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SOME PROPERTIES OF WEIGHT FUNCTIONS IN TAUBERIAN THEOREMS. I

The rate of convergence for weight functions series in Tauberian theorems is obtained. Numerical results demonstrate required rate of convergence. Some asymptotic properties of hypergeometric functions are obtained as auxiliary results.

1. Introduction

In numerous problems in actuarial and financial mathematics asymptotic behavior of random processes, fields, and limit theorems for some their functionals are of great importance. Abelian and Tauberian theorems find numerous applications in obtaining various asymptotic properties of random processes and fields. The majority results of such type (see, for example, [1]-[4]) describe relations between asymptotic behavior of spectral and correlation characteristics at the infinity and in zero. Such relations do not always exist for long memory random processes and fields. A new approach based on the idea of studying relations between behavior of a spectral function in zero and some functional of random field at the infinity was proposed in [5, 6]. In comparison with classical Tauberian theorems this functional is used as an equivalent of a correlational function. Representation of the functional was given in terms of the variance of spherical averages of random field. It also can be calculated as some integral of the correlation function.

In [7], [8] similar investigations were continued. New results on relations between local behavior of spectral functions in arbitrary point (not necessarily zero) and asymptotics of some functionals of random fields were obtained. Representations of these functionals were derived in terms of the variance of weighted averages of random fields. Various properties of weight functions in such representations were discussed in [7]-[10]. This paper continues the investigations.

Let $R^n, n \geq 2$ be an $n-$dimentional Euclidean space, $\xi(t)$, $t \in R^n$ be a real measurable mean-square continuous homogeneous isotropic random field (see, for example, [11]) with zero mean and the correlation function

$$B_n(r) = B_n(|t|) = E\xi(0)\xi(t), \quad t \in R^n.$$

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It is known that there exists a bounded nondecreasing function $\Phi(x)$, $x \geq 0$, which is called spectral function of the field $\xi(t)$, $t \in R^n$ (see [11]), such that $B_n(r)$ has the representation

$$B_n(r) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\int_0^\infty \frac{J_{\frac{n-2}{2}}(rx)}{(rx)^{\frac{n-2}{2}}}d\Phi(x),$$

where $J_\nu(z)$ is a Bessel function of the first kind, $\nu > -\frac{1}{2}$ (see [12]).

Let

$$\tilde{b}^a(r) := (2\pi)^n \int_0^\infty \frac{J^2_{\frac{n}{2}}(rx)}{(rx)^n}d\Phi^a(x),$$

where for arbitrary $a \in [0, +\infty)$

$$\Phi^a(\lambda) := \begin{cases} \Phi(a+\lambda) - \Phi(a-\lambda), & 0 \leq \lambda < a; \\ \Phi(a+\lambda), & \lambda \geq a. \end{cases}$$

In [8], it was shown the existence of a real-valued function $f_{r,a}(|t|)$, for which

$$\tilde{b}^a(r) = D\left[\int_{\mathbf{R}^n} f_{r,a}(|t|)\xi(t)dt\right].$$

The function can be defined by the formula

$$f_{r,a}(|t|) = \frac{1}{|t|^{\frac{n}{2}-1}}\int_0^\infty \underbrace{\lambda^{n/2}\frac{J_{\frac{n}{2}}(r(\lambda-a))}{(r(\lambda-a))^{n/2}}J_{\frac{n}{2}-1}(|t|\lambda)}_{F(\lambda)}\,d\lambda, \quad |t| \neq r. \tag{1}$$

2. Problem to investigate

In the following C denotes a constant, exact value of which is not important and which may be different in different places.

By asymptotic properties of the Bessel function (see §7.21, [12])

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right), \quad z \to \infty \tag{2}$$

it follows that the integrand $F(\lambda)$ in the definition (1) of $f_{r,a}(|t|)$ has the asymptotic behavior

$$F(\lambda) \sim \frac{C}{\lambda}\cos\left(r(\lambda-a) - \frac{\pi n}{4} - \frac{\pi}{4}\right)\cos\left(|t|\lambda - \frac{\pi}{2}\left(\frac{n}{2}-1\right) - \frac{\pi}{4}\right) \sim$$

$$\frac{C}{\lambda}\left(\cos\left((r+|t|)\lambda - ra - \frac{\pi n}{2}\right) - \sin\left((r-|t|)\lambda - ra\right)\right), \quad \lambda \to \infty.$$

Hence, the integral in the representation of the function $f_{r,a}(|t|)$ converges conditionally.

Therefore there arise various problems of studying properties of the function $f_{r,a}(|t|)$ and calculating its values with a given accuracy.

To calculate the integral (1) well known standard approach implies an application of the Poisson formula (§3.3, [12])

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^{1} (1-x^2)^{\nu-1/2} \cos(zx)dx.$$

Then the function $f_{r,a}(|t|)$ can be rewritten as

$$f_{r,a}(|t|) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})\sqrt{\pi}\,|t|^{\frac{n}{2}-1}} \int_0^\infty \lambda^{\frac{n}{2}} J_{\frac{n}{2}-1}(|t|\lambda) \int_{-1}^{1} (1-x^2)^{\frac{n-1}{2}} \cos(r(\lambda-a)x)dxd\lambda.$$

Changing the order of integration we obtain

$$f_{r,a}(|t|) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\,|t|^{\frac{n}{2}-1}} \left(\int_{-1}^{1} (1-x^2)^{\frac{n-1}{2}} \cos(rax) \int_0^\infty \lambda^{\frac{n}{2}} J_{\frac{n}{2}-1}(|t|\lambda) \cos(r\lambda x)d\lambda dx + \right.$$

$$\left. + \int_{-1}^{1} (1-x^2)^{\frac{n-1}{2}} \sin(rax) \int_0^\infty \lambda^{\frac{n}{2}} J_{\frac{n}{2}-1}(|t|\lambda) \sin(r\lambda x)d\lambda dx \right).$$

Next step is to calculate the inner integrals.

Unfortunately it is not allowed to change the order in our case. By the asymptotic formula (2) the inner integrals with respect to λ, $n \in N$ do not converge. That is why we must use another methods to study properties of the function $f_{r,a}(|t|)$ in (1).

In [8], it was proposed an efficient approach based on the representation of the function $f_{r,a}(|t|)$ as the series:

$$f_{r,a}(|t|) = \begin{cases} \left(\frac{2}{ar^3}\right)^{\frac{n}{2}} \sum_{m=0}^{\infty} d_m(n,r,a,|t|), & |t| < r, \\ \left(\frac{2}{ar|t|^2}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \sum_{m=0}^{\infty} s_m(n,r,a,|t|), & |t| > r, \end{cases} \tag{3}$$

where

$$d_m(n,r,a,|t|) = \frac{\left(\frac{n}{2}+m\right) C_{n+m-1}^{m} \Gamma(\frac{n+m}{2}) J_{\frac{n}{2}+m}(ra)\, {}_2F_1\left(\frac{n+m}{2}, -\frac{m}{2}, \frac{n}{2}; \left(\frac{|t|}{r}\right)^2\right)}{\Gamma(\frac{m}{2}+1)}, \tag{4}$$

$$s_m(n,r,a,|t|) = \frac{r^{2m+1} C_{n+2m}^{2m+1} \Gamma(\frac{n}{2}+m+\frac{1}{2}) J_{\frac{n}{2}+2m+1}(ra)}{|t|^{2m+1}\Gamma(-m-\frac{1}{2})\Gamma\left(\frac{n}{2}+2m+1\right)} \times$$

$$\times {}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}, \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right) \tag{5}$$

${}_2F_1(a,b,c;z)$ is a Gauss hypergeometric function (see [13]).

The rate of convergence of the series (3) is important for numerical calculations. We will study this problem in the paper.

3. Asymptotic properties of the Gauss hypergeometric function

To obtain the rate of convergence we will need some properties of the Gauss hypergeometric function ${}_2F_1$ in (4) and (5).

The function ${}_2F_1(a,b,c;z)$ is defined as

$$ {}_2F_1(a,b,c;z)=\sum_{l=0}^{\infty}\frac{(a)_l(b)_l\,z^l}{(c)_l\,l!}, \tag{6}$$

$$(a)_0=1 \ \text{ and } \ (a)_l=a(a+1)...(a+l-1), \text{ if } \ l\in N,$$

for arguments values for which the series (6) converges, and as analytical continuation for another arguments values (complex-valued) if such continuation exists.

Note, that in the cases (4) and (5) the function ${}_2F_1$ can be correctly defined by (6). Indeed for $m = 2k,\ k\in N\cup 0:$

$$ {}_2F_1\left(\frac{n}{2}+k,-k,\frac{n}{2};\left(\frac{|t|}{r}\right)^2\right)=\sum_{l=0}^{k}\frac{\left(\frac{n}{2}+k\right)_l(-k)_l}{\left(\frac{n}{2}\right)_l l!}z^l \tag{7}$$

becomes a k-degree polynomial.

It can be found in §2.1.1, [14] that for $a,b\neq\{0,-1,-2,...\}:$

$$\frac{(a)_l(b)_l}{(c)_l\,l!}=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,l^{a+b-c-1}[1+\mathrm{O}(l^{-1})],$$

and the series (6) converges absolutely for $|z|<1$. Therefore, for $m\neq 2k$, $k\in N\cup\{0\}$, $|t|<r$, the function ${}_2F_1\left(\frac{n+m}{2},-\frac{m}{2},\frac{n}{2};\left(\frac{|t|}{r}\right)^2\right)$ is correctly defined by (6). Similarly ${}_2F_1\left(\frac{n}{2}+m+\frac{1}{2},m+\frac{3}{2},\frac{n}{2}+2m+2;\left(\frac{r}{|t|}\right)^2\right)$ is also correctly defined by (6), when $|t|>r$.

Let us consider the asymptotic behavior of ${}_2F_1\left(\frac{n+m}{2},-\frac{m}{2},\frac{n}{2};\left(\frac{|t|}{r}\right)^2\right)$, when $m\to\infty$.

In the following we will use Stirling's formula (see §540, [15]) for the Gamma function

$$\Gamma(k+1)\sim\sqrt{2\pi}k^{k+1/2}e^{-k}e^{\frac{\theta_k}{k}},$$

where $\theta_k\in(0;\frac{1}{12})$. Particularly

$$k!\sim\sqrt{2\pi}k^{k+1/2}e^{-k}e^{\frac{\theta_k}{k}}.$$

To obtain asymptotical formulas we could use Watson's results (see §2.3.2 [14])

$${}_2F_1\left(a+\lambda, b-\lambda, c, 1/2-z/2\right) = \frac{\Gamma(1-b+\lambda)\Gamma(c)}{\Gamma(1/2)\Gamma(c-b+\lambda)} 2^{a+b-1}(1-e^{-\xi})^{-c+1/2}\times$$

$$(8)\quad (1+e^{-\xi})^{c-a-b-1/2}\lambda^{-1/2}\left(e^{(\lambda-b)\xi}+e^{\pm i\pi(c-1/2)}e^{-(\lambda+a)\xi}\right)\left(1+O(|\lambda^{-1}|)\right),$$

where $z \pm \sqrt{z^2-1} = e^{\pm\xi}$ according to $\mathrm{Im} z \gtrless 0$.

Let us choose

$$a = c = \frac{n}{2},\ b = 0,\ \lambda = \frac{m}{2},\ \frac{1}{2} - \frac{z}{2} = \left(\frac{|t|}{r}\right)^2$$

then

$${}_2F_1\left(\tfrac{n+m}{2}, -\tfrac{m}{2}, \tfrac{n}{2}; \left(\tfrac{|t|}{r}\right)^2\right) \sim \tfrac{\Gamma(m/2+1)\Gamma(n/2)2^{n/2-1/2}}{\Gamma(n/2+m/2)\Gamma(1/2)m^{1/2}}\times$$

$$\times\left(\tfrac{2|t|}{r}\right)^{1/2-n/2}\left(\tfrac{|t|}{r}+\sqrt{\left(\tfrac{|t|}{r}\right)^2-1}\right)^{1/2-n/2}2^{-1/2}\left(1-\left(\tfrac{|t|}{r}\right)^2-\tfrac{|t|}{r}\sqrt{\left(\tfrac{|t|}{r}\right)^2-1}\right)^{-1/2}\times$$

$$\times\left(\left(\tfrac{|t|}{r}+\sqrt{\left(\tfrac{|t|}{r}\right)^2-1}\right)^{m/2}+\left(\sin\left(\tfrac{n\pi}{2}\right)\pm i\cos\left(\tfrac{n\pi}{2}\right)\right)\left(\tfrac{|t|}{r}+\sqrt{\left(\tfrac{|t|}{r}\right)^2-1}\right)^{m/2+n/2}\right) =$$

$$= \mathrm{O}\left(\tfrac{\Gamma(\frac{m}{2}+1)}{\Gamma(\frac{n}{2}+\frac{m}{2})m^{\frac{1}{2}}}\right) \sim \mathrm{O}\left(\tfrac{C^m}{m^{\frac{n}{2}-1}}\right).$$

when $m \to \infty$.

Unfortunately the formula (8) is true only if $z \in \mathrm{C}\backslash(-\infty, 1)$ (see [16], [17]). Therefore we cannot use it directly in our case. Nevertheless we will show that the asymptotics $\mathrm{O}\left(\frac{C^m}{m^{\frac{n}{2}-1}}\right)$ is valid for our parameters case.

Lemma 1 *For* $|t| < r$ *:*

$${}_2F_1\left(\frac{n+m}{2}, -\frac{m}{2}, \frac{n}{2}; \left(\frac{|t|}{r}\right)^2\right) = O\left(\frac{C^m}{m^{\frac{n}{2}-\frac{1}{2}}}\right),\quad m \to \infty.$$

Proof. If $m = 2k,\ k \in \mathrm{N}\cup\{0\}$, then it follows from (7)

$$\left|{}_2F_1\left(\frac{n}{2}+k, -k, \frac{n}{2}; z\right)\right| = \left|\sum_{l=0}^{k}\frac{\Gamma(\frac{n}{2}+k+l)\Gamma(\frac{n}{2})(-1)^l k! z^l}{\Gamma(\frac{n}{2}+k)\Gamma(\frac{n}{2}+l)(k-l)!l!}\right| \le$$

$$(9)\qquad \le \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+k)}\sum_{l=0}^{k} C_k^l z^l \frac{\Gamma(\frac{n}{2}+k+l)}{\Gamma(\frac{n}{2}+l)} \le \frac{\Gamma(\frac{n}{2}+2k)\Gamma(\frac{n}{2})}{\Gamma^2(\frac{n}{2}+k)}(1+z)^k,$$

because

$$\frac{\Gamma(\frac{n}{2}+k+l)}{\Gamma(\frac{n}{2}+l)} \le \frac{\Gamma(\frac{n}{2}+2k)}{\Gamma(\frac{n}{2}+k)},\ l = \overline{0,k}.$$

By Stirling's formula

$$\frac{\Gamma(\frac{n}{2}+2k)}{\Gamma^2(\frac{n}{2}+k)} \sim \frac{e^{-\frac{n}{2}-2k+1}\left(\frac{n}{2}+2k-1\right)^{\frac{n}{2}+2k-\frac{1}{2}}}{\sqrt{2\pi}e^{-n-2k+2}\left(\frac{n}{2}+k-1\right)^{n+2k-1}} =$$

$$= \frac{e^{\frac{n}{2}-1}2^{\frac{n}{2}+2k-\frac{1}{2}}\left(1+\frac{\frac{n}{4}-\frac{1}{2}}{k}\right)^{\frac{k}{\frac{n}{4}-\frac{1}{2}}\cdot\frac{\frac{n}{4}-\frac{1}{2}}{k}\left(\frac{n}{2}+2k-\frac{1}{2}\right)}}{\sqrt{2\pi}\left(1+\frac{\frac{n}{2}-1}{k}\right)^{\frac{k}{\frac{n}{2}-1}\cdot\frac{\frac{n}{2}-1}{k}\cdot(n+2k-1)}\cdot k^{n+2k-1}} k^{\frac{n}{2}+2k-\frac{1}{2}} \sim \frac{2^{\frac{n}{2}+2k-1}}{\sqrt{\pi}k^{\frac{n}{2}-\frac{1}{2}}}. \tag{10}$$

By (9) and (10) we obtain

$$ {}_2F_1\left(\frac{n}{2}+k,-k,\frac{n}{2};\left(\frac{|t|}{r}\right)^2\right) = \mathrm{O}\left(\frac{C^k}{k^{\frac{n}{2}-\frac{1}{2}}}\right) = \mathrm{O}\left(\frac{C^m}{m^{\frac{n}{2}-\frac{1}{2}}}\right), \tag{11}$$

when $m = 2k \to \infty$.

Let us show that ${}_2F_1\left(\frac{n+m}{2},-\frac{m}{2};\frac{n}{2};\left(\frac{|t|}{r}\right)^2\right)$ has the same asymptotics when m is odd. Actually

$$ {}_2F_1\left(\tfrac{n+m}{2},-\tfrac{m}{2};\tfrac{n}{2};z\right) = \sum_{l=0}^{\left[\frac{m}{2}\right]+1}\frac{\left(\frac{n}{2}+\frac{m}{2}\right)_l\left(-\frac{m}{2}\right)_l}{\left(\frac{n}{2}\right)_l\, l!}z^l + \sum_{l=\left[\frac{m}{2}\right]+2}^{\infty}\frac{\left(\frac{n}{2}+\frac{m}{2}\right)_l\left(-\frac{m}{2}\right)_l}{\left(\frac{n}{2}\right)_l\, l!}z^l. \tag{12}$$

For $l = 0, ..., \left[\frac{m}{2}\right]+1:$

$$\left|\left(\tfrac{n}{2}+\tfrac{m}{2}\right)_l\left(-\tfrac{m}{2}\right)_l\right| = \left|\left(\tfrac{n}{2}+\tfrac{m}{2}\right)\cdot\ldots\cdot\left(\tfrac{n}{2}+\tfrac{m}{2}+l-1\right)\cdot\tfrac{m}{2}\cdot\ldots\cdot\left(\tfrac{m}{2}-l+1\right)\right| \le$$

$$\left|\left(\tfrac{n}{2}+\tfrac{m}{2}+\tfrac{1}{2}\right)\cdot\ldots\cdot\left(\tfrac{n}{2}+\tfrac{m}{2}+l-\tfrac{1}{2}\right)\cdot\tfrac{m+1}{2}\cdot\ldots\cdot\left(\tfrac{m+1}{2}-l+1\right)\right| =$$

$$\left|\left(\tfrac{n}{2}+\tfrac{m+1}{2}\right)_l\left(-\tfrac{m+1}{2}\right)_l\right|.$$

Hence

$$\left|\sum_{l=0}^{\left[\frac{m}{2}\right]+1}\frac{\left(\frac{n}{2}+\frac{m}{2}\right)_l\left(-\frac{m}{2}\right)_l}{\left(\frac{n}{2}\right)_l\, l!}z^l\right| \le \sum_{l=0}^{\frac{m+1}{2}}\frac{\left(\frac{n}{2}+\frac{m+1}{2}\right)_l\left|\left(-\frac{m+1}{2}\right)_l\right|}{\left(\frac{n}{2}\right)_l\, l!}z^l = \mathrm{O}\left(\frac{C^{\frac{m+1}{2}}}{\left(\frac{m+1}{2}\right)^{\frac{n}{2}-\frac{1}{2}}}\right), \tag{13}$$

where the last identity follows from (9) and (10) proven for even m.

Consider the asymptotics of the second term in (12).

$$\sum_{l=\left[\frac{m}{2}\right]+2}^{\infty}\frac{\left(\frac{n}{2}+\frac{m}{2}\right)_l\left(-\frac{m}{2}\right)_l}{\left(\frac{n}{2}\right)_l l!}z^l = \sum_{l=\left[\frac{m}{2}\right]+2}^{\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{m}{2}+l\right)\Gamma\left(\frac{n}{2}\right)\cdot\left(-\frac{m}{2}\right)\left(-\frac{m}{2}+1\right)\cdot\ldots\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\ldots\cdot\left(-\frac{m}{2}+l-1\right)z^l}{\Gamma\left(\frac{n}{2}+\frac{m}{2}\right)\Gamma\left(\frac{n}{2}+l\right)\Gamma(l+1)} =$$

$$\left|\begin{array}{l} k = l-\left[\frac{m}{2}\right]-1 \\ l = k+\frac{m}{2}+\frac{1}{2}\end{array}\right| = \frac{(-1)^m\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{n}{2}\right)z^{\frac{m}{2}+\frac{1}{2}}}{\pi\Gamma\left(\frac{n}{2}+\frac{m}{2}\right)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n}{2}+m+\frac{1}{2}+k\right)\Gamma\left(k+\frac{1}{2}\right)z^k}{\Gamma\left(\frac{n}{2}+\frac{m}{2}+\frac{1}{2}+k\right)\Gamma\left(k+\frac{m}{2}+\frac{3}{2}\right)}. \tag{14}$$

By Stirling's formula with $\theta_{k,j} \in \left(0,\frac{1}{12}\right)$, $j = \overline{1,4}$:

$$\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n}{2}+m+\frac{1}{2}+k\right)\Gamma\left(k+\frac{1}{2}\right)z^k}{\Gamma\left(\frac{n}{2}+\frac{m}{2}+\frac{1}{2}+k\right)\Gamma\left(k+\frac{m}{2}+\frac{3}{2}\right)} = \sum_{k=1}^{\infty}\frac{e^{-\frac{n}{2}-m-k+\frac{1}{2}}\left(\frac{n}{2}+m+k-\frac{1}{2}\right)^{\frac{n}{2}+m+k}e^{\frac{\theta_{k,1}}{\frac{n}{2}+m+k-\frac{1}{2}}}}{e^{-\frac{n}{2}-\frac{m}{2}-k+\frac{1}{2}}\left(\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}\right)^{\frac{n}{2}+\frac{m}{2}+k}e^{\frac{\theta_{k,2}}{\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}}}}\times$$

$$\frac{e^{-k+\frac{1}{2}}\left(k-\frac{1}{2}\right)^k e^{\frac{\theta_{k,3}}{k-\frac{1}{2}}}}{e^{-k-\frac{m}{2}-\frac{1}{2}}\left(k+\frac{m}{2}+\frac{1}{2}\right)^{k+\frac{m}{2}+1} e^{\frac{\theta_{k,4}}{k+\frac{m}{2}+1}}} z^k \le C\cdot\sum_{k=1}^{\infty}\frac{\left(\frac{n}{2}+m+k-\frac{1}{2}\right)^{\frac{n}{2}+m+k}\left(k-\frac{1}{2}\right)^k z^k}{\left(\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}\right)^{\frac{n}{2}+\frac{m}{2}+k}\left(k+\frac{m}{2}+\frac{1}{2}\right)^{k+\frac{m}{2}+1}} =$$

$$= C\cdot\sum_{k=1}^{\infty}\frac{\left(1+\frac{\frac{m}{2}}{\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}}\right)^{\frac{\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}}{\frac{m}{2}}\cdot\frac{\frac{m}{2}\cdot\left(\frac{n}{2}+\frac{m}{2}+k\right)}{\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}}}\left(1+\frac{\frac{m}{2}+\frac{n}{2}-1}{k+\frac{m}{2}+\frac{1}{2}}\right)^{\frac{m}{2}} z^k}{\left(1+\frac{\frac{m}{2}+1}{k-\frac{1}{2}}\right)^{\frac{k-\frac{1}{2}}{\frac{m}{2}+1}\cdot\frac{\frac{m}{2}+1}{k-\frac{1}{2}}\cdot k}\left(k+\frac{m}{2}+\frac{1}{2}\right)} := \Sigma.$$

To estimate the last sum we use:

Lemma 2 *For all* $x \in (0,\infty) : 0 < a \le \left(1+\frac{1}{x}\right)^x \le b < +\infty$.

Proof. The function $\left(1+\frac{1}{x}\right)^x$ is positive, continuous and it is not equal to 0 or $+\infty$ for any $x \in (0;+\infty)$. Moreover $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, $\lim\limits_{x\to+0}\left(1+\frac{1}{x}\right)^x = 1$. The statement of the lemma immediately follows from these properties. □

By application of Lemma 2

$$\Sigma \le C\cdot\sum_{k=1}^{\infty}\frac{b^{\frac{\frac{m}{2}\left(\frac{n}{2}+\frac{m}{2}+k\right)}{\frac{n}{2}+\frac{m}{2}+k-\frac{1}{2}}}\left(1+\frac{\frac{m}{2}+\frac{n}{2}-1}{k+\frac{m}{2}+\frac{1}{2}}\right)^{\frac{m}{2}} z^k}{a^{\frac{\frac{m}{2}+1}{k-\frac{1}{2}}\cdot k}\left(k+\frac{m}{2}+\frac{1}{2}\right)} \le C\cdot\sum_{k=1}^{\infty}\frac{C_1^{\frac{m}{2}}C_3^{\frac{m}{2}}z^k}{C_2^{\frac{m}{2}+1}\left(k+\frac{m}{2}+\frac{1}{2}\right)},$$

because of

$$\left(1+\frac{\frac{m}{2}+\frac{n}{2}-1}{k+\frac{m}{2}+\frac{1}{2}}\right)^{\frac{m}{2}} \le \left(1+\frac{\frac{m}{2}+\frac{n}{2}-1}{\frac{m}{2}+\frac{3}{2}}\right)^{\frac{m}{2}} = \left(2+\frac{\frac{n}{2}-\frac{5}{2}}{\frac{m}{2}+\frac{3}{2}}\right)^{\frac{m}{2}} \le \left(\frac{n+3}{4}\right)^{\frac{m}{2}}.$$

Hence

$$\Sigma \le C^{\frac{m}{2}}\sum_{k=1}^{\infty}\frac{z^k}{k+\frac{m}{2}+\frac{1}{2}} < C^{\frac{m}{2}}\sum_{k=1}^{\infty}\frac{z^k}{k+1}.$$

By (14) and Stirling's formula

$$\left|\sum_{l=\left[\frac{m}{2}\right]+2}^{\infty}\frac{\left(\frac{n}{2}+\frac{m}{2}\right)_l\left(-\frac{m}{2}\right)_l}{\left(\frac{n}{2}\right)_l\ l!}z^l\right| = \mathrm{O}\left(\frac{C^{\frac{m}{2}}\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{m}{2}\right)}\right) = \mathrm{O}\left(\frac{C^m}{m^{\frac{n}{2}-1}}\right), \tag{15}$$

when $m \to \infty$.

For odd m by (12), (13) and (15) we obtain

$$ {}_2F_1\left(\frac{n+m}{2}, -\frac{m}{2}, \frac{n}{2}; \left(\frac{|t|}{r}\right)^2\right) = \mathrm{O}\left(\frac{C^m}{m^{\frac{n}{2}-\frac{1}{2}}}\right), \quad m \to \infty.$$

Statement of Lemma 1 follows from the last asymptotics and (11). □

To investigate asymptotic properties of

$$ {}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}, \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right), \quad |t| > r, \quad m \to \infty$$

we could also use Watson's results (see §2.3.2 [14])

$$\left(\tfrac{z}{2}-\tfrac{1}{2}\right)^{-a-\lambda} {}_2F_1\left(a+\lambda, a-c+1+\lambda, a-b+1+2\lambda; 2(1-z)^{-1}\right) =$$

$$\frac{2^{a+b}\Gamma(a-b+1+2\lambda)\Gamma\left(\frac{1}{2}\right)\lambda^{-\frac{1}{2}}}{\Gamma(a-c+1+\lambda)\Gamma(c-b+\lambda)} e^{-(a+\lambda)\xi}(1-e^{-\xi})^{-c+\frac{1}{2}}\times$$

$$(1+e^{-\xi})^{c-a-b-\frac{1}{2}}\left(1+\mathrm{O}\left(|\lambda|^{-1}\right)\right), \tag{16}$$

where ξ is defined as in (8).

Choosing

$$a=\frac{n}{2}+\frac{1}{2},\ b=-\frac{1}{2},\ c=\frac{n}{2},\ \lambda=m,\ \frac{2}{1-z}=\left(\frac{r}{|t|}\right)^2$$

we obtain

$${}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}, \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right)\sim$$

$$\sim\frac{2^{\frac{n}{2}+1}\Gamma(\frac{n}{2}+2m+2)\sqrt{\pi}m^{-\frac{1}{2}}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}+m\right)}\mathrm{O}\left(C^m\right)\left(\frac{r}{|t|}\right)^{n+1+2m}\sim\mathrm{O}\left(C^m\right),$$

when $m\to\infty$. Unfortunately, similarly to the case (8) we cannot apply (16) directly. Nevertheless we will show that the asymptotics $\mathrm{O}\left(C^m\right)$ is valid for our parameters case.

Lemma 3 *For* $|t|>r$:

$${}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}, \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right) = O\left(C^m\right),\ m\to\infty.$$

Proof. By (2) we have

$${}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}, \frac{n}{2}+2m+2; z\right) = \sum_{l=0}^{\infty}\frac{\left(\frac{n}{2}+m+\frac{1}{2}\right)_l\left(m+\frac{3}{2}\right)_l}{\left(\frac{n}{2}+2m+2\right)_l l!}z^l =$$

$$\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+m+l+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+2m+2\right)\Gamma\left(m+l+\frac{3}{2}\right)z^l}{\Gamma\left(\frac{n}{2}+m+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+2m+l+2\right)\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(l+1\right)} =$$

$$1+\frac{\Gamma\left(\frac{n}{2}+2m+2\right)}{\Gamma\left(\frac{n}{2}+m+\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}\sum_{l=1}^{\infty}\frac{\Gamma\left(\frac{n}{2}+m+l+\frac{1}{2}\right)\Gamma\left(m+l+\frac{3}{2}\right)z^l}{\Gamma\left(\frac{n}{2}+2m+l+2\right)\Gamma\left(l+1\right)} \tag{17}$$

By Stirling's formula

$$\Gamma\left(\tfrac{n}{2}+2m+2\right)\sim\sqrt{2\pi}\left(\tfrac{n}{2}+2m+1\right)^{\frac{n}{2}+2m+\frac{3}{2}}e^{-\frac{n}{2}-2m-1}\sim\sqrt{2\pi}\left(2m\right)^{\frac{n}{2}+2m+\frac{3}{2}}e^{-2m},$$

$$\Gamma\left(\tfrac{n}{2}+m+\tfrac{1}{2}\right)\sim\sqrt{2\pi}\left(\tfrac{n}{2}+m-\tfrac{1}{2}\right)^{\frac{n}{2}+m}e^{-\frac{n}{2}-m+\frac{1}{2}}\sim\sqrt{2\pi}m^{\frac{n}{2}+m}e^{-m}, \tag{18}$$

$$\Gamma\left(m+\tfrac{3}{2}\right)\sim\sqrt{2\pi}\left(m+\tfrac{1}{2}\right)^{m+1}e^{-m-\frac{1}{2}}\sim\sqrt{2\pi}m^{m+1}e^{-m},\quad\text{when}\quad m\to\infty.$$

The asymptotic behaviour of the first multiplier in (17) is

$$\frac{\Gamma\left(\frac{n}{2}+2m+2\right)}{\Gamma\left(\frac{n}{2}+m+\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)} \sim \frac{2^{\frac{n}{2}+2m+\frac{3}{2}}m^{\frac{1}{2}}}{\sqrt{2\pi}}. \tag{19}$$

Using Stirling's formula with $\theta_{l,j} \in (0, \frac{1}{12})$, $j = \overline{1,4}$, we transform the series in (17):

$$\sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{n}{2}+m+l+\frac{1}{2}\right)\Gamma\left(m+l+\frac{3}{2}\right)z^l}{\Gamma\left(\frac{n}{2}+2m+l+2\right)\Gamma(l+1)} =$$

$$= \sum_{l=1}^{\infty} \frac{\left(\frac{n}{2}+m+l-\frac{1}{2}\right)^{\frac{n}{2}+m+l} e^{-\frac{n}{2}-m-l+\frac{1}{2}} e^{\frac{\theta_{l,1}}{\frac{n}{2}+m+l-\frac{1}{2}}} \left(m+l+\frac{1}{2}\right)^{m+l+1}}{\left(\frac{n}{2}+2m+l+1\right)^{\frac{n}{2}+2m+l+\frac{3}{2}} e^{-\frac{n}{2}-2m-l-1} e^{\frac{\theta_{l,2}}{\frac{n}{2}+2m+l+1}}} \times$$

$$\frac{e^{-m-l-\frac{1}{2}} e^{\frac{\theta_{l,3}}{m+l+\frac{1}{2}}}}{l^{l+\frac{1}{2}} e^{-l} e^{\frac{\theta_{l,4}}{l}}} z^l \leq C \sum_{l=1}^{\infty} \frac{\left(\frac{n}{2}+m+l-\frac{1}{2}\right)^{\frac{n}{2}+m+l}\left(m+l+\frac{1}{2}\right)^{m+l+1}}{\left(\frac{n}{2}+2m+l+1\right)^{\frac{n}{2}+2m+l+\frac{3}{2}} l^{l+\frac{1}{2}}} z^l =$$

$$= C \sum_{l=1}^{\infty} \frac{\left(1+\frac{m+\frac{1}{2}}{l}\right)^{\frac{l}{m+\frac{1}{2}}\cdot\left(m+\frac{1}{2}\right)} z^l}{\left(1+\frac{m+\frac{3}{2}}{\frac{n}{2}+m+l-\frac{1}{2}}\right)^{\frac{\frac{n}{2}+m+l-\frac{1}{2}}{m+\frac{3}{2}}\cdot\frac{\left(m+\frac{3}{2}\right)\left(\frac{n}{2}+m+l\right)}{\frac{n}{2}+m+l-\frac{1}{2}}}} \frac{\left(m+l+\frac{1}{2}\right)^{m+1}}{l^{\frac{1}{2}}\left(\frac{n}{2}+2m+l+1\right)^{m+\frac{3}{2}}} = \Sigma_1.$$

By Lemma 2

$$\Sigma_1 \leq C \sum_{l=1}^{\infty} \frac{b^{m+\frac{1}{2}}}{a^{\frac{\left(m+\frac{3}{2}\right)\left(\frac{n}{2}+2m+l\right)}{\frac{n}{2}+m+l-\frac{1}{2}}}} \cdot \frac{z^l}{l^{\frac{1}{2}}\left(\frac{n}{2}+2m+l+1\right)^{\frac{1}{2}}} \leq C \sum_{l=1}^{\infty} \frac{C_1^m z^l}{l^{\frac{1}{2}}\, m^{\frac{1}{2}}}. \tag{20}$$

By (17), (19) and (20), we get

$${}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}, \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right) = \mathrm{O}\left(C^m\right), \quad m \to \infty. \quad \square$$

4. Rate of convergence

Let us estimate the rate of convergence for the series (3). For this purpose we will study asymptotics of $d_m(n, r, a, |t|)$ and $s_m(n, r, a, |t|)$, as $m \to \infty$.

4.1 Case $|t| < r$.

Let us consider the expression (3) for $|t| < r$ and investigate asymptotics for multipliers in (4) as $m \to \infty$.

Lemma 4 *For* $|t| < r$

$$d_m(n, r, a, |t|) = O\left(\frac{C^m}{m^{m-\frac{n}{2}+\frac{1}{2}}}\right),\ m \to \infty. \tag{21}$$

Proof. By Stirling's formula

$$C_{m+n-1}^m \sim \frac{(m+n-1)^{m+n-1/2} e^{-m-n+1}}{m^{m+1/2} e^{-m} (n-1)!} \sim \frac{m^{n-1}}{(n-1)!}.$$

For Gamma functions we have

$$\Gamma\left(\frac{m}{2}+1\right) \sim \sqrt{2\pi}e^{-\frac{m}{2}}(m/2)^{\frac{m+1}{2}},$$
$$\Gamma\left(\frac{m}{2}+\frac{n}{2}\right) \sim \sqrt{2\pi}e^{-\frac{m}{2}}(m/2)^{\frac{m+n-1}{2}}.$$

Therefore

$$\frac{\left(\frac{n}{2}+m\right)C^m_{n+m-1}\Gamma(\frac{n+m}{2})}{\Gamma(\frac{m}{2}+1)} \sim \frac{m^n}{(n-1)!}\left(\frac{m}{2}\right)^{n/2-1}.$$

For $\nu > 0,\ z > 0$ due to a representation of the Bessel function as a series (see §8.1 [12]) it follows that

$$|J_\nu(z)| = \left|\sum_{m=0}^\infty \frac{(-1)^m(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)}\right| = \left|\frac{(z/2)^\nu}{\Gamma(\nu+1)}\sum_{m=0}^\infty \frac{(-1)^m(z/2)^{2m}}{m!(\nu+1)(\nu+2)...(\nu+m)}\right| \leq$$

$$\leq \frac{(z/2)^\nu}{\Gamma(\nu+1)}\sum_{m=0}^\infty \frac{(z/2)^{2m}}{m!\nu^m}. \tag{22}$$

By Stirling's formula

$$\frac{(z/2)^\nu}{\Gamma(\nu+1)}\sum_{m=0}^\infty \frac{(z/2)^{2m}}{m!\nu^m} \sim \frac{(z/2)^\nu e^\nu}{\sqrt{2\pi}\nu^{\nu+1/2}}\sum_{m=0}^\infty \frac{(\frac{z^2}{4\nu})^m}{m!} = \frac{e^{\frac{z^2}{4\nu}+\nu}}{\sqrt{2\pi\nu}}\left(\frac{z}{2\nu}\right)^\nu. \tag{23}$$

For large values of m

$$J_{\frac{n}{2}+m}(ra) < \frac{1}{\sqrt{(n+2m)\pi}}e^{\frac{(ra)^2+(n+2m)^2}{2n+4m}}\left(\frac{ra}{n+2m}\right)^{\frac{n}{2}+m} = \mathrm{O}\left(\frac{C^m}{m^{m+\frac{n}{2}+\frac{1}{2}}}\right). \tag{24}$$

Applying Lemma 1 and all previous asymptotics to (4) we obtain

$$d_m(n,r,a,|t|) = \mathrm{O}\left(\frac{C^m}{m^{m-\frac{n}{2}+1}}\right), \quad m\to\infty. \qquad \square$$

Consider the series (3) remainder for $|t| < r$.

Theorem 1 *For* $|t| < r$

$$\sum_{m=N}^\infty d_m(n,r,a,|t|) = O\left(\sum_{m=N}^\infty \frac{C^m}{m^{m-\frac{n}{2}+1}}\right),\ N\to\infty. \tag{25}$$

For any $\varepsilon > 0$:

$$\sum_{m=N}^\infty d_m(n,r,a,|t|) = O\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right),\ N\to\infty. \tag{26}$$

Proof. The asymptotic formula (25) is a direct corollary of Lemma 4. The assertion (26) follows from the the chain of estimates

$$\sum_{m=N}^\infty \frac{C^m}{m^{m-\frac{n}{2}+1}} \leq \sum_{m=N}^\infty \frac{C^m}{m^{m(1-\varepsilon)}} \leq \sum_{m=N}^\infty \left(\frac{C}{N^{1-\varepsilon}}\right)^m =$$

$$= \frac{C^N N^{1-\varepsilon}}{N^{N(1-\varepsilon)}(N^{1-\varepsilon}-C)} = \mathrm{O}\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right)$$

which is valid for large N. □

4.2 Case $|t| > r$.

Let us consider the expression (3) for $|t| > r$ and investigate asymptotic behavior for multipliers in (5) as $m \to \infty$.

Lemma 5 *For* $|t| > r$

$$s_m(n, r, a, |t|) = O\left(\frac{C^m}{m^{2m-\frac{n}{2}+2}}\right), \ m \to \infty. \tag{27}$$

Proof. By Stirling's formula

$$C_{n+2m}^{2m+1} \sim \frac{(n+2m)^{n+2m+\frac{1}{2}} e^{2m+1}}{e^{n+2m}(2m+1)^{2m+\frac{3}{2}}(n-1)!} \sim \frac{(2m)^{n-1}}{(n-1)!},$$

$$\Gamma\left(n/2 + 2m + 1\right) \sim \sqrt{2\pi} e^{-2m} (2m)^{\frac{n}{2}+2m+\frac{1}{2}}.$$

Let us find the asymptotics of $(2m+1)!!$:

$$(2m+1)!! = 2^{m+1}\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\left(\tfrac{1}{2}+2\right)...\left(\tfrac{1}{2}+m\right) =$$
$$= 2^{m+1}\frac{\Gamma(1/2+m+1)}{\Gamma(1/2)} \sim \sqrt{2}(2m+1)^{m+1}e^{-m-1/2}.$$

Using properties of Gamma functions (see §538, [15]) and Stirling's formula we obtain

$$\Gamma\left(-m-\tfrac{1}{2}\right) = (-1)^{m+1}\sqrt{\pi}\frac{2^{m+1}}{(2m+1)!!} \sim (-1)^{m+1}\sqrt{\tfrac{\pi}{2}}\frac{e^{m+\frac{1}{2}}}{\left(m+\frac{1}{2}\right)^{m+1}} \sim (-1)^{m+1}\sqrt{\tfrac{\pi}{2}}\frac{e^m}{m^{m+1}}.$$

We will use the formula (23) to study the asymptotics of the Bessel function $J_{\frac{n}{2}+2m+1}(ra)$. For large values of m

$$J_{\frac{n}{2}+2m+1}(ra) < \frac{e^{\frac{(ra)^2}{2n+8m+4}+\frac{n}{2}+2m+1}}{\sqrt{2\pi\left(\frac{n}{2}+2m+1\right)}}\left(\frac{ra}{n+4m+1}\right)^{\frac{n}{2}+2m+1} = \mathrm{O}\left(\frac{C^m}{m^{\frac{n}{2}+2m+\frac{3}{2}}}\right).$$

Taking into account (18), all previous asymptotics for (5), and Lemma 3 we obtain

$$s_m(n, r, a, |t|) = \mathrm{O}\left(\frac{C^m}{m^{2m-\frac{n}{2}+2}}\right), \ m \to \infty. \qquad \square$$

Consider the series (3) remainder for $|t| > r$.

Theorem 2 *For* $|t| > r$

$$\sum_{m=N}^{\infty} s_m(n, r, a, |t|) = O\left(\sum_{m=N}^{\infty} \frac{C^m}{m^{2m-\frac{n}{2}+2}}\right), \ N \to \infty \tag{28}$$

For any $\varepsilon > 0$:

$$\sum_{m=N}^{\infty} s_m(n, r, a, |t|) = O\left(\frac{C^N}{N^{2N(1-\varepsilon)}}\right), \ N \to \infty. \tag{29}$$

Proof. The asymptotic formula (28) is a direct corollary of Lemma 5. The assertion (29) follows from the chain of estimates

$$\sum_{m=N}^{\infty} \frac{C^m}{m^{2m-\frac{n}{2}+2}} \leq \sum_{m=N}^{\infty} \frac{C^m}{m^{2m(1-\varepsilon)}} \leq \sum_{m=N}^{\infty} \left(\frac{C}{N^{2(1-\varepsilon)}}\right)^m =$$

$$\frac{C^N N^{2(1-\varepsilon)}}{N^{2N(1-\varepsilon)}(N^{2(1-\varepsilon)}-C)} = \mathrm{O}\left(\frac{C^N}{N^{2N(1-\varepsilon)}}\right),$$

which is valid for large N. □

5. Numerical examples

In this section we give some numerical examples of our results.

Let $n = 3$. In this case $f_{r,a}(|t|)$ can be written explicitly using functions $Si(z)$, $Ci(z)$ (see §5, [8]).

Plots of the function $f_{r,a}(|t|)$ for $r = 1,\ a = 1.2,$ and $a = 15$ are shown on Fig.1 and Fig.2. To plot the function we used $N = 100$ first terms of the series (3).

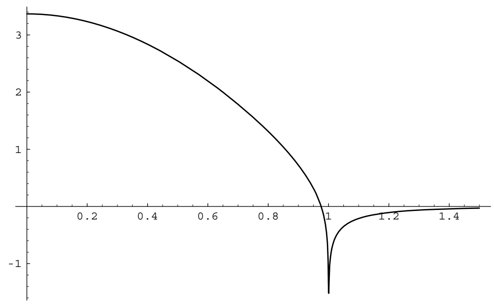

Fig.1. $f_{1,1.2}(|t|)$

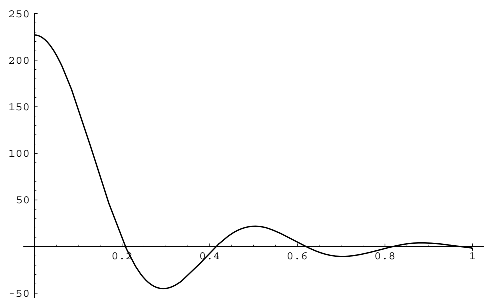

Fig.2. $f_{1,15}(|t|)$

Comparison of Fig.1 and Fig.2 with corresponding plots from §5 [8] shows their identity. The following table gives some exact numerical values of $f_{r,a}(|t|)$ calculated by formulae from §5 [8] and its approximations $\widehat{f}^N_{r,a}(|t|)$ by increasing number N of first terms in the series (3).

t	$f_{1,1.2}(t)$	$\widehat{f}^5_{1,1.2}(t)$	$\widehat{f}^{10}_{1,1.2}(t)$	$\widehat{f}^{25}_{1,1.2}(t)$
0.1	3.3346	3.33328	3.3346	3.3346
0,5	2.54618	2.54648	2.54618	2.54618
0.99	-0.327282	-0.326829	-0.327282	-0.327282
1.01	-0.759792	-0.759792	-0.759792	-0.759792
3	-0.0012482	-0.0012482	-0.0012482	-0.0012482
10	-9.25×10^{-6}	-9.25×10^{-6}	-9.25×10^{-6}	-9.25×10^{-6}

Analyzing the table we see that even for $N = 10$ the values of the function are calculated with a high accuracy. It is important to mention that the accuracy of calculations declines, when t tends to r, due to discontinuity of the function $f_{r,a}(|t|)$ in the point $|t| = r$.

Figures 3, 4, 5, and 6 show the sequence $\lg(g_N(t))$ for different t, where

$$g_N(t) := \left| f_{1,\,1.2}(t) - \widehat{f}^{N}_{1,\,1.2}(t) \right| \cdot \begin{cases} N^{\frac{3N}{4}}, & t < 1, \\ N^{\frac{3N}{2}}, & t > 1. \end{cases}$$

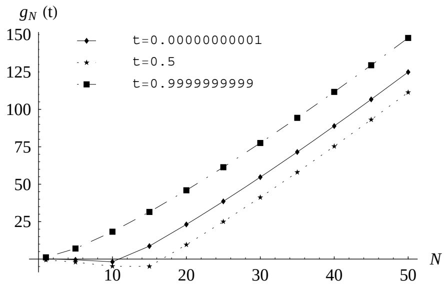

Fig.3. Plot of $\lg(g_N(t)),\ t < 1$

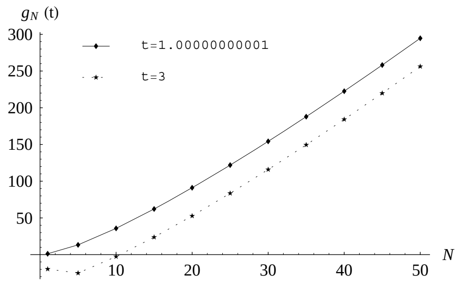

Fig.4. Plot of $\lg(g_N(t)),\ t > 1$

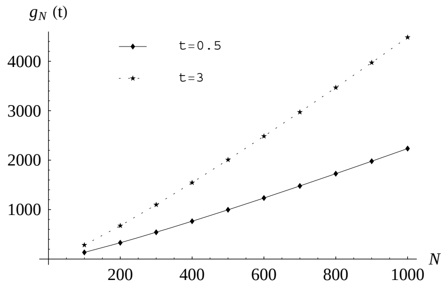

Fig.5. Plot of $\lg(g_N(t))$

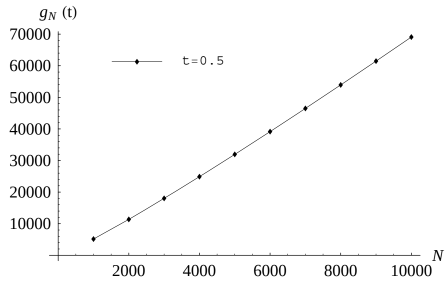

Fig.6. Plot of $\lg(g_N(t))$

Numerical results shown on the figures 3-6 are in complete accordance with estimates for the rates of convergence in Theorems 1 and 2 ($\varepsilon = \frac{1}{4}$ was chosen).

Concluding remarks

The rate of convergence for weight functions series in Tauberian theorems for random fields was obtained. Numerical results show that partial sums of the series give good approximation for weight functions and have required rate of convergence.

Some asymptotic properties of hypergeometric functions were obtained as auxiliary results.

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