Theory of Stochastic Processes
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DIFFUSION APPROXIMATION ALGORITHMS IN MERGING PHASE SPACE

Diffusion approximation algorithms for stochastic systems in split and merging phase space are represented in servey form.

The main mathematical tools of such algorithms are described in our book "Stochastic Systems in Merging Phase Space" (World Scientific Publishing, 2005).

1. Introduction

The general scheme is illustrated on the semi-Markov random evolution given by a solution of the evolutionary equation

$$\frac{dU^\varepsilon(t)}{dt} = C^\varepsilon(U^\varepsilon(t); \mathfrak{X}^\varepsilon(t/\varepsilon^2))dt. \tag{1}$$

The velocity function

$$C^\varepsilon(u;x) = C(u;x) + \varepsilon^{-1}C_0(u;x), \ u \in \mathbb{R}^d, \ x \in E \tag{2}$$

satisfies the conditions of existence of global solutions for associated deterministic evolutionary equations

$$\frac{dU_x^\varepsilon(t)}{dt} = C^\varepsilon(U_x^\varepsilon(t);x), \ x \in E. \tag{3}$$

2. The semi-Markov switching process

The semi-Markov switching process $\mathfrak{X}^\varepsilon(t)$, $t \geq 0$, is considered in the series scheme with small parameter series $\varepsilon \to 0$ $(\varepsilon > 0)$ given on the double split phase space (E, ξ) :

$$E = \cup_{k=1}^N E_k, \ E_k = \cup_{r=1}^{N_k} E_k^r, \ E_k^r \cap E_k^{r'} = \varnothing, \ r \neq r', \tag{4}$$

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by the semi-Markov kernel

$$Q^\varepsilon(x,B,t) = P^\varepsilon(x,B)F_x(t),\ x \in E,\ B \in \xi,\ t \geq 0. \tag{5}$$

The stochastic kernels

$$P^\varepsilon(x,B) = P(x,B) + \varepsilon P_1(x,B) + \varepsilon^2 P_2(x,B) \tag{6}$$

in (6) coordinated with split (4) as follows:

$$\begin{aligned} &P(x,E_k^r) = \mathbf{1}_k^r(x) := \begin{cases} 1, & x \in E_k^r, \\ 0, & x \notin E_k^r, \end{cases} \\ &P_1(x,E_k) = 0,\ \ P_2(x,E) = 0,\ \ x \in E. \end{aligned} \tag{7}$$

The associated Markov process $\mathfrak{X}^0(t)$, $t \geq 0$ given by the generator

$$\begin{aligned} &Q\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y) - \varphi(x)], \\ &q(x) := 1/g(x),\ \ g(x) := \int_0^\infty \overline{F}_x(t)dt,\ \ \overline{F}_x(t) := 1 - F_x(t), \end{aligned} \tag{8}$$

is considered uniformly ergodic in every class E_k^r, $1 \leq r \leq N_k$, $1 \leq k \leq N$, with stationary distributions

$$\pi_k^r(dx),\ \ 1 \leq r \leq N_k,\ \ 1 \leq k \leq N. \tag{9}$$

Introduce the merging functions

$$\widehat{V}(x) = V_k^r,\ \ x \in E_k^r,\ \ \widehat{\widehat{V}}(x) = k,\ \ x \in E_k \tag{10}$$

and operators

$$Q_l\varphi(x) = q(x)\int_E P_l(x,dy)\varphi(y),\ \ l = 1,2. \tag{11}$$

2. Phase merging principle

Theorem 4.3 (double merging principle) [1, Section 4.2.4]. *Under the merging conditions MD1–MD3 the following weak convergence take place*

$$\begin{aligned} &\widehat{V}(\mathfrak{X}^\varepsilon(t/\varepsilon)) \Rightarrow \widehat{\mathfrak{X}}(t),\ \varepsilon \to 0, \\ &\widehat{\widehat{V}}(\mathfrak{X}^\varepsilon(t/\varepsilon^2)) \Rightarrow \widehat{\widehat{\mathfrak{X}}}(t),\ \varepsilon \to 0. \end{aligned} \tag{12}$$

The limit Markov processes $\widehat{\mathfrak{X}}(t)$ and $\widehat{\widehat{\mathfrak{X}}}(t)$, on the merged phase spaces $\widehat{E} = \bigcup_{k=1}^N \widehat{E}_k$, $\widehat{E}_k = \{V_k^r,\ 1 \leq r \leq N_k,\ 1 \leq k \leq N\}$ and $\widehat{E} = \{1,2,\ldots,N\}$ are given by the generating matrices $\widehat{Q}_1$ and $\widehat{\widehat{Q}}_2$ correspondedly determined as follows:

$$\widehat{Q}_1\Pi = \Pi Q_1 \Pi,\ \ \widehat{\widehat{Q}}_2\widehat{\Pi} = \widehat{\Pi}\Pi Q_2 \Pi\widehat{\Pi}. \tag{13}$$

The projectors Π and $\widehat{\Pi}$ are defined as follows:

$$\begin{aligned}&\Pi\varphi(x) = \sum_{k=1}^{N}\sum_{r=1}^{N_k} \widehat{\varphi}_k^r \mathbf{1}_k^r(x), \quad \widehat{\varphi}_k^r = \int_{E_k^r} \pi_k^r(dx)\varphi(x)\\ &\widehat{\Pi}\widehat{\varphi}_k = \sum_{r=1}^{N_k} \widehat{\pi}_k^r \widehat{\varphi}_k^r.\end{aligned} \tag{14}$$

Here $\widehat{\pi}_k = (\widehat{\pi}_k^r, 1 \le r \le N_k)$, $1 \le k \le N$, are the stationary distributions of the merged Markov process $\widehat{\mathfrak{X}}(t)$, $t \ge 0$.

The uniformly ergodicity of the double merged Markov process $\widehat{\widehat{\mathfrak{X}}}(t)$, $t \ge 0$, is assumed also with stationary distribution $\widehat{\widehat{\pi}} = (\widehat{\widehat{\pi}}_k, 1 \le k \le N)$ defines the corresponding projector

$$\widehat{\widehat{\Pi}}\widehat{\varphi} = \sum_{k=1}^{N} \widehat{\widehat{\pi}}_k \widehat{\varphi}_k.$$

4. Split and merging scheme

In what follows the following generators of semigroups are used:

$$\mathbb{C}_0(x)\varphi(u) = C_0(u;x)\varphi'(u)$$

$$\mathbb{C}(x)\varphi(u) = C(u;x)\varphi'(u)$$

Here for simplicity the following equality is used:

$$C(u;x)\varphi'(u) := \sum_{i=1}^{d} C_i(u;x)\partial\varphi(u)/\partial U_i$$

4.1. Split & Merging

Theorem 4.7 [1, Section 4.4.1]. *Under the conditions of Section 4.4.1 [1] the weak convergence takes place:*

$$U^\varepsilon(t) \Rightarrow \zeta(t), \;\; \varepsilon \to 0.$$

The limit diffusion process $\zeta(t)$, $t \ge 0$, switched by the Markov process $\widehat{\mathfrak{X}}(t)$, $t \ge 0$, is defined by the generator

$$\widehat{\mathbb{L}}\varphi(u,k) = \widehat{\mathbb{L}}_0\varphi(u,\,\cdot\,) + \widehat{Q}_1\varphi(\,\cdot\,,k),$$

$$\widehat{\mathbb{L}}_0\varphi(u) = \widehat{b}(u;k)\varphi'(u) + \frac{1}{2}\widehat{B}(u;k)\varphi''(u).$$

The vector of drift $\widehat{b}(u;k)$ and the covariance matrix $\widehat{B}(u;k)$ are defined by a solution of singular perturbation problem for the operator

$$\mathbb{L}^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}\mathbb{C}_0(x)P + \mathbb{C}(x)P + Q_1,$$

given by the following formulae:

$$\widehat{\mathbb{L}} = \widehat{\mathbb{C}}_0 + \widehat{\mathbb{C}} + \widehat{Q}_1,$$

$$\widehat{\mathbb{C}}_0 = \Pi\mathbb{C}_0(x)PR_0\mathbb{C}_0(x)\Pi, \;\; \widehat{\mathbb{C}} = \Pi\mathbb{C}(x)\Pi.$$

4.2. Split & Double merging

Theorem 4.9 [1, Section 4.4.2]. *Under the conditions of Section 4.4 in [1] the weak convergence takes place*

$$U^\varepsilon(t/\varepsilon) \Rightarrow \widehat{\zeta}(t), \ \ \varepsilon \to 0.$$

The limit diffusion process $\widehat{\zeta}(t)$, $t \geq 0$, is defined by the generator

$$\widehat{\mathbb{L}}\varphi(u) = \widehat{b}(u)\varphi'(u) + \frac{1}{2}\widehat{B}(u)\varphi''(u).$$

The drift-coefficient $\widehat{b}(u)$ and the covariance matrix $\widehat{B}(u)$ are defined by a solution of singular perturbation problem for the operator

$$\mathbb{L}^\varepsilon = \varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \varepsilon^{-1}\mathbb{C}_0(x)P + \mathbb{C}(x)P,$$

given by the following formulae:

$$\widehat{\widehat{\mathbb{L}}} = \widehat{\widehat{\mathbb{C}}}_0 + \widehat{\widehat{\mathbb{C}}},$$

$$\widehat{\widehat{\mathbb{C}}}_0 = \widehat{\Pi}\widehat{\mathbb{C}}_0\widehat{R}_0\widehat{\mathbb{C}}_0\widehat{\Pi}, \ \ \widehat{\mathbb{C}}_0 = \Pi\mathbb{C}_0(x)\Pi,$$
$$\widehat{\widehat{\mathbb{C}}} = \widehat{\Pi}\widehat{\mathbb{C}}\widehat{\Pi}, \qquad \widehat{\mathbb{C}} = \Pi\mathbb{C}(x)\Pi.$$

4.3. Double split & Merging

Theorem 4.10 [1, Section 4.4.4]. *Under the conditions of Section 4.4 in [1] the weak convergence takes place*

$$U^\varepsilon(t/\varepsilon) \Rightarrow \widehat{\widehat{\zeta}}(t), \ \ \varepsilon \to 0.$$

The limit diffusion process $\widehat{\widehat{\zeta}}(t)$, $t \geq 0$, is defined by the generator

$$\widehat{\widehat{\mathbb{L}}}\varphi(u,k) = \widehat{\widehat{b}}(u;k)\varphi'(u) + \frac{1}{2}\widehat{\widehat{B}}(u;k)\varphi''(u).$$

The drift-coefficient $\widehat{\widehat{b}}(u)$ and the covariance matrix $\widehat{\widehat{B}}(u)$ are defined by a solution of singular perturbation problem for the operator

$$\mathbb{L}^\varepsilon = \varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \varepsilon^{-1}\mathbb{C}_0(x) + \mathbb{C}(x) + Q_2,$$

given by the following formulae:

$$\widehat{\widehat{\mathbb{L}}} = \widehat{\widehat{\mathbb{C}}}_0 + \widehat{\widehat{\mathbb{C}}} + \widehat{\widehat{Q}}_2,$$

$$\widehat{\widehat{\mathbb{C}}}_0 = \widehat{\Pi}\widehat{\mathbb{C}}_0\widehat{R}_0\widehat{\mathbb{C}}_0\widehat{\Pi}, \ \ \widehat{\mathbb{C}}_0 = \Pi\mathbb{C}_0(x)\Pi,$$
$$\widehat{\widehat{\mathbb{C}}} = \widehat{\Pi}\widehat{\mathbb{C}}\widehat{\Pi}, \qquad \widehat{\mathbb{C}} = \Pi\mathbb{C}(x)\Pi.$$

4.4. Double split & Double merging

Theorem 4.11 [1, Section 4.4.5]. *Under the conditions of Section 4.4 in [1] the weak convergence takes place*

$$U^\varepsilon(t/\varepsilon^2) \Rightarrow \widehat{\widehat{\widehat{\zeta}}}(t), \ \ \varepsilon \to 0.$$

The limit diffusion process $\widehat{\widehat{\widehat{\zeta}}}(t)$, $t \geq 0$, is defined by the generator

$$\widehat{\widehat{\widehat{\mathbb{L}}}}\varphi(u) = \widehat{\widehat{\widehat{b}}}(u)\varphi'(u) + \frac{1}{2}\widehat{\widehat{\widehat{B}}}(u)\varphi''(u).$$

The drift-coefficient $\widehat{\widehat{\widehat{b}}}(u)$ and the covariance matrix $\widehat{\widehat{\widehat{B}}}(u)$ are defined by a solution of singular perturbation problem for the operator

$$\mathbb{L}^\varepsilon = \varepsilon^{-4}Q + \varepsilon^{-3}Q_1 + \varepsilon^{-2}Q_2 + \varepsilon^{-1}\mathbb{C}_0(x) + \mathbb{C}(x),$$

given by the formulae:

$$\widehat{\widehat{\widehat{\mathbb{L}}}} = \widehat{\widehat{\widehat{\mathbb{C}}}}_0 + \widehat{\widehat{\widehat{\mathbb{C}}}},$$

$$\widehat{\widehat{\widehat{\mathbb{C}}}}_0 = \widehat{\widehat{\Pi}}\widehat{\widehat{\mathbb{C}}}_0\widehat{\widehat{\Pi}}, \ \ \widehat{\widehat{\mathbb{C}}}_0 = \widehat{\Pi}\widehat{\mathbb{C}}_0\widehat{\Pi},$$

$$\widehat{\widehat{\widehat{\mathbb{C}}}} = \widehat{\widehat{\Pi}}\widehat{\widehat{\mathbb{C}}}\widehat{\widehat{\Pi}}, \qquad \widehat{\widehat{\mathbb{C}}} = \widehat{\Pi}\widehat{\mathbb{C}}\widehat{\Pi}.$$

5. Additional comments

Diffusion approximation algorithms in split and merging phase space are constructed due to specific dependency of the generator of the associated Markov process on the parameter series ε. According to conditions of Section 4.4 in [1] the asymptotic extension of the compensative operator of the random evolution process is used to construct the truncated operators $\mathbb{L}^\varepsilon$ in Theorems 4.7 – 4.11. A solution of singular perturbation problems given in Chapter 5, [1], are used to construct the limit generators. Verification of weak convergences is realized on the scheme represented in Chapter 6, [1].

Bibliography

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