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ON RESELLING OF EUROPEAN OPTION

On Black and Scholes market investor buys a European call option. At each moment of time till the maturity, he is allowed to resell the option for the quoted market price. A model is proposed, under which there is no arbitrage possibility. It is shown that the optimal reselling problem is equivalent to constructing nonrandom two dimensional stopping domains. For a modified model of the market price, it is shown that the stopping domains have a threshold structure.

1. Introduction

Optimal strategies for Investor in American option were studied in the papers [2, 4, 5, 7, 9, 12]. Construction of these strategies leads to the construction of a one-dimensional stopping domain G_t for each moment t up to maturity T. For the European call option, Investor is not entitled to exercise the option before the time T and should wait until the maturity. However, it is known that on real financial markets he has an opportunity to resell the option before the maturity. Thus an investigation of the reselling problem is essential, while, to the authors' knowledge, there is no paper dealing with this problem.

In this paper, we treat the following model. On the Black–Scholes security market with an interest rate r, at the moment $t_0 = 0$, Investor buys a European call option with the strike price K and the maturity T, on the stock with initial value S_0, for the price $C_{BS}(S_0, T) = C_{BS}(S_0, T; \sigma, K, r)$ computed by the Black–Scholes formula. At any moment $t \in (0, T)$ he can resell the option for a certain market price C^m_t, which may differ from the "fair" price $C_{BS}(S_t, T - t)$.

The paper proposes a stochastic model of the market price $C_m(t)$, which does not lead to an arbitrage opportunity. It is shown that such an option, with reselling possibility, is equivalent to certain American type derivative. This allows to describe the optimal reselling time for the option in terms of nonrandom stopping sets G_t, which are subsets of the two-dimensional phase space $\mathbb{R}^+ \times \mathbb{R}^+ \ni (S_t, C^m_t)$ (hereafter $\mathbb{R}^+$ denotes the set of all non-negative real numbers).

In the paper [9], analytic structure of boundary of stopping sets for the optimal exercise of an American option is studied. For more general models

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a threshold structure of those sets is shown in [4, 5, 6], and the algorithm is proposed for constructing the sets. This algorithm is based on dynamical programming and Monte Carlo technique and relies on the threshold structure of stopping domains.

Thus it is natural to establish a threshold structure for the stopping sets in the problem of reselling of a European option, in particular, that G_t is a set of points lying above a certain curve. We establish similar threshold structure for stopping sets in the simplified model of market price for European option, where stochastic volatility process has no memory.

The paper is organized as follows. In Section 2, it is proved that an assumption about equality of a market option price and the Black–Scholes price makes the problem of reselling lose any sense. The model for a market price of an option in terms of implied volatility is introduced. Section 3 contains the main result about absence of arbitrage possibility in the proposed model. Section 4 focuses on consistent estimates of parameters of the option market price model. It is shown in Section 5 that the optimal Investor's strategy in the reselling problem is determined by nonrandom stopping sets G_t. A modified non-arbitrage model for the option price is proposed, in which the optimal sets have a threshold structure. A numerical algorithm for construction of the optimal stopping set is given. And Section 6 concludes.

2. Models for option market price

Consider the classical Black and Scholes market

$$\begin{aligned} S_t &= S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}, \\ B_t &= B_0 e^{rt}, \quad t \ge 0. \end{aligned} \tag{1}$$

Here S_t and B_t are the stock and the bond prices at the moment t, W_t is Wiener process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$; μ, σ, and r are positive parameters, which we assume to be known. Positive initial values S_0 and B_0 are nonrandom. Throughout the paper $\mathsf{E}[\cdot]$ denotes expectation w.r.t. P.

Assume that at the moment $t_0 = 0$ Investor buys a European call option with strike price K and maturity T. We suppose that the Investor buys the option for "fair" price C_0 given by the Black–Scholes formula

$$C_0 = C_{BS}(S_0, T, \sigma; K, r) := S_0 \Phi\left(\frac{\log \widetilde{S}_0}{v_0} + \frac{v_0}{2}\right) - K e^{-rT} \Phi\left(\frac{\log \widetilde{S}_0}{v_0} - \frac{v_0}{2}\right),$$

where $\widetilde{S}_0 := e^{rT} S_0 / K$, $v_0 := \sigma\sqrt{T}$, and Φ is the standard normal distribution function.

Now suppose that Investor can resell the option at any moment t for a certain random market price C_t^m. Naturally, we will assume that $C_0^m = C_0$, and $C_T^m = g(S_T) = (S_T - K)_+ = \max(S_T - K, 0)$.

The problem of optimal reselling of the option is optimization problem

$$\Psi(\tau) = \mathsf{E}[\, e^{-r\tau} C^m_\tau \,] \to \max \tag{2}$$

in the class of all (Markov) stopping times $\tau \in [0, T]$. The maximizing stopping time is called *optimal reselling time*, and we denote it by $\tau_{\rm opt}$. Later on, we will specify the filtration, under which the Markovian property is considered.

Remark 2.1. It is natural to ask what happens in a dynamical setting, i.e., when Investor is allowed to dynamically trade either a stock or the option. There are three possible cases how one can understand this dynamic trading. We assume that dynamic trading of the bond is always allowed, that is "selling the stock" "selling the option" means also immediate investment into the bond.

1. Investor is not allowed to trade the stock, and is allowed to sell his option in parts. The latter can seem meaningless, but we can understand it as ability to sell a part of a large option holding. Thus in this sense Investor is allowed to use some strategies of the form $\{\gamma_t,\, t \in [0, T]\}$, where $\gamma_t \in [0, 1]$ is a decreasing adapted process indicating part of the option, which Investor owns at the moment t. In this case Investor's gain will be

$$F(\gamma) = -\mathsf{E}\Big[\int_0^T e^{-rt} C^m_t d\gamma_t \Big]$$

and it should be maximized in the set Γ of all decreasing predictable strategies. Now we make two observations. The first is that $F(\gamma)$ is a linear functional, and it is continuous in the supremum norm on under some natural assumptions on C^m_t.

The second is that the set Γ is closed (in sup-norm) convex hull of the set Ξ of processes of the form $I_{t \le \tau}$, where τ is a stopping time. Hence the functional F attains its maximum on Ξ. Therefore in this setting the optimal reselling problem is reduced to (2).

2. Investor is allowed both to sell and buy either the option, or the stock and the option. In this case we can consider the option as a new stock, and this way we are lead to portfolio optimization in a rather standard semimartingale setting.

3. Investor is allowed to buy and sell the stock and is allowed to sell the option (possibly, in parts). This seems to be the most interesting setting. It is worth to note that, in contrast to the first case, trading of the option is truly dynamic now, because Investor can invest more to the stock after he has resold the option, and the set of strategies is not convex anymore.

2.1. The case where market price coincides with "fair price". Assume that for all moments t the market price C^m_t is equal to the Black–Scholes price $C_t = C_{BS}(S_t, T-t; \sigma, K, r)$,

$$C_{BS}(S_t, T-t) = S_t \Phi\Big(\frac{\log \widetilde{S}_t}{v_t} + \frac{v_t}{2}\Big) - K e^{-r(T-t)} \Phi\Big(\frac{\log \widetilde{S}_t}{v_t} - \frac{v_t}{2}\Big), \tag{3}$$

where

$$\widetilde{S}_t := e^{r(T-t)}S_t/K, \quad v_t := \sigma\sqrt{T-t}. \tag{4}$$

It is known that there is a measure P^* such that $S_t e^{-rt}$ is a martingale w.r.t the filtration $\mathcal{F}_t$. This is equivalent to the fact that

$$\widetilde{W}_t = W_t + \frac{\mu - r}{\sigma}t$$

is Wiener process under P^*. Then

$$C_t^m = C_{BS}(S_t, T-t) = \mathsf{E}^*[\, e^{-r(T-t)}g(S_T) \mid \mathcal{F}_t\,]. \tag{5}$$

Here $\mathsf{E}^*[\cdot]$ denotes (conditional) expectation with respect to the measure P^*. We will consider Markov property w.r.t. the filtration $\mathcal{F}_t$. Then

$$\Psi(\tau) = e^{-rT}\mathsf{E}[\, Y_\tau\,], \tag{6}$$

where $Y_t = \mathsf{E}^*[\, g(S_T) \mid \mathcal{F}_t\,]$, $0 \le t \le T$.

Lemma 2.2. *The process Y_t is a*

a) *P-supermartingale for $\mu \le r$,*
b) *P-submartingale for $\mu \ge r$.*

Consequently, Y_t is a P-martingale for $\mu = r$.

Proof. a) Suppose $\mu \le r$. First note that

$$S_t \ge S_t' := S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\widetilde{W}_t \right\},$$

hence $g(S_T) \ge g(S_T')$. Since the distribution of S_t' w.r.t. P^* is the same as of S_t w.r.t. P, we can write for $t \ge s$

$$\begin{aligned}\mathsf{E}[\, Y_t \mid \mathcal{F}_s\,] &= \mathsf{E}\big[\, \mathsf{E}^*[\, g(S_T) \mid \mathcal{F}_t\,] \mid \mathcal{F}_s\,\big] \\ &\ge \mathsf{E}\big[\, \mathsf{E}^*[\, g(S_T') \mid \mathcal{F}_t\,] \mid \mathcal{F}_s\,\big] = \mathsf{E}\big[\, \mathsf{E}[\, g(S_T) \mid \mathcal{F}_t\,] \mid \mathcal{F}_s\,\big] = Y_s,\end{aligned}$$

which proves the statement a).

Statement b) is proved similarly.

Corollary 2.3. *If an option market price coincides with the Black–Scholes price, then*

a) $\tau_{opt} = 0$ *for* $\mu < r$,
b) $\tau_{opt} = T$ *for* $\mu > r$,
c) *any stopping time is optimal for* $\mu = r$.

Proof. This follows immediately from formula (6) and Lemma 2.2, and from the observation that Y_t is a strict P-sub- or P-supermartingale in case of strict inequalities in Lemma 2.2.

In other words, if the market price is given by the "fair" Black–Scholes price, then

b) it makes no difference when to resell it for $\mu = r$,
b) it should be resold immediately for $\mu < r$,
c) it should be held till the maturity for $\mu > r$.

Thus under conditions of Corollary 2.3 the problem (2) has no practical sense.

2.2. Stochastic model for the market price. For arbitrary moment $t \in [0, T)$, an *implied volatility* σ_t is defined as a solution to the equation

$$C_{BS}(S_t, T-t; \sigma_t, K, r) = C_t^m, \quad \sigma_t > 0. \tag{7}$$

If the right-hand side satisfies

$$(S_t - Ke^{-r(T-t)})_+ < C_t^m < S_t \tag{8}$$

then this equation has a unique solution, since the left-hand side is continuous, increasing in σ_t, tends to $(S_t - Ke^{-r(T-t)})_+$, as $\sigma_t \to 0$, and tends to S_t, as $\sigma_t \to +\infty$. It is natural to assume that inequalities (8), being universal bounds for the option price, are always valid. Thus we can construct a model for the market price in terms of the implied volatility σ_t. Note that $C_0^m = C_0$ implies $\sigma_0 = \sigma$.

We model this implied volatility σ_t as a *stochastic volatility.* Here we assume that it satisfies a linear stochastic differential equation

$$\frac{d\sigma_t}{\sigma_t} = \alpha\, dt + \beta\, dW_t^1, \tag{9}$$

where W_t^1 is Wiener process on $(\Omega, \widetilde{\mathcal{F}}_t, P)$ with a new filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$. The model (9) is a standard model of stochastic volatility, see, e.g., [1].

Also we will assume the following:

$$\text{Wiener processes } W_t \text{ and } W_t^1$$
$$\text{are jointly Gaussian and positively correlated.} \tag{10}$$

This condition can be understood as follows. If S_t grows, then so does σ_t, which makes C_t^m go beyond the "fair" price $C_t = C_{BS}(S_t, T-t, \sigma)$. This corresponds to Investor's aim to hold an option if the stock price is growing. On the other hand, when the stock price drops, Investor is willing to get rid of an option, which makes C_t^m go below its "fair" price.

Set $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$. In the reselling model (1), (7), (9) the optimal stopping time τ_{opt} is defined as maximizer of $\Psi(\tau)$ in a class of all stoping times w.r.t. the filtration $\mathcal{G}_t$.

3. Absence of arbitrage

We start with standard definition of arbitrage.

Definition 3.1. In the model (1), (7), (9) a stopping time τ is said to provide an *arbitrage possibility*, if

a) $P(e^{-r\tau}C_\tau^m \geq C_0) = 1$,
b) $P(e^{-r\tau}C_\tau^m > C_0) > 0$.

If there is no such stopping time, then the model is called *arbitrage-free.*

Before stating the main result, it is convenient to introduce dimension-free variables. Denote

$$\widetilde{C}_t = e^{r(T-t)} C_t / K, \quad \widetilde{C}^m_t = e^{r(T-t)} C^m_t / K,$$
$$v^m_t = \sigma_t \sqrt{T-t}.$$

Recall that $\widetilde{S}_t$ and v_t were introduced in (4). Then the Black–Scholes formula is reduced to

$$\widetilde{C}_t = f_{BS}(\widetilde{S}_t, v_t), \tag{11}$$

where

$$f_{BS}(s,v) = \widetilde{S}_t \Phi\Big(\frac{\log \widetilde{S}_t}{v_t} + \frac{v_t}{2}\Big) - \Phi\Big(\frac{\log \widetilde{S}_t}{v_t} - \frac{v_t}{2}\Big),$$

while v^m_t and $\widetilde{C}^m_t$ are related by an equality

$$\widetilde{C}^m_t = f_{BS}(\widetilde{S}_t, v^m_t).$$

Now, the universal bounds (8) take the form

$$(\widetilde{S}_t - 1)_+ < \widetilde{C}^m_t < \widetilde{S}_t.$$

We can assume that the the observed process $(\widetilde{S}_t, \widetilde{C}^m_t)$ takes values in the phase space

$$V := \{(s,c) : (s-1)_+ \le c \le s\}.$$

Theorem 3.2. *There is no arbitrage possibility in the model* (1), (7), (9).

Proof. Step 1. First we study the properties of the function $f = f_{BS}$. We are particularly interested in the properties of an (implicit) function $s(v)$ such that $f(s(v), v) = f(s_0, v_0)$, $s_0 = \widetilde{S}_0$ and v_0 is volatility at the moment 0. We have

$$f'_s(s,v) = \Phi\Big(\frac{\log s}{v} + \frac{v}{2}\Big),$$
$$f'_v(s,v) = \frac{s^{1/2}}{\sqrt{2\pi}} \exp\Big\{-\Big(\frac{v^2}{8} + \frac{\log^2 s}{2v^2}\Big)\Big\}.$$

This implies that the function $s(v)$ is decreasing. Moreover, we claim that it is infinitely differentiable due to such property of f and the implicit function theorem. Define for $\alpha > 0$ $g_\alpha(u) = s(u^\alpha)$, $u_0 = v_0^{1/\alpha}$. Now we are going to show that for some $\alpha \in (0,1)$

$$g_\alpha(u) > l_\alpha(u) := g'_\alpha(u_0)(u - u_0) + s_0, \text{ for all } u > 0,\ u \ne u_0. \tag{12}$$

First note that $s(0+) = f(s_0, v_0) + 1 > s_0$, thus for some $\delta > 0$ we have $s(v) > s_0$ on $(0, \delta]$, and $g_\alpha(u) > s_0$ on $(0, \delta^{1/\alpha}]$. Further,

$$l_\alpha(0) = g'_\alpha(u_0)(-u_0) + s_0 = s_0 - \alpha s'(u_0^\alpha) u_0^\alpha = s_0 - \alpha s'(v_0) v_0 \to s_0,\ \alpha \to 0+.$$

Thus for some $\alpha_0 > 0$ we have $g_\alpha(u) > l_\alpha(0) \ge l_\alpha(u)$ for all $\alpha \in (0, \alpha_0)$ and $u \in (0, \delta^{1/\alpha}]$.

Next,

$$g''_\alpha(u) = \alpha u^{\alpha-2}\big(\alpha s''(u^\alpha)u^\alpha + (\alpha-1)s'(u^\alpha)\big) = \\ = -\alpha u^{\alpha-2}s'(u^\alpha)\Big(1-\alpha-\alpha\frac{u^\alpha s''(u^\alpha)}{s'(u^\alpha)}\Big)$$

and the last term here can be evaluated by implicit differentiation

$$a(v) := \frac{vs''(v)}{s'(v)} = -\frac{v^2}{4} + \frac{\log^2 s}{v^2} + \frac{\exp\Big\{-\frac{\log^2 s}{2v^2} - \frac{v^2}{8}\Big\}}{\Phi\big(\frac{\log s}{v}+\frac{v}{2}\big)}\sqrt{\frac{2}{\pi s}}\Big(\frac{\log s}{v} - \frac{v}{2}\Big) \\ + \frac{\exp\Big\{-\frac{\log^2 s}{v^2} - \frac{v^2}{4}\Big\}}{\pi s \Phi^2\big(\frac{\log s}{v}+\frac{v}{2}\big)},$$

where s is an abbreviation for $s(v)$. Clearly, $a(v) \to -\infty$ as $v \to +\infty$. This means that it is bounded from above on $[\delta, +\infty)$ by some number M. Then for $\alpha \in (0, \frac{1}{M+1})$ and $u \in [\delta^{1/\alpha}, +\infty)$ it holds $g''_\alpha(u) < 0$, and the function g_α is strictly convex on $[\delta^{1/\alpha}, +\infty)$, therefore $g_\alpha(u) > l_\alpha(u)$, $u \in [\delta^{1/\alpha}, +\infty) \setminus \{u_0\}$.

Thus we have (12) for all $\alpha \in (0, \min(\alpha_0, \frac{1}{M+1}))$.

Step 2. Take $\alpha > 0$ such that (12) holds. Recall that $\widetilde{S}_t$ and σ_t are geometric Brownian motions, therefore $\sigma_t^{1/\alpha}$ is also a geometric Brownian motion, thus there exists an equivalent measure P^α, such that $\widetilde{S}_t$ and $\sigma_t^{1/\alpha}$ are martingales w.r.t. P^α. Consequently, $(v_t^m)^{1/\alpha}$ is a supermartingale w.r.t. P^α:

$$d(v_t^m)^{1/\alpha} = d(\sigma_t^{1/\alpha}(T-t)^{1/2\alpha}) = (T-t)^{1/2\alpha}d(\sigma_t^{1/\alpha}) - \frac{1}{2\alpha}\sigma_t^{1/\alpha}(T-t)^{1/2\alpha-1}dt.$$

Now assume that τ is such that it satisfies Definition . In terms of $\widetilde{C}_t^m$, this is equivalent to the following conditions:

a′) $P(\widetilde{C}_\tau^m \ge \widetilde{C}_0) = 1$,

b′) $P(\widetilde{C}_\tau^m > \widetilde{C}_0) > 0$.

Condition a′) means that $f(\widetilde{S}_\tau, v_\tau^m) \ge f(s_0, v_0)$. But $f(s, v)$ is increasing in s, hence

$$\widetilde{S}_\tau \ge s(v_\tau^m) = g_\alpha((v_\tau^m)^{1/\alpha}).$$

By (12), we can write further

$$\widetilde{S}_\tau \ge l_\alpha((v_\tau^m)^{1/\alpha}).$$

Taking expectations w.r.t. P^α, we get by martingale property of $\widetilde{S}_t$ and supermartingale property of $(v_t^m)^{1/\alpha}$, that

$$s_0 \ge s_0 + g'_\alpha(u_0)\mathsf{E}^\alpha[\,(v_\tau^m)^{1/\alpha} - u_0\,] \ge s_0,$$

with equality possible only if $\widetilde{S}_\tau = l_\alpha((v_\tau^m)^{1/\alpha})$ (mod P^α). The last means in particular that $g_\alpha((v_\tau^m)^{1/\alpha}) = l_\alpha((v_\tau^m)^{1/\alpha})$ (mod P^α). But then from (12) it follows that $v_\tau^{1/\alpha} = u_0 \pmod{P^\alpha}$, i.e., $v_\tau = v_0 \pmod{P^\alpha}$. On the other hand,

$$\widetilde{S}_\tau = g_\alpha((v_\tau^m)^{1/\alpha}) = s(v_0) = s_0 \pmod{P^\alpha}.$$

This, however, contradicts b′), as P^α is equivalent to P.

Remark 3.3. It is easy to see that Theorem 3.2 remains valid if only the following is assumed: for each α small enough there exists probability measure P^α such that S_t and $\sigma_t^{1/\alpha}$ are martingales w.r.t. P^α. This allows to consider a wide class of semimartingale models of the stock price and volatility, which satisfy the Novikov condition.

4. Estimation of model parameters

Estimation of α, β, and correlation coefficient ρ. According to the condition (10), the processes W_t and W_t^1 are positively correlated and we denote the correlation coefficient by ρ. Then W_t^1 can be decomposed as

$$W_t^1 = \rho W_t + \gamma W_t^2, \quad \gamma = \sqrt{1-\rho^2}, \tag{13}$$

where W_t^2 is a Wiener process independent of W_t. Hence the processes $\widetilde{S}_t$ and W_t^2 are independent as well, moreover,

$$\log\frac{\sigma_t}{\sigma} = x\log\frac{\widetilde{S}_t}{\widetilde{S}_0} + yt + zW_t^2, \tag{14}$$

where

$$x := \frac{\beta\rho}{\sigma}, \quad y := \alpha - \frac{\beta^2}{2} - (\mu - r - \frac{\sigma^2}{2})\frac{\beta\rho}{\sigma}, \quad z := \beta\gamma. \tag{15}$$

Suppose that the processes $\widetilde{S}_t$ and σ_t^m are observed at the moments $t_0 < t_1 < t_2 < \cdots < t_n$. Then by (14) we have for $k = 0, \ldots, n-1$

$$\frac{1}{b_k}\log\frac{\sigma_{t_{k+1}}^m}{\sigma_{t_k}^m} = x\frac{1}{b_k}\log\frac{\widetilde{S}_{t_{k+1}}}{\widetilde{S}_{t_k}} + yb_k + z\varepsilon_k, \tag{16}$$

where $\Delta t_k := t_{k+1} - t_k$, $b_k := \sqrt{\Delta t_k}$, $\varepsilon_k := (W_{t_k+1}^2 - W_{t_k}^2)/b_k$. Random variables $\{\varepsilon_k\}$ are independent and have standard Gaussian distribution. Denote

$$U_k = \frac{1}{b_k}\log\frac{\sigma_{t_{k+1}}}{\sigma_{t_k}}, \quad a_k = \frac{1}{b_k}\log\frac{\widetilde{S}_{t_{k+1}}}{\widetilde{S}_{t_k}}.$$

Then

$$u_k = xa_k + yb_k + z\varepsilon_k, \quad k = 0, \ldots, n-1. \tag{17}$$

This is a linear multiple regression model with a random regressor a_k and a nonrandom regressor b_k; $z\varepsilon_k$ are the observation errors with variance z^2.

Define the design matrix and the response vector

$$A = \begin{bmatrix} a_0 & b_0 \\ \vdots & \vdots \\ a_{n-1} & b_{n-1} \end{bmatrix}, \quad u = (u_0, \dots, u_{n-1})^\top.$$

Then, see e.g. [8], the maximum likelihood estimator for x and y in the model (17) coincides with a least squares estimator and is given by the formula

$$(\hat{x}, \hat{y})^\top = (A^\top A)^{-1} A^\top u. \tag{18}$$

An unbiased estimator for parameter z is

$$\hat{z} = \left(\frac{\left\| u - A\binom{\hat{x}}{\hat{y}} \right\|}{n-2} \right)^{1/2}. \tag{19}$$

Under fairly mild conditions estimates (18), (19) are strongly consistent and asymptotically normal, as $n \to \infty$. We substitute $\hat{x}$, $\hat{y}$ instead of x, y into (15), solve the system for α, β, and ρ, and get consistent estimators of unknown parameters. Remember that the parameters μ, σ, and r are assumed to be known.

Thus the proposed model (1), (7) is identifiable, i.e., additional parameters α, β, ρ are uniquely determined by the processes S_t and C_t^m.

5. Investor's strategy in the reselling problem

5.1. Stopping sets. In dimension-free variables, the problem (2) is equivalent to the optimization problem

$$\widetilde{\Psi}(\tau) := \mathsf{E}[\,\widetilde{C}_\tau^m\,] \to \max \tag{20}$$

in the class of all $\mathcal{G}_t$-stopping times. This problem is a problem of optimal realization of American type option with pay-off function

$$\widetilde{g}(\widetilde{S}_t, \widetilde{C}_t^m) := \widetilde{C}_t^m, \tag{21}$$

this is an option with maturity T on (correlated) stocks $\widetilde{S}_t$ and $\widetilde{C}_t^m$. As far as there is no discounting factor in (20), the interest rate for such an option is $\tilde{r} = 0$. Then (see e.g. [11]) the optimal reselling (or option exercise) time τ_{opt} is given by the formula

$$\tau_{\text{opt}} = \inf\{t \in [0, T] \mid \widetilde{C}_t^m \in G_t\}, \tag{22}$$

where the nonrandom stopping sets are given by

$$G_t = \{(s, c) \in V \mid c = f_t(s, c)\}, \tag{23}$$

the function $f_t(s, c)$ is the reward function,

$$f_t(s, c) := \sup_{\tau \in [t,T]} \mathsf{E}[\,\widetilde{C}_\tau^m \mid \widetilde{S}_t = s, \widetilde{C}_t^m = c\,], \tag{24}$$

the supremum is taken over all $\mathcal{G}_t$-stopping times, which belong to $[0,T]$. Since f_t is jointly continuous, G_t is a closed subset of V.

Unfortunately, we failed to prove that G_t has a threshold structure, i.e., that it consists of points lying beyond a certain curve. We thus propose a modification of the model (7), which has the stopping sets with the required property.

5.2. Modified model for option market price. For the model (1), (7) we can rewrite (14) in a form

$$\frac{\sigma_t}{\sigma} = \Big(\frac{\widetilde{S}_t}{\widetilde{S}_0}\Big)^x e^{yt+zW_t^2}, \quad t \in [0,T]. \tag{25}$$

We assume that a transaction can be made only at one of a finite number of moments

$$t \in \Pi_N := \{t_0 := 0 < t_1 < \cdots < t_N := T\}, \tag{26}$$

and that the stock price S_t and the option market price C_t^m are observed only at these moments. Instead of the relation (25), we adopt the following:

$$\frac{\sigma_t}{\sigma} = \Big(\frac{\widetilde{S}_t}{\widetilde{S}_0}\Big)^x e^{yt+z\sqrt{t}\varepsilon_t}, \quad t \in \Pi_N, \tag{27}$$

where $\varepsilon_{t_k},\ k = 0,\dots,N$ are i.i.d. variables with standard Gaussian distribution, which are independent of S_t. The conditional distribution $\mathcal{L}(\sigma_t \mid \widetilde{S}_t)$ in the models (25) and (27) is the same. The relation (27) means that the additional randomness on the reselling market has no memory and is renewed at each new moment, while in the model (25) the randomness is accumulated from the previous trading periods.

It can be shown by reasoning similar to the proof of Theorem 3.2 that in the discrete time model (1), (27) there is no arbitrage possibility as well (with slightly modified definition of an arbitrage, which involves Π_N-valued stopping times). Then the optimal reselling problem (2) is formulated in this class of stopping times.

Remark 5.1. In the model (1), (27) the parameters can be estimated similarly to the discussion of subsection 4.2, if we consider linear regression of the response variable $\frac{1}{\sqrt{t}}\log\frac{\sigma_t^m}{\sigma}$ to the covariates $\frac{1}{\sqrt{t}}\log\frac{\widetilde{S}_t}{S_0}$ and $\sqrt{t}$.

The optimal reselling problem in the model (1), (27) can be reduced to the problem of optimal exercise of the corresponding American option with discrete time, similarly to subsection 5.1. Then the optimal reselling time equals

$$\tau_{\text{opt}} = \min\{t_k : (\widetilde{S}_{t_k}, \widetilde{C}_{t_k}^m) \in F_k\}, \tag{28}$$

where $F_k,\ k = 0, 1, \dots, N$ are some nonrandom stopping sets. For $k = N$ we have $F_N = V$, and for $k = 0, \dots, N$

$$F_k := \{(s, c) \in V : c \geq f_k(s)\}, \tag{29}$$

$$f_k(s) := \sup_{\tau \in [t_{k+1}, T] \cap \Pi_N} \mathsf{E}[\, \widetilde{C}^m_t \mid \widetilde{S}_{t_k} = s\,], \tag{30}$$

where the upper bound is taken over all $\mathcal{G}_t$-stopping times.

Note that the reward function (30), in contrast to (24), depends only on s. The reason is independence of ε_t in (27): indeed, for $(s, c) \in V$ one has

$$\mathsf{E}[\, \widetilde{C}^m_\tau \mid \widetilde{S}_{t_k} = s, \widetilde{C}^m_{t_k} = c\,] = \mathsf{E}[\, \widetilde{C}^m_\tau \mid \widetilde{S}_{t_k} = s\,]$$

almost surely for $\tau \geq t_{k+1}$.

We see that in the model (1), (27) the stopping sets have threshold structure. If in the model (27) the parameter $x > 0$, then the premium function (30) is increasing and monotone. Moreover, at the moment t_{N-1} preceding the maturity,

$$\begin{aligned} f_{N-1}(s) &= \mathsf{E}[\, \widetilde{C}^m_T \mid \widetilde{S}_{t_{N-1}} = s\,] = \mathsf{E}[\, (\widetilde{S}_T - 1)_+ \mid \widetilde{S}_{t_{N-1}} = s\,] \\ &= f_{BS}(s, \sigma\sqrt{T - t_{N-1}}), \end{aligned}$$

where the function f_{BS} is defined by (11). Thus for the moment t_{N-1} the stopping set is known:

$$F_{N-1} = \{(s, c) \in V \mid c \geq f_{BS}(s, \sigma\sqrt{T - t_{N-1}})\}. \tag{31}$$

The function f_{BS} is strictly convex in s, therefore the threshold curve for F_{N-1} is strictly convex.

5.3. Construction of the stopping sets in the modified model. Following [3] and [10], we apply the dynamical programming and Monte Carlo technique to construct stopping sets F_k in the model (1), (27). The set F_{N-1} is already constructed in (31). The stopping set F_{N-2} is then built in the following manner.

Fix the vertical lines grid $s = s_i,\ i \geq 1$, in the phase space V. On the line $s = s_i$ we have to find a threshold point $f_{N-2}(s_i)$. In order to do that, we first take a point $M_1 = (s_i, c_1) \in V$, and we should decide, whether or not $M_1 \in V$. First we simulate a path $(\widetilde{S}_t, \widetilde{C}^m_t)$, $t = t_{N-2}$, t_{N-1}, t_N, which starts at M_1. If it gets to F_{N-1} at the moment t_{N_1}, we put $\tau(1) = t_{N-1}$ and stop, otherwise we put $\tau(1) = t_N$, and we calculate $\widetilde{C}^m_{\tau(1)}$ for this path. Repeating this procedure for M paths, we obtain values $\widetilde{C}^m_{\tau(k)}$, $k = 1, \dots, M$. If $c_1 \geq \frac{1}{M} \sum_{k=1}^M \widetilde{C}^m_{\tau(k)}$, then we make a decision that $M_1 \in F_{N-2}$, otherwise we decide that $M_1 \notin F_{N-2}$. The threshold point $M(s_i, f_{N-2}(s_i))$ is found by the dichotomy procedure.

The similar operations are made for all vertical lines of the fixed grid, and we use the fact that the function f_{N-2} is increasing. This way we have obtained a discrete approximation of the threshold curve. Then we make a linear interpolation. The points lying on this curve and above form a set $\widehat{F}_{N-2}$, which is an approximation to F_{N-2}.

Next we construct an approximation $\widehat{F}_{N-3}$ of F_{N-3} similarly. Again, we fix a grid of vertical lines in the phase space V. Then a trial point $M_1 = (s_i, c_1) \in V$ is taken, and we generate a path $(\widetilde{S}_t, \widetilde{C}_t^M)$, $t = t_{N-3}, \dots, t_N$, starting at M_1. If it gets to $\widehat{F}_{N-2}$ at the moment t_{N-2}, we set $\tau(1) = t_{N-2}$ and stop, if it gets to F_{N-1} at t_{N-1}, we set $\tau(1) = t_{N-1}$ and stop, otherwise we set $\tau(1) = t_N$. Repeating this procedure M times, we get corresponding values $\widetilde{C}^m_{\tau(k)}$, $k = 1, \dots, M$. If $c_1 \geq \frac{1}{M}\sum_{k=1}^{M} \widetilde{C}^m_{\tau(k)}$, we decide that $M_1 \in F_{N-3}$, otherwise $M_1 \notin F_{N-3}$. Then we again use dichotomy, get approximation of the threshold curve and corresponding approximation $\widehat{F}_{N-3}$ of the stopping set F_{N-3}. Next we construct $\widehat{F}_{N-4}, \dots, \widehat{F}_0$. Thus we utilize the dynamic programming to construct stopping sets backwards in time, using Monte Carlo method to calculate the reward function (24).

6. Conclusion

We have considered the problem of European option reselling and proposed a stochastic model for option market price. For a wide class of models, which includes the proposed one, absence arbitrage opportunities is shown. Optimal strategy for Investor in this model is described by nonrandom stopping sets in the phase space of possible stock prices and option market prices. For the modified model the threshold structure is established.

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