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OLGA V. ARYASOVA AND MYKOLA I. PORTENKO

ONE CLASS OF MULTIDIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS HAVING NO PROPERTY OF WEAK UNIQUENESS OF A SOLUTION

A class of stochastic differential equations in a multidimensional Euclidean space such that the property of a solution to be unique (in a weak sense) fails for it is considered. We present the correct formulation of the corresponding martingale problem and prove the uniqueness of its solution.

Introduction

Let ν be a fixed unit vector in a d-dimensional Euclidean space $\mathbb{R}^d$. By S, denote the hyperplane in $\mathbb{R}^d$ that is orthogonal to the vector ν. By D_+ and D_-, denote the half-spaces into which the hyperplane S divides the space $\mathbb{R}^d$: $D_+ = \{x \in \mathbb{R}^d : (x, \nu) > 0\}$ and $D_- = \{x \in \mathbb{R}^d : (x, \nu) < 0\}$, and put $D = D_- \cup D_+$. The indicator function of a set $\Gamma \subset \mathbb{R}^d$ is denoted by $1\!\!1_\Gamma(x), x \in \mathbb{R}^d$, and the identity operator in $\mathbb{R}^d$ is denoted by I.

For a given real-valued continuous function $A(x), x \in S$, consider the stochastic differential equation in $\mathbb{R}^d$

$$dx(t) = \nu A(x(t))1\!\!1_S(x(t))dt + d\xi(t), \tag{1}$$

where $(\xi(t))_{t\geq 0}$ is an $\mathbb{R}^d$-valued continuous square integrable martingale, whose characteristic is given by

$$\langle\xi\rangle_t = I\int_0^t 1\!\!1_D(x(s))ds,\ t \geq 0. \tag{2}$$

We will construct infinitely many solutions to this equation. Each solution corresponds to a representation of the function $A(\cdot)$ in the form

$$A(x) = \frac{q(x)}{r(x)},\ x \in S, \tag{3}$$

where $q(\cdot)$ is a continuous function on S with its values in the interval $[-1, 1]$, and $r(\cdot)$ is a continuous bounded function on S with positive values (it is clear that such a representation is not unique). For such a pair of the functions $q(\cdot)$ and $r(\cdot)$, the desired solution will be constructed as a continuous Markov process in $\mathbb{R}^d$ obtained from a d-dimensional Wiener process by two transformations: its skewing on S and random changing time (these transformations are characterized by the functions $q(\cdot)$ and $r(\cdot)$ respectively).

In particular, if $A(x) \equiv 0$ (there is no skewing) we should put $q(x) \equiv 0$ and choose a non-negative function $r(\cdot)$ arbitrarily. The corresponding solution to Eq. (1) (and even a more general one, see Section 1) is a Wiener process in $\mathbb{R}^d$ for which the points of the hyperplane S are sticky.

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Another particular case assumes $A(x)$ to be strictly positive for all $x \in S$. One can choose $q(x) \equiv +1$ and $r(x) \equiv A(x)^{-1}$ in this case; the corresponding solution to Eq. (1) has the following property: its part in the region $D_+ \cup S$ is a Wiener process slowly reflected on S in the direction ν (maximal skewing is equal to reflecting); this solution can be singled out by the requirement $(x(t), \nu) \geq 0$ for all $t > 0$ if $(x(0), \nu) \geq 0$; such a solution in the one-dimensional situation was investigated by R. Chitashvili, see [2].

A.V. Skorokhod was the first one who used the random change of time in order to construct a slowly reflected process (in the one-dimensional case) from such a one, for which the reflection was instantaneous (see [5], §24). The main distinction of our construction is admitting the boundary to be permeable. The one-dimensional situation was expounded in [7] in a manner quite similar to that of Section 1 of this article. Some analytical methods were used in [8] for constructing much more general processes than those considered here.

The problem of solving Eq. (1) can be formulated as the martingale problem: for any $x \in \mathbb{R}^d$, we search for a probabilistic measure $\mathbb{P}_x$ on the space of all continuous functions $x(\cdot) : [0, +\infty) \to \mathbb{R}^d$ with usual filtration (see Section 2 for details). Moreover, $\mathbb{P}_x(\{x(0) = x\}) = 1$, and the process

$$\xi(t) = x(t) - x(0) - \nu \int_0^t A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau, \ t \geq 0, \tag{4}$$

is a square integrable martingale with its characteristic given by (2). The above discussion shows that such a problem is not a well-posed one: there are infinitely many solutions to it.

In Section 2, we show the correct form of the martingale problem [it involves the functions $q(\cdot)$ and $r(\cdot)$ from representation (3)]. The existence of a solution to this problem (and even to more general one) was given in [1]. We prove the uniqueness theorem for our problem making use of the method by Stroock–Varadhan from [11] that has been become a classical one.

We emphasize that Eq. (1) is a degenerate one: its diffusion operator vanishes at the points of the hyperplane S. As mentioned in [4], p. 153, "the equation does not determine the amount of time which the solution spends at the zeros of the diffusion coefficient. Thus the sojourn time at the zeros of this coefficient appears as additional degree of freedom for the solution". We now can add to these words the following ones: the extent of skewing is another degree of freedom for the solution; the equation determines only the ratio of these two coefficients.

1. Constructing the solutions to Eq. (1).

1.1. The case of $A(x) \equiv 0$.

Let $(w(t))_{t \geq 0}$ be a standard Wiener process in $\mathbb{R}^d$. By $g_0(t, x, y)$ for $t > 0, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$, we denote its transition probability density (with respect to the Lebesgue measure in $\mathbb{R}^d$)

$$g_0(t, x, y) = (2\pi t)^{-d/2} \exp\left\{ -\frac{1}{2t} |y - x|^2 \right\}. \tag{5}$$

For a given continuous bounded function $r(\cdot) : S \to [0, +\infty)$, define a W-functional (see [3], Chapter 6) $(\widetilde{\eta}_t)_{t \geq 0}$ of the process $w(\cdot)$ such that for $t \geq 0$ and $x \in \mathbb{R}^d$

$$\mathbb{E}_x \widetilde{\eta}_t = \int_0^t d\tau \int_S g_0(\tau, x, y) r(y) d\sigma_y$$

(here, the inner integral on the right-hand side is a surface integral). For a given $\lambda \geq 0$, we determine the function $g_\lambda(t,x,y)$ from the relation

$$\mathbb{E}_x\varphi(w(t))\exp\{-\lambda\widetilde{\eta}_t\} = \int_{\mathbb{R}^d}\varphi(y)g_\lambda(t,x,y)dy$$

valid for $t>0, x\in\mathbb{R}^d$, and any real-valued bounded measurable function φ on $\mathbb{R}^d$ (the set of all such functions with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d}|\varphi(x)|$ forms a Banach space denoted by $\mathbb{B}$; the subspace of $\mathbb{B}$ consisting of all continuous functions also forms a Banach space denoted by $\mathbb{C}$). The function g_λ can be found as a solution to each of two integral equations (see [10])

$$g_\lambda(t,x,y) = g_0(t,x,y) - \lambda\int_0^t d\tau\int_S g_0(\tau,x,z)g_\lambda(t-\tau,z,y)r(z)d\sigma_z, \tag{6}$$

$$g_\lambda(t,x,y) = g_0(t,x,y) - \lambda\int_0^t d\tau\int_S g_\lambda(\tau,x,z)g_0(t-\tau,z,y)r(z)d\sigma_z, \tag{7}$$

where $t>0, \lambda\geq 0, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$. Moreover, there is exactly one solution to (6) and (7) satisfying the inequality

$$0\leq g_\lambda(t,x,y)\leq g_0(t,x,y)$$

for all $t>0, x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$.

For $t\geq 0$, we now put

$$\widetilde{\zeta}_t = \inf\{s : s+\widetilde{\eta}_s\geq t\},\ \widetilde{w}(t) = w(\widetilde{\zeta}_t).$$

It is well known (see [3], Theorem 10.11) that $(\widetilde{w}(t))_{t\geq 0}$ is a standard Markov process. Denote its transition probability by $\widetilde{P}(t,x,\Gamma)$:

$$\mathbb{E}_x\varphi(\widetilde{w}(t)) = \int_{\mathbb{R}^d}\varphi(y)\widetilde{P}(t,x,dy),\ t>0, x\in\mathbb{R}^d, \varphi\in\mathbb{B}.$$

The following calculation is similar to that given in [6], Chapter II, §6 (we assume that $\varphi\in\mathbb{C}$):

$$\int_0^\infty e^{-\lambda t}\left(\int_{\mathbb{R}^d}\varphi(y)\widetilde{P}(t,x,dy)\right)dt = \mathbb{E}_x\int_0^\infty e^{-\lambda t}\varphi(w(\widetilde{\zeta}_t))dt =$$
$$= \int_0^\infty e^{-\lambda t}\mathbb{E}_x(\varphi(w(t))\exp\{-\lambda\widetilde{\eta}_t\})dt + \mathbb{E}_x\int_0^\infty e^{-\lambda(t+\widetilde{\eta}_t)}\varphi(w(t))d\widetilde{\eta}_t =$$
$$= \int_0^\infty e^{-\lambda t}\left(\int_{\mathbb{R}^d}\varphi(y)g_\lambda(t,x,y)dy\right)dt +$$
$$+\int_0^\infty e^{-\lambda t}\left(\int_S\varphi(y)r(y)g_\lambda(t,x,y)d\sigma_y\right)dt.$$

Putting

$$Q_\lambda(x,y) = \int_0^\infty e^{-\lambda t}g_\lambda(t,x,y)dt$$

for $\lambda>0, x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$, we arrive at the formula

$$\int_0^\infty e^{-\lambda t}\left(\int_{\mathbb{R}^d}\varphi(y)\widetilde{P}(t,x,dy)\right)dt =$$
$$= \int_{\mathbb{R}^d}Q_\lambda(x,y)\varphi(y)dy + \int_S Q_\lambda(x,y)r(y)\varphi(y)d\sigma_y \tag{8}$$

valid for all $\lambda > 0, x \in \mathbb{R}^d$, and $\varphi \in \mathbb{B}$. In addition, Eqs. (6) and (7) imply the relations for the function Q_λ

$$Q_\lambda(x,y) = \widetilde{g}_0(\lambda,x,y) - \lambda \int_S \widetilde{g}_0(\lambda,x,z) Q_\lambda(z,y) r(z) d\sigma_z,$$

$$Q_\lambda(x,y) = \widetilde{g}_0(\lambda,x,y) - \lambda \int_S Q_\lambda(x,z) \widetilde{g}_0(\lambda,z,y) r(z) d\sigma_z \tag{9}$$

which are true for $\lambda > 0, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$. Here, we put

$$\widetilde{g}_0(\lambda,x,y) = \int_0^\infty e^{-\lambda t} g_0(t,x,y) dt.$$

These formulae allow us to calculate $\mathbb{E}_x \varphi(\widetilde{w}(t))$ for some functions φ. For example, if we put $\varphi_0(y) \equiv 1$, then (9) implies

$$\int_{\mathbb{R}^d} Q_\lambda(x,y) dy = \frac{1}{\lambda} - \int_S Q_\lambda(x,z) r(z) d\sigma_z.$$

Taking into account (8), this yields the identity $\widetilde{P}(t,x,\mathbb{R}^d) \equiv 1$.

For a fixed $\theta \in \mathbb{R}^d$, we put $\varphi_1(x) = (x,\theta)$. It follows from (8) and (9) that

$$\int_{\mathbb{R}^d} \varphi_1(y) \widetilde{P}(t,x,dy) = \varphi_1(x), t > 0, x \in \mathbb{R}^d.$$

This means that $\mathbb{E}_x(\widetilde{w}(t) - \widetilde{w}(0)) \equiv 0$. Therefore, the process $\xi(t) = \widetilde{w}(t) - \widetilde{w}(0), t \geq 0$, is a martingale. Let us find out its square characteristic. With that end in view, we calculate $\mathbb{E}_x(\widetilde{w}(t),\theta)^2$. Putting $\varphi_2(x) = (x,\theta)^2$ for a fixed $\theta \in \mathbb{R}^d$ and taking into account that

$$\int_{\mathbb{R}^d} \varphi_2(y) \widetilde{g}_0(\lambda,x,y) dy = \lambda^{-1} \varphi_2(x) + \lambda^{-2} |\theta|^2, \lambda > 0, x \in \mathbb{R}^d,$$

relations (8) and (9) yield

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \varphi_2(\widetilde{w}(t)) dt = \lambda^{-1} \varphi_2(x) +$$

$$+\lambda^{-1} |\theta|^2 \left[\lambda^{-1} - \int_S Q_\lambda(x,z) r(z) d\sigma_z \right]. \tag{10}$$

Denote, by $S_+(r)$, the set of those $x \in S$ for which $r(x) > 0$ and, by $S_+(r)^c$, its complement: $S_+(r)^c = \mathbb{R}^d \setminus S_+(r)$. Using formula (8), we can write

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \mathbb{I}_{S_+(r)^c}(\widetilde{w}(t)) dt = \int_{S_+(r)^c} Q_\lambda(x,y) dy =$$

$$= \left[\int_{\mathbb{R}^d} Q_\lambda(x,y) dy + \int_S Q_\lambda(x,y) r(y) d\sigma_y \right] - \int_S Q_\lambda(x,y) r(y) d\sigma_y =$$

$$= \lambda^{-1} - \int_S Q_\lambda(x,y) r(y) d\sigma_y.$$

This equality and (10) imply the relation

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \varphi_2(\widetilde{w}(t)) dt = \lambda^{-1} \varphi_2(x) +$$

$$+|\theta|^2 \int_0^\infty e^{-\lambda t} E_x \left(\int_0^t \mathbb{I}_{S_+(r)^c}(\widetilde{w}(s)) ds \right) dt$$

that can be rewritten in the form

$$\mathbb{E}_x(\widetilde{w}(t)-\widetilde{w}(0),\theta)^2 = |\theta|^2\mathbb{E}_x\int_0^t 1\!\!1_{S_+(r)^c}(\widetilde{w}(s))ds.$$

Since this equality holds true for all $t>0, x\in\mathbb{R}^d$, and $\theta\in\mathbb{R}^d$, we can assert that the square characteristic of the martingale $\xi(t)=\widetilde{w}(t)-\widetilde{w}(0), t\ge 0$, is given by

$$\langle\xi\rangle_t = I\int_0^t 1\!\!1_{S_+(r)^c}(\widetilde{w}(s))ds, t\ge 0. \tag{11}$$

So, we have just proved that in the case of $S_+(r)\ne\emptyset$ the stochastic integral equation

$$\widetilde{w}(t)=\widetilde{w}(0)+\xi(t), t\ge 0, \tag{12}$$

driven by a square integrable martingale $\xi(\cdot)$ with characteristic (11) has infinitely many solutions.

Indeed, let us consider another continuous bounded function $\widehat{r}(\cdot)$ on S with non-negative values such that $S_+(\widehat{r})=S_+(r)\ne\emptyset$ (for example, $\widehat{r}(x)=\beta r(x)$ for $x\in S$ with an arbitrary positive constant $\beta\ne 1$) and construct the process $\widehat{w}(\cdot)$ for it in the same way as the process $\widetilde{w}(\cdot)$ has been constructed for the function $r(\cdot)$. The process $\widehat{w}(\cdot)$ is also a solution to Eq. (12).

1.2. The general case.

We now try to do something like that in Section 1.1, but this time we start from the process that can be called a Wiener process skewed on the hyperplane S. Fix a continuous function $q(\cdot)$ on S with its values in the interval $[-1,1]$ and put

$$G_0(t,x,y)=g_0(t,x,y)+\int_0^t d\tau\int_S g_0(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial\nu_z}q(z)d\sigma_z$$

for $t>0, x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$. It is well known that there exists a continuous Markov process $(x_0(t))_{t\ge 0}$ such that

$$\mathbb{E}_x\varphi(x_0(t))=\int_{\mathbb{R}^d}\varphi(y)G_0(t,x,y)dy$$

for $t>0, x\in\mathbb{R}^d$, and $\varphi\in\mathbb{B}$ (see, for example, [9]). Moreover, for a given continuous bounded function $r(\cdot)$ on S with non-negative values, there exists a W-functional $(\eta_t)_{t\ge 0}$ of the process $x_0(\cdot)$ such that for all $t>0$ and $x\in\mathbb{R}^d$

$$\mathbb{E}_x\eta_t=\int_0^t d\tau\int_S G_0(\tau,x,y)r(y)d\sigma_y=\int_0^t d\tau\int_S g_0(\tau,x,y)r(y)d\sigma_y$$

(it is not difficult to see that the equality $G_0(t,x,y)=g_0(t,x,y)$ is held for $t>0, x\in\mathbb{R}^d$, and $y\in S$). We now define a function $G_\lambda(t,x,y)$ for $\lambda\ge 0$, $t>0, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ in such a way that the relation

$$\mathbb{E}_x\varphi(x_0(t))\exp\{-\lambda\eta_t\}=\int_{\mathbb{R}^d}\varphi(y)G_\lambda(t,x,y)dy$$

holds true for $t>0, x\in\mathbb{R}^d$, and $\varphi\in\mathbb{B}$. It is known (see, for example, [10]) that such a function exists and it can be found as a solution to each one of the following pair of equations

$$G_\lambda(t,x,y)=G_0(t,x,y)-\lambda\int_0^t d\tau\int_S g_0(\tau,x,z)G_\lambda(t-\tau,z,y)r(z)d\sigma_z, \tag{13}$$

$$G_\lambda(t,x,y) = G_0(t,x,y) - \lambda \int_0^t d\tau \int_S G_\lambda(\tau,x,z)G_0(t-\tau,z,y)r(z)d\sigma_z. \tag{14}$$

Moreover, there is no more than one solution to these equations satisfying the inequality

$$G_\lambda(t,x,y) \le G_0(t,x,y) \tag{15}$$

for all $t > 0, \lambda \ge 0, x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

Proposition 1. *For $t > 0, \lambda > 0, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$, the formula*

$$G_\lambda(t,x,y) = g_\lambda(t,x,y) + \int_0^t d\tau \int_S g_\lambda(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial \nu_z}q(z)d\sigma_z \tag{16}$$

holds true.

The proof of this statement is elementary, and we omit it.

Remark 1. Proposition 1 means that the transformations of skewing and killing considered above are commutative.

Remark 2. One can see that $G_\lambda(t,x,y)$ coincides with $g_\lambda(t,x,y)$ for $t > 0, x \in \mathbb{R}^d$, and $y \in S$, because of $\dfrac{\partial g_0(t,z,y)}{\partial \nu_z} = 0$ for $z \in S$ and $y \in S$. Nevertheless, the well-known theorem on the jump of the normal derivative of a single-layer potential shows that

$$G_\lambda(t,x,y\pm) = (1 \pm q(y))g_\lambda(t,x,y)$$

for $y \in S$, where $G_\lambda(t,x,y+)$ and $G_\lambda(t,x,y-)$ are the non-tangent limits of $G_\lambda(t,x,z)$, as $z \to y$ from the sides D_+ and D_-, respectively.

Putting

$$R_\lambda(x,y) = \int_0^\infty e^{-\lambda t} G_\lambda(t,x,y)dt$$

for $\lambda > 0, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$ and taking into account Remark 2, relation (16) yields the formula

$$R_\lambda(x,y) = Q_\lambda(x,y) + \int_S Q_\lambda(x,z)\frac{\partial \widetilde{g}_0(\lambda,z,y)}{\partial \nu_z}q(z)d\sigma_z \tag{17}$$

valid for $\lambda > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d$, and if $y \in S$ we have $R_\lambda(x,y) = Q_\lambda(x,y)$.

For $t \ge 0$, we now put

$$\zeta_t = \inf\{s : s + \eta_s \ge t\}, x(t) = x_0(\zeta_t).$$

Theorem 10.11 from [3] allow us, as above, to assert that the process $(x(t))_{t\ge 0}$ is a standard Markov process. Denote, by $P(t,x,dy)$, its transition probability

$$\mathbb{E}_x\varphi(x(t)) = \int_{\mathbb{R}^d} \varphi(y)P(t,x,dy), t \ge 0, x \in \mathbb{R}^d, \varphi \in \mathbb{B}.$$

Using the calculations similar to those in Section 1.1, we arrive at the formula

$$\int_0^\infty e^{-\lambda t}\mathbb{E}_x\varphi(x(t))dt = \int_{\mathbb{R}^d} R_\lambda(x,y)\varphi(y)dy +$$

$$+ \int_S Q_\lambda(x,y)\varphi(y)r(y)d\sigma_y \tag{18}$$

that is fulfilled for $\lambda > 0, x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}$.

Formulae (17) and (18) allow us to give a martingale characterization of the process $x(\cdot)$. In what follows up to the end of this section, we assume that the function $r(\cdot)$ takes on only strictly positive values.

For a fixed $\theta \in \mathbb{R}^d$, we put $\varphi_1(x) = (x, \theta)$, as above. Since

$$\int_{\mathbb{R}^d} (y,\theta) \frac{\partial \widetilde{g}_0(\lambda, z, y)}{\partial \nu_z} dy = \lambda^{-1}(\theta, \nu),$$

we have

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \varphi_1(x(t)) dt = \int_{\mathbb{R}^d} R_\lambda(x,y)\varphi_1(y)dy +$$
$$+ \int_S Q_\lambda(x,y)\varphi_1(y) r(y) d\sigma_y = \int_{\mathbb{R}^d} Q_\lambda(x,y)\varphi_1(y)dy +$$
$$+ \int_S Q_\lambda(x,y)\varphi_1(y) r(y) d\sigma_y + \lambda^{-1}(\theta,\nu)\int_S Q_\lambda(x,z) q(z) d\sigma_z.$$

As shown in Section 1.1, the sum of the first two terms here is equal to $\lambda^{-1}(x,\theta)$. So we obtain

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \varphi_1(x(t)) dt =$$
$$= \lambda^{-1}\varphi_1(x) + \lambda^{-1}(\theta,\nu) \int_S Q_\lambda(x,z) A(z) r(z) d\sigma_z.$$

This equality means that

$$\mathbb{E}_x(x(t) - x(0), \theta) = (\theta, \nu)\mathbb{E}_x \int_0^t A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau, \tag{19}$$

because of the equality

$$\int_0^\infty e^{-\lambda t} \left(\mathbb{E}_x \int_0^t A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau \right) dt =$$
$$= \lambda^{-1} \int_S Q_\lambda(x,y) A(y) r(y) d\sigma_y$$

that is a simple consequence of (18).

If we put, for $t \geq 0$,

$$\xi(t) = x(t) - x(0) - \nu \int_0^t A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau,$$

then equality (19) will mean that the process $(\xi(t))_{t\geq 0}$ is a martingale. We now find out the square characteristic of this martingale.

Since

$$\mathbb{E}_x(\xi(t),\theta)^2 = \mathbb{E}_x(x(t)-x(0),\theta)^2 + 2(\nu,\theta)(x,\theta)\mathbb{E}_x \int_0^t A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau -$$
$$-2(\nu,\theta)\mathbb{E}_x \int_0^t (x(\tau),\theta) A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau$$

and

$$\mathbb{E}_x(x(t)-x(0),\theta)^2 = E_x(x(t),\theta)^2 - (x,\theta)^2 -$$
$$-2(x,\theta)(\nu,\theta)\mathbb{E}_x \int_0^t A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau,$$

we obtain the formula

$$\mathbb{E}_x(\xi(t),\theta)^2 = E_x(x(t),\theta)^2 - (x,\theta)^2 -$$
$$-2(\nu,\theta)\mathbb{E}_x \int_0^t (x(\tau),\theta) A(x(\tau)) 1\!\!1_S(x(\tau)) d\tau. \tag{20}$$

Let us calculate $\mathbb{E}_x(x(t),\theta)^2$. We have

$$\int_0^\infty e^{-\lambda t}\mathbb{E}_x(x(t),\theta)^2 dt = \int_{\mathbb{R}^d} R_\lambda(x,y)(y,\theta)^2 dy +$$

$$+\int_S Q_\lambda(x,y)(y,\theta)^2 r(y) d\sigma_y = \int_{\mathbb{R}^d} Q_\lambda(x,y)(y,\theta)^2 dy +$$

$$+\frac{2(\nu,\theta)}{\lambda}\int_S Q_\lambda(x,z)(z,\theta)q(z)d\sigma_z + \int_S Q_\lambda(x,y)(y,\theta)^2 r(y)d\sigma_y. \tag{21}$$

It is not difficult to establish the relation

$$\int_0^\infty e^{-\lambda t}\left(\mathbb{E}_x\int_0^t (x(\tau),\theta)A(x(\tau))\mathbb{1}_S(x(\tau))d\tau\right)dt =$$

$$= \lambda^{-1}\int_S Q_\lambda(x,y)(y,\theta)q(y)d\sigma_y.$$

Taking into account this equality and (21), relation (20) yields

$$\int_0^\infty e^{-\lambda t}\mathbb{E}_x(\xi(t),\theta)^2 dt = \int_{\mathbb{R}^d} Q_\lambda(x,y)(y,\theta)^2 dy - \lambda^{-1}(x,\theta)^2 +$$

$$+\int_S Q_\lambda(x,y)(y,\theta)^2 r(y)dy. \tag{22}$$

We can now write

$$\int_{\mathbb{R}^d} Q_\lambda(x,y)(y,\theta)^2 dy + \int_S Q_\lambda(x,y)(y,\theta)^2 r(y)d\sigma_y =$$

$$= \int_0^\infty e^{-\lambda t}\mathbb{E}_x(\widetilde{w}(t),\theta)^2 dt = \frac{(x,\theta)^2}{\lambda} + \frac{|\theta|^2}{\lambda}\left[\frac{1}{\lambda} - \int_S Q_\lambda(x,z)r(z)d\sigma_z\right] =$$

$$= \frac{|\theta|^2}{\lambda}\int_{\mathbb{R}^d} Q_\lambda(x,z)dz + \frac{(x,\theta)^2}{\lambda}.$$

Substituting this into (22), we get

$$\int_0^\infty e^{-\lambda t}\mathbb{E}_x(\xi(t),\theta)^2 dt = \frac{|\theta|^2}{\lambda}\int_{\mathbb{R}^d} Q_\lambda(x,z)dz =$$

$$= \frac{|\theta|^2}{\lambda}\int_{\mathbb{R}^d} \mathbb{1}_D(z)Q_\lambda(x,z)dz =$$

$$= |\theta|^2\int_0^\infty e^{-\lambda t}\left(\mathbb{E}_x\int_0^t \mathbb{1}_D(x(\tau))d\tau\right)dt,$$

because of

$$\int_0^\infty e^{-\lambda t}\left(\mathbb{E}_x\int_0^t \mathbb{1}_D(x(\tau))d\tau\right)dt = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\mathbb{E}_x\mathbb{1}_D(x(t))dt =$$

$$= \frac{1}{\lambda}\left[\int_{\mathbb{R}^d} R_\lambda(x,y)\mathbb{1}_D(y)dy + \int_S Q_\lambda(x,y)\mathbb{1}_D(y)r(y)d\sigma_y\right] =$$

$$= \frac{1}{\lambda}\int_{\mathbb{R}^d} Q_\lambda(x,y)\mathbb{1}_D(y)dy.$$

We have thus proved that, for $t > 0, \theta \in \mathbb{R}^d$, and $x \in \mathbb{R}^d$, the equality

$$\mathbb{E}_x(\xi(t),\theta)^2 = |\theta|^2\mathbb{E}_x\int_0^t \mathbb{1}_D(x(\tau))d\tau$$

is held true. This means that the square characteristic of the martingale $\xi(\cdot)$ is given by formula (2). Let us formulate the obtained result as a proposition.

Proposition 2. *For a continuous function $q(\cdot) : S \to [-1, 1]$ and a continuous bounded function $r(\cdot) : S \to (0, +\infty)$, the Markov process $(x(t))_{t\geq 0}$ constructed above is such that its trajectories satisfy Eq. (1) with $A(x) = q(x)/r(x)$ for $x \in S$.*

Remark 3. As shown in [9], the process $(x_0(t))_{t\geq 0}$ is a solution to the stochastic differential equation

$$dx_0(t) = \nu q(x_0(t))\delta_S(x_0(t))dt + dw(t),$$

where $w(\cdot)$ is a standard Wiener process in $\mathbb{R}^d$, and $\delta_S(x), x \in \mathbb{R}^d$, is a generalized function determined by the relation

$$\langle \delta_S, \varphi \rangle = \int_S \varphi(x) d\sigma$$

valid for any compactly supported function $\varphi \in \mathbb{C}$. The process $(x(t))_{t\geq 0}$ is obtained from $(x_0(t))_{t\geq 0}$ by a random change of time generated by the functional

$$\int_0^t v(x_0(s))ds, t \geq 0,$$

where $v(x) = 1 + r(x)\delta_S(x), x \in \mathbb{R}^d$. It is well known (see [3], Chapter 10) that in order to obtain the local characteristics of the process $x(\cdot)$, the corresponding ones of the process $x_0(\cdot)$ should be multiplied by $v(x)^{-1}$. So, we have that the diffusion coefficients of the process $x(\cdot)$ should be given, for $x \in \mathbb{R}^d$, by

$$\nu \frac{q(x)\delta_S(x)}{1 + r(x)\delta_S(x)} = \nu \frac{q(x)}{r(x)} \mathbb{I}_S(x),$$

$$I \frac{1}{1 + r(x)\delta_S(x)} = I \cdot \mathbb{I}_D(x).$$

Exactly these characteristics were obtained as a result of the calculations in proving Proposition 2. As we can see, the diffusion coefficients of the process $x(\cdot)$ are not generalized functions. The arguments of Remark 3 are heuristical, but they lead to the right formulae.

Remark 4. For a given $\varphi \in \mathbb{C}$, the function

$$U_\lambda(x, \varphi) = \int_0^\infty e^{-\lambda t}\mathbb{E}_x\varphi(x(t))dt, \lambda > 0, x \in \mathbb{R}^d,$$

is continuous in the argument $x \in \mathbb{R}^d$, satisfies the equation

$$\lambda U_\lambda(x, \varphi) - \varphi(x) = \frac{1}{2}\Delta U_\lambda(x, \varphi)$$

in the region D (Δ stands for the Laplace operator) and the equation

$$\frac{1 + q(x)}{2r(x)} \frac{\partial U_\lambda(x+, \varphi)}{\partial \nu} - \frac{1 - q(x)}{2r(x)} \frac{\partial U_\lambda(x-, \varphi)}{\partial \nu} = \lambda U_\lambda(x, \varphi) - \varphi(x)$$

for $x \in S$, where $\frac{\partial U_\lambda(x+,\varphi)}{\partial \nu}$ and $\frac{\partial U_\lambda(x-,\varphi)}{\partial \nu}$ mean the non-tangent limits of $\frac{\partial U_\lambda(z,\varphi)}{\partial \nu}$, as $z \to x, x \in S$, from the sides D_+ and D_-, respectively.

2. The uniqueness theorem.

2.1. The martingale problem.

Throughout this section a continuous function $q(\cdot) : S \to [-1, 1]$ and a continuous bounded function $r(\cdot) : S \to [0, +\infty)$ will be fixed.

We say that a function $f : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^1$ satisfies condition F if

1) f is a continuous and bounded function in the arguments $(t, x) \in [0, +\infty) \times \mathbb{R}^d$;

2) the first t-derivative of f on $[0, +\infty) \times \mathbb{R}^d$ and the first two x-derivatives on $[0, +\infty) \times D$ are continuous and bounded;

3) for all $t \in [0, +\infty)$ and $x \in S$, there exist $\frac{\partial f(t,x\pm)}{\partial \nu}$ such that the function

$$Kf(t, x) = \frac{1 + q(x)}{2} \frac{\partial f(t, x+)}{\partial \nu} - \frac{1 - q(x)}{2} \frac{\partial f(t, x-)}{\partial \nu}$$

is continuous and bounded on $[0, +\infty) \times S$.

Denote, by Ω, the space of all continuous functions $x(\cdot) : [0, +\infty) \to \mathbb{R}^d$ and, by $\mathcal{M}_t = \sigma\{x(u) : 0 \le u \le t\}$, the smallest σ-algebra of subsets of Ω which makes each of the maps $x \to x(u)$ from Ω to $\mathbb{R}^d$ $\mathcal{M}_t$-measurable for $u \in [0, t]$ and put $\mathcal{M} = \sigma\{x(u) : 0 \le u < +\infty\}$.

Definition 1. Given $x \in \mathbb{R}^d$, a probability measure $\mathbb{P}_x$ on $\mathcal{M}$ is a solution of the submartingale problem starting from x if

1) $\mathbb{P}_x\{x(0) = x\} = 1$;

2) the process

$$X_f(t) = f(t, x(t)) - \int_0^t 1\!\!\mathrm{I}_D(x(u))(f_u + \frac{1}{2}\Delta f)(u, x(u))du, t \ge 0,$$

is a $\mathbb{P}_x$-submartingale for any f satisfying condition F and the inequality

$$r(x)f_t(t, x) + Kf(t, x) \ge 0 \text{ for } t \ge 0 \text{ and } x \in S.$$

Here and below, we put $f_t(t, x) = \frac{\partial f(t,x)}{\partial t}$.

It is not difficult to verify that the measure $\mathbb{P}_x$ on Ω that corresponds to the Markov process $(x(t))_{t\ge0}$ constructed in Section 1 is a solution to this submartingale problem.

Define the function ϕ on $\mathbb{R}^d$ by the equality $\phi(x) = |(x, \nu)|, x \in \mathbb{R}^d$. Then

1) ϕ satisfies condition F;

2) $S = \{x \in \mathbb{R}^d : \phi(x) = 0\}$, $D = \{x \in \mathbb{R}^d : \phi(x) > 0\}$,

3) $K\phi(x) \equiv 1$ on S.

Proposition 3. *Given $x \in \mathbb{R}^d$, the probability measure $\mathbb{P}_x$ on $\mathcal{M}$ solves the submartingale problem starting from x iff $\mathbb{P}_x\{x(0) = x\} = 1$ and there exists a continuous non-decreasing $(\mathcal{M}_t)$-adapted process $\alpha(t), t \ge 0$, such that*

1) $\alpha(0) = 0$, $\mathbb{E}\alpha(t) < +\infty$ for all $t \ge 0$;

2) $\alpha(t) = \int_0^t 1\!\!\mathrm{I}_S(x(u))d\alpha(u), t \ge 0$;

3) the process

$$f(t, x(t)) - \int_0^t 1\!\!\mathrm{I}_D(x(u))(f_u + \frac{1}{2}\Delta f)(u, x(u))du-$$

$$-\int_0^t (rf_u + Kf)(u, x(u))d\alpha(u), t \ge 0,$$

is a $\mathbb{P}_x$-martingale for any f satisfying condition F.

If $\mathbb{P}_x$ is such a solution, then $\alpha(t)$ is uniquely determined, up to $\mathbb{P}_x$-equivalence, by the condition that

$$\phi(x(t)) - \alpha(t)$$

is a $\mathbb{P}_x$*-martingale.*

Proof. The proof is similar to that of Theorem 2.5 in [11].

Corollary 1. *For each* $x \in \mathbb{R}^d, t \geq 0$, *the equality*

$$\int_0^t 1\!\!\mathrm{I}_S(x(u))du = \int_0^t r(x(u))d\alpha(u)$$

is held $\mathbb{P}_x$*-almost surely.*

Proof. Let now $f(t) = t$. Then the process

$$Y(t) = t - \int_0^t 1\!\!\mathrm{I}_D(x(u))du - \int_0^t r(x(u))d\alpha(u) =$$

$$= \int_0^t 1\!\!\mathrm{I}_S(x(u))du - \int_0^t r(x(u))d\alpha(u), t \geq 0,$$

is a $\mathbb{P}_x$-martingale. It has continuous paths of bounded variation, and $Y(0) = 0$. Therefore, the process $Y(t)$ is almost surely zero.

Corollary 2. *If* $x \in S$, *then* $\mathbb{P}_x\{\alpha(t) > 0, t > 0\} = 1$.

Proof. The process $X(t) = \phi(x(t)) - \alpha(t)$ is a martingale relative to $\mathcal{M}_t$. It is easy to see that $X(t)$ is a $\mathbb{P}_x$-martingale with respect to $\mathcal{M}_{t+} = \cap_{\varepsilon>0}\mathcal{M}_{t+\varepsilon}$. Indeed, for all $A \in \mathcal{M}_{s+}, \varepsilon > 0, s \geq 0, t > 0, s + \varepsilon \leq t$, we have

$$\int_A \mathbb{E}^{\mathbb{P}_x}(X(t)/\mathcal{M}_{s+})d\mathbb{P}_x = \int_A X(t)d\mathbb{P}_x = \int_A X(s+\varepsilon)d\mathbb{P}_x. \tag{23}$$

The process $X(t)$ is continuous and locally bounded. Passing to the limit in (23) as $\varepsilon \downarrow 0$, we obtain

$$\mathbb{E}^{\mathbb{P}_x}(X(t)/\mathcal{M}_{s+}) = X(s).$$

Define $\tau_0 = \sup\{t > 0 : \alpha(t) = 0\}$. Since $\{\alpha(t) > 0\} \in \mathcal{M}_t$, we get $\{\tau_0 < t\} \in \mathcal{M}_t$, and hence τ_0 is a stopping time relative to $\mathcal{M}_{t+}$.

The process

$$X(t \wedge \tau_0) = \phi(x(t \wedge \tau_0)) - \alpha(t \wedge \tau_0) = \phi(x(t \wedge \tau_0)), t \geq 0,$$

is a $\mathbb{P}_x$-martingale with respect to $\mathcal{M}_{(t\wedge\tau_0)+}$ such that $\mathbb{E}^{\mathbb{P}_x}X(t \wedge \tau_0) = 0$ for all $t \geq 0$. This implies $x(t) \in S$ for $t \in [0, \tau_0]$. But

$$\int_0^{\tau_0} 1\!\!\mathrm{I}_S(x(u))du = \int_0^{\tau_0} r(x(u))d\alpha(u) = 0 \ \ \mathbb{P}_x\text{-a.s.}$$

by Corollary 1. Therefore, $\tau_0 = 0$ almost surely.

2.2. A boundary process.

Let $\mathbb{P}_x$ be a solution to the submartingale problem starting from $x \in S$. Then there exists the process $\alpha(t), t \geq 0$, with properties stated in Proposition 3. For $\theta \geq 0$, we put $\tau(\theta) = \sup\{t \geq 0 : \alpha(t) \leq \theta\}$. It is well known that $\lim_{t\to+\infty} \alpha(t) = +\infty$, so $\tau(\theta) < +\infty$ for all $\theta \geq 0$. In addition, $\tau(\theta)$ is a stopping time relative to $\mathcal{M}_{t+}$. Since the starting point is on S we have $\alpha(t) > 0$ for $t > 0$ almost surely, i.e. $\tau(0) = 0$ and $x(\tau(0)) = x$. Moreover, $x(\tau(\theta))$ must be on S for all $\theta \geq 0$.

Define the $(d + 1)$-dimensional process $(\tau(\theta), x(\tau(\theta))), \theta \geq 0$, we call the boundary process starting from x. This process is a right-continuous and has no discontinuities of the second kind.

We denote, by $C_0^{1,2}([0, +\infty) \times S)$, the class of functions on $[0, +\infty) \times S$ having compact support which together with their first t-derivative and two x-derivatives are continuous and, by $C_0^\infty([0,+\infty)\times S)$, the class of infinitely differentiable functions on $[0,+\infty)\times S$ with compact support.

For $f \in C_0^{1,2}([0,+\infty)\times S)$, we put

$$Hf(t,x) = \int_t^\infty d\tau \int_S \frac{\partial g_0(\tau - t, x, y)}{\partial \nu_x} f(\tau, y) d\sigma_y,$$

where $t \geq 0, x \in \mathbb{R}^d, g_0(t,x,y)$ defined by formula (5).

Proposition 4. *The function Hf satisfies condition F.*

Proof. It is easy to observe that the function Hf together with its first t-derivative are continuous and bounded on $[0,+\infty)\times\mathbb{R}^d$, and its first two x-derivatives are continuous and bounded on $[0,+\infty)\times D$. We will show that $\frac{\partial (Hf)(t,x\pm)}{\partial \nu}$, $t\geq 0, x\in S$, are continuous and bounded.

We can write $Hf = J_1 + J_2$, where

$$J_1 = \int_t^\infty d\tau \int_S \frac{\partial g_0(\tau-t,x,y)}{\partial \nu_x}[f(\tau,y) - f(t,x_S) - (\nabla_{x_S} f(t,x_S), y - x_S)]d\sigma_y,$$

$$J_2 = f(t,x_S)\int_t^\infty \frac{|(x,\nu)| e^{-\dfrac{(x,\nu)^2}{2(\tau-t)}}}{\sqrt{2\pi(\tau-t)^3}} d\tau.$$

Here, $t \geq 0, x \in D, x_S = x - \nu(x,\nu)$ is a projection of x on S.

It is easily to see that $J_2 = f(t,x_S)$. Formally for $t\geq 0, x\in D$,

$$\frac{\partial J_1}{\partial \nu} = \int_t^\infty \left(1 - \frac{(x,\nu)^2}{\tau - t}\right) \frac{\operatorname{sign}(x,\nu)}{\sqrt{2\pi(\tau-t)^3}} e^{-\dfrac{(x,\nu)^2}{2(\tau-t)}} \Phi(t,\tau,x_S) d\tau,$$

where

$$\Phi(t,\tau,x_S) = \int_S \frac{e^{-\dfrac{|y-x_S|^2}{2(\tau-t)}}}{\sqrt{(2\pi(\tau-t))^{d-1}}}[f(\tau,y) - f(t,x_S) - (\nabla_{x_S} f(t,x_S), y-x_S)]d\sigma_y.$$

Since f has a compact support in t, there exists $T>0$ such that $\Phi(t,\tau,x_S) = 0$ for $t \geq T$, and the estimate

$$|\Phi(t,\tau,x_S)| \leq C \int_S \frac{e^{-\dfrac{|y-x_S|^2}{2(\tau-t)}}}{\sqrt{(2\pi(\tau-t))^{d-1}}}(|y-x_S|^2 + (\tau - t))d\sigma_y \leq C(\tau - t) \tag{24}$$

is valid for $t \in [0,T], \tau > t$, and some positive constant C.

Using (24), we arrive at the conclusion that, for $t\geq 0, x \in D$, there exists $\frac{\partial I}{\partial \nu}$, and we can pass to the limit as $(x,\nu) \to 0\pm$.

The statement of Proposition 4 follows immediately.

Remark 5. The function Hf is a solution of the equation

$$\frac{\partial U}{\partial t} + \frac{1}{2}\Delta U = 0 \text{ in the region } [0,+\infty)\times D,$$

and the relation $(Hf)(t,x\pm) = f(t,x)$ is true for all $t \geq 0, x \in S$.

According to Proposition 4, we can define

$$(\widetilde{K}f)(t,x) = r(x)(Hf)_t(t,x) +$$

$$+\left[\frac{1+q(x)}{2}\frac{\partial(Hf)(t,x+)}{\partial\nu}-\frac{1-q(x)}{2}\frac{\partial(Hf)(t,x-)}{\partial\nu}\right]$$

as a continuous and bounded function on $[0,+\infty)\times S$.

Proposition 5. *Let a probability measure $\mathbb{P}_x$ solve the submartingale problem for a given $x\in S$. Then the relation $\mathbb{P}_x\{(\tau(0),x(\tau(0)))=(0,x)\}=1$ is held, and the process*

$$f(\tau(\theta),x(\tau(\theta)))-\int_0^\theta(\widetilde{K}f)(\tau(u),x(\tau(u)))du$$

is a $\mathbb{P}_x$-martingale with respect to the filtration $\left(\mathcal{M}_{\tau(\theta)}\right)_{\theta\geq 0}$ for any function

$$f\in C_0^{1,2}([0,+\infty)\times S).$$

Proof. Proposition 5 can be proved analogously to Theorem 4.1 of [11].

Denote, by $\mathcal{D}([0,+\infty),[0,+\infty)\times S)$, the class of $[0,+\infty)\times S$-valued right-continuous functions on $[0,+\infty)$ with no discontinuities of the second kind.

Definition 2. The uniqueness theorem is valid for the boundary process if, for any given $x\in S$, there is only one probability measure $\mathbb{Q}_x$ on the space $\mathcal{D}([0,+\infty),[0,+\infty)\times S)$ such that

1) $\mathbb{Q}_x\{\tau(0)=0,x(\tau(0))=x\}=1$;

2) the process

$$f(\tau(\theta),x(\tau(\theta)))-\int_0^\theta(\widetilde{K}f)(u,x(\tau(u)))du,\theta\geq 0,$$

is a $\mathbb{Q}_x$-martingale relative to $\left(\mathcal{M}_{\tau(\theta)}\right)_{\theta\geq 0}$ for any function $f\in C_0^{1,2}([0,+\infty)\times S)$.

Proposition 6. *Let r and q be given continuous real-valued functions on S such that r is bounded and non-negative, $|q|\leq 1$. Then the uniqueness theorem is valid for the boundary process.*

Proof. We follow the proof of Theorem 5.2 in [11].

Given $x\in S$, let $\mathbb{R}_x$ be a solution of the submartingale problem starting from x. Then

$$f(\tau(\theta),x(\tau(\theta)))-\int_0^\theta(\widetilde{K}f)(\tau(u),x(\tau(u)))du$$

is a $\mathbb{R}_x$-martingale with respect to $\left(\mathcal{M}_{\tau(\theta)}\right)_{\theta\geq 0}$, and

$$\mathbb{E}^{\mathbb{R}_x}[f(\tau(\theta),x(\tau(\theta)))]=f(0,x)+\mathbb{E}^{\mathbb{R}_x}\left[\int_0^\theta(\widetilde{K}f)(\tau(u),x(\tau(u)))du\right].$$

Performing the Laplace transformation, we get the equality

$$\int_0^\infty e^{-\lambda u}\mathbb{E}^{\mathbb{R}_x}[\lambda f(\tau(u),x(\tau(u)))-(\widetilde{K}f)(\tau(u),x(\tau(u)))]du=f(0,x)$$

for $\lambda>0$.

If we know that the equation $\lambda f-\widetilde{K}f=g$ has the unique solution for each $g\in C_0^\infty([0,\infty)\times S)$ (this fact is proved in Lemma 1 below), then the integral

$$\int_0^\infty e^{-\lambda u}\mathbb{E}^{\mathbb{R}_x}g(\tau(u),x(\tau(u)))du$$

is uniquely determined. This yields the uniqueness of $\mathbb{R}_x$ in the same way as in Corollary 6.2.4 of [12].

Lemma 1. *For each $\lambda > 0$, $g \in C_0^\infty([0,+\infty) \times S)$, there is only one solution of the equation*

$$\lambda f - \widetilde{K} f = g. \tag{25}$$

Proof. We prove Lemma 1 in two steps. At the first one, we deal with r being equal to zero on S identically. Denote, by $G^1_\lambda(t,x,y)$, the solution to the pair of equations (13), (14), in which we put $r(x) \equiv 1$, satisfying inequality (15). Let

$$V_\lambda(t,x) = \int_t^\infty d\tau \int_S G^1_\lambda(\tau - t, x, y)\psi(\tau, y) d\sigma_y,$$

where $t \geq 0, x \in \mathbb{R}^d, \psi$ is a continuous function on $[0,+\infty) \times S$ with compact support. As a consequence of (13), we can write the following relation for the function $V_\lambda(t,x)$:

$$V_\lambda(t,x) = \int_t^\infty d\tau \int_S g_0(\tau - t, x, y)\psi(\tau, y) d\sigma_y -$$

$$-\lambda \int_t^\infty d\tau \int_S g_0(\tau - t, x, y) V_\lambda(\tau, y) d\sigma_y. \tag{26}$$

It is not hard to verify that the function $V_\lambda(t,x)$ has the properties:

1) $V_\lambda(t,x)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 0 \quad \text{in the region} \quad [0,+\infty) \times D;$$

2) $V_\lambda(t,x)$ is a continuous function of (t,x) in the region $t \geq 0, x \in \mathbb{R}^d$;

3) for each $\lambda \geq 0$, $\sup_{x\in\mathbb{R}^d} |V_\lambda(t,x)| \to 0$ as $t \to \infty$;

4) for each $t \geq 0$, $V_\lambda(t,x) \to 0$ as $|x| \to \infty$.

Using the theorem on the jump of the normal derivative of a single-layer potential (mentioned above), relation (26) yields

5) the equality

$$\frac{\partial V_\lambda(t, x\pm)}{\partial \nu} = \mp\psi(t,x) \pm \lambda V_\lambda(t,x) \tag{27}$$

valid for $t \geq 0$ and $x \in S$.

The maximum principle implies the uniqueness of a function satisfying conditions 1) – 5).

Equality (27) is equivalent to the equation

$$\lambda V_\lambda(t,x) - \left[\frac{1+q(x)}{2}\frac{\partial V_\lambda(t,x+)}{\partial \nu} - \frac{1-q(x)}{2}\frac{\partial V_\lambda(t,x-)}{\partial \nu}\right] = \psi(t,x), \tag{28}$$

where $t \geq 0, x \in S$.

Moreover, the function $V_\lambda(t,x)$ on $[0,+\infty) \times \mathbb{R}^d$ is defined by its values on $[0,+\infty) \times S$ according to the formula

$$V_\lambda(t,x) = \int_t^\infty d\tau \int_S \frac{\partial g_0(\tau - t, x, y)}{\partial \nu_x} V_\lambda(\tau, y) d\sigma_y.$$

Thus, Eq. (28) coincides with (25) in the case where r is equal to zero on S, and we get the statement of Lemma 1 in this case.

If r is non-negative, we get the result from the previous one in the same way as in [11], pp.194–196.

Theorem. *Let q and r be given continuous functions on S with their values on $[-1, 1]$ and $[0, +\infty)$, respectively, and r is bounded. Then, for each $x \in \mathbb{R}^d$, there is only one solution to the submartingale problem.*

Proof. This assertion follows from Proposition 5 by the arguments of Theorem 4.2 in [11].

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Institute of Geophysics, National Academy of Sciences of Ukraine, Palladina pr. 32, 03680, Kiev-142, Ukraine
E-mail: oaryasova@mail.ru

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska str. 3, 01601, Kiev, Ukraine
E-mail: portenko@imath.kiev.ua