Theory of Stochastic Processes
Vol. 12 (28), no. 3–4, 2006, pp. 187–202

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NECESSARY AND SUFFICIENT CONDITIONS FOR WEAK CONVERGENCE OF FIRST-RARE-EVENT TIMES FOR SEMI-MARKOV PROCESSES. II

Necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes, obtained in the first part of this paper [66], are applied to counting processes generating by flows of rare events controlled by semi-Markov processes, random geometric sums, and risk processes. In particular, necessary and sufficient conditions for stable approximation of ruin probabilities including the case of diffusion approximation are given.

4. Introduction

In the first part of the present paper [66], necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes. In the second part of the paper, we apply these results to counting processes generating by flows of rare events controlled by semi-Markov processes, random geometric sums, and risk processes. In particular, we give necessary and sufficient conditions for stable approximation of ruin probabilities including the case of diffusion approximation are given.

We use all notations and conditions introduced in the first part of the present paper as well as continue numbering of sections, theorems and lemmas began there.

First-rare-event times for semi-Markov processes considered in the first part of the present paper reduce to random geometric sums in the case of degenerated imbedded Markov chain.

In this way, our results are connected with asymptotical results for geometric sums. Here, we would like first to mention originating papers by Rényi (1956), who first formulated conditions of convergence of geometrical sums to exponential law, and works by Kovalenko (1965), Gnedenko and Fraier (1969), Szynal (1976), Korolev (1989), Kalashnikov and Vsekhsvyatski (1989), Melamed (1989), Kruglov and Korolev (1990), Kartashov (1991), Gnedenko and Korolev (1996), Kalashnikov (1997), Bening and Korolev (2002).

It should be noted that the class of possible limiting laws for standard geometric random sums, with summands independent on geometric random

2000 *Mathematics Subject Classification.* Primary 60K15, 60F17, 60K20.
Key words and phrases. Weak convergence, semi-Markow processes, first-rare-event time, limit theorems, necessary and sufficient conditions.

indices and possessing regularly varying tail probabilities, was described by Kovalenko (1965) who also credited for necessary and sufficient conditions for weak convergence of such sums. These results were generalized by Kruglov and Korolev (1990), in particular, to the case of triangular array mode. Our results yield necessary and sufficient conditions for weak convergence summands have regularly varying tail probabilities but for more general geometric sums with summands that can depend on geometric random indices via indicators of rare events. Also, our conditions have different and, as we think, more convenient form than in the works mentioned above.

We give also necessary and sufficient conditions for weak convergence of counting processes generating by the corresponding flows of rare events to standard renewal flows. These results are connected with the results related to studies of convergence for rareficated stochastic flows given in Rényi (1956), Belyaev (1963), Kovalenko (1965), Mogyoródi (1971, 1972a, 1972b), Szantai (1971a, 1971b), Råde (1972a, 1972b), Zakusilo (1972a, 1972b), Jagers (1974), Jagers and Lindvall (1974), Tomko (1974), Kallenberg (1975) and Lindvall (1976), Serfozo (1976, 1984a, 1984b), Gasanenko (1980), and Böker and Serfozo (1983).

As possible areas of applications, we can point out limit theorems for different lifetime functionals such as occupation times or waiting times in queuing systems, lifetime in reliability models, extinction times in population dynamic models, ruin times for insurance models, etc. We mention here works by Gnedenko (1964a, 1964b), Gnedenko and Kovalenko (1964, 1987), Silvestrov (1969), Masol (1973), Kovalenko (1975, 1977, 1980, 1994), Solov'ev (1971), 1983), Ivchenko, Kashtanov, and Kovalenko (1979), Kovalenko and Kuznetsov (1988), Kovalenko, Kuznetsov, Pegg (1997), Englund (1999a, 1999b), Anisimov, Zakusilo, and Donchenko (1987), Assmussen (1987, 2003), Kalashnikov and Rachev (1990), Gyllenberg and Silvestrov (1994).

We apply our results to asymptotical analysis of non-ruin probabilities for risk processes. We study asymptotics for non-ruin probabilities in the critical case when the initial capital of an insurance company tends to infinity and simultaneously the safety loading coefficient tends to 1. Two cases are considered, when the claim distribution belongs to the domain of attraction of a degenerated or a stable law. The first case corresponds to the classical model of diffusion approximation, the second one can be referred as the case of stable approximation for non-ruin probabilities.

As was mention above, we give necessary and sufficient conditions for stable approximation for non-ruin probabilities, including the case of diffusion approximation.

Our results are connected with the results related to various sufficient conditions for diffusion approximation given in Iglehart (1969), Siegmund (1975), Harrison (1977), Grandell (1977, 1991), Gerber (1979), Asmussen

(1987, 1989, 2000, 2003), Glynn (1990), Schmidli (1992, 1997), Gyllenberg and Silvestrov (1999, 2000b), and Silvestrov (2000b).

It is not out of picture to note that asymptotical relations related to diffusion approximation for ruin probabilities can be also interpreted in terms of the queue theory as variants of so-called heavy traffic approximation for waiting times (see, for instance, Asmussen (1987)).

The results presented in the paper are also connected with results related to asymptotics of ruin probabilities for risk processes with heavy tailed claim distributions given by von Bahr (1975), Thorin and Wikstad (1976), Embrechts, Goldie and Veraverbeke (1979), Embrechts and Veraverbeke (1982), Assmussen (1996, 2000, 2003), Embrehts (1997), Klüppelberg and Stadmular (1998), Zinchenko (1999). It should be noted, however, that the latter group of works relates to so-called sub-critical case when the safety loading coefficient less than and separated of 1.

The paper is organized in the following way. In Section 2 we give necessary and sufficient of weak convergence of counting processes generating by the corresponding flows of rare events. As was mentioned above, the model of first-rare-event times for semi-Markov processes reduces to the model of geometrical random sums in the case of degenerated imbedded Markov chain. In Section 3, we present applications of results obtained in the first part of the present paper to random geometric sums as well as give necessary and sufficient conditions for stable approximation for non-ruin probabilities.

5. Flows of rare events

In this section we study conditions of convergence for the flows of rare events connected with the model considered above.

Let us define recursively random variables

$$\nu_\varepsilon(k) = \min(n \geq \nu_\varepsilon(k-1) : \zeta_n \in D_\varepsilon), \ k = 1, 2, \dots,$$

where $\nu_\varepsilon(0) = 0$. A random variable $\nu_\varepsilon(k)$ counts the number of transitions of the imbedded Markov chain η_n up to the k-th appearance of the "rare" event $\{\zeta_n \in D_\varepsilon\}$. Obviously, $\nu_\varepsilon(1) = \nu_\varepsilon$.

Let us also define inter-rare-event times,

$$\varkappa_\varepsilon(k) = \sum_{n=\nu_\varepsilon(k-1)}^{\nu_\varepsilon(k)} \varkappa_n, \ k = 1, 2, \dots.$$

Let us also introduce random variables showing positions of the imbedded Markov chain η_n at moments $\nu_\varepsilon(k)$,

$$\eta_\varepsilon(k) = \eta_{\nu_\varepsilon(k)}, \ k = 0, 1, \dots.$$

Obviously $(\eta_\varepsilon(k), \varkappa_\varepsilon(k)), k = 0, 1, \dots$ (here $\varkappa_\varepsilon(k) = 0$) is a Markov renewal process, i.e. a homogeneous Markov chain with the phase space

$X \times [0, \infty)$ and transition probabilities,

$$(1)\quad \begin{aligned} &\mathsf{P}\{\eta_\varepsilon(k+1) = j, \varkappa_\varepsilon(k+1) \le t / \eta_\varepsilon(k) = i, \varkappa_\varepsilon(k) = s\} \\ &\quad = \mathsf{P}\{\eta_\varepsilon(k+1) = j, \varkappa_\varepsilon(k+1) \le t / \eta_\varepsilon(k) = i\} \\ &\quad = Q_{ij}^{(\varepsilon)}(t) = \mathsf{P}_i\{\eta_{\nu_\varepsilon} = j, \xi_\varepsilon \le t\},\ i, j \in X,\ s, t \ge 0. \end{aligned}$$

Let us now define random variables,

$$\xi_\varepsilon(k) = \sum_{n=1}^{\nu_\varepsilon(k)} \varkappa_n = \sum_{n=1}^{k} \varkappa_\varepsilon(n),\ k = 0, 1, \ldots .$$

Random variable $\xi_\varepsilon(k)$ can be interpreted as the time of k-th appearance of the rare event time for the semi-Markov process $\eta(t)$. Obviously, $\xi_\varepsilon(0) = 0$ and $\xi_\varepsilon(1) = \xi_\varepsilon$.

Now we can define a counting stochastic process that describes the flow of rare events,

$$N_\varepsilon(t) = \max(k \ge 0 : \xi_\varepsilon(k) \le t u_\varepsilon),\ t \ge 0.$$

Note that the time scale for this counting process is stretched with the use of the scale parameter u_ε according the asymptotic results given in Theorem 1.

Let us also define the corresponding limiting counting process. Let $\varkappa(k)$, $k = 1, 2, \ldots$ be a sequence of positive i.i.d. random variables with the distribution $F_{a,\gamma}(u)$, and

$$\xi(k) = \sum_{n=1}^{k} \varkappa(n), \qquad n = 0, 1, \ldots .$$

Let us also define the standard renewal counting process with i.i.d. inter-renewal times $\varkappa(k), k = 1, 2, \ldots$,

$$N(t) = \max(k \ge 0 : \xi(k) \le t),\ t \ge 0.$$

Theorem 2. *Let conditions* **A**, **B**, **C**, *and* **D** *hold. Then, the class of all possible non-zero limiting counting processes* (*in the sense of weak convergence of finite-dimensional distributions*) *for the counting processes* $N_\varepsilon(t), t \ge 0$ *coincides with the class of standard renewal counting process* $N(t), t \ge 0$ *with the distribution function of inter-renewal time* $F_{a,\gamma}(u)$ *where* $0 < \gamma \le 1$, $a > 0$. *Conditions* $\mathbf{E}_\gamma$ *and* $\mathbf{F}_{a,\gamma}$ *are necessary and sufficient for such convergence in the case when the corresponding limiting counting process has the distribution function of inter-renewal time* $F_{a,\gamma}(u)$.

Proof. Obviously,

$$F_i^{(\varepsilon)}(u) = \mathsf{P}_i\{\xi_\varepsilon \le u\} = \sum_{j \in X} Q_{ij}^{(\varepsilon)}(t),\ u \ge 0.$$

Using Markov property of the Markov renewal process $(\eta_\varepsilon(k), \varkappa_\varepsilon(k))$ we get the following formula for joint distributions of the properly normalized inter-renewal times for the counting process $N_\varepsilon(t)$,

$$\begin{aligned}
&\mathsf{P}_i\{\varkappa_\varepsilon(k)/u_\varepsilon \le t_k, k = 1, \dots, n\} \\
&\quad = \sum_{j \in X} \mathsf{P}_i\{\varkappa_\varepsilon(k)/u_\varepsilon \le t_k, k = 1, \dots, n-1, \eta_\varepsilon(n-1) = j\} \\
&\quad \times F_j^{(\varepsilon)}(t_n u_\varepsilon),\ i \in X, t_1, \dots, t_n \ge 0,\ n = 1, 2, \dots.
\end{aligned} \tag{2}$$

According Theorem 1, under **A**, **B**, **C**, and **D**, conditions $\mathbf{E}_\gamma$ and $\mathbf{F}_{a,\gamma}$ imply that (**a**) $F_j^{(\varepsilon)}(t_n u_\varepsilon) \to F_{a,\gamma}(t_n)$ as $\varepsilon \to 0$, $t_n \ge 0, j \in X$.

Using (**a**) and relation (2) we get that, under **A**, **B**, **C**, and **D**, conditions $\mathbf{E}_\gamma$ and $\mathbf{F}_{a,\gamma}$ imply that, for every $i \in X, n = 1, 2, \dots, t_1, \dots, t_n \ge 0$,

$$\begin{aligned}
&\mathsf{P}_i\{\varkappa_\varepsilon(k)/u_\varepsilon \le t_k, k = 1, \dots, n\} \\
&\quad \to \prod_{k=1}^{n} F_{a,\gamma}(t_k) \text{ as } \varepsilon \to 0.
\end{aligned} \tag{3}$$

Relation (3) means that the inter-renewal times $\varkappa_\varepsilon(k)/u_\varepsilon, k = 1, 2, \dots$ are asymptotically independent. Note that the multivariate distribution function on the right hand side in (3) is continuous. Due to this fact, relation (3) implies in an obvious way that, for every $i \in X$, real $0 = t_0 \le t_1 \le \cdots \le t_n$, integer $0 \le r_1 \le \cdots \le r_n$, $n = 1, 2, \dots$,

$$\begin{aligned}
&\mathsf{P}_i\{N_\varepsilon(t_k) \ge r_k, k = 1, \dots, n\} \\
&\quad = \mathsf{P}_i\{\xi_\varepsilon(r_k)/u_\varepsilon \le t_k, k = 1, \dots, n\} \\
&\quad \to \mathsf{P}_i\{\xi(r_k) \le t_k, k = 1, \dots, n\} \\
&\quad = \mathsf{P}_i\{N(t_k) \ge r_k, k = 1, \dots, n\} \text{ as } \varepsilon \to 0,
\end{aligned} \tag{4}$$

The statement of necessity follows is trivial and follows from the following formula

$$\mathsf{P}_i\{N_\varepsilon(t) \ge 1\} = \mathsf{P}_i\{\xi_\varepsilon/u_\varepsilon \le t\}, \qquad t \ge 0.$$

The proof is complete. □

What is interesting, that under **A**, **B**, **C**, and **D**, conditions $\mathbf{E}_\gamma$ and $\mathbf{F}_{a,\gamma}$ are not sufficient for weak convergence of transition probabilities of the Markov renewal process $(\eta_\varepsilon(k), \varkappa_\varepsilon(k))$ that forms the counting process $N_\varepsilon(t)$.

It follows from the following lemma which describes the asymptotic behavior of so-called “absorbing” probabilities,

$$Q_{ij}^{(\varepsilon)}(\infty) = \mathsf{P}_i\{\eta_{\nu_\varepsilon} = j\},\ i, j \in X.$$

Let us denote

$$p_{i\varepsilon}(r) = \mathsf{P}_i\{\zeta_1 \in D_\varepsilon, \eta_1 = r\},\ i, r \in X,$$

and

$$p_\varepsilon(r) = \sum_{i=1}^{m} \pi_i p_{i\varepsilon}(r), \ j \in X.$$

By the definition,

$$p_\varepsilon = \sum_{i \in X} \pi_i \mathsf{P}_i\{\zeta_1 \in D_\varepsilon\} = \sum_{r \in X} p_\varepsilon(r). \tag{5}$$

Lemma 7. *Let conditions* **B**, **C** *hold. Then, for every* $i \in X$,

$$Q_{ir}^{(\varepsilon)}(\infty) - \frac{p_\varepsilon(r)}{p_\varepsilon} \to 0 \ \textit{as} \ \varepsilon \to 0, \ r \in X. \tag{6}$$

Proof. Let us define the probability that the first rare event will occur when the state of the imbedded Markov chain will be r and before the first hitting of the imbedded Markov chain in the state i, under condition that the initial state of this Markov chain $\eta_0 = j$,

$$q_{ji\varepsilon}(r) = \mathsf{P}_j\{\nu_\varepsilon \le \tau_i, \eta_{\nu_\varepsilon} = r\}, \ i, j, r \in X.$$

Taking into account that the Markov renewal process $(\eta_n, \varkappa_n, \zeta_n)$ regenerates at moments of return to every state i and ν_ε is a Markov moment for this process, we can get following cyclic representation for absorbing probabilities $Q_{ir}^{(\varepsilon)}(\infty)$,

$$\begin{aligned} Q_{ir}^{(\varepsilon)}(\infty) &= \sum_{n=0}^{\infty} \mathsf{P}_i\{\tau_i(n) < \nu_\varepsilon \le \tau_i(n+1), \eta_{\nu_\varepsilon} = j\} \\ &= \sum_{n=0}^{\infty} (1 - q_{i\varepsilon})^n q_{ii\varepsilon}(r) = \frac{q_{ii\varepsilon}(r)}{q_{i\varepsilon}}, \ i, r \in X. \end{aligned} \tag{7}$$

The probabilities $q_{ji\varepsilon}(r), j \in X$ satisfy, for every $i, r \in X$, the following system of linear equations similar with system (20) given in the proof of Lemma 1,

$$\begin{cases} q_{ji\varepsilon}(r) = p_{j\varepsilon}(r) + \sum_{k \ne i} p_{jk}^{(\varepsilon)} q_{ki\varepsilon}(r) \\ j \in X \end{cases} \tag{8}$$

This system has the matrix of coefficients ${}_i\mathbf{P}^{(\varepsilon)}$ as the system of linear equations (20) mentioned above and differs of this system only by the free terms. Thus by repeating reasoning given in the proof of Lemma 1 we can get the following formula similar with formula (24) also given in the proof of Lemma 1,

$$q_{ii\varepsilon}(r) = \sum_{k=1}^{m} \mathsf{E}_i \delta_{ik\varepsilon} \, p_{k\varepsilon}(r). \tag{9}$$

Recall that it was shown in the proof of Lemma 1 that (**b**) $\mathsf{E}_i \delta_{ik\varepsilon} \to \pi_k / \pi_i$ as $\varepsilon \to 0$, for $i, k \in X$. Using (**b**), relation (**c**) $\pi_i q_{i\varepsilon} / p_\varepsilon \to 1$ given in

Lemma 1, and inequality (**d**) $p_\varepsilon(r) \leq p_\varepsilon$, following from formula (5), we get, for every $i, r \in X$,

$$\begin{aligned} \frac{\left| q_{ii\varepsilon}(r) - \frac{p_\varepsilon(r)}{\pi_i} \right|}{q_{i\varepsilon}} &\leq \sum_{k=1}^{m} \left| \mathsf{E}_i \delta_{ik\varepsilon} - \frac{\pi_k}{\pi_i} \right| \cdot \frac{\pi_i p_{k\varepsilon}(r)}{\sum_{j=1}^{m} \pi_j p_{j\varepsilon}} \cdot \frac{p_\varepsilon}{\pi_i q_{i\varepsilon}} \\ &\leq \sum_{k=1}^{m} \left| \mathsf{E}_i \delta_{ik\varepsilon} - \frac{\pi_k}{\pi_i} \right| \cdot \frac{\pi_i}{\pi_k} \cdot \frac{p_\varepsilon}{\pi_i q_{i\varepsilon}} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{10}$$

Using (**c**) and (**d**) ones more time we get,

$$\begin{aligned} \left| \frac{p_\varepsilon(r)}{\pi_i q_{i\varepsilon}} - \frac{p_\varepsilon(r)}{p_\varepsilon} \right| &\leq \frac{p_\varepsilon(r)}{p_\varepsilon} \cdot \frac{\left| q_{i\varepsilon} - \frac{p_\varepsilon}{\pi_i} \right|}{\pi_i q_{i\varepsilon}} \\ &\leq \frac{\left| q_{i\varepsilon} - \frac{p_\varepsilon}{\pi_i} \right|}{p_\varepsilon} \cdot \frac{p_\varepsilon}{\pi_i q_{i\varepsilon}} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{11}$$

Formula (7) together with relations (10) and (11) imply in an obvious way relation (6). This completes the proof. □

Let us introduce the following balancing condition:

L:: $\frac{p_\varepsilon(j)}{p_\varepsilon} \to Q_j$ as $\varepsilon \to 0, \ j \in X$.

Constants Q_j, automatically satisfy the following conditions ($\mathbf{e}_1$) $Q_j \geq 0, j \in X$, and ($\mathbf{e}_2$) $\sum_{j \in X} Q_j = 1$.

Lemma 7 implies the following statement.

Lemma 8. *Let conditions* **B**, **C** *hold. Then, condition* **L** *is necessary and sufficient for the following relation to hold* (*for some or every* $i \in X$, *respectively, in the statements of necessity and sufficiency*),

$$Q_{ir}^{(\varepsilon)}(\infty) \to Q_r \text{ as } \varepsilon \to 0, \ r \in X. \tag{12}$$

The following theorem shows that the-first-rare-event times ξ_ε and random functional η_{ν_ε} are asymptotically independent, and completes the description of the asymptotic behavior of the transition probabilities $Q_{ij}^{(\varepsilon)}(t)$ for the Markov renewal process $(\eta_\varepsilon(k), \varkappa_\varepsilon(k))$.

Theorem 3. *Let conditions* **A**, **B**, **C**, *and* **D** *hold. Then, conditions* $\mathbf{E}_\gamma$, $\mathbf{F}_{a,\gamma}$ *and* **L** *are necessary and sufficient for the asymptotic relation* (2) *given in Theorem* 1 *and the asymptotic relation* (12) *given in Lemma* 8 *to hold. In this case, for every* $u \in [0, \infty], i, r \in X$,

$$Q_{ir}^{(\varepsilon)}(uu_\varepsilon) \to F_{a,\gamma}(u) Q_r \textit{ as } \varepsilon \to 0. \tag{13}$$

Proof. The first statement of the theorem follows from Theorem 1 and Lemma 8. Let us prove that conditions $\mathbf{E}_\gamma$, $\mathbf{F}_{a,\gamma}$ and $\mathbf{L}$ imply the asymptotic relation (13).

Let us introduce, for $i, r \in X$, Laplace transforms,

$$\Phi_{ir\varepsilon}(s) = \mathsf{E}_i \exp\{-s\xi_\varepsilon\}\chi(\eta_{\nu_\varepsilon} = r), \ s \geq 0,$$

and

$$\widetilde{\psi}_{ir\varepsilon}(s) = \mathsf{E}_i\{\exp\{-s\tilde{\beta}_{i\varepsilon}\}\chi(\eta_{\nu_\varepsilon} = r)/\nu_\varepsilon \leq \tau_i\}, \ s \geq 0.$$

Analogously to formula (4) given in the proof of Theorem 1 the following representation can be written down for Laplace transforms $\Phi_{ir\varepsilon}(s)$,

$$\begin{aligned} \Phi_{ir\varepsilon}(s) &= \sum_{n=0}^{\infty}(1-q_{i\varepsilon})^n q_{i\varepsilon}\psi_{i\varepsilon}(s)^n\widetilde{\psi}_{ir\varepsilon}(s) \\ &= \frac{q_{i\varepsilon}\widetilde{\psi}_{ir\varepsilon}(s)}{1-(1-q_{i\varepsilon})\psi_{i\varepsilon}(s)} \\ &= \frac{1}{1+(1-q_{i\varepsilon})\frac{(1-\psi_{i\varepsilon}(s))}{q_{i\varepsilon}}}\cdot\widetilde{\psi}_{ir\varepsilon}(s), \ s \geq 0. \end{aligned} \tag{14}$$

Let us now show that under, conditions **A**, **B**, and **C**, for every $s \geq 0$ and $i, r \in X$,

$$\widetilde{\psi}_{ir\varepsilon}(s/u_\varepsilon) - q_{ii\varepsilon}(r)/q_{i\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{15}$$

Indeed, using Lemma 2 we get, for any $\delta > 0$,

$$\begin{aligned} &\frac{q_{ii\varepsilon}(r)}{q_{i\varepsilon}} - \widetilde{\psi}_{ir\varepsilon}(\frac{s}{u_\varepsilon}) \\ &= \mathsf{E}_i\{(1-\exp\{-s\frac{\tilde{\beta}_{i\varepsilon}}{u_\varepsilon}\})\chi(\eta_{\nu_\varepsilon} = r)/\nu_\varepsilon \leq \tau_i\} \\ &\leq (1-e^{s\delta}) + \mathsf{P}_i\{\frac{\tilde{\beta}_{i\varepsilon}}{u_\varepsilon} \geq \delta/\nu_\varepsilon \leq \tau_i\} \to (1-e^{s\delta}) \text{ as } \varepsilon \to 0. \end{aligned} \tag{16}$$

Relation (16) implies, due to possibility of an arbitrary choice of $\delta > 0$, relation (15).

Using formula (14) and relation (29), given in Lemma 2, and (15), Theorem 1 and Lemma 8 we get, for every $s \geq 0$ and $i, r \in X$,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\Phi_{ir\varepsilon}(s/u_\varepsilon) \\ &= \lim_{\varepsilon\to 0}\frac{1}{1+(1-q_{i\varepsilon})\frac{(1-\psi_{i\varepsilon}(s))}{q_{i\varepsilon}}}\cdot\widetilde{\psi}_{ir\varepsilon}(s/u_\varepsilon) \\ &= \lim_{\varepsilon\to 0}\Phi_{i\varepsilon}(s/u_\varepsilon)\cdot Q_{ir}^{(\varepsilon)}(\infty) \\ &= \frac{1}{1+as^\gamma}\cdot Q_r. \end{aligned} \tag{17}$$

Relation (17) equivalent to relation (13). □

6. Geometric sums and stable approximation for non-ruin probabilities

In this section we apply our results to so-called geometric random sums. This is reduction of our model for the case when the imbedded Markov chain η_n has the degenerated set of states $X = \{1\}$.

In this case, the first-rare-event time $\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n$ is a geometric sum.

Indeed, $(\varkappa_n, \zeta_n), n = 1, 2, \dots$ is a sequence of i.i.d. random vectors. Therefore the random variable

$$\nu_\varepsilon = \min(n \geq 1 : \zeta_n \in D_\varepsilon)$$

has a geometric distribution with the success probability

$$p_\varepsilon = \mathsf{P}\{\zeta_n \in D_\varepsilon\}.$$

However, the geometric random index ν_ε and random summands $\varkappa_n, n = 1, 2, \dots$ are, in this case, dependent random variables. They depend via the indicators of rare events $\chi_{n\varepsilon} = \chi(\zeta_n \in D_\varepsilon), n = 1, 2, \dots$. More precisely, $(\varkappa_n, \chi_{n\varepsilon}, n = 1, 2, \dots)$ is a sequence of i.i.d. random vectors.

Conditions **A** and **B** take, in this case, the following form:

A′:: $\lim_{t\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\varkappa_1 > t / \zeta_1 \in D_\varepsilon\} = 0$;

and

B′:: $0 < p_\varepsilon = \mathsf{P}\{\zeta_1 \in D_\varepsilon\} \to 0$ as $\varepsilon \to 0$.

Condition **C** holds automatically.

Condition **D** remains and should be imposed on the function p_ε^{-1}, defined in condition **B′**, and the normalization function u_ε.

Conditions $\mathbf{E}_\gamma$ and $\mathbf{F}_{a,\gamma}$ remain and should be imposed on the distribution function $G(t) = \mathsf{P}\{\varkappa_1 \leq t\}$ (no averaging is involved).

A standard geometric sum is particular case of the model described above, which corresponds to the case, when two sequences of random variables $\varkappa_n, n = 1, 2, \dots$ and $\zeta_n, n = 1, 2, \dots$ are independent. In this case, the random index ν_ε and summands $\varkappa_n, n = 1, 2, \dots$ are also independent.

Note that a standard geometric sum with any distribution of summands $G(t)$ and parameter of geometric random index $p_\varepsilon \in (0, 1]$ can be modelled in this way. Indeed, it is enough to consider the geometric sum

$$\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n$$

defined above, where $(\mathbf{a}_1)$ $\varkappa_n, n = 1, 2, \dots$ is a sequence of i.i.d. random variables with the distribution function $G(t)$;

$(\mathbf{a}_2)$ $$\nu_\varepsilon = \max(n \geq 1 \colon \zeta_n \in D_\varepsilon),$$

where $\zeta_n, n = 1, 2, \dots$ is a sequence of i.i.d. random variables uniformly distributed in the interval $[0, 1]$ and domains $D_\varepsilon = [0, p_\varepsilon)$; $(\mathbf{a}_3)$ two sequences of random variables $\varkappa_n, n = 1, 2, \dots$ and $\zeta_n, n = 1, 2, \dots$ are independent.

In the case of standard geometric sums, condition **A** holds automatically. Theorem 1 reduces in this case to the result equivalent to those obtained by Kovalenko (1965). The difference is in the form of necessary and sufficient conditions. Conditions $\mathbf{G}_\gamma$ and $\mathbf{H}_{a,\gamma}$, based on Laplace transforms of distributions $G(t)$, were used in this work. Our results are based on conditions $\mathbf{E}_\gamma$ and $\mathbf{F}_{a,\gamma}$, based on distributions $G(t)$, and have, as we think, a more transparent form.

Let us illustrate applications of Theorem 1 by giving necessary and sufficient conditions for stable approximation of non-ruin probabilities. Let us consider a process used in classical risk theory to model the business of an insurance company,

$$X_\varepsilon(t) = c_\varepsilon t - \sum_{n=1}^{N_\lambda(t)} Z_n,\ t \geq 0.$$

Here, a positive constant c_ε (depending on parameter $\varepsilon > 0$) is the gross premium rate, $N_\lambda(t)$, $t \geq 0$ is a Poissonian process with parameter λ counting the number of claims on an insurance company in the time-interval $[0, t]$, and $Z_n, n = 1, 2, \dots$ is a sequence of nonnegative i.i.d. random variables, which are independent on the process $N_\lambda(t), t \geq 0$. The random variable Z_k is the amount of the k^{th} claim.

An important object for studies in this model is the non-ruin probabilities on infinite time interval for a company with an initial capital $u \geq 0$,

$$F_\varepsilon(u) = \mathsf{P}\left\{u + \inf_{t \geq 0} X(t) \geq 0\right\}, \qquad u \geq 0.$$

Let $H(x) = \mathsf{P}\{Z_1 \leq x\}$ be a claim distribution function. We assume the standard condition:

M:: $\mu = \int_0^\infty sH(ds) < \infty$.

The crucial role in is plaid by the so-called safety loading coefficient

$$\alpha_\varepsilon = \lambda\mu / c_\varepsilon.$$

If $\alpha_\varepsilon \geq 1$ then $F_\varepsilon(u) = 0, u \geq 0$. The only non-trivial case is when $\alpha_\varepsilon < 1$. We assume the following condition:

N:: $\alpha_\varepsilon < 1$ for $\varepsilon > 0$ and $\alpha_\varepsilon \to 1$ as $\varepsilon \to 0$.

According to Pollaczek–Khinchine formula (see, for example, Asmussen (2000)), the non-ruin distribution function $F_\varepsilon(u)$ coincides with distribution function of a geometric random sum which is slightly differ on the standard geometric sums considered above. Namely,

$$F_\varepsilon(u) = \mathsf{P}\{\xi'_\varepsilon = \sum_{n=1}^{\nu_\varepsilon - 1} \varkappa_n \leq u\},\ u \geq 0, \tag{18}$$

where (**a**) $\varkappa_n, n = 1, 2, \ldots$ is a sequence of non-negative i.i.d. random variables with distribution function

$$G(u) = \frac{1}{\mu}\int_0^u (1 - H(s))ds, \qquad u \geq 0$$

(so-called steady claim distribution);

$$\textbf{(b)} \qquad \nu_\varepsilon = \min(n \geq 1, \qquad \chi_{n\varepsilon} = 1);$$

and (**c**) $\chi_{n\varepsilon}$, $n = 1, 2, \ldots$, is a sequence of i.i.d. random variables taking values 1 and 0 with probabilities $p_\varepsilon = 1 - \alpha_\varepsilon$ and $1 - p_\varepsilon$; (**d**) random sequences $\varkappa_n, n = 1, 2, \ldots$ and $\chi_{n\varepsilon}, n = 1, 2, \ldots$ are independent.

As was mentioned above condition **A** holds automatically in this case, and, therefore, due to Remark 12, Theorem 1, which specification to the geometric sums was described above, can be applied to the geometric random sums ξ'_ε.

Conditions **A** and **C** can be omitted. Condition **B** is equivalent to condition **N**. Condition **D** takes the following form:

$\mathbf{D'}$:: $u_\varepsilon, p_\varepsilon^{-1} = (1 - \alpha_\varepsilon)^{-1} \in \mathbb{W}$.

Conditions $\mathbf{E}_\gamma$ and $\mathbf{F}_{a,\gamma}$ ($a > 0$ and $0 < \gamma \leq 1$) take, in this case, the following form:

$\mathbf{E}'_\gamma$:: $\frac{t\int_t^\infty (1-H(s))ds}{\int_0^t s(1-H(s))ds} \to \frac{1-\gamma}{\gamma}$ as $t \to \infty$.

$\mathbf{F}'_{a,\gamma}$:: $\frac{\int_0^{u_\varepsilon} s(1-H(s))ds}{(1-\alpha_\varepsilon)\mu u_\varepsilon} \to a\frac{\gamma}{\Gamma(2-\gamma)}$ as $\varepsilon \to 0$.

Let us summarize the discussion above in the form of the following theorem which gives necessary and sufficient conditions for stable approximation of non-ruin probabilities.

Theorem 4. *Let conditions* **M**, **N**, *and* $\mathbf{D'}$ *hold. Then the class of all possible non-concentrated in zero limiting* (*in the sense of weak convergence*) *distribution functions* $F(u)$, *such that the non-ruin distribution functions* $F_\varepsilon(uu_\varepsilon) \Rightarrow F(u)$ *as* $\varepsilon \to 0$, *coincides with the class of distributions* $F_{a,\gamma}(u)$ *with Laplace transforms* $\frac{1}{1+as^\gamma}$, $0 < \gamma \leq 1$, $a > 0$. *Conditions* $\mathbf{E}'_\gamma$ *and* $\mathbf{F}'_{a,\gamma}$ *are necessary and sufficient for weak convergence* $F_\varepsilon(uu_\varepsilon) \Rightarrow F_{a,\gamma}(u)$ *as* $\varepsilon \to 0$.

In conclusion let us give and comment some sufficient conditions providing diffusion and stable approximations for ruin probabilities.

The case $\gamma = 1$ corresponds to so called diffusion approximations of risk processes. The traditional way for obtaining a diffusion type asymptotic is based on approximation of risk process in a proper way by a Wiener process with a shift. Typical conditions assume finiteness of the second moment of the claim distribution $H(t)$ (or equivalently, finiteness of expectation for the steady claim distribution $G(t) = \frac{1}{\mu}\int_0^t (1 - H(s))ds$):

O:: $\mu_2 = \int_0^\infty z^2 H(ds) < \infty$.

Obviously, condition **O** is sufficient for condition $\mathbf{E}'_1$ to hold. Note however that the necessary and sufficient condition $\mathbf{E}'_1$ does not require the finiteness of the second moment of the claim distribution.

In this case, condition $\mathbf{F}'_{a,1}$ takes the following simple equivalent form:

$\mathbf{P}_a$:: $(1 - \alpha_\varepsilon)u_\varepsilon \to b = \mu_2/2\mu a$ as $\varepsilon \to 0$.

As was mentioned in Remark 1, the corresponding limiting distribution is in this case the exponential one, with the parameter a. This is consistent with the classical form of diffusion approximation for ruin probabilities.

Case $\gamma \in (0, 1)$ corresponds to so-called stable approximation for ruin probabilities. Remark 1 explains the sense of using the term “stable”.

Remind that, according Lemma 6, condition $\mathbf{E}'_\gamma$ is equivalent to the following condition which require regular variation for the tail probabilities of the steady claim distribution $G(t) = \frac{1}{\mu}\int_0^t (1 - H(s))ds$:

$\mathbf{R}_\gamma$:: $\frac{1}{\mu}\int_t^\infty (1 - H(s))ds \sim \frac{t^{-\gamma}L(t)}{\Gamma(1-\gamma)}$ as $t \to \infty$.

As follows from theorems about regularly varying functions (see, for example, Feller (1966)), the following condition imposing requirement of regular variation for the tail probabilities of the claim distribution $H(t)$ is sufficient for condition $\mathbf{R}_\gamma$ to hold:

$\mathbf{R}'_\gamma$:: $1 - H(t) \sim \frac{t^{-(\gamma+1)}L(t)\gamma\mu}{\Gamma(1-\gamma)}$ as $t \to \infty$.

As far as condition $\mathbf{F}'_{a,\gamma}$ is concerned, it, in this case, can be formulated in the following form equivalent to condition $\mathbf{H}_{a,\gamma}$:

$\mathbf{S}_{a,\gamma}$:: $\frac{L(u_\varepsilon)}{(1-\alpha_\varepsilon)u_\varepsilon^\gamma} \to a$ as $\varepsilon \to 0$.

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