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SPECTRAL ANALYSIS OF SOME CLASSES OF MULTIVARIATE RANDOM FIELDS WITH ISOTROPIC PROPERTY

In this paper we consider two classes of generalized random fields of second order on $\mathbb{R}^n$ with values in Hilbert space H with isotropic property: first - exponentially convex and isotropic random fields and second - homogeneous and isotropic random fields. The spectral representations for such fields and their covariance are obtained.

1. Introduction

It is well-known that spectral theory of random functions of second-order is strictly related with theory of positive definite kernel and functions (see, for instance [1], [2]).

The spectral theory of one-dimensional homogeneous and isotropic complex-valued random fields on n-dimensional vector space $\mathbb{R}^n$ was studied in the works of M.I. Yadrenko and represented at his monograph [2]. In the foundation of this theory lies the classical theorem of I. Shoenberg about integral representation of continuous positive definite radial functions on $\mathbb{R}^n$ [3].

In this paper we obtain the operator analogue of Shoenberg theorem about integral representation of operator-valued weak continuous positive definite radial functions in Hilbert space H and spectral representation for strong continuous generalized random field of second order in H defined on $\mathbb{R}^n$ with homogeneous and isotropic properties.

The classes of one-dimensional exponentially convex random functions (processes and fields) were considered in the papers [4]-[6]. The covariance functions such random functions depend on sum of their arguments. Multivariate exponentially convex random functions with values in Hilbert space H were considered in the paper [7], [8]. The analogue of Shoenberg theorem about integral representation of positive definite exponentially convex radial complex-valued continuous functions was obtained by A. Nussbaum [9].

In this paper we consider operator version of A. Nussbaum theorem about integral representation operator-valued exponentially convex radial function in Hilbert space and use this result for studying of new class of random fields

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with isotropic property - the class of exponentially convex and isotropic random fields in Hilbert space H. The spectral representations of such random fields are obtained.

2. Spectral representations of multivariate homogeneous and isotropic random fields

Let $L_2(\Omega)$ be Hilbert space of all complex-valued random variable of second order, which defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and H be a complex Hilbert space. The set $L(H, L_2(\Omega))$ of all linear continuous mappings of the space H into $L_2(\Omega)$ may be considered as the set of generalized random elements of second order in H which defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Such generalized random elements may be realized as usual random elements in some extension of the space H (see [10]).

Denote by $B(H)$ Banach algebra of all linear bounded operators in H and denote by $B_+(H)$ the convex cone of all nonnegative Hermitian operators in $B(H)$.

The expectation $m = \mathbb{E}\Xi \in H$ and covariance operator $[\Xi, \Psi] \in B(H)$ for generalized random elements $\Xi, \Psi \in L(H, L_2(\Omega))$ are uniquely determinated by the relations

$$(u|m) = \mathbb{E}(\Xi u), ([\Xi, \Psi]u|\nu) = \mathbb{E}(\Xi u)\overline{(\Psi \nu)}, u, \nu \in H,$$

where $(\cdot|\cdot)$ is an inner product in H. Note that $[\Xi, \Psi]$ is sesquilinear form on $L(H, L_2(\Omega))$ and $[\Xi, \Xi] \in B_+(H)$.

If $dimH = d < \infty$ and $\{e_j\}_{j=1}^d$ is orthonormal basis in H, then random element $\Xi \in L(H, L_2(\Omega))$ may be identified with random vector $\xi = \{\xi_j\}_{j=1}^d$, with $\xi_j = \Xi e_j$ and $\mathbb{E}\Xi$ - with vector $\mathbb{E}\xi = \{\mathbb{E}\xi_j\}_{j=1}^d$ and covariance operator $[\Xi, \Psi], \Psi \in L(H, L_2(\Omega))$ - with matrix $\{\mathbb{E}\xi_j\varphi_k\}_{j,k=1}^d$, where $\varphi_k = \Psi e_k, k = 1, .., d$. The family of generalized random elements $\{\Xi x, x \in \mathbb{R}^n\}, \Xi x \in L(H, L_2(\Omega))$ is called a (generalized) random field of second order in H. The field Ξ_x is continuous if it is continuous as function $\widetilde{\Xi} : \mathbb{R}^n \to L(H, L_2(\Omega))$ with respect to strong operator topology of the space $L(H, L_2(\Omega))$. Then its mean $m_x = \mathbb{E}\Xi_x, x \in \mathbb{R}^n$ and covariance kernel

$$Q(x, y) = [\Xi_x, \Xi_y], x, y \in \mathbb{R}^n$$

are continuous respectively in norm of space H and in weak topology of $B(H)$.

Note that Q is positive definite operator kernel on $\mathbb{R}^n$, i.e. for all integers $m \in \mathbb{N}$, vectors $x^k \in \mathbb{R}^n, u^k \in H, k = 1, \dots, m$

$$\sum_{k=1}^{m}\sum_{j=1}^{m}(Q(x^k, x^j)u^k|u^j) \geq 0$$

Let us suppose that $m_x \equiv 0$. Denote by $SO(n)$ the group of all rotations of $\mathbb{R}^n$ around zero point.

Definition 2.1. *Random feld $\Xi_x, x \in \mathbb{R}^n$ in H is called an isotropic field if its covariance kernel Q is invariant with respect to all rotations $g \in SO(n)$:*

$$Q(gx, gy) = Q(x, y), x, y \in \mathbb{R}^n.$$

Random field $\Xi_x, x \in \mathbb{R}^n$ in H is called homogeneous if its covariance kernel is invariant with respect to all shifts in $\mathbb{R}^n$: for all $z, x, y \in \mathbb{R}^n$

$$Q(x + z, y + z) = Q(x, y).$$

From this definition it follows that:

1) the covariance kernel $Q(x, y)$ of homogeneous random field Ξ_x depends only on difference of its arguments $x - y$;
2) the covariance kernel $Q(x, y)$ of homogeneous and isotropic random field depends only on distance $|x - y|$ of its arguments where $|\cdot|$ is Eucledian norm in $\mathbb{R}^n$, i.e. it exists such $B(H)$-valued function K on $[0, \infty)$
$$Q(x, y) = K(|x - y|), x, y \in \mathbb{R}^n.$$
Such function K is called a covariance function of field Ξ_x. The last fact is consequence of invariance of Q with respect to all hard motions of space $\mathbb{R}^n$.

Theorem 2.1. *If $\Xi_x, x \in \mathbb{R}^n$ is continuous homogeneous and isotropic random field in Hilbert space H, then its covariance function $K : \mathbb{R}_+ \to B(H)$ admits the following spectral representation*

$$K(r) = \int_0^\infty Y_n(\lambda r) G(d\lambda), r \geq 0, \tag{1}$$

where $Y_n(t)$ is spherical Bessel function

$$Y_n(t) = (\frac{2}{t})^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) J_{\frac{n-2}{2}}(t) \tag{2}$$

($J_m(t)$ denotes there the Bessel function of first kind of order m and Γ denotes gamma function), G is operator $B_+(H)$-valued measure on σ-algebra of Borel sets $\mathcal{B}(\mathbb{R}_+)$ on semiline $\mathbb{R}_+$ and $G(\mathbb{R}_+) = K(0)$.

Proof. Because the field Ξ_x is homogeneous its covariance function

$$Q(x - y, 0) = [\Xi_x, \Xi_y]$$

is Fourier transformation of uniquely defined operator $B_+(H)$-valued Radon measure F on σ-algebra of Borel sets $\mathcal{B}(\mathbb{R}^n)$:

$$Q(x - y, 0) = \int_{\mathbb{R}^n} exp\{i(z|x - y)\} F(dz), x, y \in \mathbb{R}^n, \tag{3}$$

where $(\cdot|\cdot)$ is inner product in $\mathbb{R}^n$ and i is imaginary unit (see, for example [7], [10]).

Isotropic property of the field Ξ_x is equivalent then to invariance of the spectral measure F of field Ξ_x with respect to all rotation $g \in SO(n)$:

$F(\Delta) = F(g\Delta), \Delta \in \mathcal{B}(\mathbb{R}^n)$. This fact is cosequence of uniqueness of measure F and exchange of arguments in integral (3).

The passage to spherical coordinates in representation (3) gives through invariantness of F the following equality

$$F(dz) = \frac{1}{W_n(|z|)} dM_n(|z|)G(d|z|), \tag{4}$$

where $W_n(|z|)$ is area of sphere $S_n(|z|)$ of radius $|z|$ in $\mathbb{R}^n$, $dM_n(|z|)$ is element of area for surface of this sphere and $G(d|z|)$ is value of measure F of spherical layer $\{x \in \mathbb{R}^n : |z| < |x| < |z| + d|z|\}$.

Then representation (1) follows from the equality

$$\frac{1}{W_n(\lambda)} \int_{S_n(\lambda)} exp\{i(z|x-y)\}M_n(dz) = Y_n(\lambda|x-y|), \lambda > 0. \tag{5}$$

Remark 2.1. *Operator $B(H)$-valued weakly continuous positive definite function $V(x)$ on $\mathbb{R}^n$ is radial, i.e. depends only on norm of its arguments, $V(x) = K(|x|), x \in \mathbb{R}^n, K : \mathbb{R}_+ \to B(H)$, if and only if it has integral representation (1).*

This consequence is operator analogue of Shoenberg theorem [3] about integral representation of complex-valued positive definite radial function on $\mathbb{R}^n$ and follows from the fact that class of covariance functions of homogeneous random fields coinsides with class of $B(H)$-valued positive definite functions on $\mathbb{R}^n$ (see [10]).

Remark 2.2. *The spectral measure G of homogeneous and isotropic random field $\Xi_x \in \mathbb{R}^n$ in H is uniquely reproduced through covariance function $K(r), r \geq 0$ of Ξ_x by the formula*

$$G([0,\lambda]) = \int_0^\infty J_{\frac{n}{2}}(\lambda r)(\lambda r)^{\frac{n}{2}}[\frac{K(r)}{r2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})}]dr, \lambda \geq 0 \tag{6}$$

The equality (6) may be obtained by the same way as in one-dimensional case (see [2]).

Remark 2.3 *In the case $n = 2$ the representation (1) has the following simple form*

$$K(r) = \int_0^\infty J_0(\lambda r)G(d\lambda)$$

and in the case $n = 3$ the form

$$K(r) = 2\int_0^\infty \frac{sin\lambda r}{\lambda r}G(d\lambda).$$

Theorem 2.2. *Continuous homogeneous and isotropic random field* $\Xi_x \in \mathbb{R}^n$ *in space* H *admits the spectral representation*

$$\Xi_x = c_n \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(\theta_1, \dots, \theta_{n-2}, \varphi) \int_0^{\infty} (\lambda r)^{\frac{2-n}{2}} J_{m+\frac{n-2}{2}}(\lambda r) \Phi_m^l(d\lambda), \tag{7}$$

where $(r, \theta_1, \dots, \theta_{n-2}, \varphi)$ *are spherical coordinates of vector* x *in* $\mathbb{R}^n$,

$$S_m^l(\theta_1, \dots, \theta_{n-2}, \varphi)$$

are orthogonal spherical harmonics of degree m *and*

$$h(m,n) = \frac{(2m+n-2)(m+n-3)!}{(n-1)!\, m!}$$

is a number of such harmonics, Φ_m^l *is a sequence of random* $L(H, L_2(\Omega))$*-valued measures on* $\mathcal{B}(\mathbb{R}_+)$ *such that* $\mathbb{E}\Phi_m^l(\Delta) = 0$ *and*

$$[\Phi_m^l(\Delta_1), \Phi_p^q(\Delta_2)] = \delta_{mp}\delta_{lq} G(\Delta_1 \cap \Delta_2), \tag{8}$$

δ_{mq} *is a Croneker symbol and* $c_n = 2^{n-1}\Gamma(\frac{n}{2})\pi^{\frac{n}{2}}$.

Proof. Using the formula (1) for $[\Xi_x, \Xi_y] = K(|x-y|)$ and the formula of summation for spherical harmonics [11] we have that $K(|x-y|)$ may by represented through spherical coordinates of vectors x and y,

$$x = (r, \theta_1, \dots, \theta_{n-2}, \varphi), \qquad y = (r', \theta'_1, \dots, \theta'_{n-2}, \varphi')$$

by the equality

$$K(|x-y|) = c_n^2 \int_0^{\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(\theta_1, \dots, \theta_{n-2}, \varphi) S_m^l(\theta'_1, \dots, \theta'_{n-2}, \varphi') \times$$

$$[(\lambda r)^{\frac{2-n}{2}} J_{m+\frac{n-2}{2}}(\lambda r)][(\lambda r')^{\frac{2-n}{2}} J_{m+\frac{n-2}{2}}(\lambda r')] G(d\lambda). \tag{9}$$

Now the spectral representation (7) follows from (9) by application of theorem 3 [12] about integral representation of generalized random functions in locally convex spaces.

Remark 2.4. *In the case of plane,* $n = 2$ *with polar system of coordinate* (r, φ) *we have that planar continuous homogeneous and isotropic random field admits the spectral representation*

$$\Xi_{(r,\varphi)} = \sum_{m=-\infty}^{\infty} exp\{im\varphi\} \int_0^{\infty} J_m(\lambda r) \Phi_m(d\lambda). \tag{10}$$

The representation (2) follows from (9) because $c_2 = \sqrt{2\pi}, h(m,2) = 2$, and for given m spherical harmonics have the form $(2\pi)^{\frac{-1}{2}} exp\{im\varphi\}$ and $(2\pi)^{\frac{-1}{2}} exp\{-im\varphi\}$.

3. Spectral representation of exponentially convex and isotropic random fields

Definition 3.1 *Random field $\Xi_x \in \mathbb{R}^n$ in space H is said to be an exponentially convex field (or additively stationary or symmetric field in other terminology) if its covariance kernel $Q(x,y), x, y \in \mathbb{R}^n$ depends only on sum of arguments $x+y$.*

Continuous exponentially convex random field Ξ_x in H and its covariance function $K(x+y) = Q(x,y)$ admit the following spectral representations

$$(11)\quad \Xi_x = \int_{\mathbb{R}^n} exp\{(\lambda|x)\}\Phi(d\lambda),\ K(z) = \int_{\mathbb{R}^n} exp\{(\lambda|z)\}F(d\lambda), x, z \in \mathbb{R}^n$$

where Φ is $L(H, L_2(\Omega))$-valued orthogonal random measure on $\mathcal{B}(\mathbb{R}^n)$, F is finite $B_+(H)$-valued Radon measure on $\mathcal{B}(\mathbb{R}^n)$ and

$$[\Phi(\Delta_1), \Phi(\Delta_2)] = F(\Delta_1 \cap \Delta_2), \Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}^n)$$

(see [7], [8]).

Its easy to see that if exponentially convex random field $\Xi_x, x \in \mathbb{R}^n$ in H is also isotropic then its covariance function $K(z)$ depends only on $|z|, K(z) = \overline{K}(|z|)$ and its spectral measure F in (11) is invariant with respect to all rotations $g \in SO(n)$.

Theorem 3.1. *If $\Xi_x, x \in \mathbb{R}^n$ is continuous exponentially convex and isotropic random field in Hilbert space H, then its covariance function $K(z)$, $z \in \mathbb{R}^n$ admits the following spectral representation*

$$(12)\qquad K(z) = \int_0^\infty Y_n(i\lambda|z|)G(d\lambda), z \in \mathbb{R}^n$$

where Y_n is spherical Bessel function (2) and G is uniquely defined operator $B_+(H)$-valued measure on $\mathcal{B}(\mathbb{R}_+)$.

In the case of plane, $n = 2$

$$(13)\qquad K(z) = \int_0^\infty Y_0(i\lambda|z|)G(d\lambda),$$

and in the case $n = 3$

$$(14)\qquad K(z) = 2\int_0^\infty \frac{sini\lambda|z|}{i\lambda|z|}G(d\lambda).$$

Proof. One way of proof of this theorem is to use method analogous to the proof of theorem 2.1. We represent the other way.

Let us consider complex-valued quadratic form $k_u(z) = (K(z)u|u), u \in H$ on space H which generated by function $K : \mathbb{R}^n \to B(H)$. The function $k_u(z), z \in \mathbb{R}^n$ under each fixed vector $u \in H$ is continuous complex-valued positive definite exponentially convex radial function on $\mathbb{R}^n$ (which depend only on $|z|$) and by Nussbaum theorem [9] it admits the following representation

$$k_u(z) = \int_0^\infty Y_n(i\lambda|z|)\mu_u(d\lambda)$$

μ_u is uniquely defined positive finite measure on $\mathcal{B}(\mathbb{R}_+)$ for every $u \in H$.

By polarization formula for space H quadratic form $k_u(z), u \in H$ uniquely defined sesquilinear form $k_{u,\nu}(z) = (K(z)u|\nu), u, \nu \in H$ for which

$$k_{u,\nu}(z) = \int_0^\infty Y_n(i\lambda|z|)\mu_{u,\nu}(d\lambda), \tag{15}$$

where $\mu_{u,\nu}(d\lambda)$ as function of $(u, \nu) \in H \times H$ is sesquilinear form for which

$$\mu_{u,\nu}(\Delta) = \frac{1}{4}\{\mu_{u+\nu}(\Delta) - \mu_{u-\nu}(\Delta) + i\mu_{u+i\nu}(\Delta) - i\mu_{u-i\nu}(\Delta)\}$$

for $\Delta \in \mathcal{B}(\mathbb{R}_+)$. The form $\mu_{u,\nu}(\Delta)$ is bounded because

$$|\mu_{u,\nu}(\Delta)| \le |(K(0)u|\nu)| \le \|K(0)\|\|u\|\|\nu\|.$$

Then it exists uniquely defined $B_+(H)$-valued measure G on $\mathcal{B}(\mathbb{R}_+)$ such that

$$\mu_{u,\nu}(\Delta) = (G(\Delta)u|\nu), u, \nu \in H, \Delta \in \mathcal{B}(\mathbb{R}_+)$$

Then (12) follows from (15).

The equalities (13) and (14) are consequence of (12).

Remark 3.1 *Weakly continuous operator function $K : \mathbb{R}^n \to B(H)$ is positive definite exponentially convex and radial function if and only if it admits the integral representation (12).*

This consequence is operator version of foregoing Nussbaum theorem about integral description of complex-valued exponentially convex and radial functions. The result follows from the fact that class of such operator functions is identical to the class of covariance function of exponentially convex and isotropic random fields in space H.

Theorem 3.2 *Under the assumption of theorem 3.1 random field $\Xi_x, x \in \mathbb{R}^n$ admits the spectral representation*

(16)
$$\Xi_x = c_n \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(\theta_1, \dots, \theta_{n-2}, \varphi) \int_0^\infty (i\lambda r)^{\frac{2-n}{2}} J_{m+\frac{n-2}{2}}(i\lambda r)\Phi_m^l(d\lambda),$$

which are related to spectral representation (7) of homogeneous and isotropic random field with the same notations (in particular, $(r, \theta_1, \dots, \theta_{n-2}, \varphi)$ is spherical coordinates of vector x in $\mathbb{R}^n$).

In the case of plane, $n = 2$, with polar coordinates (r, φ) spectral representation (16) have the form

$$\Xi_{(r,\varphi)} = \sum_{m=-\infty}^{\infty} exp\{im\varphi\} \int_0^\infty J_m(i\lambda r)\Phi_m(d\lambda). \tag{17}$$

The proof of representation (16) is analogous to the proof of representation (7) with using instead of spectral expansion (1) the spectral expansion (12). The formula (17) is consequence of representation (16) because in the case of $n = 2$ we have that $c_2 = \sqrt{2\pi}, h(m, 2) = 2$ and for each m corresponding spherical harmonics have the form $(2\pi)^{\frac{-1}{2}} e^{im\varphi}$ and $(2\pi)^{\frac{-1}{2}} e^{-im\varphi}$.

Let $\Xi_{t,x}$ be a continuous exponentially convex random field in H on $(-\infty, \infty) \times \mathbb{R}^n$ where t is time variable and $x \in \mathbb{R}^n$ is space variable and field $\Xi_{t,x}$ is isotropic with respect to space variable x. We may obtain the following result by using arguments similar to proofs of theorem 2.1, 2.2, 3.1, 3.2.

Theorem 3.3 *The covariance function of random field $\Xi_{t,x}$ admits the spectral representation*

$$[\Xi_{t,x}, \Xi_{s,y}] = \int_{-\infty}^{+\infty} \int_0^{+\infty} exp\{\mu(t+s)\} Y_n(i\lambda|x-y|)F(d\mu d\lambda) \tag{18}$$

where F is uniquely defined $B_+(H)$-valued measure on $\mathbb{R}^{n+1} = (-\infty, \infty) \times \mathbb{R}^n$. Random field $\Xi_{t,x}$ admits the following spectral representation

$$\Xi_{t,x} = c_n \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(\theta_1, \ldots, \theta_{n-2}, \varphi) \times$$

$$\int_{-\infty}^{+\infty} \int_0^\infty e^{t\mu} (i\lambda r)^{\frac{2-n}{2}} J_{m+\frac{n-2}{2}}(i\lambda r)\Phi_m^l(d\mu d\lambda), \tag{19}$$

where $(r, \theta_1, \ldots, \theta_{n-2}, \varphi)$ is spherical coordinates of vector x and Φ_m^l is sequence of random $L(H, L_2(\Omega))$- valued measures with structural measure F, i.e.the equality (8) take place with $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_+)$.

References

1. Yaglom, A.M., *Second-order homogeneous random fields.* Proc. of IV Berkely Symp. on Math. Statist. and Probab. **2**, (1961), p.593–622.
2. Yadrenko M.I., *Spectral theory of random fields.* Optimization software, ING, Publication division, New York, (1983).
3. Shoenberg I.J., *Metric space and completely monotone functions.*, J. Anal. Math., **39**, (1938), p.811–841.
4. Loève M., *Fonctions aléatoires a charactére exponentiel.*, In P. Levy, processus stochastiques et mouvement Brownien, 2nd td. Gauthier-Villars, Paris, (1965), p.409–420.
5. Ehm W., Genton M.G., Gneiting T., *Stationary covariance assosiated with exponentially convex functions.*, Bernoulli, 9, (2003), p.607–615.

6. Michálek J., *Spectral decomposition of locally stationary random processes.*, Kybernetika, **22**, (1986), p.244–255.
7. Ponomarenko O.I., Perun Yu.D., *Multivariate weak stationary random functions on semigroup.*, Theor. Probab. and Math. Stat., **73**, (2005), p.136-147(in Ukrainian).
8. Ponomarenko O.I., Perun Yu.D., *Multivariate additively stationary random functions on convex structures.* Theory. Probab. and Math. Stat., **74**, (2006), p.118-130 (in Ukrainian).
9. Nussbaum A. E., *Radial exponentially convex functions*, J. Anal. Math., **25**, (1972), p.277–288.
10. Ponomarenko O.I., *To spectral theory of infinite-dimensional homogeneous in wide sense random fields and groups.* Vysnik of Kiev Univer.,sec. math. and mech., **11**, (1969), p.144–121.
11. Beteman H., Erdeyi A., *Higher transcendental functions.* vol 2, Mc-Grow-Hill Book Company, New York, (1953).
12. Ponomarenko O.I., *Integral representation of random functions with values in locally convex spaces.* Theory. Probab. and Math. Stat., **46**, (1992), p.132–141.

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