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PARAMETER ESTIMATORS OF NONLINEAR QUANTILE REGRESSION

We have obtained the asymptotic normality of parameter estimators of a nonlinear quantile regression with nonsymmetric random noise.

Introduction

Here, we examine the asymptotic normality of Koenker and Basset estimators [1] or the generalized least moduli estimators (GLME) of nonlinear regression model parameters that generalize least moduli estimators for non-symmetric observation errors.

The consistency property of GLME has been considered in [2].

1. Assumptions and the main result

Suppose that an observation X_j is a r.v. with values in $(\mathbb{R}^1, \mathcal{B}^1)$ ($\mathbb{R}^1$ is a real line, $\mathcal{B}^1$ - σ-algebra of its Borel subsets) and distribution P_j . We also assume that the unknown distribution P_j belongs to a certain parametric family $\{P_{i\theta},\ \theta \in \Theta\}$. We call the triple $\mathcal{E}_j = \{\mathbb{R}^1, \mathcal{B}^1, P_{j\theta},\ \theta \in \Theta\}$ a statistical experiment generated by the observation X_j.

We say that a statistical experiment $\mathcal{E}^n = \{\mathbb{R}^n, \mathcal{B}^n, P^n_\theta,\ \theta \in \Theta\}$ is the product of the statistical experiments $\mathcal{E}_i,\ i = 1, ..., n$, if $P^n_\theta = P_{1\theta} \times ... \times P_{n\theta}$ ($\mathbb{R}^n$ - n-dimensional Euclidean space and $\mathcal{B}^n$ - σ-algebra of its Borel subsets). We say that the experiment $\mathcal{E}^n$ is generated by n independent observations $X = (X_1, .., X_n)$.

Let the observations have the form

$$X_j = g(j, \theta) + \varepsilon_j,\ j = 1, ..., n\ , \tag{1.1}$$

where $g(j, \theta)$ is a non-random sequence of functions defined on Θ^c, Θ^c is the closure of an open convex set $\Theta \subset \mathbb{R}^q$ in $\mathbb{R}^q$, and

A1. ε_j are independent identically distributed random variables (r.v.) with zero mean, distribution function $\mathcal{P}$, and

$$\mathcal{P}(0) = \beta,\ \beta \in (0, 1). \tag{1.2}$$

It is not supposed that the functions $g(j, \theta)$ are the linear forms of coordinates of the vector θ.

Definition. *GLME of the parameter $\theta \in \Theta$ obtained by the observations X_j, $j = 1, ..., n$ of the form* (1.1) *is said to be any random vector $\widehat{\theta}_n = \widehat{\theta}_n(X_j,\ j = 1, ..., n) \in \Theta^c$ having the property*

$$S_\beta(\widehat{\theta}_n) = \inf_{\tau \in \Theta^c} S_\beta(\tau),\quad S_\beta(\tau) = \sum \rho_\beta(X_j - g(j, \tau)), \tag{1.3}$$

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where $\sum = \sum_{j=1}^{n}$ *and*

$$\rho_\beta(x) = \begin{cases} \beta x, & x \geq 0 \\ (\beta - 1)x, & x < 0 \end{cases}, \quad \beta \in (0,1). \tag{1.4}$$

Since $P_\theta^n\{X_j < g(j,\theta)\} = P_\theta^n\{\varepsilon_j < 0\} = F(0) = \beta$, the observation model (1.1) can be interpreted as a nonlinear quantile regression [1]. Indeed, $\widehat{\theta}_n$ estimates the β-quantile $g(j,\theta)$ of observations $X_j,\ j = 1, ..., n$.

Let us impose some restrictions on r.v. ε_j:

A2. $\mu_s = E|\varepsilon_j|^s < \infty$ for some natural s.

A3. R.v. ε_j has a bounded density $p(x) = \mathcal{P}'(x)$ with the property

$$|p(x) - p(0)| \leq H|x|, \quad p(0) > 0,$$

where $H < \infty$ is a certain constant.

Example. *A r.v.* $\xi = \chi^2_{2m} - 2m$, *where* χ^2_{2m} *has chi-squared distribution with even degrees of freedom, satisfies conditions* **A1-A3**.

Denote, by $\mathcal{C}^q \subset \mathcal{B}^q$, the class of all convex Borel subsets of $\mathbb{R}^q$ and, by $T \subset \Theta$, some compact.

Let us introduce the notation

$$g_i(j,\tau) = \frac{\partial}{\partial \tau^i} g(j,\tau), \quad g_{il}(j,\tau) = \frac{\partial^2}{\partial \tau^i \partial \tau^l} g(j,\tau),$$
$$d^2_{in}(\theta) = \sum g_i^2(j,\theta), \quad d^2_{il,n}(\tau) = \sum g^2_{il}(j,\tau), \ \tau \in \Theta^c, \ i,l = 1, ..., q.$$

Here, $d^2_n(\theta)$ is a diagonal matrix with elements $d^2_{in}(\theta),\ i = 1, ..., q$ on the diagonal.

Consider the change of variables $u = n^{-1/2} d_n(\theta)(\tau - \theta)$, i.e.

$$g(j,\tau) = g(j, \theta + n^{1/2} d_n^{-1}(\theta) u) = f(j,u),$$

assuming that θ is a true value of the parameter. Under this change of variables, the set Θ turns to the set $\widetilde{U}_n(\theta) = n^{-1/2} U_n(\theta)$, where $U_n(\theta) = d_n(\theta)(\Theta - \theta)$, and GLME $\widehat{\theta}_n$ turns to a normed random vector $\widehat{u}_n = n^{-1/2} d_n(\theta)(\widehat{\theta}_n - \theta)$.

We will denote positive constants by the letter k. Suppose that

B1. Functions $g(j,\theta),\ j \geq 1$ are continuous on Θ^c together with all the first partial derivatives, and $g_i(j,\theta),\ i = 1, ..., q,\ j \geq 1$, are continuously differentiable in Θ. Moreover, for any $R \geq 0$,

$$\text{(i)} \quad \sup_{\theta \in T} \sup_{u \in v(R) \cap \widetilde{U}_n^c(\theta)} \max_{1 \leq j \leq n} \frac{|f_i(j,u)|}{d_{in}(\theta)} \leq k^i(R) n^{-1/2}, \ i = 1, ..., q, \tag{1.5}$$

$$\text{(ii)} \quad \sup_{\theta \in T} \sup_{u \in v(R) \cap \widetilde{U}_n^c(\theta)} \frac{d_{il,n}(\theta + n^{1/2} d_n^{-1/2}(\theta) u)}{d_{in}(\theta) d_{ln}(\theta)} \leq k^{il}(R) n^{-1/2}, \ i,l = 1, ..., q. \tag{1.6}$$

It follows from (1.5) that

$$\sup_{\theta \in T} \sup_{u_1,\, u_2 \in v^c(R) \cap \widetilde{U}_n^c(\theta)} n^{-1} \frac{\Phi_n(u_1,u_2)}{|u_1 - u_2|^2} \leq k(R), \tag{1.7}$$

where $\Phi_n(u_1,u_2) = \sum (f(j,u_1) - f(j,u_2))^2$.

Similarly, relation (1.6) yields the inequality

$$\sup_{\theta \in T} \sup_{u_1,\, u_2 \in v^c(R) \cap \widetilde{U}_n^c(\theta)} \frac{\Phi_n^{(i)}(u_1,u_2)}{d^2_{in}(\theta)|u_1 - u_2|^2} \leq \tilde{k}^{(i)}(R), \tag{1.8}$$

with $\Phi_n^{(i)}(u_1,u_2) = \sum((f_i(j,u_1) - f_i(j,u_2))^2, \quad i = 1,...,q$.

Suppose that GLME is consistent, namely:

C. For any $r > 0$

$$\sup_{\theta\in T} P_\theta^n\{|n^{-1/2}d_n(\theta)(\widehat{\theta}_n - \theta)| \geq r\} = \begin{cases} O(n^{-s+1}), & s \geq 2, \\ o(1), & s = 1. \end{cases}.$$

The sufficient conditions for **C** to be fulfilled are stated in [2].

Let us denote

$$I(\theta) = \left(d_{in}^{-1}(\theta)d_{ln}^{-1}(\theta)\sum g_i(j,\theta)g_l(j,\theta)\right)_{i,l=1}^q, \quad \theta \in \Theta.$$

The matrix $I(\theta)$ is symmetric and non-negative definite. Let $\lambda_{\min}(I(\theta))$ be the smallest eigenvalue of $I(\theta)$. Assume that

B2. For $n > n_0$, $\inf_{\theta\in T}\ \lambda_{\min}(I(\theta)) \geq \lambda_0 > 0$.

Let l be an arbitrary direction in $\mathbb{R}^q$, and $\tau \in \Theta$. Then

$$\frac{\partial}{\partial l}S_\beta(\tau) = \sum \langle\nabla g(j,\tau), l\rangle\,(\chi\{X_j * g(j,\tau) - \beta\}),$$

where "$*$" denotes "$\leq$" if $\langle\nabla g(j,\tau), l\rangle \geq 0$ and "$<$" if $\langle\nabla g(j,\tau), l\rangle < 0$. Let r_0 be a distance between T and $\mathbb{R}^q\backslash\Theta$. If an event $\{|\hat{\theta}_n - \theta| < r\}$ occurs for $\theta \in T$ and $r < r_0$, then, for any direction l,

$$\frac{\partial}{\partial l}S_\beta(\hat{\theta}_n) \geq 0.$$

This remark will be used in the proof of the main result.

Theorem. *If conditions* **A1 - A3**, **B1**, **B2**, *and* **C** *are fulfilled, then*

$$\sup_{\theta\in T}\ \sup_{C\in\mathcal{C}^q}\left|P_\theta^n\left\{\frac{p(0)}{\sqrt{\beta(1-\beta)}}\ I^{1/2}(\theta)d_n(\theta)(\widehat{\theta}_n - \theta) \in C\right\} - \Phi(C)\right| \underset{n\to\infty}{\longrightarrow} 0, \tag{1.9}$$

where $\Phi(C) = \int_C \dfrac{1}{(2\pi)^{q/2}}\ e^{-\frac{\|x\|^2}{2}}dx.$

In other words, the normal distribution $N\left(0, \frac{\beta(1-\beta)}{p^2(0)}\ I^{-1}(\theta)\right)$ is the accompanying law for the distribution of the normed estimator $d_n(\theta)(\widehat{\theta}_n - \theta)$.

2. Auxiliary assertions

We carry out the proof by the scheme of the theorem on asymptotic normality of the least moduli estimators [3], by using the method of partitioning a parametric set [4,5].

Let $l_1,...,l_q$ be the positive directions of the coordinate axes. Let us consider the vectors $S_\beta^\pm(\tau)$ with coordinates

$$S_{i\beta}^\pm(\tau) = d_{in}^{-1}(\theta)\left(\frac{\partial}{\partial(\pm l_i)}\right)S_\beta(\tau), \quad i = 1,...,q,$$

and the vectors $E_\theta^n S_\beta^\pm(\theta)$ with coordinates

$$E_\theta^n S_{i\beta}^\pm(\tau) = \pm d_{in}^{-1}(\theta)\sum g_i(j,\tau)[\mathcal{P}(g(j,\tau) - g(j,\theta)) - \beta], \quad i = 1,...,q.$$

Clearly,

$$E_\theta^n S_\beta^\pm(\theta) = 0,$$

due to assumption **A1**. Let us denote $S_\beta^{*\pm}(u) = S_\beta^{\pm}(\theta + n^{1/2} d_n^{-1}(\theta)u)$ and

$$z_n^{\pm}(\theta, u) = \frac{\left| S_\beta^{*\pm}(u) - S_\beta^{*\pm}(0) - E_\theta^n S_\beta^{*\pm}(u) \right|}{1 + \left| E_\theta^n S_\beta^{*\pm}(u) \right|}.$$

Lemma 1. *Under the conditions of the theorem, for any $\epsilon > 0$ and sufficiently small $r > 0$,*

$$\sup_{\theta \in T} P_\theta^n \left\{ \sup_{u \in v^c(r) \cap \tilde{U}_n^c(\theta)} z_n^{\pm}(\theta, u) > \epsilon \right\} \underset{n \to \infty}{\longrightarrow} 0. \tag{2.1}$$

Proof. We will proof the statement for $z_n^+(\theta, u)$. Assume, for simplicity, that $r = 1$ and the inner supremum in (2.1) is defined in a cube

$$C_0 = \left\{ u : |u|_0 = \max_{1 \le i \le q} |u_i| \le 1 \right\} \supset v(1).$$

Let us cover the cube C_0 with $N_0 = O(\ln n)$ cubes $C_{(1)}, ..., C_{(N_0)}$ in the following way. For the number $t \in (0, 1)$, we consider a concentric system of sets

$$\begin{aligned} C^{(m)} =& \{u : |u|_0 \in [(1-t)^{m+1}, (1-t)^m]\}, \quad m = 0, \dots, m_0 - 1, \\ C^{(m_0)} =& \{u : |u|_0 \le (1-t)^{m_0}\}. \end{aligned}$$

We cover each of the sets $C^{(m)}$ by identical cubes with sides

$$a_m = (1-t)^m - (1-t)^{m+1} = t(1-t)^m$$

and enumerate these cubes. They form the required covering

$$C_{(1)}, \dots, C_{(N_0 - 1)}, C_{(N_0)} =^{\text{def}} C^{(m_0)}.$$

Let us choose $m_0 = m_0(n)$ from the condition $(1-t)^{\tilde{m}_0} = n^{-\gamma}$, $\quad m_0 = [\tilde{m}_0]$, $\quad \gamma \in (\frac{1}{2}, 1)$.

We denote, by $|\cdot|_0$, the distance from $C_{(j)}$ to 0 which is equal to

$$r(j) = (1-t) n^{-\gamma m / \tilde{m}_0},$$

and, by $|\cdot|_0$, the diameter of $C_{(j)}$ which is equal to

$$a(j) = t n^{-\gamma m / \tilde{m}_0}$$

for some $m = m(j)$, $j = 1, ..., N_0 - 1$. Moreover, if the cube $C_{(j)}$ is an element of the covering of the sets $C^{(m)}$, then

$$a(j) = a_m, \quad r(j) = t(1-t)^{m+1} + ... + t(1-t)^{m_0 - 1} + (1-t)^{m_0}.$$

The number of cubes $C_{(j)}$ covering each set $C^{(m)}$ can be made not depending on m and, consequently, on n. In order to verify this, let us consider any octant in $\mathbb{R}^q$. The volume occurring in its part of the set $C^{(m)}$ is $(1-t)^{mq} - (1-t)^{(m+1)q}$, and the volume of the sets $C_{(j)}$ is equal to $a^q(j) = t^q(1-t)^{mq}$. In this way, the maximum number of cubes $C_{(j)}$ that can be "placed" in the part of $C^{(m)}$ that belongs to the given octant is equal to

$$\frac{(1-t)^{mq} - (1-t)^{(m+1)q}}{t^q (1-t)^{mq}} = \frac{1 - (1-t)^q}{t^q}$$

cubes. Since $m_0 = O(\ln n)$, $N_0 = O(\ln n)$ as well. Let us fix $\theta \in T$. Then

$$P_\theta^n \left\{ \sup_{u \in C_0} z_n^+(\theta, u) > \epsilon \right\} \le \sum_{j=1}^{N_0} P_\theta^n \left\{ \sup_{u \in C_{(j)}} z_n^+(\theta, u) > \epsilon \right\}. \tag{2.2}$$

Let us estimate each term in (2.2). The general element of the derivative matrix $D_n(u)$ of the mapping

$$u \longrightarrow E_\theta^n S_\beta^{*+}(u)$$

has the form

$$\begin{aligned} D_n^{il}(u) &= \frac{\partial}{\partial u_l} E_\theta^n S_{i\beta}^{*+}(u) \\ &= n^{1/2} d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \sum f_{il}(j,u)[\mathcal{P}(g(j,\tau)-g(j,\theta))-\beta] \\ &\quad + n^{1/2} d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \sum f_i(j,u) f_l(j,u)\, p(g(j,\tau)-g(j,\theta)) \\ &= {}_1D_n^{il}(u) +_2 D_n^{il}(u). \end{aligned}$$

Taking into account (1.6), (1.7), and the inequality

$$\sup_{x\in\mathbb{R}^1} p(x) = p_0 < \infty,$$

we obtain, for $|u| < r$,

$$\begin{aligned} n^{-1/2}|_1D_n^{il}(u)| \;\le\; & n^{1/2} d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) d_{il,n}(\theta + n^{1/2} d_n^{-1}(\theta)u) \times \\ \times & \left(n^{-1}\sum(\mathcal{P}(f(j,u)-f(j,0))-\mathcal{P}(0))^2\right)^{1/2} \le k^{(il)}(r) k^{1/2}(r) p_0 |u|. \end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned} & \left| n^{-1/2}\ {}_2D_n^{il}(u) - p(0) I_{il}(\theta) \right| \le \\ & \le\ p_0 \left[d_{in}^{-1}(\theta)\ d_{in}(\theta + n^{1/2} d_n^{-1}(\theta)u)\ d_{ln}^{-1}(\theta) \left(\Phi_n^{(l)}(u,0)\right)^{1/2} + d_{in}^{-1}(\theta) \left(\Phi_n^{(i)}(u,0)\right)^{1/2} \right] \\ & + d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \left| \sum g_i(j,\theta) g_l(j,\theta)(p(f(j,u)-f(j,0))-p(0)) \right|. \end{aligned} \tag{2.4}$$

It follows from (1.5) and (1.8) that the terms in square brackets are bounded by the quantity

$$p_0 \left((\tilde{k}^{(i)})^{1/2} + k^{(i)}(r) (\tilde{k}^{(l)})^{1/2} \right) |u|.$$

For another term on the right-hand side of (2.4), we can find, by using condition **A3** and (1.5), the upper bound

$$\begin{aligned} & n^{1/2} d_{in}^{-1}(\theta) \max_{1\le j\le n} |g_i(j,\theta)| \left(n^{-1}\sum(p(f(j,u)-f(j,0))-p(0))^2\right)^{1/2} \\ & \le k^{(i)}(r) H k^{1/2}(r) |u|. \end{aligned} \tag{2.5}$$

Since the matrix $n^{-1/2} D_n(0) = p(0) I(\theta)$ is positive definite by condition **B2**, it follows from the above-presented considerations that, for sufficiently small u (for simplicity we assume that $u \in C_0$) and some $k_0 > 0$,

$$\inf_{\theta\in T} \left| E_\theta^n S_\beta^+(\theta + n^{1/2} d_n^{-1}(\theta)u) \right| \ge k_0 n^{1/2} |u|_0. \tag{2.6}$$

Let $l \ne N_0$, and let $v \in C_{(l)}$ be an arbitrary point. Then, in view of (2.6), we can write

$$\sup_{u\in C_{(l)}} z_n^+(\theta,u) \le \left(\sup_{u\in C_{(l)}} M_n^{(l)}(\theta,u,v) + L_n^{(l)}(\theta,v) \right) (1 + k_0 n^{1/2} r(l))^{-1},$$

$$M_n^{(l)}(\theta,u,v) = \sum_{\lambda=1}^{4} M_{\lambda n}^{(l)}(\theta,u,v) \quad (\mod P_\theta^n)$$

$$M_{1n}^{(l)}(\theta,u,v) = \left| d_n^{-1}(\theta)\sum \nabla f(j,u)\left(\chi\{X_j * f(j,u)\} - \chi\{X_j < f(j,v)\}\right)\right|$$
$$M_{2n}^{(l)}(\theta,u,v) = \left| d_n^{-1}(\theta)\sum (\nabla f(j,u) - \nabla f(j,v))(\chi\{X_j < f(j,v)\} - \beta)\right|$$
$$M_{3n}^{(l)}(\theta,u,v) = \left| d_n^{-1}(\theta)\sum \nabla f(j,u)\left(\mathcal{P}(f(j,u)-f(j,0)) - \mathcal{P}(f(j,v)-f(j,0))\right)\right|$$
$$M_{4n}^{(l)}(\theta,u,v) = \left| d_n^{-1}(\theta)\sum (\nabla f(j,u) - \nabla f(j,v))(\mathcal{P}(f(j,v)-f(j,0)) - \beta)\right|$$
$$L_n^{(l)}(\theta,v) = \Big| d_n^{-1}(\theta)\sum (\nabla f(j,v)(\chi\{X_j < f(j,v)\} - \beta) - \nabla f(j,0)(\chi\{\varepsilon_j * 0\} - \beta)$$
$$-\nabla f(j,v)(\mathcal{P}(f(j,v)-f(j,0)) - \beta)| \quad (\mod P_\theta^n).$$

By (1.8) and for $u,\ v \in C_{(l)}$, we obtain

$$n^{-1/2}M_{2n}^{(l)}(\theta,u,v) \le \beta' \left(\sum_{i=1}^q d_{in}^{-2}(\theta)\Phi_n^{(i)}(u,v)\right)^{1/2} \le k_1 a(l). \tag{2.7}$$

Furthermore, in accordance with (1.5), (1.7), and **A3**, we get

$$n^{-1/2}M_{3n}^{(l)}(\theta,u,v) \le p_0 n^{-1/2}\Phi_{2n}^{1/2}(u,v)\left(\sum_{i=1}^q \frac{d_{in}^2(\theta + n^{1/2}d_n^{-1}(\theta)u)}{d_{in}^2(\theta)}\right)^{1/2} \le k_2 a(l). \tag{2.8}$$

Analogously,

$$n^{-1/2}M_{4n}^{(l)}(\theta,u,v) \le p_0 n^{-1/2}\Phi_{2n}^{1/2}(v,0)\left(\sum_{i=1}^q d_{in}^{-2}(\theta)\Phi_n^{(i)}(u,v)\right)^{1/2} \le k_3 a(l). \tag{2.9}$$

Let us estimate $M_{1n}^{(l)}(\theta,u,v)$. For any $u,v \in C_{(l)}$,

$$|\chi\{X_j * f(j,u)\} - \chi\{X_j < f(j,v)\}|$$
$$\le \chi\left\{\inf_{u\in C_{(l)}} f(j,u) - f(j,0) \le \varepsilon_j \le \sup_{u\in C_{(l)}} f(j,u) - f(j,0)\right\} = \chi_j \quad (\mod P_\theta^n).$$

Consequently, by (1.5),

$$n^{-1/2}M_{1n}^{(l)}(\theta,u,v) \le n^{-1/2}\left(\sum_{i=1}^q \left(d_{in}^{-1}(\theta)\max_{1\le j\le n}|f_i(j,u)|\right)^2\right)^{1/2}\sum \chi_j$$
$$\le k_4 n^{-1}\sum \chi_j. \tag{2.10}$$

Using the formula for finite increments, we find

$$n^{-1}\sum E_\theta^n \chi_j = n^{-1}\sum\left(\mathcal{P}\left(\sup_{u\in C_{(l)}} f(j,u) - f(j,0)\right) - \mathcal{P}\left(\inf_{u\in C_{(l)}} f(j,u) - f(j,0)\right)\right)$$
$$\le p_0 n^{-1}\sum \sup_{u_1,u_2\in C_{(l)}} |f(j,u_1) - f(j,u_2)|$$

$$\le p_0 q^{1/2}\left(\sum_{i=1}^q\left(n^{1/2}d_{in}^{-1}(\theta)\sup_{u\in C_{(l)}}\max_{1\le j\le n}|f_i(j,u)|\right)^2\right)^{1/2} a(l) \le k_5 a(l). \tag{2.11}$$

Estimates (2.7)-(2.11) show that there exist constants k_6 and k_7 such that

$$P_\theta^n\left\{\sup_{u\in C_{(l)}} M_n^{(k)}(\theta,u,v)(1 + k_0 n^{1/2} r(l))^{-1} > \frac{\epsilon}{2}\right\}$$

$$\leq P_\theta^n \left\{ k_6 n^{-1} \sum (\chi_j - E_\theta^n \chi_j) > \frac{\epsilon}{2} r(l) - k_7 a(l) \right\}. \tag{2.12}$$

Note that $\frac{\epsilon}{2} r(l) - k_7 a(l) = \left(\frac{\epsilon}{2}(1-t) - k_7 t \right) n^{-\gamma m/\tilde{m}_0} > 0$, if t is chosen sufficiently small. Therefore, probability (2.12) can be estimated, with the help of the Chebyshev inequality and (2.11), by the quantity

$$\frac{4k_6^2}{(\epsilon(1-t) - 2k_7 t)^2} n^{-2+2\gamma m/\tilde{m}_0} \sum E_\theta^n \chi_j \leq k_8 n^{-1+\gamma m/\tilde{m}_0}. \tag{2.13}$$

Using the notation

$$L_{1i}(j) = (f_i(j,v) - f_i(j,0))(\chi\{X_j < f(j,v)\} - \beta),$$
$$L_{2i}(j) = f_i(j,0)(\chi\{X_j < f(j,v)\} - \chi\{\varepsilon_j * 0\}), \ i = 1, ..., q,$$

we obtain

$$P_1 \ = \ P_\theta^n \left\{ L_n^{(k)}(\theta, v)(1 + k_0 n^{1/2} r(l))^{-1} > \frac{\epsilon}{2} \right\}$$

$$\leq \ \frac{4}{n(k_0\epsilon)^2 r^2(l)} \sum_{i=1}^q d_{in}^{-2}(\theta) \sum_{\lambda=1}^2 E_\theta^n \left(\sum (L_{\lambda i}(j) - E_\theta^n L_{\lambda i}(j)) \right)^2, \tag{2.14}$$

$$D_\theta^n(\sum L_{1i}(j)) \ \leq \ \Phi_{2n}^{(i)}(v, 0), \tag{2.15}$$

$$D_\theta^n(\sum L_{2i}(j)) \ \leq \ \sum f_i^2(j,0) |\mathcal{P}(f(j,v) - f(j,0)) - \mathcal{P}(0)|$$

$$\leq p_0 \max_{1 \leq j \leq n} |g_i(j,\theta)| d_{in}(\theta) \Phi_{2n}^{1/2}(v,0). \tag{2.16}$$

It follows from relations (2.14)-(2.16) and the conditions of the theorem that

$$P_1 \leq \frac{4n^{-1}}{(k_0\epsilon)^2} \left[\frac{(r(l) + a(l))^2}{r^2(l)} \sum_{i=1}^q \tilde{k}^{(i)}(1) + \frac{r(l) + a(l)}{r^2(l)} p_0 k^{1/2}(1) \sum_{i=1}^q k^{(i)}(1) \right]$$

$$\leq k_9 n^{-1} \left[(1-t)^{-2} + (1-t)^{-2} n^{\gamma m/\tilde{m}_0} \right] = O\left(n^{-1+\gamma m/\tilde{m}_0} \right). \tag{2.17}$$

Inequalities (2.13) and (2.17) show that, for $l = 1, ..., N_0 - 1$ and some $m = m(l) < m_0$,

$$\sup_{\theta \in T} P_\theta^n \left\{ \sup_{u \in C_{(l)}} z_n^+(\theta, u) > \epsilon \right\} = O\left(n^{-1+\gamma m/\tilde{m}_0} \right). \tag{2.18}$$

Let us consider the case $l = N_0$. Clearly,

$$P_\theta^n \left\{ \sup_{u \in C_{(N_0)}} z_n^+(\theta, u) > \epsilon \right\} \leq$$

$$\leq P_\theta^n \left\{ \sup_{|u|_0 < n^{-\gamma m/\tilde{m}_0}} \left| S_\beta^{*+}(u) - S_\beta^{*+}(0) - E_\theta^n S_\beta^{*+}(u) \right| > \epsilon \right\}. \tag{2.19}$$

Let us rewrite the expression standing under the sign of supremum in (2.19) in the form of $\nu_1(\theta,u) + \nu_2(\theta,u) + \nu_3(\theta,u)$, where

$$\nu_1(\theta, u) = d_n^{-1}(\theta) \sum (\nabla f(j,u) - \nabla f(j,0))(\chi\{X_j * f(j,u)\} - \beta),$$
$$\nu_2(\theta, u) = d_n^{-1}(\theta) \sum \nabla f(j,0)(\chi\{X_j * f(j,u)\} - \chi\{\varepsilon_j * 0\}),$$
$$\nu_3(\theta, u) = d_n^{-1}(\theta) \sum \nabla f(j,u)(\mathcal{P}(f(j,u) - f(j,0)) - \beta).$$

It is easy to show that, for $|u|_0 < n^{-\gamma m/\tilde{m}_0}$,

$$|\nu_1(\theta,u)| \le \beta' n^{\frac{1}{2}} \left(\sum_{i=1}^{q} d_{in}^{-2}(\theta)\Phi_{2n}^{(i)}(u,0)\right)^{1/2} \le k_1 n^{\frac{1}{2}-\frac{\gamma m}{\tilde{m}_0}}, \tag{2.20}$$

$$|\nu_3(\theta,u)| \le p_0 \Phi_{2n}^{\frac{1}{2}}(u,0) \left(\sum_{i=1}^{q} \frac{d_{in}^2(\theta + n^{1/2} d_n^{-1}(\theta)u)}{d_{in}^2(\theta)}\right)^{1/2} \le k_2 n^{\frac{1}{2}-\frac{\gamma m}{\tilde{m}_0}}, \tag{2.21}$$

where k_1 and k_2 are the same as in (2.7) and (2.8), correspondingly.

If $\gamma > \frac{1}{2}$, then the exponents in (2.20) and (2.21) are negative for $n > n_0$. That is, for $\epsilon' < \epsilon$, it remains to estimate the probability

$$\begin{aligned} P_\theta^n &\left\{ \sup_{|u|_0 < n^{-\gamma m/\tilde{m}_0}} |\nu_2(\theta,u)| > \epsilon' \right\} \\ &\le P_\theta^n \left\{ \left(\sum_{i=1}^{q} \left(d_{in}^{-1}(\theta) \max_{1\le j\le n} |g_i(j,\theta)| \right)^2 \right)^{1/2} \sum \tilde{\chi}_j > \epsilon' \right\}, \\ &\le P_\theta^n \left\{ k_4 n^{-1/2} \sum \tilde{\chi}_j > \epsilon' \right\}, \end{aligned} \tag{2.22}$$

$$\tilde{\chi}_j = \chi \left\{ \inf_{|u|_0 \le n^{-\gamma m/\tilde{m}_0}} f(j,u) - f(j,0) \le \varepsilon_j \le \sup_{|u|_0 \le n^{-\gamma m/\tilde{m}_0}} f(j,u) - f(j,0) \right\}.$$

From the conditions of the theorem,

$$\sum E_\theta^n \tilde{\chi}_j \le k_5 n^{-\gamma m/\tilde{m}_0}, \qquad j = 1,\dots,n.$$

Hence, instead of (2.22), it is sufficient to estimate, for any $\epsilon'' > 0$, the probability

$$P_\theta^n \left\{ n^{-1/2} \sum (\tilde{\chi}_j - E_\theta^n \tilde{\chi}_j) > \epsilon'' \right\} \le (\epsilon'')^2 k_5 n^{-\gamma m/\tilde{m}_0}.$$

Taking into account the fact that all the bounds are uniform in $\theta \in T$, we obtain that the lemma is proved for $z_n^+(\theta,u)$. The case of $z_n^-(\theta,u)$ is investigated similarly. □

Let us set

$$E_\theta^n S_\beta^\pm(\hat{\theta}_n) = (E_\theta^n S_\beta^\pm(\tau))_{\tau=\hat{\theta}_n}.$$

Lemma 2. *Under the conditions of the theorem, for any $\epsilon > 0$,*

$$\sup_{\theta\in T} P_\theta^n \left\{ |S_\beta^\pm(\theta) + E_\theta^n S_\beta^\pm(\hat{\theta}_n)| > \epsilon \right\} \underset{n\to\infty}{\longrightarrow} 0. \tag{2.23}$$

Proof. Let us introduce the events

$$A_i^\pm(\theta) = \{S_{i\beta}^\pm(\theta) + E_\theta^n S_{i\beta}^\pm(\hat{\theta}_n) - S_{i\beta}^\pm(\hat{\theta}_n) \ge -\epsilon(1 + |E_\theta^n S_\beta^\pm(\hat{\theta}_n)|)\},$$
$$i = 1,\dots,q.$$

It follows from (1.11) and the previous lemma that

$$\inf_{\theta\in T} P_\theta^n\{A_i^\pm(\theta)\} \underset{n\to\infty}{\longrightarrow} 1, \quad i = 1,\dots,q. \tag{2.24}$$

For the events $\{|\hat{\theta}_n - \theta| < r\},\ r < r_0,\ S_\beta^\pm(\hat{\theta}_n) \ge 0$. Therefore, relation (2.24) is true for the events

$$B_i^\pm(\theta) = \{S_{i\beta}^\pm(\theta) + E_\theta^n S_{i\beta}^\pm(\hat{\theta}_n) \ge -\epsilon(1 + |E_\theta^n S_\beta^\pm(\hat{\theta}_n)|)\} \supset A_i^\pm(\theta).$$

On the other hand,

$$S_{i\beta}^{+}(\theta) + S_{i\beta}^{-}(\theta) = \sum |g_i(j,\theta)|\chi\{\varepsilon_j = 0\} = 0 \qquad (\text{mod } P_\theta^n),$$

and the events $B_i^-(\theta)$ are equally like to the events

$$C_i^+(\theta) = \{S_{i\beta}^+(\theta) + E_\theta^n S_{i\beta}^+(\hat\theta_n) \le \epsilon(1 + |E_\theta^n S_\beta^+(\hat\theta_n)|)\}.$$

Furthermore, for $\epsilon < q^{-1}$, the events $D_i^+(\theta) = B_i^+(\theta) \cap C_i^+(\theta)$, $i = 1, ..., q$,

$$D_i^+(\theta) = \left\{\left|S_{i\beta}^+(\theta) + E_\theta^n S_{i\beta}^+(\hat\theta_n)\right| \le \epsilon(1 + |E_\theta^n S_\beta^+(\hat\theta_n)|)\right\}, \tag{2.25}$$

$$\begin{aligned}\bigcap_{i=1}^{q} D_i^+(\theta) &\subseteq \left\{\left|S_\beta^+(\theta) + E_\theta^n S_\beta^+(\hat\theta_n)\right| \le q\epsilon(1 + |E_\theta^n S_\beta^+(\hat\theta_n)|)\right\} \\ &\subseteq \left\{\left|E_\theta^n S_\beta^+(\hat\theta_n)\right| \le (1 - q\epsilon)^{-1}(q\epsilon + |S_\beta^+(\theta)|)\right\} = X^+(\theta),\end{aligned}$$

i.e.,

$$\inf_{\theta\in T} P_\theta^n\{X^+(\theta)\} \underset{n\to\infty}{\longrightarrow} 1. \tag{2.26}$$

Let us note that

$$P_\theta^n\{|E_\theta^n S_\beta^+(\hat\theta_n)| > M\} \le P_\theta^n\{\overline{X^+(\theta)}\} + P_\theta^n\{|S_\beta^+(\theta)| > M(1 - q\epsilon) - q\epsilon\}, \tag{2.27}$$

where $\overline{X^+(\theta)}$ is a complement of the event $X^+(\theta)$. Let us denote

$$\eta_j = \chi\{\varepsilon_j < 0\} - \beta, \quad j \ge 1,$$
$$I_{in}(\theta) = \{1, \dots, n\} \cap \{j : g_i(j,\theta) > 0\}.$$

Then P_θ^n- a.s.

$$S_\beta^+(\theta) - d_{in}^{-1}(\theta)\sum g_i(j,\theta)\eta_j = d_{in}^{-1}(\theta)\sum_{j\in I_{in}(\theta)} g_i(j,\theta)\chi\{\varepsilon_j = 0\} = 0.$$

Therefore, by the Chebyshev inequality,

$$P_\theta^n\{|S_\beta^+(\theta)| > M(1 - q\epsilon) - q\epsilon\} \le q(M(1 - q\epsilon) - q\epsilon)^{-2} \underset{M\to\infty}{\longrightarrow} 0,$$

i.e., the vector $S_\beta^+(\theta)$ is bounded in probability. It follows from (2.26) and (2.27) that the vector $E_\theta^n S_\beta^+(\hat\theta_n)$ is also bounded in probability uniformly in $\theta \in T$.

According to (2.25),

$$\sup_{\theta\in T} P_\theta^n\left\{|S_\beta^+(\theta) + E_\theta^n S_\beta^+(\hat\theta_n)| > \epsilon\left(1 + |E_\theta^n S_\beta^+(\hat\theta_n)|\right)\right\} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, (2.23) holds. We remark that the boundedness in probability of the r.v. $E_\theta^n S_\beta^+(\hat\theta_n)$ can also be obtained immediately from condition **C**, the explicit form of $E_\theta^n S_\beta^+(\hat\theta_n)$, and from the conditions of the theorem. □

Lemma 3. *Under the conditions of the theorem, for any $\epsilon > 0$,*

$$P_\theta^n\left\{|E_\theta^n S_\beta^+(\hat\theta_n) - p(0)I(\theta)d_n(\theta)(\hat\theta_n - \theta)| > \epsilon\right\} \underset{n\to\infty}{\longrightarrow} 0. \tag{2.28}$$

Proof. If the quantity $n^{-1/2}|d_n(\theta)(\hat\theta_n - \theta)|$ is small, then it follows from inequality (2.6) and the boundedness of the r.v. $E_\theta^n S_\beta^+(\hat\theta_n)$ in probability that the norm of the vector $d_n(\theta)(\hat\theta_n - \theta)$ is bounded in probability. The statement of Lemma 3 follows from condition **C** and inequalities (2.3)-(2.5). □

3. Proof of the theorem

Relations (2.23) and (2.28) show that, for any $\epsilon > 0$,

$$P_\theta^n \left\{ |(p(0))^{-1}\Lambda(\theta)S_\beta^+(\theta) + d_n(\theta)(\hat\theta_n - \theta)| > \epsilon \right\} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.1}$$

As was noted above,

$$S_\beta^+(\theta) = d_n^{-1}(\theta) \sum \nabla g(j,\theta)\eta_j \qquad (\text{mod } P_\theta^n).$$

Let us apply Corollary 17.2 in ([5], p. 165) to the random vectors

$$\xi_{jn} = n^{1/2} d_n^{-1}(\theta)\nabla g(j,\theta)\eta_j, \quad j = 1,\dots,n.$$

It follows from (1.5) that

$$n^{-1}\sum E_\theta^n |\xi_{jn}|^3 \le q^{1/2}\sum_{i=1}^q n^{-1}\sum d_{in}^{-3}(\theta)|g_i(j,\theta)|^3 n^{3/2} \le k_{10} < \infty$$

uniformly in $\theta \in T$. Then

$$\sup_{\theta\in T}\sup_{C\in\mathcal{C}^q}\left| P_\theta^n \left\{ I^{-1/2}(\theta)S_\beta^+(\theta) \in C \right\} - \Phi(C)\right| = O(n^{-1/2}). \tag{3.2}$$

Let us find the correlation matrix of $S_\beta^+(\theta)$. Clearly, $ES_\beta^+(\theta) = 0$. Then, taking into account **A1**, we get

$$E_\theta^n S_{i\beta}^+(\theta)S_{l\beta}^+(\theta) = d_{in}^{-1}(\theta)d_{ln}^{-1}(\theta)\sum g_i(j,\theta)g_l(j,\theta)E\eta_j^2, \quad i,l = 1,...,q.$$

It follows from the form of η_j that $E\eta_j^2 = \beta(1-\beta)$. Then

$$E_\theta^n S_\beta^+(\theta)(S_\beta^+(\theta))^T = \beta(1-\beta)I(\theta). \tag{3.3}$$

Relations (3.1)-(3.3) yield that, for any $\epsilon > 0$ and $C \in \mathcal{C}^q$,

$$-\Delta_n + \Phi(C_{-\epsilon}) \le P_\theta^n \left\{ \frac{p(0)}{\sqrt{\beta(1-\beta)}} I^{1/2}(\theta)d_n(\theta)(\hat\theta_n - \theta) \in C \right\} \le \Delta_n + \Phi(C_\epsilon), \tag{3.4}$$

where $C_{-\epsilon}$ and C_ϵ are the exterior and interior sets parallel to C, and $\Delta_n \underset{n\to\infty}{\longrightarrow} 0$ uniformly in $\theta \in T$ and $C \in \mathcal{C}^q$. The statement of the theorem follows from (3.4) and the theorem from Section 3 in [6] which state that, for any $\epsilon > 0$,

$$\sup_{C\in\mathcal{C}^q} |\Phi(C_{\pm\epsilon}) - \Phi(C)| \le k\epsilon,$$

where k is a constant that does not depend on ϵ.

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