$ we have*

$$f_{0,r,a}(|t|) = \frac{2r}{\pi}\int_{\min(1,\,|t|/r)}^1 \frac{x\cos(arx)\,\mathrm{d}x}{\sqrt{r^2x^2-|t|^2}\sqrt{1-x^2}} - J_0(ar) -$$

$$\frac{2r}{\pi}\int_0^{\min(1,\,|t|/r)} \frac{x\sin(arx)\,\mathrm{d}x}{\sqrt{|t|^2-r^2x^2}\sqrt{1-x^2}}\,. \tag{6}$$

[1]All *Mathematica* 5.0 computations in this paper were verified by *Maple* 9.5 and tables [6], when it was possible

Proof. To evaluate $f_{0,r,a}(|t|)$ we use the relation $J_{-n}(z) = (-1)^n J_n(z)$ (see §2.1, [5]) and Poisson's integral (see §2, [1]):

$$f_{0,r,a}(|t|) = |t| \int_0^\infty J_0(r(\lambda - a))J_{-1}(|t|\lambda)\mathrm{d}\lambda = \tag{7}$$

$$-\frac{|t|}{\Gamma^2(1/2)} \int_0^\infty J_1(|t|\lambda) \int_{-1}^1 \frac{\cos(r(\lambda-a)x)}{\sqrt{1-x^2}}\mathrm{d}x\mathrm{d}\lambda =$$

$$-\frac{|t|}{\pi} \int_{-1}^1 (1-x^2)^{-1/2} \int_0^\infty J_1(|t|\lambda)\cos(r(\lambda-a)x)\mathrm{d}\lambda\mathrm{d}x\,. \tag{8}$$

We need to prove the correctness of interchanging the integration order in the last identity.

Let us rewrite the integral in (8) as

$$\int_0^\infty \int_{-1}^1 G_{|t|,r,a}(x,\lambda)\mathrm{d}x\mathrm{d}\lambda = \int_0^C \int_{-1}^1 G_{|t|,r,a}(x,\lambda)\mathrm{d}x\mathrm{d}\lambda +$$

$$\int_C^\infty \int_{-1}^1 G_{|t|,r,a}(x,\lambda)\mathrm{d}x\mathrm{d}\lambda\,, \tag{9}$$

where

$$G_{|t|,r,a}(x,\lambda) := J_1(|t|\lambda)\frac{\cos(r(\lambda-a)x)}{\sqrt{1-x^2}}\,.$$

The first integral in (9) has compact integration domain and it converges absolutely. Therefore, interchanging the integration order in it is correct and it does not change its value. Hence, to prove (8) it is enough to show that

$$\int_C^\infty \int_{-1}^1 G_{|t|,r,a}(x,\lambda)\mathrm{d}x\mathrm{d}\lambda \to 0\,, \quad \int_{-1}^1 \int_C^\infty G_{|t|,r,a}(x,\lambda)\mathrm{d}\lambda\mathrm{d}x \to 0, \tag{10}$$

when $C \to +\infty$.

The first statement in (10) follows from the representation (7)

$$\int_C^\infty \int_{-1}^1 G_{|t|,r,a}(x,\lambda)\mathrm{d}x\mathrm{d}\lambda = -\pi \int_C^\infty J_0(r(\lambda-a))J_{-1}(|t|\lambda)\mathrm{d}\lambda$$

and Bessel functions asymptotic behavior, see (2) in [1].

To prove the second statement in (10) we note that

$$\int_C^\infty J_1(|t|\lambda)\cos(r(\lambda-a)x)\mathrm{d}\lambda = \cos(rax)\underbrace{\int_C^\infty J_1(|t|\lambda)\cos(r\lambda x)\mathrm{d}\lambda}_{I_1} +$$

$$\sin(rax)\underbrace{\int_C^\infty J_1(|t|\lambda)\sin(r\lambda x)\mathrm{d}\lambda}_{I_2}\,.$$

We will use the relation (see (1) in §3.61 [5])

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2},$$

where $H_\nu^{(k)}(z),\ k = 1, 2$ are Bessel functions of the third kind.

By asymptotic expansions (3) and (4) for Bessel functions of the third kind given in §7.2 [5], for parameters values $\nu = p = 1$, we have

$$H_1^{(k)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{\pm \mathrm{i}(z - \frac{3\pi}{4})} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right),$$

for $k = 1$ and 2, respectively.

Hence

$$|I_1| \le \left|\int_C^\infty \frac{e^{\mathrm{i}|t|\lambda}\cos(r\lambda x)}{\sqrt{2\pi|t|\lambda}} \mathrm{d}\lambda\right| + \left|\int_C^\infty \frac{e^{\mathrm{i}|t|\lambda}\cos(r\lambda x)}{\sqrt{2\pi|t|\lambda}} \mathcal{O}\left(\frac{1}{|t|\lambda}\right) \mathrm{d}\lambda\right| +$$

$$\left|\int_C^\infty \frac{e^{-\mathrm{i}|t|\lambda}\cos(r\lambda x)}{\sqrt{2\pi|t|\lambda}} \mathrm{d}\lambda\right| + \left|\int_C^\infty \frac{e^{-\mathrm{i}|t|\lambda}\cos(r\lambda x)}{\sqrt{2\pi|t|\lambda}} \mathcal{O}\left(\frac{1}{|t|\lambda}\right) \mathrm{d}\lambda\right| \le$$

$$\left|\int_C^\infty \frac{\sqrt{2}\cos(|t|\lambda)\cos(r\lambda x)}{\sqrt{\pi|t|\lambda}} \mathrm{d}\lambda\right| + \left|\int_C^\infty \frac{\sqrt{2}\sin(|t|\lambda)\cos(r\lambda x)}{\sqrt{\pi|t|\lambda}} \mathrm{d}\lambda\right| +$$

$$\int_C^\infty \frac{C_1 \mathrm{d}\lambda}{2\lambda^{3/2}} \le \underbrace{\left|\int_C^\infty \frac{\cos(\lambda(|t| - rx))}{\sqrt{2\pi|t|\lambda}} \mathrm{d}\lambda\right|}_{I_3} + \underbrace{\left|\int_C^\infty \frac{\cos(\lambda(|t| + rx))}{\sqrt{2\pi|t|\lambda}} \mathrm{d}\lambda\right|}_{I_4} +$$

$$\underbrace{\left|\int_C^\infty \frac{\sin(\lambda(|t| - rx))}{\sqrt{2\pi|t|\lambda}} \mathrm{d}\lambda\right|}_{I_5} + \underbrace{\left|\int_C^\infty \frac{\sin(\lambda(|t| + rx))}{\sqrt{2\pi|t|\lambda}} \mathrm{d}\lambda\right|}_{I_6} + \frac{C_1}{\sqrt{C}}$$

and

$$\left|\int_{-1}^1 \frac{\cos(rax)}{\sqrt{1 - x^2}}\, I_1\, \mathrm{d}x\right| \le \int_{-1}^1 \left(|I_3| + |I_4| + |I_5| + |I_6| + \frac{C_1}{\sqrt{C}}\right) \frac{\mathrm{d}x}{\sqrt{1 - x^2}}.$$

Note that

$$\int_{-1}^1 \frac{|I_3|}{\sqrt{1 - x^2}}\, \mathrm{d}x = \int_{-1}^1 \left|\int_{C||t| - rx|}^\infty \frac{\cos z}{\sqrt{z}}\, \mathrm{d}z\right| \frac{\mathrm{d}x}{\sqrt{1 - x^2}\sqrt{||t| - rx|}}.$$

If $|t| \ne r$ it is easy to check that:

$$\text{for each } x \in (-1, 1),\ x \ne \tfrac{|t|}{r} : \frac{\left|\int_{C||t| - rx|}^\infty \frac{\cos z}{\sqrt{z}}\, \mathrm{d}z\right|}{\sqrt{1 - x^2}\sqrt{||t| - rx|}} \to 0, \text{ when } C \to +\infty;$$

$$\frac{\left|\int_{C||t|-rx|}^{\infty}\frac{\cos z}{\sqrt{z}}\,\mathrm{d}z\right|}{\sqrt{1-x^2}\sqrt{||t|-rx|}} \le \frac{\sup\limits_{C\ge 0}\left|\int_{C}^{\infty}\frac{\cos z}{\sqrt{z}}\,\mathrm{d}z\right|}{\sqrt{1-x^2}\sqrt{||t|-rx|}} \in L_1([-1,1]).$$

This means that conditions of Lebesgue dominated convergence theorem hold true and

$$\int_{-1}^{1}\frac{|I_3|}{\sqrt{1-x^2}}\,\mathrm{d}x \to 0, \quad \text{when} \quad C\to\infty\,.$$

For $k=4,5,6$ proofs of $\displaystyle\int_{-1}^{1}\frac{|I_k|}{\sqrt{1-x^2}}\,\mathrm{d}x \to 0$, as $C\to\infty$ are similar.

Therefore

$$\left|\int_{-1}^{1}\frac{\cos(rax)}{\sqrt{1-x^2}}\,I_1\,\mathrm{d}x\right| \to 0, \quad \text{when} \quad C\to\infty\,. \tag{11}$$

Similarly

$$\left|\int_{-1}^{1}\frac{\sin(rax)}{\sqrt{1-x^2}}\,I_2\,\mathrm{d}x\right| \to 0, \quad \text{when} \quad C\to\infty\,. \tag{12}$$

By (11) and (12) we obtain the second statement in (10).

Computations by *Mathematica* 5.0 give

$$\int_0^{\infty} J_1(|t|\lambda)\cos(r\lambda x)\mathrm{d}\lambda = \begin{cases} \frac{1}{|t|} - \frac{r|x|}{|t|\sqrt{r^2x^2-|t|^2}} & , \ |t|<r|x|; \\ \frac{1}{|t|} & , \ |t|>r|x|\,, \end{cases}$$

$$\int_0^{\infty} J_1(|t|\lambda)\sin(r\lambda x)\mathrm{d}\lambda = \begin{cases} 0 & , \ |t|<r|x|; \\ \frac{rx}{|t|\sqrt{|t|^2-r^2x^2}} & , \ |t|>r|x|\,, \end{cases}$$

and (6) as a corollary. □

Remark. Let us stress that in comparison with representation (7) integrals in the formula (6) converge absolutely. Hence the representation (6) is preferable for investigations and computations functions $f_{0,r,a}(|t|)$, and $f_{2k,r,a}(|t|),\ k\in\mathbf{N}$.

5. Main result

By Lemmas 1-3 and the Corollary we obtain

Theorem 1. *For* $n\ge 1$, $|t|\ne r$,

(i) *if* n *is even, then*

$$f_{n,r,a}(|t|) = \frac{g_{r,a}(|t|)}{r^n} + \sum_{k=1}^{\frac{n}{2}}\frac{F_{2k-2}(r,|t|,a)}{r^{n-2k}}\,;$$

(ii) *if n is odd, then*

$$f_{n,r,a}(|t|) = \frac{f_{1,r,a}(|t|)}{r^{n-1}} + \sum_{k=1}^{\frac{n-1}{2}} \frac{F_{2k-1}(r,|t|,a)}{r^{n-2k-1}},$$

where $F_n(r,|t|,a)$ is given by (1), *$f_{1,r,a}(|t|)$ can be calculated by* (4) *and* (5),

$$g_{r,a}(|t|) := \frac{2r}{\pi}\int_{\min(1,\,|t|/r)}^{1} \frac{x\cos(arx)\,\mathrm{d}x}{\sqrt{r^2x^2-|t|^2}\sqrt{1-x^2}} -$$

$$\frac{2r}{\pi}\int_{0}^{\min(1,\,|t|/r)} \frac{x\sin(arx)\,\mathrm{d}x}{\sqrt{|t|^2-r^2x^2}\sqrt{1-x^2}}.$$

6. On properties of weight functions

In this section we apply Theorem 1 to study properties of weight functions.

Theorem 2. *For $n \geq 1$*

$$\lim_{|t|\to+\infty} f_{n,r,a}(|t|) = 0.$$

Proof. Hence, by (3) and the representation (4) from Lemma 2, for $|t| > r$, we have

$$|f_{1,r,a}(|t|)| = \frac{1}{\pi r}\left| \cos(|t|a)\int_{a(|t|-r)}^{a(|t|+r)} \frac{\sin(u)}{u}\,\mathrm{d}u - \sin(|t|a)\int_{a(|t|-r)}^{a(|t|+r)} \frac{\cos(u)}{u}\,\mathrm{d}u \right| =$$

$$\frac{1}{\pi r}\left|\int_{|t|-r}^{|t|+r} \frac{\sin(a(u-|t|))}{u}\,\mathrm{d}u\right| \leq \frac{1}{\pi r}\int_{|t|-r}^{|t|+r}\frac{\mathrm{d}u}{u} = \ln\left(1+\frac{2r}{|t|-r}\right) \to 0,$$

when $|t| \to +\infty$.

Recalling the definition of $g_{r,a}(|t|)$, it is easy to verify that

$$\lim_{|t|\to+\infty} g_{r,a}(|t|) = \frac{2r}{\pi}\lim_{|t|\to+\infty}\int_0^1 \frac{x\sin(arx)}{\sqrt{|t|^2-r^2x^2}\sqrt{1-x^2}} = 0.$$

By (1), for $n \geq 0$,

$$|F_n(r,|t|,a)| = \frac{a}{|t|^{n/2}}\left|\int_0^a \underbrace{\frac{J_{\frac{n}{2}+1}(r\lambda)}{(r\lambda)^{n/2+1}}(a-\lambda)^{n/2} J_{\frac{n}{2}}(|t|(a-\lambda))\,\mathrm{d}\lambda}_{Q_1(|t|,r,a,\lambda)} + \right.$$

$$\int_0^\infty \underbrace{\frac{J_{\frac{n}{2}+1}(r\lambda)}{(r\lambda)^{n/2+1}}(\lambda+a)^{n/2}J_{\frac{n}{2}}(|t|(\lambda+a))}_{Q_2(|t|,r,a,\lambda)}\,\mathrm{d}\lambda\Bigg| \to 0, \tag{13}$$

when $|t| \to \infty$. To see this, we can apply Lebesgue dominated convergence theorem. Due to Bessel functions behavior at infinity (see (2) in [1]) and at zero (see Poisson's integral in §2, [1]) conditions of Lebesgue theorem hold true:

$$\frac{a}{|t|^{n/2}}\,|Q_1(|t|,r,a,\lambda)| \le C\frac{|J_{\frac{n}{2}+1}(r\lambda)|}{(r\lambda)^{n/2+1}}(a-\lambda)^{n/2} \in L_1([0,a]), \tag{14}$$

$$\frac{a}{|t|^{n/2}}\,|Q_2(|t|,r,a,\lambda)| \le C\frac{|J_{\frac{n}{2}+1}(r\lambda)|}{(r\lambda)^{n/2+1}}(\lambda+a)^{n/2} \in L_1([0,+\infty]), \tag{15}$$

$$\frac{a}{|t|^{n/2}}\,|Q_1(|t|,r,a,\lambda)| \to 0, \quad \frac{a}{|t|^{n/2}}\,|Q_2(|t|,r,a,\lambda)| \to 0, \tag{16}$$

when $|t| \to \infty$.

Finally, we use the representation of $f_{n,r,a}(|t|)$ from Theorem 1. □

Remark. Theorem 2 explains decreasing magnitude of $f_{n,r,a}(|t|)$ on plots in [1, 3, 4], when $|t|$ grows.

Theorem 3. *For $n \ge 1$*

$$\lim_{r\to+\infty} f_{n,r,a}(|t|) = 0.$$

Proof. By the representation (5) from Lemma 2, for $r > |t|$, we have

$$|f_{1,r,a}(|t|)| = \frac{1}{\pi r}\Bigg|\cos(|t|a)\left(\int_0^{a(r+|t|)}\frac{\sin(u)}{u}\,\mathrm{d}u + \int_0^{a(r-|t|)}\frac{\sin(u)}{u}\,\mathrm{d}u\right) -$$

$$\sin(|t|a)\int_{a(r-|t|)}^{a(r+|t|)}\frac{\cos(u)}{u}\,\mathrm{d}u + \pi\cos(|t|a)\Bigg| \le$$

$$\frac{2}{\pi r}\left|\int_0^{a(r+|t|)}\frac{\sin(u)}{u}\,\mathrm{d}u\right| + \frac{1}{\pi r}\left|\int_{a(r-|t|)}^{a(r+|t|)}\frac{1}{u}\,\mathrm{d}u\right| + \frac{1}{r} \to 0,$$

when $|r| \to +\infty$.

By the definition of $g_{r,a}(|t|)$, for $r > |t|$,

$$\frac{|g_{r,a}(|t|)|}{r^n} \le \frac{2}{\pi r^{n-1}}\left(\int_{|t|/r}^{1}\frac{x\,\mathrm{d}x}{\sqrt{r^2x^2-|t|^2}\sqrt{1-x^2}} + \int_0^{|t|/r}\frac{x\,\mathrm{d}x}{\sqrt{|t|^2-r^2x^2}\sqrt{1-x^2}}\right) =$$

$$\frac{1}{\pi r^n}\left(\ln\left(1+\frac{2|t|}{r-|t|}\right)+\pi\right) \to 0, \quad \text{as } r \to \infty.$$

For $n \geq 0$, (13) is also true, when $r \to \infty$. To see this, we can apply Lebesgue dominated convergence theorem again. Due to Bessel functions asymptotic behavior (16) holds, when $r \to \infty$, $\lambda \neq 0$. It is also easy to check the another theorem's conditions:

$$\frac{a}{|t|^{n/2}}\,|Q_1(|t|, r, a, \lambda)| \leq C(a-\lambda)^{n/2}|J_{\frac{n}{2}}(|t|(a-\lambda))| \in L_1([0,a]),$$

$$\frac{a}{|t|^{n/2}}\,|Q_2(|t|, r, a, \lambda)| \leq C(\lambda+a)^{n/2}|J_{\frac{n}{2}}(|t|(\lambda+a))| \in L_1([0,C_1]),$$

$$\frac{a}{|t|^{n/2}}\,|Q_2(|t|, r, a, \lambda)| \leq C\frac{(\lambda+a)^{n/2}}{\lambda^{n/2+1}}|J_{\frac{n}{2}}(|t|(\lambda+a))| \in L_1([C_1,+\infty]).$$

Finally, we use $f_{n,r,a}(|t|)$ representation from Theorem 1. □

Example. The assertion of Theorem 3 is illustrated in Figures 1 and 2 in the cases $n = 3$, $a = 1.2$, $|t| = 1$ and $|t| = 5$.

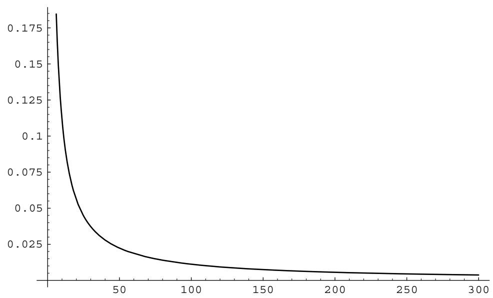

Fig.1 Plot of $f_{3,r,1.2}(1)$

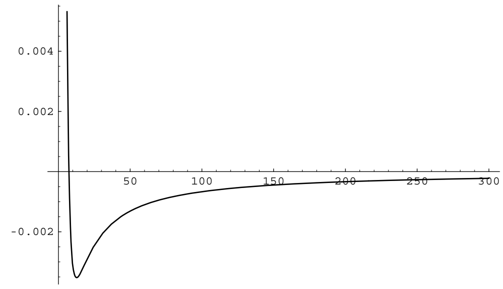

Fig.2 Plot of $f_{3,r,1.2}(5)$

Theorem 4. *For $n \geq 1$ the weight function $f_{n,r,a}(|t|)$ has a discontinuity at $|t| = r$. If $\frac{ar}{\pi} \in \mathbb{N}$ there exists a jump discontinuity at $|t| = r$. In other cases $f_{n,r,a}(|t|)$ has an infinite discontinuity.*

Proof. From the representation (1) and conditions (14) and (15) it follows that $F_n(r, |t|, a)$, $n \geq 0$ is a continuous function of $|t|$. It is easy to show that for each $|t| \neq r$ both functions $f_{1,r,a}(|t|)$ and $g_{r,a}(|t|)$ are continuous. Therefore, we need to investigate $f_{1,r,a}(|t|)$ and $g_{r,a}(|t|)$ behavior in $|t| = r$.

By Lemma 2

$$\lim_{|t|\to r+} f_{1,r,a}(|t|) = \frac{1}{\pi r}\Big(\mathrm{Si}(2ar)\cos(ra) - \mathrm{Ci}(2ar)\sin(ra) + \lim_{|t|\to r+}\mathrm{Ci}(a(|t|-r))\times$$

$$\sin(|t|a)\Big) = \frac{\mathrm{Si}(2ar)\cos(ra) + (\gamma - \mathrm{Ci}(2ar))\sin(ra)}{\pi r} + \lim_{|t|\to r+}\frac{\ln(a(|t|-r))}{\pi r}\times$$

$$\sin(|t|a) = \frac{\mathrm{Si}(2ar)\cos(ra) + (\gamma - \mathrm{Ci}(2ar))\sin(ra)}{\pi r} + \begin{cases} 0, & \text{if } \frac{ar}{\pi} \in \mathbb{N}; \\ \infty, & \text{otherwise;} \end{cases}$$

$$\lim_{|t|\to r-} f_{1,r,a}(|t|) = \frac{\mathrm{Si}(2ar)\cos(ra) + (\gamma - \mathrm{Ci}(2ar))\sin(ra)}{\pi r} +$$

$$\frac{\cos(ra)}{r} + \begin{cases} 0, & \text{if } \frac{ar}{\pi} \in \mathbb{N}; \\ \infty, & \text{otherwise.} \end{cases}$$

This implies Theorem's assertion for odd n.

By application of Theorem 1 and computations by *Mathematica* 5.0 it is easy to derive

$$\lim_{|t|\to r+} g_{r,a}(|t|) = -\frac{2r}{\pi} \lim_{|t|\to r+} \int_0^1 \frac{x\sin(arx)\,\mathrm{d}x}{\sqrt{|t|^2 - r^2x^2}\sqrt{1-x^2}} =$$

$$\begin{cases} (-1)^{ar/\pi}\mathrm{Si}(2ar\pi)/\pi, & \text{if } \frac{ar}{\pi} \in \mathbb{N}; \\ \infty, & \text{otherwise;} \end{cases}$$

$$\lim_{|t|\to r-} g_{r,a}(|t|) = \frac{2r}{\pi} \lim_{|t|\to r-} \int_{|t|/r}^1 \frac{x\cos(arx)\,\mathrm{d}x}{\sqrt{r^2x^2 - |t|^2}\sqrt{1-x^2}} -$$

$$\frac{2r}{\pi} \lim_{|t|\to r-} \int_0^1 \frac{x\sin(a|t|x)\,\mathrm{d}x}{\sqrt{1 - |t|^2x^2/r^2}\,\sqrt{1-x^2}} =$$

$$\cos(ar) + \begin{cases} \frac{(-1)^{ar/\pi}\mathrm{Si}(2ar\pi)}{\pi}, & \text{if } \frac{ar}{\pi} \in \mathbb{N}; \\ \infty, & \text{otherwise.} \end{cases}$$

This implies Theorem's assertion for even n. □

Remark. For the case $a = 0$ it is well known that

$$f_{n,r,0}(|t|) = \begin{cases} r^{-n}, & \text{if } |t| < r\,; \\ 0, & \text{if } |t| > r \end{cases}$$

has a jump discontinuity at $|t| = r$.

Remark. Theorem 4 explains behavior of $f_{n,r,a}(|t|)$ in the neighborhood of $|t| = r$ on plots in [1, 3, 4].

Example. The assertion of Theorem 4 is illustrated in Figures 3 and 4 in the cases $n = 3$, $r = 1$, $a = \pi$ and $a = 2$.

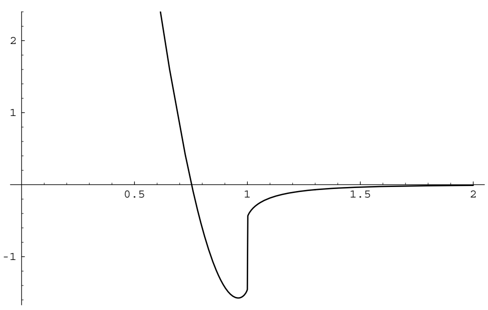

Fig.3 Plot of $f_{3,1,\pi}(|t|)$

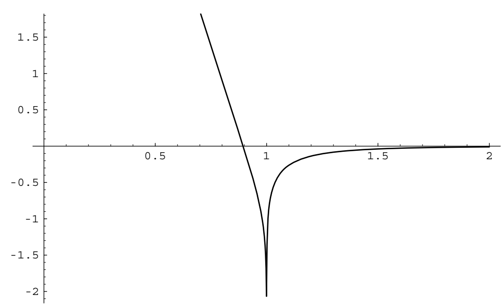

Fig.4 Plot of $f_{3,1,2}(|t|)$

7. Concluding remarks

New recurrent formulae for weight functions in Tauberian theorems are derived. In comparison with previous representations the formulae are written in terms of absolutely convergent integrals and functions $\mathrm{Si}(\cdot)$, $\mathrm{Ci}(\cdot)$. It gives a new possibility to investigate the weight functions and to obtain their new properties. The formulae are convenient for numerical computations of weight functions values for small space dimensions n. For large n, representations proposed and studied in [1, 3] are preferable.

It would be interesting to apply the formulae exposed here and in [1, 3] to establish new properties (for example, oscillations which are observed on plots in [1, 3, 4]) of the weight functions.

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