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RESULTS ON FRACTAL MEASURE OF SOME SETS

The fractal dimensions are very important characteristics of fractal sets. A problem which arises in the study of fractal sets is the determination of their dimensions. The Hausdorff dimension is difficult to be determined, even if the box dimensions can be computed. In this article we present some relations between these types of measures and we estimate them for some sets.

1. Introduction

The dimensions calculus is fundamental in the fractals study. The Hausdorff measures, the box and packing dimensions are widely used and in many papers the relations between them are given ([5] - [7], [11]). In [1] - [4] we gave some boundedness conditions for a class of fractal sets, in $\mathbf{R}^n$. This type of conditions is important in order to prove theorems concerning the module and the capacities and the relations between them ([10]) or to determine the dimensions of fractal sets ([8], [9], [12], [13]).

In what follows we shall work with the following basic notions.

Definition 1. *Let $\mathbf{R}^n$ be the Euclidean n-dimensional metric space, E a subset of $\mathbf{R}^n$ and $r_0 > 0$.*
A continuous function $h(r)$, defined on $[0, r_0)$, nondecreasing and such that $\lim\limits_{r\to 0} h(r) = 0$ is called a *measure function.*

If $0 < \beta < \infty$ and h is a measure function, then, the Hausdorff h-measure of E is defined by:

$$H_h(E) = \lim_{\beta\to 0} \inf \left\{ \sum_i h(|U_i|) : E \subseteq \bigcup_i U_i : 0 < |U_i| < \beta \right\}.$$

where $|U_i|$ is the diameter of U_i.

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Remark. There are definitions where the cover of the set E is made with balls. The relation between these "spherical" Hausdorff h - measure, denoted by H'_h and H_h is:

$$H_h(E) \leq H'_h(E). \tag{1}$$

Definition 2. *Let $\varphi_1, \varphi_2 > 0$ be functions defined in $D \subset \mathbf{R}^n$. We say that φ_1 and φ_2 are similar and we denote by: $\varphi_1 \sim \varphi_2$, if there exists $Q > 0$, satisfying:*

$$\frac{1}{Q}\varphi_1(x) \leq \varphi_2(x) \leq Q\varphi_1(x), \forall x \in D.$$

Definition 3. *Let $\delta > 0$ and $f : D \subset \mathbf{R}^n \to \overline{\mathbf{R}}$. f is said to be a δ - class Lipschitz function if there exists $M > 0$ such that:*

$$|f(x+\alpha) - f(x)| \leq M\|\alpha\|^{\delta}, \forall x \in D, \forall \alpha \in \mathbf{R}^n \ with x + \alpha \in D,$$

where for $\alpha \in \mathbf{R}^n$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\|\alpha\| = \sum_{i=1}^n \alpha_i^2$.

f is said to be a Lipschitz function if $\delta = 1$.

Definition 4. *A set $E \subset \mathbf{R}^n$ is called k - rectifiable if there are a Lipschitz function $f : \mathbf{R}^k \to \mathbf{R}^n$ and a bounded subset F of $\mathbf{R}^k$, such that $f(F) = E$.*

If $f : [0,1] \to \mathbf{R}$, we denote by $R_f[t_1, t_2]$ the oscillation of f on $[t_1, t_2] \subset [0,1]$ and by $\Gamma(f)$ the graph of the function f.

2. Results

In ([1] – [3]), the following functions were studied:

$$g(x) = \begin{cases} 2x & , 0 \leq x < \frac{1}{2} \\ -2(x-1), & \frac{1}{2} \leq x < \frac{3}{2} \\ 2(x-2) & , \frac{3}{2} \leq x < 2 \end{cases} \tag{2}$$

$$f(x) = \sum_{i=1}^{\infty} \lambda_i^{s-2} g(\lambda_i x), (\forall) x \in [0,1], \tag{3}$$

where g is given in (2), $s > 0$ and $\{\lambda_i\}_{i \in \mathbf{N}^*}$ is a sequence such that

$$(\exists)\, \varepsilon > 1 : \lambda_{i+1} \geq \varepsilon \lambda_i > 0, (\forall)\ i \in \mathbf{N}^*. \tag{4}$$

Theorem 1. ([2]) *If h is a measure function, $h(t)\tilde{\ }t^p$, $p \geq 2$, $f : [0,1] \to \overline{\mathbf{R}}$ is a δ- class Lipschitz function, $\delta \geq 0$, then $H_h(\Gamma(f)) < +\infty$. The result* *remains true if* $p \geq 1$ *and* $\delta > 1$.

Theorem 2. *If* h *is a measure function,* $h(t)\tilde{\ } t^p$, $p \geq 2$, f *is the function defined in* (3), *with* $s \in [0\,,\,2)$ *and* $\{\lambda_i\}_{i\in\mathbf{N}^*} \in \mathbf{R}_+$ *is a sequence that satisfies* (4), *then* $H_h(\Gamma(f)) < +\infty$.

Proof. For $s \in [1\,,\,2)$, the proof is analogous to that of theorem 2 [1]. It remains to prove the result for $s \in [0\,,\,1)$.

We consider $0 < \alpha < 1$, small enough and $k \in \mathbf{N}^*$ such that:

$$\lambda_{k+1}^{-1} \leq \alpha < \lambda_k^{-1}. \tag{5}$$

Then:

$$|f(x+\alpha) - f(x)| = \left|\sum_{i=1}^{\infty} \lambda_i^{s-2}\left\{g(\lambda_i(x+\alpha)) - g(\lambda_i x)\right\}\right| \leq$$

$$\leq \sum_{i=1}^{k} \lambda_i^{s-2}\left|g(\lambda_i(x+\alpha)) - g(\lambda_i x)\right| + \sum_{i=k+1}^{\infty} \lambda_i^{s-2}\left|g(\lambda_i(x+\alpha)) - g(\lambda_i x)\right|.$$

From the definition of g it results:

$$|g(\lambda_i(x+\alpha)) - g(\lambda_i x)| \leq 2.$$

Thus:

$$|f(x+\alpha) - f(x)| \leq \sum_{i=1}^{k} \lambda_i^{s-2}\left|g(\lambda_i(x+\alpha)) - g(\lambda_i x)\right| + 2\sum_{i=k+1}^{\infty} \lambda_i^{s-2} \Rightarrow$$

$$|f(x+\alpha) - f(x)| \leq 2\alpha \sum_{i=1}^{k} \lambda_i^{s-1} + 2\sum_{i=k+1}^{\infty} \lambda_i^{s-2}. \tag{6}$$

Using (4) we have:

$$\varepsilon^{i-1}\lambda_1 < \lambda_i\,,\; s < 1 \Rightarrow \lambda_i^{s-1} < \lambda_1^{s-1}\left(\varepsilon^{i-1}\right)^{s-1} \Rightarrow$$

$$\sum_{i=1}^{k} \lambda_i^{s-1} < \lambda_1^{s-1}\sum_{i=1}^{k}\left(\varepsilon^{s-1}\right)^{i-1} = \lambda_1^{s-1}\frac{1-\left(\frac{1}{\varepsilon}\right)^{k(1-s)}}{1-\left(\frac{1}{\varepsilon}\right)^{1-s}} \Rightarrow$$

$$\sum_{i=1}^{k} \lambda_i^{s-1} < \lambda_1^{s-1} \cdot \frac{1}{1-\left(\frac{1}{\varepsilon}\right)^{1-s}}. \tag{7}$$

The relations (6) and (7) give:

$$|f(x+\alpha) - f(x)| \leq \alpha\frac{2\lambda_1^{s-1}}{1-\left(\frac{1}{\varepsilon}\right)^{1-s}} + 2\sum_{i=k+1}^{\infty} \lambda_i^{s-2}. \tag{8}$$

$$\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\leq\sum_{j=0}^{\infty}\left(\varepsilon^j\lambda_{k+1}\right)^{s-2}=\lambda_{k+1}^{s-2}\sum_{j=0}^{\infty}\left(\varepsilon^j\right)^{s-2}=\lambda_{k+1}^{s-2}\sum_{j=0}^{\infty}\left(\varepsilon^{s-2}\right)^j$$

Since $s-2<0,\varepsilon>1$, the series $\sum\limits_{j=0}^{\infty}\left(\varepsilon^{s-2}\right)^j$ is convergent and

$$\sum_{i=k+1}^{\infty}\lambda_i^{s-2}<\lambda_{k+1}^{s-2}\cdot\frac{1}{1-\left(\frac{1}{\varepsilon}\right)^{2-s}} \tag{9}$$

From (8) and (9), it results:

$$|f(x+\alpha)-f(x)|\leq\alpha\frac{2\lambda_1^{s-2}}{1-\left(\frac{1}{\varepsilon}\right)^{1-s}}+\frac{2\lambda_{k+1}^{s-2}}{1-\left(\frac{1}{\varepsilon}\right)^{2-s}} \tag{10}$$

The relation (5) implies: $\lambda_{k+1}^{s-2}\leq\alpha^{2-s}<\alpha$ because $\alpha\in[0,\ 1)$. Thus:

$$|f(x+\alpha)-f(x)|\leq\alpha\left(\frac{2\lambda_1^{s-1}}{1-\left(\frac{1}{\varepsilon}\right)^{1-s}}+\frac{2}{1-\left(\frac{1}{\varepsilon}\right)^{2-s}}\right)=\alpha M\Leftrightarrow$$

$$|f(x+\alpha)-f(x)|<\alpha M, \tag{11}$$

where $M=\frac{2\lambda_1^{s-1}}{1-\left(\frac{1}{\varepsilon}\right)^{1-s}}+\frac{2}{1-\left(\frac{1}{\varepsilon}\right)^{2-s}}$.

From (11) it results that f is a Lipschitz function. Since the hypothesis of the theorem 1 is satisfied, then $H_h(\Gamma(f))<+\infty$.

Theorem 3. *Let h be a measure function, such that*

$$h(t)\tilde{}\,P(t)e^{T(t)},t\geq0, \tag{12}$$

where P and T are polynomials:

$$P(t)=a_1t+a_2t^2+...+a_pt^p,\ p\geq1,$$

$$T(t)=b_0+b_1t+...+a_mt^m,$$

with the property

$$P'(t)+P(t)\cdot T(t)>0,\ t\geq0. \tag{13}$$

If $f:[0,1]\to\overline{\mathbf{R}}$ is a δ - class Lipschitz function, $\delta\geq1$, then $H_h(\Gamma(f))<+\infty$.

The result remains true if $p\geq2,a_1=0$ and $\delta\in[0,1]$.

Proof. The condition (13) means that $P(t)e^{T(t)},t\geq0$ is itself a measure function.

The first part of the proof follows that of [5].

We suppose that the Lipschitz constant is $M = 1$.

To any x corresponds an interval $(x - k, x + k)$ such that, for any $x + \alpha$ of this interval:

$$|f(x + \alpha) - f(x)| \leq |\alpha|^{\delta}.$$

Since $[0, 1]$ is a compact set, there exists a finite set of overlapping intervals covering (0, 1):

$$(0, k_0), (x_1 - k_1, x_1 + k_1), ..., (x_{n-1} - k_{n-1}, x_{n-1} + k_{n-1}), (1 - k_n, 1).$$

If c_i are arbitrary points, satisfying:

$$\begin{aligned} c_1 &\in (0, x_1), c_i \in (x_{i-1}, x_i), i = 2, ..., n-1, c_n \in (x_{n-1}, 1) \\ c_i &\in (x_{i-1} - k_{i-1}, x_{i-1} + k_{i-1}) \bigcap (x_i - k_i, x_i + k_i), i = 2, ..., n-1. \end{aligned}$$

we have: $0 < c_1 < x_1 < c_2 < x_2 < ... < x_{n-1} < c_n < 1\cdot$

The oscillation of $f(x)$ in the interval (c_{i-1}, c_i) is less than $2\left(c_i - c_{i-1}\right)^{\delta}$ and thus the part of the curve corresponding to the interval (c_{i-1}, c_i) can be enclosed in a rectangle of height 2 $\left(c_i - c_{i-1}\right)^{\delta}$ and of base $c_i - c_{i-1}$, and consequently in $\left[2\left(c_i - c_{i-1}\right)^{\delta-1}\right] + 1$ squares of side $c_i - c_{i-1}$ or in the number of circles of radius $\frac{c_i - c_{i-1}}{\sqrt{2}}$ circumscribed about each of these squares.

The integer part of x was denoted by $[x]$.

Given an arbitrary $r \in \left(0, \frac{1}{2}\right)$ it can always be assumed that: $c_i - c_{i-1} < r$, $i = 2, 3, ..., n$.

Denote by C_r the set of all the above circles and consider

$$\sum_{C_r} h(2r) = \sum_{C_r} \left\{ \frac{h(2r)}{e^{T(2r)} \cdot P(2r)} \cdot e^{T(2r)} \cdot P(2r) \right\}.$$

From (12) it results that:

$$(\exists)\, Q > 0 : \frac{h(2r)}{e^{T(2r)} \cdot P(2r)} \leq Q.$$

Then,

$$\sum_{C_r} h(2r) \leq Q \sum_{C_r} \left\{ e^{T(2r)} \cdot P(2r) \right\} \Leftrightarrow$$

$$\sum_{C_r} h(2r) \leq Q \sum_{C_r} P(2r) \cdot e^{\sum_{k=0}^{m} b_k \cdot (2r)^k}.$$

$$r \in \left(0, \frac{1}{2}\right) \Rightarrow \sum_{C_r} h(2r) \leq Q \cdot e^{\sum_{k=0}^{m} |b_k|} \cdot \sum_{C_r} P(2r)\cdot$$

We have to estimate $\sum_{C_r} P(2r)$.

The sum of the terms corresponding to the interval $(c_{i-1}, c_i), i = 2, ..., n$ is:

$$S_i = \left\{ \left[2\left(c_i - c_{i-1}\right)^{\delta-1} \right] + 1 \right\} \cdot \sum_{k=1}^{p} a_k \left\{ \left(c_i - c_{i-1}\right) \sqrt{2} \right\}^k,$$

where $[x]$ is the integer part of x.

$$S_i \leq \left\{ 2\left(c_i - c_{i-1}\right)^{\delta-1} + 1 \right\} \cdot \sum_{k=1}^{p} \left\{ a_k \cdot \left(c_i - c_{i-1}\right)^k \cdot 2^{k/2} \right\} \Rightarrow$$

$$S_i \leq 2^{\frac{p}{2}} \cdot \max_{k \in \overline{1,p}} |a_k| \cdot \sum_{k=1}^{p} \left\{ 2\left(c_i - c_{i-1}\right)^{k+\delta-1} + \left(c_i - c_{i-1}\right)^k \right\}$$

$$c_i - c_{i-1} < 1, k + \delta - 1 \geq 1 \Rightarrow \left(c_i - c_{i-1}\right)^{k+\delta-1} \leq c_i - c_{i-1} \Rightarrow$$

$$S_i \leq 3 \cdot 2^{\frac{p}{2}} \cdot p \cdot \max_{k \in \overline{1,p}} |a_k| \left(c_i - c_{i-1}\right) \Rightarrow$$

$$\sum_{C_r} P(2r) \leq 3 \cdot 2^{\frac{p}{2}} \cdot p \cdot \max_{k \in \overline{1,p}} |a_k| \sum_{i=2}^{n} \left(c_i - c_{i-1}\right) \leq 3 \cdot 2^{\frac{p}{2}} \cdot p \cdot \max_{k \in \overline{1,p}} |a_k| \Rightarrow$$

$$\sum_{C_r} h(2r) \leq 3 \cdot 2^{\frac{p}{2}} \cdot p \cdot Q \cdot \max_{k \in \overline{1,p}} |a_k| \cdot e^{\sum_{k=0}^{m} |b_k|}.$$

Then $H_h'(\Gamma(f)) < +\infty \Rightarrow H_h'(\Gamma(f)) < +\infty$.

If $M \neq 1$, then

$$\sum_{C_r} h(2r) \leq 3 \cdot 2^{\frac{p}{2}} \cdot p \cdot Q \cdot M \cdot \max_{k \in \overline{1,p}} |a_k| \cdot e^{\sum_{k=0}^{m} |b_k|} \Rightarrow H_h(\Gamma(f)) < +\infty.$$

If $p \geq 2$ and $\delta > 0$, then $k \geq 2$,$\left(c_i - c_{i-1}\right)^{k+\delta-1} < c_i - c_{i-1}$ and the proof is the same as above.

Theorem 4. *If* $\Gamma(f)$ *is the graph of the function defined in* (3)*,* $s \in [0, 2)$*,* $\{\lambda_i\}_{i \in \mathbf{N}^*} \in \mathbf{R}_+$ *is a sequence of numbers, that satisfies* (4) *and* h *is a measure function satisfying* (12)*, then* $H_h(\Gamma(f)) < +\infty$.

Proof. The proof is analogous to that of the Theorem 3.

The following lemma will be used:

Lemma 1. ([6]) *Let* $f \in C[0, 1]$, $0 < \beta < 1$ *and* m *be the least integer greater than or equal to* $1/\beta$. *If* $N_\beta(\Gamma(f))$ *is the number of the squares of the* $\beta-$ *mesh that intersects* $\Gamma(f)$, *then*

$$\beta^{-1} \sum_{j=0}^{m-1} R_f\left[j\beta, \, (j+1)\beta\right] \leq N_\beta(\Gamma(f)) \leq 2m + \beta^{-1} \sum_{j=0}^{m-1} R_f\left[j\beta, \, (j+1)\beta\right].$$

Let us consider $\delta > 0$ and a δ - class Lipschitz function, $f : [0,1] \to \overline{\mathbf{R}}$. Then:

$$|f(x) - f(y)| \leq M|x-y|^{\delta}, (\forall)x, y \in [0,1].$$

$$R_f\,[j\beta,\,(j+1)\,\beta] = \sup_{j\beta \leq t,\, u \leq (j+1)\beta} |f\,(t) - f\,(u)| \Rightarrow$$

$$R_f\,[j\beta,\,(j+1)\,\beta] \leq M \sup_{j\beta \leq t,\, u \leq (j+1)\beta} |t-u|^{\delta} \Rightarrow$$

$$R_f\,[j\beta,\,(j+1)\,\beta] \leq M\beta^{\delta}.$$

Let $N'_{\beta}(\Gamma(f))$ be the number of β - mesh squares that cover the set $\Gamma(f)$. Denoting by $[x]$, the integer part of $x \in \mathbf{R}$ and using lemma 1, it can be deduced that:

$$N'_{\beta}(\Gamma(f)) \leq 2\left[\frac{1}{\beta}\right] + \beta^{-1}\left[\frac{1}{\beta}\right] M\beta^{\delta} \Rightarrow$$

$$N'_{\beta}(\Gamma(f)) < M\beta^{\delta-2} + \frac{2}{\beta}. \tag{14}$$

But,

$$N_{\beta\sqrt{n}}(\Gamma(f)) \leq N'_{\beta}(\Gamma(f)) \leq 2^{n} N_{\beta}(\Gamma(f)), \tag{15}$$

where $N_{\beta}(\Gamma(f))$ is the smallest number of discs of diameters at most β that cover $\Gamma(f)$.
The relations (14) and (15) give for $n = 2$:

$$N_{\beta}(\Gamma(f)) \leq N'_{\frac{\beta}{\sqrt{2}}}(\Gamma(f)) < M\left(\frac{\beta}{\sqrt{2}}\right)^{\delta-2} + \frac{2\sqrt{2}}{\beta} \Rightarrow$$

$$N_{\beta}(\Gamma(f)) \leq N'_{\frac{\beta}{\sqrt{2}}}(\Gamma(f)) < \frac{3}{\beta} + M'\beta^{\delta-2}, \tag{16}$$

with $M' = \frac{M}{\sqrt{2}^{\delta-2}}$.

Theorem 5. *If $f : [0,1] \to \mathbf{R}$ is a δ-class Lipschitz function, $\delta > 0$ and h is a measure function such that $h(t) \sim t^{p}$, $p > 2$, then $H_h(\Gamma(f)) = 0$. The assertion remains true if $p \geq 1$ and $\delta > 1$.*

Proof. By hypotheses, $\Gamma(f)$ is a compact set. Therefore, if $\beta > 0$, for every cover of $\Gamma(f)$ with open discs $U_i, i \in \mathbf{N}^{*}$, with diameters $d_i \leq \beta$, there is a finite number of discs, n_{β}, that covers $\Gamma(f)$.

$$H'_h(\Gamma(f)) = \lim_{\beta \to 0} \inf\left\{\sum_i h(|U_i|) : E \subseteq \bigcup_i U_i : 0 < |U_i| \leq \beta\right\} =$$

$$= \liminf_{\beta \to 0} \left\{ \sum_{i=1}^{n_\beta} h(|U_i|) \right\} \leq \liminf_{\beta \to 0} \left\{ h(\beta) n_\beta \right\},$$

since h is nondecreasing. Then

$$H'_h(\Gamma(f)) \leq \lim_{\beta \to 0} \left\{ h(\beta) N_\beta(\Gamma(f)) \right\},$$

where $N_\beta(\Gamma(f))$ is the smallest number of open discs of diameters at most β that cover $\Gamma(f)$.

Denoting by $N'_\beta(\Gamma(f))$ the number of β - mesh squares that cover $\Gamma(f)$ and using the relations (1) and (16), the previous inequality becomes:

$$H_h(\Gamma(f)) \leq H'_h(\Gamma(f)) \leq \lim_{\beta \to 0} \left\{ h(\beta) N'_{\frac{\beta}{\sqrt{2}}}(\Gamma(f)) \right\} \Rightarrow$$

$$H_h(\Gamma(f)) \leq \lim_{\beta \to 0} \left\{ h(\beta)(3\beta^{-1} + M'\beta^{\delta-2}) \right\} \Rightarrow$$

$$H_h(\Gamma(f)) \leq \lim_{\beta \to 0} \left\{ \frac{h(\beta)}{\beta^p}(3\beta^{p-1} + M'\beta^{p+\delta-2}) \right\}.$$

Since $h(t) \sim t^p$, $p > 2$, there is $Q > 0$ such that:

$$\frac{1}{Q} t^p \leq h(t) \leq Q t^p,$$

and then

$$H_h(\Gamma(f)) \leq Q \lim_{\beta \to 0} (3\beta^{p-1} + M'\beta^{p+\delta-2}) = 0,$$

because $p - 1 > 0$ and $p + \delta - 2 > 0$.

So, $H_h(\Gamma(f)) = 0$.

If $p \geq 1$ and $\delta > 1$ the proof is the same because $p + \delta - 2 > 0$.

Remark. Theorem 5 gives a better result as the theorem 6 [2], where it was proved in the same hypotheses, that $H'_h(\Gamma(f)) < \infty$.

Lemma 2. ([4]) *If $E \subset \mathbf{R}^m$, $F \subset \mathbf{R}^n$, $f : E \to F$ is a surjective Lipschitz function, with the Lipschitz constant M and h is a measure function, then: $H_h(F) \leq H_h(M \cdot E)$.*

Theorem 6. *If $E \subset \mathbf{R}^n$ is a k-rectifiable set and h is a measure function such that $h(t) \sim t^p$, $p > 2$, then $H_h(E) = 0$.*

Proof. If $E \subset \mathbf{R}^n$ is k - rectifiable, there exists a bounded set $G \subset \mathbf{R}^k$ and $f : \mathbf{R}^k \to \mathbf{R}^n$ such that:

$$\|f(x) - f(y)\| < M\|x - y\|, (\forall) x, y \in \mathbf{R}^k$$

and $f(G) = E$.

The restriction of f at G is a surjection and, by Lemma 2,

$$H_h(f(G)) \leq H_h(M \cdot G) \Leftrightarrow H_h(E) \leq H_h(M \cdot G).$$

$M \cdot G$ is bounded, so there is a disc $B(z_0, r), (r > 0)$ such that $M \cdot G \subset \overline{B(z_0, r)}$. Therefore,

$$H_h(E) \leq H'_h(E) \leq H'_h(B(z_0, r)) =$$

$$= \lim_{\beta \to 0} \inf \left\{ \sum_i h(|U_i|) : B(z_0, r) \subseteq \bigcup_i U_i : 0 < |U_i| \leq \beta \right\} =$$

$$= \lim_{\beta \to 0} \inf \left\{ n_\beta \cdot h(|U_i|) \right\},$$

where n_β is the number of the open discs U_i with the diameters $|U_i| \leq \beta$, that covers $B(z_0, r)$.

$$n_\beta \geq \frac{\pi \cdot r^2}{\frac{\pi \cdot |U_i|^2}{4}} = 4 \cdot \frac{r^2}{|U_i|^2} \Rightarrow$$

$$H_h(E) \leq H'_h(B(z_0, r)) \leq \lim_{\beta \to 0} \inf \left\{ n_\beta \cdot h(\beta) \right\} =$$

$$= \lim_{\beta \to 0} h(\beta) \inf n_\beta = \lim_{\beta \to 0} h(\beta) \cdot \frac{4r^2}{\beta^2} = 4r^2 \lim_{\beta \to 0} \frac{h(\beta)}{\beta^2} =$$

$$= 4r^2 \lim_{\beta \to 0} \left\{ \frac{h(\beta)}{\beta^p} \cdot \beta^{p-2} \right\} \leq 4Qr^2 \lim_{\beta \to 0} \beta^{p-2} = 0,$$

where it was used that $h(t) \sim t^p, p > 2$. So, $H_h(E) = 0$.

Remark. If in Theorem 6, $p = 2$, it results that $H_h(E) \leq 4Qr^2$, so the Hausdorff h-measure of the k - rectifiable set E is finite.

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