ISSN 1028-7507 Акустичний вісник. 2000. Том 3, N 4. С. 4–10

517.52

NONLINEAR DIFFERENTIAL EQUATIONS IN WAVELET BASES

C. CATTANI*, M. PECORARO**

*Department of Mathematics "G. Castelnuovo", University of Rome "La Sapienza"

**Department of Engineering and Applied Mathematics, University of Salerno

Received 16.06.2000

In this paper we discuss the problem of finding numerical solution of some nonlinear differential equations using wavelet bases. It is also given a short outline of the method in dealing with the nonlinear heat equation.

У статті обговорюється проблема чисельного розв'язання деяких нелінійних диференційних рівнянь із використанням вейвлет-базисів. У ній також дано короткий опис методу розв'язання нелінійного рівняння теплопровідності.

В статье обсуждается проблема численного решения некоторых нелинейных дифференциальных уравнений с использованием вейвлет-базисов. В ней также дано краткое описание метода решения нелинейного уравнения теплопроводности.

INTRODUCTION

It has been recently shown that wavelet theory [1–5] is a powerful method for the numerical modelling of linear and nonlinear differential equations [6–10]. However human experience tell us that the physics of the real word cannot be interpreted with linear equations. Mostly, there are nonlinear equations, which give us a real modelling of nature. On the other hand phenomena usually have a very short range of existence and a very high intensity in order to make influence on the environment. Thus, a localized analysis [1–5] seems to be the most convenient approach to the study of phenomena. For instance functions, like a delta Dirac or a more general distribution or simply an impulse function have been hardly accepted within a mathematical model because of the peculiarity in treating functions vanishing nearly everywhere but in a discrete set of points. In correspondence of the singularity, very often a wrong practice suggest us to remove the singularity. However, some phenomena are interesting just when the singularities "appears", therefore removing the singularity means canceling the phenomenon. Wavelets are very well localized functions, and, since they are zero nearly everywhere, they can be easily treated in numerical applications.

In this paper, the numerical approach to a differential equation is based on the representation of the unknown function in a given basis of fundamental functions, together with its derivatives. Like in the so-called collocation methods, the value of the unknown function will be a linear combination of known discrete values with some coefficients to be determined. We will choose the Haar family of wavelets as basis, and the coefficients will be computed by the fast Haar transform (see e. g. [5, 11, 12]). The problem of representing the derivatives of the unknown function will be solved smoothing the Haar wavelets with suitable order splines.

The problem of numerically solving nonlinear equations is given in section 1, then after some preliminary definitions on the Haar and on the periodic wavelets (section 2), the wavelets representations are defined in section 3. The wavelet (discrete) representation is discussed in section 4 and the fast Haar transform is described in section 5. A smoothing process, suitable for the wavelets is given in section 6, and a final example is sketched in section 7.

1. NUMERICAL SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS: L_2-APPROXIMATION

A class of methods for the numerical solution of differential equations consists in the so-called collocation (pointwise) methods. The solution is assumed to be known in a discrete set of points (nodes), and extended by a series containing the known values and a basis set of functions. This basis should be carefully chosen in order to obtain the convergence [6, 8, 9, 13, 14] of the numerical method and the completeness of the functional space.

Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right), \tag{1}$$

and assume without restrictions that $u = u(x,t)$ with $x \in \Omega \subseteq \mathsf{R}$ and $f \in \mathsf{R}$. If the domain Ω is unbounded, a simple transformation (see e. g. [14]) of the variable x leads us to an equivalent equation on a bounded domain thus we consider $\Omega = [a,b]$ where $a < \infty$, $b < \infty$. Moreover, if the function f, in the right hand side of (1), depends on higher order derivatives of u with respect to t, the equation (1) can be transformed into a system of equations containing only first order derivatives with respect to t, therefore we assume that, with respect to t, f depends only on $\partial u/\partial t$. Together with (1) it comes a set of conditions (initial and/or boundary)

$$\begin{aligned} & u(x,t)|_{t=0} = u_0(x) && \forall x \in [a,b], \\ & \begin{cases} L(u(x,t))|_{x=a} = u_a(t) \\ L(u(x,t))|_{x=b} = u_b(t) \end{cases} && \forall t \in \mathsf{R}, \end{aligned} \tag{2}$$

where L is a linear operator; u_a, u_b, and u_0 are given functions. According to the values of L, equations (1) – (2) give rise to the so called Cauchy, Dirichlet, Neumann, or Robin problem [14]. If we are looking for a solution $u(x,t)$ of (1), (2) to be expressed as a series of independent functions such as polynomials $\{x^\alpha\}$, or trigonometric functions $\{\sin(\alpha x)\}$ or other kind of functions we expect also u_a, u_b, u_0 to belong to the same class of functions as $u(x,t)$. When at least one of the "starting" functions u_a, u_b, u_0 is a discontinuous function, the best approximation of the singularity is given by the wavelets, therefore a representation of $u(x,t)$ as series of wavelets is the most suitable way to approach problems (1), (2) with starting discontinuities.

In order to build up a numerical approximation of the solution of a given problem (1), (2) let us first discretize the x-domain Ω into a set of M points x_i, with $x_0 = a$, $x_{M-1} = b$, so that in each point x_i, $i = 0, \dots, M-1$ the unknown function $u(x,t)$ becomes the unknown function of t: $u(x_i,t)$. Assume that the solution is L_2 (square summable function) and can be expressed as (see e. g. [14])

$$u(x,t) \cong \sum_{n=0}^{N} c_n(t) F_n(x) \tag{3}$$

with M (in general) independent on N, and $F_n(x)$ orthonormal functions:

$$\int_\Omega \omega(x) F_r(x) F_s(x) dx = d_{rs} \delta_{rs}$$

being δ_{rs} the Kronecker symbol; $\omega(x)$ a weight function; d_{rs} the suitable coefficients to get the orthonormality. By multiplying (3), integrating over Ω and taking into account of the above equation, there follow the coefficients

$$c_n(t) = \int_\Omega F_n(x) u(x,t) dx, \tag{4}$$

so that in a discrete time evolution problem:

$$0 \le t, \quad t_j = j\Delta t, \quad j = 0, 1, \dots$$

it is

$$c_n(0) = \int_\Omega F_n(x) u_0(x) dx,$$

and, since $u_0(x)$ is assumed to be given, the representation (3) of $u(x,t)$, at the initial time, gives the input set of discrete values for a numerical recursive formula in order to evaluate $u(x,t)$ at subsequent time steps t_j.

The space derivatives can be evaluated as follows [14]:

$$\begin{aligned} \frac{\partial u}{\partial x} &\cong \sum_{n=0}^{N} c_n(t) \frac{dF_n(x)}{dx}, \\ \frac{\partial^2 u}{\partial x^2} &\cong \sum_{n=0}^{N} c_n(t) \frac{d^2 F_n(x)}{dx^2}, \end{aligned} \tag{5}$$

and in each point x_i

$$\begin{cases} \dfrac{\partial u_i}{\partial x} \stackrel{\text{def}}{=} \dfrac{\partial u}{\partial x}\Big|_{x=x_i} \cong \displaystyle\sum_{n=0}^{N} c_n(t) \left.\frac{dF_n(x)}{dx}\right|_{x=x_i}, \\ \dfrac{\partial^2 u_i}{\partial x^2} \stackrel{\text{def}}{=} \dfrac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} \cong \displaystyle\sum_{n=0}^{N} c_n(t) \left.\frac{d^2 F_n(x)}{dx^2}\right|_{x=x_i}. \end{cases} \tag{6}$$

The partial derivatives with respect to x, become functions of time t only:

$$\begin{aligned} \frac{\partial u_i}{\partial x} &= \sum_{n=0}^{N} A_{in} c_n(t), \\ \frac{\partial^2 u_i}{\partial x^2} &= \sum_{n=0}^{N} B_{in} c_n(t), \end{aligned} \tag{7}$$

being

$$A_{in} \stackrel{\text{def}}{=} \left.\frac{dF_n(x)}{dx}\right|_{x=x_i},$$

$$B_{in} \stackrel{\text{def}}{=} \left.\frac{d^2 F_n(x)}{dx^2}\right|_{x=x_i}$$

independent on t. Thus, at time step t_j equation (1) becomes

$$\frac{du_n}{dt} = f_n(t_j, u_n; x_i, A_{in}, B_{in}),$$
$$i = 0, \ldots, M-1, \quad n = 0, \ldots, N, \quad h = 0, 1, \ldots$$

The integration of equation (1) is transformed into an ordinary system of differential equations, but still there are some crucial points in the application of this method, namely:

- since there are some integrals (4) to be evaluated one has to carefully discretize the domain Ω, in order to optimize the efficiency of the calculations (see section 5).
- we are going to use the Haar wavelets, which are either zero or constant, therefore we must consider some interpolating functions of the Haar wavelet in order to enable the calculation of (at least) the second order derivatives with respect to x (see section 6).

2. HAAR AND PERIODIC WAVELETS

In the spectral representation (3) we consider as functions $F_n(x)$ the Haar family of wavelets [1–5], based on two fundamental functions the so-called scaled function $\Phi(x)$ and the corresponding wavelet $\Psi(x)$. The Haar function has a constant value in a suitable interval and vanishes in its complement in R.

Let $A \subset \mathsf{R}$ and $\mathbf{1}_A(x)$ the function defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & (x \in A), \\ 0 & (x \notin A), \end{cases}$$

the Haar scaling function, in the interval $[0, 1]$, is

$$\Phi(x) \stackrel{\text{def}}{=} \mathbf{1}_{[0,1)}(x) \tag{8}$$

and satisfies the integral conditions

$$\int_{-\infty}^{\infty} \Phi(x)dx = 1, \qquad \int_{-\infty}^{\infty} \mid \Phi(x) \mid dx = 1. \tag{9}$$

Strictly related to Φ, the Haar wavelet $\Psi(x)$ (in $[0, 1]$) is

$$\Psi(x) = \mathbf{1}_{[0,1/2)}(x) - \mathbf{1}_{(1/2,1]}(x). \tag{10}$$

It is a function with compact support in $[0, 1]$, that trivially vanishes outside $[0, 1]$, and satisfies the conditions

$$\int_{-\infty}^{\infty} \Psi(x)dx = 0, \qquad \int_{-\infty}^{\infty} |\Psi(x)|dx = 1. \tag{11}$$

Both functions Φ and Ψ are related by a recursive formula

$$\begin{cases} \Phi(x) = \Phi(2x) + \Phi(2x-1), \\ \Psi(x) = \Psi(2x) - \Psi(2x-1). \end{cases} \tag{12}$$

Function $\Psi(x)$, defined on $[0, 1)$, as well, as $\Phi(x)$ can be easily prolonged to R by scaling and translating.

The functions

$$\Psi_{n,k}(x) \stackrel{\text{def}}{=} \sum_{j \in Z} 2^{n/2}\Psi(2^n(x-j)-k), \qquad n, j \in Z \tag{13}$$

are periodic wavelets of period 1, zero everywhere but in the interval

$$\left[\frac{k}{2^n} - j, \frac{k+1/2}{2^n} - j\right),$$
$$\left[\frac{k+1/2}{2^n} - j, \frac{k+1}{2^n} - j\right).$$

In the same intervals we can define the family of scaling functions

$$\Phi_{n,k}(x) \stackrel{\text{def}}{=} \sum_{j \in Z} 2^{n/2}\Phi(2^n(x-j)-k), \qquad j \in Z.$$

It can be easily shown that (13) vanish identically when $n < 0$. It is

$$\Psi_{n,k} = \Psi_{n,k+2^n},$$

and, if we restrict to the interval $[0, 1)$, the property

$$\sum_{j \in Z} 2^{n/2}\Psi(2^n(x-j)-k) = \Psi(2^n(x-j)-k), \qquad j \in Z, n \geq 0, \quad k = 0, 1, \ldots, 2^n - 1$$

holds. The family of wavelets are bases for L_2 functions, so that each point $(\overline{x}, \overline{y})$ of $\mathsf{R}\times\mathsf{R}$ might be expressed by $(\overline{x}, 2^{n/2}\Psi(2^n\overline{x} - k))$ for suitable (n, k). Thus, each point of $\mathsf{R}\times\mathsf{R}$ becomes function of two parameters (n, k), in the functional space whose generator are $\{2^{n/2}\Psi(2^n x - k)\}$, which are orthonormal bases for L_2.

3. MULTISCALE ANALYSIS AND HAAR REPRESENTATION

Let $\{V_n\}_{n \in Z}$ be the subset of L_2 defined as the set of functions $f(x)$ of compact support

$$V_n \stackrel{\text{def}}{=} \left\{ f(x) \in L_2 : f(x) = \begin{cases} \text{const} & \forall x \in A_{k,n}, \\ 0 & \forall x \notin A_{k,n} \end{cases} \right\},$$

$$A_{k,n} \stackrel{\text{def}}{=} \left[\frac{k}{2^n}, \frac{(k+1)}{2^n}\right), k \in Z.$$

Functions $f(x)$ and subsets V_n fulfill the following relations:

$$\begin{cases} f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}, \\ V_n \subset V_{n+1}, \\ \{\cup_{n\in Z} V_n\} = L_2, \\ \cap_{n\in Z} V_n = \emptyset. \end{cases} \tag{14}$$

Furthermore, there exists a function $\Phi(x)$ such that $\{\Phi(x-k)\}_{k\in Z}$ is an orthonormal basis in V_0. This axiom together with the first equation from (14) implies that also $\{2^{n/2}\Phi(2^n x-k)\}_{k\in Z}$ is an orthonormal basis in V_n. However, the set $\{2^{n/2}\Phi(2^n x-k)\}_{k,n\in Z}$ is not a basis for the funcions of L_2, because the subspaces V_n are not mutually orthogonal. They become orthogonal adding the complementary subspace (of wavelet) W_n, so that

$$V_{n+1} = V_n \oplus W_n,$$

where $\oplus$ is the direct sum of orthogonal subspaces. As a consequence the space L_2 is "reconstructed" as

$$L_2 = V_0 \oplus (\oplus_{n=0}^{\infty} W_n) \tag{15}$$

when $n>0$, and $\forall n$

$$L_2 = \oplus_{n\in Z} W_n,$$

so that L_2 is the direct sum of orthogonal subspaces of wavelets W_n. In each W_n the basis functions are (n is fixed)

$$\Psi_{n,k}(x) \stackrel{\text{def}}{=} \left\{2^{n/2}\Psi(2^n x - k)\right\}_{k,n\in Z}, \tag{16}$$

and for variable n the all set of functions $\Psi_{n,k}$ represents an orthonormal basis for L_2.

Any L_2-function $f(x)$ can be written in a scaling function representation as

$$f(x) = \sum_{n,k\in Z} \alpha_k^n 2^{n/2}\Phi(2^n x - k), \tag{17}$$

where

$$\alpha_k^n \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} f(x) 2^{n/2}\Phi(2^n x - k)\,dx. \tag{18}$$

The wavelet (Haar) representation is

$$f(x) = \sum_{n,k\in Z} \beta_k^n 2^{n/2}\Psi(2^n x - k), \tag{19}$$

where

$$\beta_k^n \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} f(x) 2^{n/2}\Psi(2^n x - k)\,dx. \tag{20}$$

Although, equation (19) is called wavelet representation of the function $f(x)$, the right hand side contains both scaling functions and wavelet functions.

Once $\Psi(x)$ is given, the functional dependence on the factor 2^n, through the parameter n, produces a scaling of $\Psi(x)$ while a non-trivial k translates $\Psi(x)$ either rightward ($k>0$), or leftward ($k<0$).

4. DISCRETE HAAR SERIES

In order to show the application of this method we resctrict ourselves to a L_2 function $f(x)$ defined in the unit interval $[0,1)$. Consider a domain Ω discretized in M sub-intervals at points

$$x_j = \frac{j}{M}, \qquad j = 0,\ldots,M-1.$$

According to equation (3), and using a wavelet basis, our approximation formula at a given time $\overline{t}$ will be

$$u(x,\overline{t}) \cong \sum_{n=0}^{N} c_n(\overline{t})\Psi_n(x),$$

and since $\Psi_n(x)$ is a function of two parameters n, k we have

$$u(x,\overline{t}) \cong \alpha_0^0(\overline{t}) + \\ + \sum_{n=0}^{N}\sum_{k=0}^{2^M-1} d_M \beta_k^n(\overline{t}) 2^{n/2}\Psi(2^n x - k), \tag{21}$$

so that the value of $u(x,\overline{t})$ at the nodal points x_j is

$$u_j \stackrel{\text{def}}{=} u(x_j,\overline{t}) \cong \alpha_0^0(\overline{t}) + \\ + \sum_{n=0}^{N}\sum_{k=0}^{2^M-1} d_M \beta_k^n(\overline{t}) 2^{n/2}\Psi(2^n x_j - k), \tag{22}$$

where d_M is the correction factor. The coefficients α_0^0, β_k^n at time $\overline{t}$ will be computed by a FHT, or fast Haar transform (see section 5).

5. FAST HAAR TRANSFORM

In this section, we will show how to compute the coefficients α_0^0, β_k^n of the wavelets in equation (21) in order to obtain the wavelet series of a function

$f(x) \stackrel{\text{def}}{=} u(x,\overline{t})$. In fact, for a fixed $\overline{t}$ we assume that $u(x,\overline{t})$ depends only on x. Furthermore, according to axioms (14) together with the completeness of the space L_2, we assume that after N approximation steps it is

$$L_2(\mathsf{R}) \cong V_0 \oplus (\oplus_{n=0}^{N} W_n),$$

that is (when $n+1=N$)

$$\begin{gathered} \alpha_k^N \cong f_k, \qquad f_k \stackrel{\text{def}}{=} f(x)|_{x=x_k}, \\ 0 \le k \le M-1. \end{gathered} \tag{23}$$

It is known that in order to optimize the efficiency of calculus one has to choose a number of points equal to a power of 2 [5, 11, 12, 15], for we take $M=2^R$, which implies a number of basis wavelets $N=\log_2 M=R$. In other words, we should discretize Ω in either $1,2,4,8,\ldots,2^R$ intervals, to which correspond $1,2,3,\ldots,R$ (fundamental) wavelets and $2,3,5,9,\ldots,2^R+1$ nodal points x_j.

Equations (12) imply a set of relations among the coefficients of the spectral decomposition of $f(x)$ with respect to $\Phi(x)$ and $\Psi(x)$. From the definition

$$\begin{cases} \alpha_k^n \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} f(x) 2^{n/2} \Phi(2^n x - k) dx, \\ \beta_k^n \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} f(x) 2^{n/2} \Psi(2^n x - k) dx, \end{cases}$$

taking into account equations (12), to arbitray n and k there follows

$$\begin{gathered} \alpha_k^n = \frac{1}{\sqrt{2}} \alpha_{2k}^{n+1} + \frac{1}{\sqrt{2}} \alpha_{2k+1}^{n+1}, \\ \beta_k^n = \frac{1}{\sqrt{2}} \alpha_{2k}^{n+1} - \frac{1}{\sqrt{2}} \alpha_{2k+1}^{n+1}, \\ k = 0, \ldots, M/2 - 1. \end{gathered} \tag{24}$$

These equations can be written in a symbolic form:

$$\mathbf{B}_{2M}^n = \mathbf{O}\mathbf{B}_{2M}^{n+1} \tag{25}$$

with

$$\begin{gathered} \mathbf{O} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{B}_s^r \stackrel{\text{def}}{=} \begin{bmatrix} \alpha_s^r \\ \beta_s^r \end{bmatrix}, \\ s = 0, \ldots, M-1. \end{gathered} \tag{26}$$

In order to derive the coefficients (26) (the second equation) of the matrix $\mathbf{B}_s^r$, we assume, according to (23), that the M values f_k are known ($k=0,\ldots,M-1$), so that α_{2M}^R are known.

Let

$$\mathbf{f} \stackrel{\text{def}}{=} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_M \end{bmatrix}, \qquad M = 2^R - 1. \tag{27}$$

It is

$$\mathbf{f} = \mathbf{B}_{2M}^R,$$

or explicitly

$$\mathbf{B}_{2M}^R = \left[\alpha_0^R, \alpha_1^R, \ldots, \alpha_{M/2-1}^R, \beta_0^R, \beta_1^R, \ldots, \beta_{M/2-1}^R\right]^T,$$

where T stands for transpose. From the knowledge of the matrix $\mathbf{B}_{2M}^R$ one has to construct the matrix $\mathbf{B}_{2M}^{R-1}$ and so forth until we get $\mathbf{B}_{2M}^0$. Once $\mathbf{B}_{2M}^r$, $0 \le s \le R-1$ is determined one should extract the submatrix of the coefficients β_h^r, $0 \le h \le M-1$. To this purpose let us define the direct sum of matrices

$$\mathbf{S} \oplus \mathbf{Q} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix},$$

$$\mathbf{0} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix},$$

and the $M \times M$ orthogonal matrix

$$\mathbf{H}_M = \oplus_{s=1}^{M/2} \mathbf{O}.$$

Then, taking into account of the orthogonality of $\mathbf{H}_M$ we can solve equation (25) to have (see e. g. [11])

$$\mathbf{H}_M \mathbf{f} = \mathbf{X}_{2M}^{R-1}, \qquad \mathbf{f} = \mathbf{B}_{2M}^R,$$

where $\mathbf{X}_{2M}^{R-1}$ contains the coefficients α_h^{R-1} mixed with β_h^{R-1} in the form

$$\begin{gathered} \mathbf{X}_{2M}^{R-1} = [\alpha_0^{R-1}, \beta_0^{R-1}, \alpha_1^{R-1}, \beta_1^{R-1}, \ldots \\ \ldots, \alpha_{M/2-1}^{R-1}, \beta_{M/2-1}^{R-1}]^T. \end{gathered}$$

From the above, by multiplying for a permutation matrix $\mathbf{P}_M$ we get

$$\mathbf{B}_M^{R-1} = \mathbf{P}_M \mathbf{X}_M^{R-1}.$$

From the knowledge of the vector $\mathbf{B}_M^{R-1}$, we are intersted only of the first M coefficients, namely α_s^{R-1}, $s=0,\ldots,M$, thus if we multilpy $\mathbf{B}_M^{R-1}$ by the $M \times M$ unit matrix $\mathbf{I}_M$ we obtain a new vector to replace

f in the same routine. We restart the same procedure based in two steps: (product by $\mathbf{H}_M$ and subsequent permutation by $\mathbf{P}_M$) to obtain the following $\mathbf{B}_M^{R-2}$ and so on until we get $\mathbf{B}_M^2$. In other words we repeat the same procedure until the number of the coefficients, to be evaluated, is two. At this stage $R=\log 2M$, $\Phi(x)$ is characterized by only one coefficient as well as the corresponding wavelet for $R=0$.

Shortly the fast Haar transformation $\mathcal{H}$ of a vector **f** is [11]

$$\mathcal{H}\ \mathbf{f} \stackrel{\text{def}}{=} (\mathbf{H}_2 \oplus \mathbf{I}_{M-2}) \dots$$

$$\dots (\mathbf{P}_{M/2} \oplus \mathbf{I}_{M/2})(\mathbf{H}_{M/2} \oplus \mathbf{I}_{M/2})\mathbf{P}_M \mathbf{H}_M \mathbf{f},$$

and the reconstruction formula of $f(x)$ is based on the values set

$$\alpha_0^0 = [\mathcal{H}\ \mathbf{f}_M]_1^1\,, \qquad \{\beta_k^n\}_{k=0,\dots,n}^{n=0,\dots,\log M} = [\mathcal{H}\ \mathbf{f}_M]_{i=1,\dots,M}^1\,, \tag{28}$$

that is the first components of $\mathcal{H}\ \mathbf{f}_M$ is α_0^0, the remaining components are β_k^n.

The inverse transformation is [11]

$$\mathcal{H}^{-1} \stackrel{\text{def}}{=} \mathbf{H}_M^T \mathbf{P}_M^T (\mathbf{H}_{M/2}^T \oplus \mathbf{I}_{M/2})(\mathbf{P}_{M/2}^T \oplus \mathbf{I}_{M/2}) \dots$$

$$\dots (\mathbf{H}_2^T \oplus \mathbf{I}_{M-2}).$$

It can be easily proven that, from numerical point of view, $\mathcal{H}$ requires only $O(M)$ operations, i.e., is linear on M, while, for example, the celebrated FFT requires $O(M \log M)$, showing the advantage (among others) of using the discrete Haar transformation (also called forward Haar wavelet transform [7]).

6. SPLINES

Once we have a wavelet representation (19) of the function $f(x)$, since we are using Haar wavelets (which are piecewise constant functions) we can easily integrate $f(x)$. Unfortunately the derivative of Haar wavelets, and then $f(x)$, vanishes trivially. Thus, if we want to apply equations (6), in order to get (7), to the wavelet representation (22) of the approximated function $u(x)$ we should "transform" the piecewise functions in non-trivially differentiable functions. This can be done, for instance, with spline interpolation. An m-order spline, in a discretize interval, is a family of m-polynomials each one defined in a subinterval that meet some smoothness conditions at the nodes.

Equation (22), in short, is a piecewise function of the kind

$$f(x,t) \cong \sum_s p_s(x,t)\delta_{x-s},$$

where

$$p_s(x,t) \stackrel{\text{def}}{=} \begin{cases} \delta_{sk} = \begin{cases} 0, & k \neq s, \\ 1, & k = s, \end{cases} & x = x_k, \\ \alpha_0^0(t) + \sum\limits_{n=0}^{N} \sum\limits_{k=0}^{2^M-1} d_M \times & x \neq x_k. \\ \times \beta_k^n(t) 2^{n/2} \Psi(2^n x - k), \end{cases} \tag{29}$$

Therefore, in order to derive a spline interpolation, we might restrict ourselves to the Kronecker delta function δ_{i-j}, defined as

$$\delta_{i-j} \stackrel{\text{def}}{=} \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i,j \in Z.$$

The only non-vanishing value of δ_{i-j}, in the interval $[-1,1]$, is in 0. Nodal points x in $[-1,1]$ are $\{-1,0,1\}$,with corresponding y-values, $\{0,1,0\}$. Thus, if we want a linear spline, we divide the interval $[-1,1]$ into $[-1,0]\cup[0,1]$ with nodal points $\{-1,0\}\cup\{0,1\}$. In each subinterval we construct a linear spline (order 1). If we divide further, we have $[-1,-1/2]\cup[-1/2,0]\cup[0,1/2]\cup[1/2,1]$, with nodal points $\{-1,-1/2,0,1/2,1\}$, and we can define a second order polynomial in each sub interval. Doing so, we can divide each interval as much as we need in order to obtain an m-order spline interpolation of the initial piecewise function.

7. THE NONLINEAR HEAT EQUATION

The nonlinear heat equation is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[k(u)\frac{\partial u}{\partial x}\right]. \tag{30}$$

If the conductivity is assumed to be

$$k(u) = 1 + hu, \qquad h = \text{const},$$

the equation (30) becomes

$$\frac{\partial u}{\partial t} = h\left(\frac{\partial u}{\partial x}\right)^2 + (1+hu)\frac{\partial^2 u}{\partial x^2}\,. \tag{31}$$

Consider for the unknown function $u(x,t)$, the wavelet representation (21) with $\overline{t}$ ranging in $[0,\overline{T}]$. According to equations (5), (7), and using the Galerkin method, the equation (31) becomes a system of binary differential equations in the coefficients β_k^n:

$$\frac{du_i}{dt} = h\left[\sum_{j=1}^{M} A_{ij}\,u_j\right]^2 + (1+hu_j)\sum_{j=1}^{M} B_{ij}\,u_j,$$

which can be easily solved by ordinary numerical methods.

CONCLUSION

We have discussed the wavelet representation method applied to the solution of nonlinear partial differential equations. In particular, the numerical solution of the nonlinear heat equation is sketched.

1. *Daubechies I.* Ten lectures on wavelets // CBMS-NSF Regional Conference Series in Applied Mathematics.– Philadelphia: SIAM, 1992.
2. *David G.* Wavelets and Singular Integrals on Curves and Surfaces, Lecture Notes in Mathematics.– Berlin, Heidelberg: Springer Verlag, 1991.
3. *Meyer Y.* Ondelettes.– Paris: Hermann, 1990.
4. *Meyer Y.* Wavelets and applications.– Masson: RMA, **20**, 1992.
5. *Meyer Y.* Wavelets: algorithms and applications.– Philadelphia: SIAM, 1993.
6. *Fröhlich J., Schneider K.* An adaptive wavelet Galerkin algorithm for one- and two- dimensional flame computations // Eur. J. Mech. B / Fluids.– 1994.– **13**, N 4.– P. 439–471.
7. *Goedecker S., Ivanov O.* Solution of multiscale partial differential equations using wavelets // Comput. in Phys.– 1998.– **12**, N 6.– P. 548–555.
8. *Lazaar S., Ponenti Pj., Liandrat J., Tchamitchian P.* Wavelet algorithms for numerical resolution of partial differential equations // Comput. Methods Appl. Mech. Eng.– 1994.– **116**.– P. 309–314.
9. *Lazaar S., Liandrat J., Tchamitchian P.* Algorithme à base d'ondelettes pour la résolution numérique d'équations aux dérivés partielles à coefficents variables // C. R. Acad. Sci. Paris.– 1994.– **319**, S. I.– P. 1101–1194.
10. *Tchamitchian P.* Wavelets and differential operators // Proceedings of Symposia in Applied Mathematics.– 1993.– **47**.– P. 77–88.
11. *Walker J. S.* Fourier analysis and wavelet analysis // Notices of the AMS.– 1997.– **44**, N 6.– P. 658–670.
12. *Strichartz R.* How to make wavelets // MAA Monthly.– 1993.– **100**, N 6.
13. *Mallat S.* Multiresolution approximation and wavelet orthonormal bases of $L2(R)$ // Trans. Amer. Math. Soc.– 1989.– **315**.– P. 69–87.
14. *Monaco R. Pandolfi M.* On a class of methods for the approximated solutions to nonlinear intial boundary value problems.– Politecnico di Torino, Internal Report, 1999.
15. *Briggs P., Henson V. E.* The DFT. An owner's manual for the Discrete Fourier Transform.– Philadelphia: SIAM, 1992.