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Local wellposedness and global regularity results for biharmonic wave maps

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Abstract

This thesis is concerned with biharmonic wave maps, i.e. a bi-harmonic version of the wave maps equation, which is a Hamiltonian equation for a higher order energy functional and arises variationally from an elastic action functional for a manifold valued map.

In the first part we present local and global results from energy estimates for biharmonic wave maps into compact, embedded target manifolds. This includes local wellposedness in high regularity and global regularity in subcritical dimension $n = 1, 2$. The results rely on the use of careful a priori energy estimates, compactness arguments in weak topologies and sharp Sobolev embeddings combined with energy conservation in the proof of global regularity.

In part two, we extend these results to global regularity in dimension $n \geq 3$ for biharmonic wave maps into spheres and initial data of small size in a scale invariant Besov norm. This follows from a small data global wellposedness and persistence of regularity result for more general systems of biharmonic wave equations with non-generic nonlinearity. In contrast to part one, the arguments in part two of the thesis rely on the analysis of bilinear frequency interactions based on Fourier restriction methods and Strichartz estimates.

The results in both parts of the thesis fundamentally depend on the non-generic form of the nonlinearity that is introduced by our biharmonic model problem.

Contents

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Introduction

A well known nonlinear wave equation is the $(2+1)$ *wave maps equation*,

$$-\partial_t^2 u + \Delta u = (|\partial_t u|^2 - |\nabla u|^2)u, \tag{1}$$

which is obtained by the Lagrangian formalism for the action

$$W(u) = \frac{1}{2}\int (|\partial_t u|^2 - |\nabla u|^2)\, d(x,t).$$

for (smooth) maps $u : (-T,T)\times\mathbb{R}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ into the round sphere. Since this is a covariant version of the free wave equation

$$\Box u = -\partial_t^2 u + \Delta u = 0, \;\; \textit{i.e.}\;\; D^\alpha \partial_\alpha u = 0,$$

where D^α is the covariant derivative along u, the equation (1) can be understood as generalizing free wave propagation to the evolution of a hypersurface in the sphere $\mathbb{S}^3$.

In this thesis, we introduce a *bi-harmonic version* of the classical wave maps equation (1) (for general targets N and dimensions $n \in \mathbb{N}$), by considering the following *elastic* action functional

$$\Phi(u) = \frac{1}{2}\int (|\partial_t u|^2 - |\Delta u|^2)\, d(x,t),$$

where the potential energy can be considered a (linearized) bending energy. Hence, in the Euler Lagrange equation, $\Box u$ will be replaced by the *biharmonic wave operator*

$$Lu = \partial_t^2 u + \Delta^2 u,$$

where $\Delta^2 = \Delta(\Delta\cdot) = \partial_{ij}\partial^{ij}$ is the (iterated) *bi-Laplacien*. In the case of the sphere $N = \mathbb{S}^L$, a critical map $u : (-T,T)\times\mathbb{R}^n \to \mathbb{S}^L$ of Φ satisfies

$$\begin{aligned}\partial_t^2 u + \Delta^2 u = &- |\partial_t u|^2 u - \Delta(|\nabla u|^2)u \\ &- (\nabla\cdot\langle\Delta u, \nabla u\rangle)u - \langle\nabla\Delta u, \nabla u\rangle u.\end{aligned} \tag{2}$$

The operator L finds applications in the description of e.g. elastic beams ($n = 1$) or thin, stiff elastic plates ($n = 2$) and hence introduces a rather *rigid* evolution. In particular, (2) can be understood as generalizing the free propagation of L into the sphere.

In the following sections of the thesis, we introduce the *biharmonic wave maps equation* for general targets and aim to prove local wellposedness with high initial regularity (Chapter 1) and global regularity with small data in a scaling critical Besov space (Chapter 2). This is an attempt to structurally extend the work on the wave maps equation (1) to equations of the type (2).

The wave maps equation

Let $N \subset \mathbb{R}^L$ be a smooth Riemannian (sub-)manifold with induced metric and $T \in (0, \infty]$. Then, taking smooth, compact variations $u_\delta : (0,T) \times \mathbb{R}^n \to N$ of the action functional

$$S(u) = \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} |\partial_t u(t)|^2 - |\nabla u(t)|^2 dx\, dt,$$

a smooth critcial map $u : (0,T) \times \mathbb{R}^n \to N$ satisfies the constraint

$$\Box u \perp T_u N \tag{3}$$

on $(0,T) \times \mathbb{R}^n$, i.e. u satisfies the equation

$$\Box u = \mathcal{A}(u)(\partial^\alpha u, \partial_\alpha u), \tag{4}$$

where $\Box = \partial^\alpha \partial_\alpha = -\partial_t^2 + \Delta$ is the d'Alembert operator, $\partial^\alpha u \partial_\alpha u = \partial_t u \partial_t u - \nabla u \cdot \nabla u$ and $\mathcal{A}$ denotes the second fundamental form of N. Calculating (4) from (3), we use the smooth family of orthogonal tangent projector

$$P_p : \mathbb{R}^L \to T_p N, \;\; p \in N,$$

such that (3) reads $\Box u = (I - P_u)(\Box u)$ by which (4) follows since u maps into N, whence $Du \in T_u N$ and $\mathcal{A}(u)(\cdot,\cdot) = -dP_u(\cdot,\cdot)$ (on the tangent bundle).

The wave equation (4) is the extrinsic formulation of the general *wave maps equation*. In the physics literature, (4) is known as *nonlinear sigma model* (for homogeneous N) and was introduced in high energy physics [16] by Gell-Mann and Lévy in 1960. Since the $(3 + 1)$ wave maps equation has scaling supercritical energy (see below), the singularity formation of (4) is considered to be a relevant toy model, for instance in general relativity.

Concerning the regularity theory, over the past decades until now, (4) served as a model equation for geometric field equations due to the subtle theory of its Cauchy problem involving e.g. the gauge invariance and the null structure of the nonlinearity. Especially, the energy critical $(2+1)$ wave maps equation admits an intriguing threshold behaviour and allows for rich classes of singular solutions. We refer to the surveys [56] and [32]. Further, [15] and [53] contain an overview over many advances in the local and global theory of the past decades, concerning which we also give a few more details below.

The biharmonic wave maps equation

As mentioned in the beginning, we take the *elastic* action functional

$$\Phi(u) = \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} |\partial_t u(t)|^2 - |\Delta u(t)|^2 \, dx \, dt,$$

where $\Delta = \partial_i \partial^i$ denotes the Laplace operator. Under smooth, compact variations $u_\delta : (0,T)\times\mathbb{R}^n \to N$, we obtain that a critical map u satisfies

$$\partial_{tt}^2 u + \Delta^2 u \perp T_u N, \tag{5}$$

pointwise on $(0,T)\times\mathbb{R}^n$. The derivation of (5) will be given rigorously below, including its expansion into a semi-linear *biharmonic wave equation* of the form

$$\begin{aligned}\partial_t^2 u + \Delta^2 u = dP_u(u_t,u_t) + dP_u(\Delta u,\Delta u) + 4dP_u(\nabla u,\nabla\Delta u)\\ + 2dP_u(\nabla^2 u,\nabla^2 u) + \mathcal{N}(u,\nabla u,\nabla^2 u),\end{aligned} \tag{6}$$

where $dP_u(\cdot,\cdot)$ is the differential of the tangent projector $P_p : \mathbb{R}^L \to T_pN,\ p \in N$ as explained above. In fact, from the expansion of (5), the derivatives in the nonlinearity $\mathcal{N}$ appear in tri-linear and quadri-linear forms (more precisely d^2P_u and d^3P_u).

The equation (6) is Hamiltonian with energy

$$\mathcal{E}(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t u(t)|^2 + |\Delta u(t)|^2 \, dx, \ \ t \in (0,T), \tag{7}$$

and thus $\mathcal{E}(u)$ is (formally) conserved along a solution u. In contrast to (3) and (4), much less is known about (5) and (6). In fact, this thesis will present the first local wellposedness and global regularity results for (6). We now proceed by a brief overview of the wave maps theory, wich serves as a motivation for the attempt of *extending the wave maps theory to the higher order equation* (6).

Motivation

In the field of nonlinear partial differential equations, the class of *geometric field equations*, such as e.g. the *Yang-Mills equation*, *Einstein's field equations*, *Dirac's equation* or *Maxwell's equation* (and couplings thereof) are of great interest, both in the physics and mathematics literature. Besides describing fundamental laws of nature, the mathematical interest in such equations is dedicated to the connection of the Cauchy problem to e.g. renormalization (via gauge invariance) or the underlying geometry/topology of the background manifolds (e.g. via the theory of solitary solutions).

A particular, simpler model problem that shares this features with the above class of equations and attracted substantial interest in the past decades, is the wave maps equation

$$\Box u = m^{\alpha\beta}\mathcal{A}(u)(\partial_\alpha u, \partial_\beta u), \tag{8}$$

for maps $u : (M^{n,1}, m) \to N \subset \mathbb{R}^N$ on an $(n+1)$ dimensional Lorentzian manifold $(M^{n,1}, m)$, i.e. the signature of the metric tensor is $sig(m) = (-, +, \ldots, +)$, into an embedded target manifold $N \subset \mathbb{R}^N$. Here

$$\sqrt{-|m|}\Box u = \partial_\alpha(m^{\alpha\beta}\sqrt{-|m|}\partial_\beta u)$$

denotes the Laplace-Beltrami operator with $|m| = \det(m_{\alpha,\beta})_{\alpha,\beta}$. Especially, in this introductory text, we assume that we can choose a global chart for the domain manifold $(M^{n,1}, m)$. As mentioned before, (8) is the Euler-Lagrange equation of the action

$$S(u) = \frac{1}{2}\int_{M^{n,1}} \delta_{ij} m^{\alpha\beta}\partial_\alpha u^i \partial_\beta u^j dV_g, \; dV_g = \sqrt{-|m|}dx, \tag{9}$$

where dV_g is the volume form. The above condition (3), resp. its expansion (8), corresponds to the *extrinsic point of view*, i.e. $N \hookrightarrow \mathbb{R}^L$ is embedded.

However, since (9) does not depend on the embedding for N, we can choose for instance Nash's isometric embedding, for which

$$u^*(\nabla_{\partial_\alpha u} X)^T = \mathcal{D}_\alpha X,$$

with $\mathcal{D}_\alpha$ denoting the *covariant derivative* (wrt. x_α) in the pull-back bundle u^*TN, X a section of TN along u and $\nabla_{\partial_\alpha u}$ the directional derivative in the ambient euclidean space. Then, in particular, equation (8) reads as

$$\mathcal{D}_\alpha(m^{\alpha\beta}\sqrt{-|m|}\partial_\beta u) = 0, \tag{10}$$

and can hence be seen as a *free wave equation in* N. This corresponds to the *intrinsic viewpoint* of the wave maps equation. In particular, if u is *localized*, i.e. the image of u is contained in the domain of a coordinate chart, this is fomulated via local coordinates $u = u^j\partial_j$ as

$$\Box u^j = m^{\alpha\beta}\Gamma^j_{kl}(u)\partial_\alpha u^k \partial_\beta u^l, \tag{11}$$

where Γ is the Christoffel symbol of the Levi-Civita connection on N. The formulations (8) and (11) of this wave equation indicate that, unlike in the case of a flat target N, the presence of curvature affects the asymptotic behaviour of solutions and might even lead to singularities. Indeed, the wave maps equation admits a rich theory of global regularity results (e.g. for small initial data) and singular solutions (in the equivariance reduction). We restrict to the Minkowski domain $(\mathbb{M}^{n+1}, m)$, for which (8) is Hamiltonian with energy

$$E(u(t)) = \frac{1}{2}\int |\partial_t u(t)|^2 + |\nabla u(t)|^2 \, dx.$$

Remark: In fact the *stress-energy tensor*

$$T^{\alpha\beta}(u) = \langle \partial_\alpha u, \partial^\beta u\rangle - \frac{1}{2} m_{\alpha\beta}\langle \partial^\gamma u, \partial_\gamma u\rangle, \ \ \alpha, \beta = 0, \dots d,$$

is *divergence free*

$$\partial_\alpha T^{\alpha\beta}(u) = 0, \ \ \beta = 0, \dots d,$$

which corresponds to the conservation of *energy* for $\beta = 0$.

A specific invariance of (8), resp. (11) is introduced by an isometry on N, which leads to a *conservation law* (Noether's law). A related property (even if the target N is not symmetric) is obtained by renormalization from the coordinate invariance of the wave maps equation. We now give a heuristic argument for (11).

Gauge invariance: Let $P : \mathbb{M}^{n+1} \to SO(L)$ and u solve (11), then (formally)

$$\Box_P u = m^{\alpha\beta}(P\Gamma(u)\partial_\alpha u P^{-1} + (\partial_\alpha P)P^{-1})P\partial_\beta u,$$

where $\Box_P u = P\Box u + \langle dP, du\rangle_m = \mathrm{div}(PDu)$. It is verified that $\Gamma(u)\partial_\alpha u$ corresponds to the connection form in the pull-back bundle u^*TN, respresented by a *coordinate frame* along u. Especially, $(P\Gamma(u)\partial_\alpha u P^{-1} + (\partial_\alpha P)P^{-1})$ corresponds to this representaion in the coordinates of $\Psi_\alpha = P\partial_\alpha u$ and (11) is *underdetermined* up to the gauge orbit of Du. For instance, this has been exploited in the intrinsic formulation (10), in the case where no localization to (11) is possible (e.g. with rough data).

The gauge invariance introduces the freedom of fixing a *good gauge transform*, for example a Coulomb gauge satisfying $\sum_{k=1}^n \partial_k A_k = 0$, and is essential in Hélein's moving frame method developed for harmonic maps in [17] (see [18]). It thus recovers (e.g. in the Coulomb gauge) a similar Hodge structure as available in the presence of symmetry (for N). We refer to [15, chapter 1.2.5 - 1.2.8] for details.

The wave maps equation has hyperbolic scaling $u_\lambda(t,x) = u(t/\lambda, x/\lambda), \ \lambda > 0$ and hence we have the scaling laws

$$E(u_\lambda(t)) = \lambda^{n-2}E(u(t/\lambda)),$$
$$\|u_\lambda(t)\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{\frac{n}{2}-s}\|u(t/\lambda)\|_{\dot{H}^s(\mathbb{R}^n)}.$$

Especially, the dimension $n = 2$ is scaling critical (for the energy E) and heuristically (11) is expected to be locally wellposed in $H^s \times H^{s-1}$for $s > \frac{n}{2}$ and globally regular for small initial data in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$ (including the energy space in dimension $n = 2$).

The null condition

One particularly relevant property of (11) is the *null condition*, i.e. we consider

$$\Box u = \Gamma(u)_{kl} Q(Du^k, Du^l), \ \ Q(u,v) := q^{\alpha\beta} u_\alpha v_\beta,$$

where $Du = (\partial_\alpha u)_\alpha$. Then $Q(u,v) = q^{\alpha\beta} u_\alpha v_\beta$ is said to be a *null form*, if Q degenerates whenever $u \perp_m v$,

$$\forall\, \xi_1\, \xi_2 \in \mathbb{R}^{n+1} \ : \ \langle \xi_1, \xi_2 \rangle_m = 0 \quad \Rightarrow \quad Q(\xi_1, \xi_2) = 0. \tag{12}$$

For such equations, Klainerman, proved the existence of $(3+1)$ global smooth solutions starting from smooth L^∞-small initial data in [28] (in fact the null condition in [28] applies to quasilinear wave equations). This follows by the use of energy methods (for invariant vector fields) and the Klainerman Sobolev inequality (see e.g. Sogge's book [48]), where the null condition is relevant for stronger decay in the vector field approach. This improves $(3+1)$ almost global existence for equations of type (11), which was proven in [23] and [37] for weaker classes of wave equations.

Remark: It is shown that any null form decomposes into the linear combination of the RHS of (11) and

$$Q_{\alpha\beta}(u,v) = u_\alpha v_\beta - u_\beta v_\alpha, \ \ \alpha, \beta = 0, \dots, n.$$

Another way of representing the null form of (11), is via the following commutator identity

$$\mathcal{Q}(u) = \Gamma_{kl}(u)(\Box(u^k u^l) - u^k \Box u^l - u^l \Box u^k), \quad \Box = \partial^\alpha \partial_\alpha. \tag{13}$$

Further, as explained in [53, chapter 6] via planar wave solutions, interacting spatial frequencies are cancelled by the null form (13) if they are parallel and enhanced if they are perpendicular.

The local wellposedness of (11) in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ for $s > \frac{n}{2}$ was proved by Klainerman-Machedon in [29] ($n = 3$) and Klainerman-Selberg in [31] ($n \geq 2$). The results depend crucially on the null condition in (11) via multilinear estimates in the *Wave-Sobolev*, or $X^{s,b}$, spaces. Especially, an example of Lindblad in [38] shows that the sharp regularity index $s_0 > 0$ for local existence satisfies $s_0 > \frac{n}{2}$ if the null condition is absent and the dependence on Du of the nonlinearity is generic. A proof of the local wellposedness is outlined in [15], estimating (13) and following the approach presented by D'Ancona-Georgiev [11].

The division problem

Small data global regularity in the scaling critical Besov space

$$\dot{B}^{2,1}_{\frac{n}{2}}(\mathbb{R}^n) \times \dot{B}^{2,1}_{\frac{n}{2}-1}(\mathbb{R}^n),$$

was first proven by Tataru in [54] ($n \geq 4$). The solution is based on estimating (13) in the F, $\Box F$ spaces, by a global bound for $\Box u \in L^1_t L^2_x$ correcting the $X^{s,b}$ approach (which is necessary in the threshold case of $b = 1/2$). The low dimensional case ($n = 2, 3$) was similarly solved by Tataru in [55] via smoothing estimates in *null-frame* coordinates and is more involved than the high dimensional case. Recently, the division problem has also been solved in a U^2 based space by Candy-Herr in [8] ($n \geq 2$) using (recent) advances in bilinear Fourier restriction theory.

The space $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$

In [51] ($n \geq 5$), Tao found a *microlocal renormalization procedure* which, in a crucial step toward regularity in the scaling critical space $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$, recovers well-behaved pertubations of *Noether's law* in the sphere $N = \mathbb{S}^L$. Subsequently, in [30] ($n \geq 5$), this was generalized to target manifolds of bounded geometry by Klainerman-Rodnianski, following Hélein's moving frame technique and Tao's renormalization idea (this was suggested by Tao in [51]).

The gauge invariance was e.g. exploited in the work of Shatah-Struwe [46] ($n \geq 4$, alternative proof of [30]) and Nahmod-Stefanov-Uhlenbeck in [42] ($n \geq 4$, N homogeneous), by Héleins moving frame approach (resp. a direct Hodge structure in [42]). The latter works are on the physical side and depend on Strichartz estimates that control the (intrinsic) system a priori, whereas the work in [51] and [30] is based on the control of frequency envelopes in a bootstrap argument. In both cases the proof deals with limits of smooth approximations, which is due to ill-posedness, see e.g. [25] and [11] for the results on the data-to-solution map.

For the particularly challanging case of small data in the energy space $\dot{H}^1 \times L^2$, global regularity was proved by Tao for $N = \mathbb{S}^2$ in [52], by Krieger for $N = \mathbb{H}^2$ in [33] and Tataru for general targets in [57]. As seen above, the $(2 + 1)$ wave maps equation has a scale invariant energy and one can identify a more precise *threshold* for global regularity. This was confirmed to be the *energy gap* (of harmonic maps) by Sterbenz-Tataru in [50] (along a Sacks-Uhlenbeck type theorem) and Krieger-Schlag for the hyperbolic target in [34] by concentration-compactness methods (in this case the threshold for global regularity is infinite).

The literature is vast and this brief overview is not complete. There are many aspects (respectively details) that are left out and for which we refer to [53], [45], [15].

Biharmonic wave maps

As mentioned above, we aim to model the free, rigid movement in an ambient curved space. As a simplified model in linear elasticity theory, the *bending energy* of a thin, stiff elastic surface (i.e. deflections are only marginal) is well approximated by the *bi-harmonic* energy

$$\tilde{\mathcal{E}}(u) = \frac{1}{2}\int |\Delta u|^2\, dx. \tag{14}$$

The classical book of Courant and Hilbert [9] gives a general variational description the evolution of elastic membranes and thin plates. For instance, in case of *small deformations* $|\nabla u| \sim \varepsilon$, the Lagrangian of the potential energy of *elastic membranes* is simplified from the area density of the graph to the Dirichlet energy (on domains)

$$A(u) = \sqrt{1 + |\nabla u|^2} = 1 + \frac{1}{2}|\nabla u|^2 + o(|\nabla u|^2) \sim 1 + \frac{1}{2}|\nabla u|^2.$$

For the evolution of a two-dimensional plate, we apply a similar reasoning. However, the *movement is rather rigid* and the Lagrangian for the bending energy of a thin plate is proportional to

$$H(u) = \frac{1}{2}\big(\kappa_1 + \kappa_2\big)^2,$$

where κ_1, κ_2 are the principle curvature functions of the graph of u. Similar as above in the case of *small deformations*, we approximate the sum $\kappa_1 + \kappa_2$

$$\begin{aligned}\partial_j\bigg(\frac{\partial^j u}{\sqrt{1+|\nabla u|^2}}\bigg) &= \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{\partial^i u \partial_j u}{(1+|\nabla u|^2)^{\frac{3}{2}}}\partial_i \partial^j u\\ &= \Delta u + \nabla^2 u \cdot \mathcal{O}(|\nabla u|^2).\end{aligned}$$

Hence, if $\Delta u \in L^2$, we observe that $H(u) \sim \frac{1}{2}|\Delta u|^2$ justifies the potential energy (14) in this situation.

We now calculate the first variation $\delta\Phi$ of the (extrinsic) elastic action

$$\Phi(u) = \frac{1}{2}\int_{-T}^{T}\int_{\mathbb{R}^n} |\partial_t u|^2 - |\Delta u|^2 \, dx \, dt \tag{15}$$

for maps $u : (-T,T) \times \mathbb{R}^n \to (N,h)$, where $(N,h) \hookrightarrow \mathbb{R}^L$ is isometrically embedded.

We denote by $\Pi : \mathbb{R}^L \to N$ the nearest point projector map, i.e. such that

$$|\Pi(p) - p| = \inf_{q \in N} |q - p|, \ \ \operatorname{dist}(p, N) \ll 1,$$

and give a reference for the existence of this map in Chapter 1. Then we take

$$\frac{d}{d\varepsilon}_{|\varepsilon=0}\Pi(u + \varepsilon\Psi) = d\Pi_u(\Psi) = \Psi^T \in T_uN, \ \ \Psi \in C_c^\infty((-T,T)\times\mathbb{R}^n, \mathbb{R}^L).$$

Hence setting $P_u := d\Pi_u : \mathbb{R}^L \to T_uN$ (the orthogonal projector) and integrating by parts

$$0 = \frac{d}{d\varepsilon}_{|\varepsilon=0}\Phi(\Pi(u+\varepsilon\Psi)) = -\int_{-T}^{T}\int_{\mathbb{R}^n} \langle P_u(\partial_t^2 u(t) + \Delta^2 u(t)), \Psi\rangle dx \, dt,$$

which gives

$$\partial_t^2 u + \Delta^2 u \perp T_uN \quad \text{on} \ \ (-T,T)\times\mathbb{R}^n, \tag{16}$$

respectively

$$\partial_t^2 u + \Delta^2 u = (I - P_u)(\partial_t^2 u + \Delta^2 u) \quad \text{on} \ \ (-T,T)\times\mathbb{R}^n\,.$$

In order to derive (6), we calculate using $Du = (\nabla u, \partial_t u) \in T_uN$

$$\begin{aligned}[(I-P_u)(\partial_t^2 u)]^l &= dP_u^l(\partial_t u, \partial_t u) = \partial_{p_k}P^l_m(u)\partial_t u^m \partial_t u^k\\ [(I-P_u)(\Delta u)]^l &= dP_u^l(\partial_i u, \partial^i u) = \partial_{p_k}P^l_m(u)\partial_i u^m \partial^i u^k,\end{aligned}$$

where $k, l, m = 1, \dots, L$ and $i = 1, \dots, n$. Further, we obtain

$$(I - P_u)(\Delta^2 u) = \Delta(dP_u(\partial_i u, \partial^i u)) + \partial_j(dP_u(\partial^j u, \Delta u)) + dP_u(\partial_j u, \partial^j \Delta u).$$

We then calculate, writing $d^k P_p = \partial_p \cdots \partial_p P_p : \mathbb{R}^{kL} \to \mathbb{R}^L$ for the k-th order derivatives,

$$\begin{aligned} \Delta(dP_u(\partial_i u, \partial^i u)) &= 2dP(u)(\partial_i \Delta u, \partial^i u) + 2dP(u)(\partial_j \partial^i u, \partial_i \partial^j u) \\ &\quad + 2d^2 P(u)(\partial_i u, \partial^i \partial_j u, \partial^j u) + 2d^2 P(u)(\Delta u, \partial_i u, \partial^i u) \\ &\quad + 2d^3 P(u)(\partial_i u, \partial^i u, \partial_j u, \partial^j u), \end{aligned} \tag{17}$$

$$\begin{aligned} \partial_j(dP_u(\partial^j u, \Delta u)) &= d^2 P(u)(\Delta u, \partial_j u, \partial^j u) + dP(u)(\Delta u, \Delta u) \\ &\quad + dP(u)(\partial_j u, \partial^j \Delta u). \end{aligned} \tag{18}$$

Hence by (17) and (18), we write (5) into

$$\begin{aligned} \partial_t^2 u + \Delta^2 u = {} & dP_u(\partial_t u, \partial_t u) + dP_u(\Delta u, \Delta u) + 4dP_u(\nabla u, \nabla \Delta u) \\ & + 2dP_u(\nabla^2 u, \nabla^2 u) + \mathcal{N}(u, \nabla u, \nabla^2 u), \end{aligned} \tag{19}$$

where $\mathcal{N}(u)$ is defined by

$$\mathcal{N}(u) = 2d^2 P_u(\nabla u, \nabla u, \Delta u) + 4d^2 P_u(\nabla u, \nabla u, \nabla^2 u) + d^3 P_u(\nabla u, \nabla u, \nabla u, \nabla u).$$

Here, we sum the derivatives as explicitly given by (17) and (18). The evaluation of $dP_u,\ d^2 P_u,\ d^3 P_u$ is rather intuitive in the coordinates of u. However, we will state the formulation in standard coordinates of $\mathbb{R}^L$ in Chapter 1.

A major difference of (15) compared to the action (9) is that Φ *depends on the embedding* of $N \hookrightarrow \mathbb{R}^L$ such that (19) is not an *intrinsic* Hamiltonian flow. However, there is an intrinsic version of (15), namely

$$\begin{aligned} \tilde{\Phi}(u) = & \frac{1}{2} \int_{-T}^{T} \int_{\mathbb{R}^n} |\partial_t u|^2 - |(\Delta u)^T|^2 \, dx \, dt, \\ = & \Phi(u) + \frac{1}{2} \int_{-T}^{T} \int_{\mathbb{R}^n} |\mathcal{A}(u)(\nabla u, \nabla u)|^2 \, dx \, dt. \end{aligned} \tag{20}$$

where $(\Delta u)^T = D^j \partial_j u$ is the (spatial) tension field of the map $u(-T, T) \times \mathbb{R}^n \to N$. Clearly $\tilde{\Phi}$ does not require an embedding for (N, h). The draw back of considering the intrinsic action $\tilde{\Phi}$, is that an energy bound will only account for regularity of $D^j \partial_j u$ (which is a nonlinear expression) instead of the differential Du. Without further constraints, it is not clear how to make sense of the energy topology or use the conservation of energy properly. In the following, we restrict to the extrinsic version (19). The results of Chapter 1 apply to the Euler Lagrange equation of (20) with data in the extrinsically defined Sobolev spaces.

Using that $\partial_t u \in T_u N$, we obtain formal conservation of the elastic energy (7)

$$\frac{d}{dt} \mathcal{E}(u(t)) = \int_{\mathbb{R}^n} \langle P_u(\partial_t^2 u + \Delta^2 u), \partial_t u \rangle \, dx = 0,$$

by (5). Further, (19) admits *parabolic scaling* $u_\lambda(t,x) = u(t/\lambda^2, x/\lambda),\ \lambda > 0$. Hence

$$\begin{aligned} \mathcal{E}(u_\lambda(t)) &= \lambda^{n-4}\mathcal{E}(u(t/\lambda^2)), \\ \|u_\lambda(t)\|_{\dot H^s(\mathbb{R}^n)} &= \lambda^{\frac{n}{2}-s} \left\|u(t/\lambda^2)\right\|_{\dot H^s(\mathbb{R}^n)}. \end{aligned}$$

In particular, the scaling critical dimension is $n = 4$ and heuristically we expect the following

(i) Dimension $n \leq 3$: The biharmonic wave maps equation (19) has a global smooth solution starting from smooth (compactly supported) data.

(ii) The biharmonic wave maps equation (19) is locally wellposed in $H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ for $s > \frac{n}{2}$.

(iii) The biharmonic wave maps equation (19) is globally regular with small data in $\dot H^{\frac{n}{2}} \times \dot H^{\frac{n}{2}-2}$.

We give partial answers for this expectations, i.e. prove (slightly) weaker results such as the solution of the division problem in chapter 2.

However, we stress that (19) is *genuinely different from the wave maps equation* (8), resp (11). Especially, the formal decomposition

$$Lu = \partial_t^2 u + \Delta^2 u = (i\partial_t + \Delta)(-i\partial_t + \Delta)u,$$

shows that the place of the half-wave propagator $e^{\pm it\sqrt{-\Delta}}$ is taken by the Schrödinger groups $e^{\pm it\Delta}$ for L. As a consequence, solutions to (19) are not expected to propagate with finite speed and admit neither Lorentzian, nor a Galilean invariance, which imposes general restrictions in the analysis for L.

In [20] the authors prove the existence of a global weak solution of (19) into the round sphere $N = \mathbb{S}^L$ by a Ginzburg -Landau approximation, which allows passing to a solution in the limit by Noether's conservation law. In [19], the authors prove local wellposedness from energy methods, which is part of this thesis and presented in Chapter 1. At the end of Chapter 1, we present an argument for the global existence of such solutions in dimension $n = 1, 2$, which was published as a preprint in [44].

Further related models

For the past decades, many evolution equations with important applications to physics and geometry have been found to propagate by higher-order terms (compared to 2nd order differential operator).
We close this introduction by giving two examples that involve the operator $Lu = \partial_t^2 + \Delta^2 u$. The first is related to effective thin plate equations, such as Kirchhoff or Von Kármán equations, that usually take the form

$$\partial_t^2 u + \Delta(h(\Delta u)) - \gamma\Delta\partial_t^2 u = F(u, u_t, \nabla u, \nabla^2 u, \nabla^3 u).$$

Hence L is relevant when internal deformations ($h = id$) and rotational forces ($\gamma = 0$) are neglected. This has been considered explicitly in [14] on a domain with dissipative boundary conditions.

The second example is concerned with the following Klein-Gordon type equation

$$\partial_t^2 u + \Delta^2 u + mu + |u|^{p-1}u = 0. \tag{21}$$

For instance, if $m > 0$ and $1 + \frac{8}{n} < p < \frac{n+4}{n-4}$, global existence and scattering of solutions of (21) has been proved by Pausader in [43], as conjectured by Levandosky and Strauss in [36]

Main results and outline of the thesis

In the first Chapter 1, we prove local wellposedness of the biharmonic wave maps equation corresponding to the condition

$$\partial_t^2 u + \Delta^2 u \perp T_u N,$$

derived from compact euclidean submanifolds $N \subset \mathbb{R}^L$. Here we restrict to high regularity, i.e. we chose initial data in $H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$ for $k \in \mathbb{Z},\ k > \lfloor \frac{n}{2} \rfloor + 2$. The result relies on a compactness argument and energy estimates, thus we first prove wellposedness of a regularized (dissipative) equation

$$\partial_t^2 u^\varepsilon + \Delta^2 u^\varepsilon - \varepsilon \Delta \partial_t u^\varepsilon \perp T_{u^\varepsilon} N, \quad \varepsilon \in (0, 1]$$

from a standard energy argument in Section 1.3.1. We then observe an improved a apriori estimate (independent of the viscosity parmeter ε) in Section 1.3.2 by exploiting the geometric form of the nonlinearity in (19). Especially, this gives the possibilty to obtain a weak* limit on a uniform local existence interval for the solutions, which is proven in Section 1.3.2. Further in Section 1.3.3, we prove the uniqueness of the limit and recover a energy estimate for the difference of two solutions, which becomes useful for proving continuous dependence on the initial data. This is carried out in Section 1.3.4.

In the last Section 1.4 of Chapter 1, we prove global energy bounds in the energy subcritical dimensions $n = 1, 2$, which exclude finite time blow up of the local solutions from the previous sections.

In the second Chapter 2, we construct the analogue of Tataru's $F,\ \Box F$ solution of the division problem (from [54]) for wave maps adapted to a generalized Cauchy problem of type (6) into the sphere $N = \mathbb{S}^L$. In particular, we prove the existence of global solutions of a biharmonic wave equation with a non-generic nonlinearity and small initial data in the scaling critical space $\dot{B}^{2,1}_{\frac{n}{2}}(\mathbb{R}^n) \times \dot{B}^{2,1}_{\frac{n}{2}-2}(\mathbb{R}^n)$ for any dimension $n \geq 3$ in Section 2.4. We first prove corresponding Strichartz estimates in Section 2.3. This includes lateral estimates that recover a local smoothing effect known for the Schrödinger equation, which is outlined in the Appendix 2.A. A proof of dyadic bilinear estimates is stated in Section 2.4.2, which also applies to conclude that the solution persists initial regularity in the space $\dot{H}^s(\mathbb{R}^n) \times \dot{H}^{s-2}(\mathbb{R}^n)$. We finally give the proof of the main results in Section 2.4.4 and deduce a small data global regularity result for biharmonic wave maps in Section 2.5.

CHAPTER 1

Local and global results from energy estimates

The following Chapter (except for Section 1.4) is based on a local wellposedness result obtained in joint work with S. Herr, T. Lamm and R. Schnaubelt and has been published in [19]. The author of this thesis hereby ensures that he has contributed a significant part to this publication. Section 1.4 has been prepublished in [44].

We briefly outline the structure of this chapter. In Section 1.3.1, we use a vanishing viscosity approximation and solve the corresponding Cauchy problem for the damped problem

$$\partial_t^2 u + \Delta^2 u - \varepsilon \Delta \partial_t u \perp T_u N, \ \ \varepsilon \in (0, 1].$$

In order to obtain a limiting solution for (1.1.1) as $\varepsilon \searrow 0$, we prove a priori energy estimates which are uniform in ε in Section 1.3.2. As a by-product we obtain the blow-up criterion in Theorem 1.1.2. The existence part in Theorem 1.1.1 is also shown at the end of Section 1.3.2, and in Section 1.3.3 we prove that the solutions are unique. Finally we establish the continuity of the flow map in Section 1.3.4.

1.1 Introduction

As calculated in the introduction, smooth critical maps of

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^n} (|\partial_t u(t)|^2 - |\Delta u(t)|^2) d(x, t)$$

satisfy

$$\partial_t^2 u + \Delta^2 u \perp T_u N, \tag{1.1.1}$$

which by the smooth family of orthogonal projectior

$$P_p : \mathbb{R}^L \to T_p N, \quad p \in N,$$

onto the tangent space T_pN can thus be written as

$$\partial_t^2 u + \Delta^2 u = (I - P_u)(\partial_t^2 u + \Delta^2 u).$$

Exploiting that u takes values in N, we have

$$\begin{aligned}\partial_t^2 u + \Delta^2 u &= dP_u(u_t, u_t) + dP_u(\Delta u, \Delta u) + 4dP_u(\nabla u, \nabla \Delta u) \\ &\quad + 2dP_u(\nabla^2 u, \nabla^2 u) + 2d^2 P_u(\nabla u, \nabla u, \Delta u) + 4d^2 P_u(\nabla u, \nabla u, \nabla^2 u) \\ &\quad + d^3 P_u(\nabla u, \nabla u, \nabla u, \nabla u) \\ &=: \mathcal{N}(u),\end{aligned} \tag{1.1.2}$$

where the tensors $d^j P$ are explicitly described in the Appendix 1.A.4. The goal of this chapter is the proof of local wellposedness for the Cauchy problem corresponding to (1.1.1) in Sobolev spaces with high regularity (for an energy estimate) and a proof that local smooth solutions extend globally in time in dimension $n = 1, 2$. From now on, let N be a compact Riemannian manifold, isometrically embedded into $\mathbb{R}^L$.

A local well-posedness result as in Theorem 1.1.1 is standard for second-order wave equations with derivative nonlinearities such as wave maps. It can be found for example in the book of Shatah-Struwe [45] and the book of Sogge [48].

In contrast to this case, *our nonlinearity $\mathcal{N}(u)$ depends on $\nabla_x^3 u$, which cannot directly be controlled by the energy of* (1.1.2) that bounds $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ (on compact time intervals). Heuristically, $\nabla^3 u \in H^{-1}$ if u is in the energy space, so that it is unclear if the equation can be tested by velocity $\partial_t u$ for an a priori estimate.

In our proof we have to exploit the geometric form of $\mathcal{N}(u)$ repeatedly in order to recover an indirect energy argument (via compactness of approximate solutions). To be specific, the core property will be

$$\mathcal{N}(u) \perp T_u N, \quad Du = (\nabla u, \partial_t u) \in T_u N,$$

in case u is a solution of (1.1.2) maps to N.

Concerning the continuous dependence of the solution on the initial data, as the nonlinearity $\mathcal{N}(u)$ depends on third spatial derivatives, no Lipschitz estimate in the norm $H^k \times H^{k-2}$ is expected from the energy method (as we observe e.g. from the a priori estimates in Section 1.3.3).

Summing up, the energy method for (1.1.2) is more involved than for comparable geometric wave equations, due to the dependence $\mathcal{N}(u) = \mathcal{N}(u, \nabla^3 u)$ and we overcome the difficulties of the energy approach in the following sections.

We briefly note that our result applies to the intrinsic version of a biharmonic wave map defined by (20) and remark that, compared with the right hand side of (1.1.2), the Euler-Lagrange equation for the intrinsic biharmonic wave maps problem (20) differs by

$$P_u(dP_u(\nabla u, \nabla u) \cdot d^2 P_u(\nabla u, \nabla u, \cdot)) + P_u(\text{div}(dP_u(\nabla u, \nabla u) \cdot dP_u(\nabla u, \cdot))). \qquad (1.1.3)$$

Since hence the Euler-Lagrange equation differs only by lower order terms (see (1.1.3) in Section 1.2 below), we can prove the existence of local unique intrinsic biharmonic wave maps with initial data as in Theorem 1.1.1. However, we do not have a result for initial data with (only) *covariant* derivatives in L^2.

1.1.1 The main results

We prove the existence of a unique local solution in Sobolev spaces $H^k \times H^{k-2}$ with sufficiently high initial regularity $k > \lfloor \frac{n}{2} \rfloor + 2$ in order to employ energy estimates. First we have the following

Theorem 1.1.1 ([19], Local existence & uniqueness)**.** *Let* $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ *satisfy* $u_0(x) \in N$ *and* $u_1(x) \in T_{u_0(x)}N$ *for a.e.* $x \in \mathbb{R}^n$ *as well as*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ *with* $k > \lfloor \frac{n}{2} \rfloor + 2$*. Then there exists a maximal existence time*

$$T_m = T_m(u_0, u_1) > T = T(\|\nabla u_0\|_{H^{k-1}}, \|u_1\|_{H^{k-2}}) > 0$$

and a unique solution $u : \mathbb{R}^n \times [0, T_m) \to N$ *of* (1.1.1) *with* $u(0) = u_0$*,* $\partial_t u(0) = u_1$*, and*

$$u - u_0 \in C^0([0, T_m), H^k(\mathbb{R}^n)) \cap C^1([0, T_m), H^{k-2}(\mathbb{R}^n)).$$

The wellposedness then holds in the sense of the following Theorem

Theorem 1.1.2 ([19], Continuous flow map & Blow up criterion)**.**
For the solution $u : \mathbb{R}^n \times [0, T_m) \to N$ *in Theorem 1.1.1, there further holds the following.*

(a) For $T_0 \in (0, T_m)$ *there exists a (sufficiently small) radius* $R_0 > 0$ *such that for all initial data* (v_0, v_1) *as above that satisfy*

$$\|(u_0, u_1) - (v_0, v_1)\|_{H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)} \leq R_0,$$

the unique solution $v : \mathbb{R}^n \times [0, T_m(v_0, v_1)) \to N$ *exists on* $\mathbb{R}^n \times [0, T_0]$*. Further, for such initial data the map* $(v_0, v_1) \to (v(t), \partial_t v(t))$ *is continuous in* $H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$ *for* $t \in [0, T_0]$*.*

(b) If $T_m < \infty$, then

$$\int_0^{T_m} \|\nabla u(s)\|_{L^\infty}^{2k} + \|u_t(s)\|_{L^\infty}^{2k} \, ds = \infty. \tag{1.1.4}$$

In particular, for smooth initial data $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for $x \in \mathbb{R}^n$ having compact supp$(\nabla u_0) \subset \mathbb{R}^n$ and supp$(u_1) \subset \mathbb{R}^n$, there exist $T_m > 0$ and a smooth solution $u : \mathbb{R}^n \times [0, T_m) \to N$ of (1.1.1).

We remark that both u_0 and $u(t)$ do not necessarily belong to $L^2(\mathbb{R}^n)$ and it is only the difference of these two functions which is contained in this space. We further mention that the lower bound $k > \lfloor \frac{n}{2} \rfloor + 2$ ensures the existence of L^∞ bounds for $\partial_t u \in H^{k-2}(\mathbb{R}^n)$ from Sobolev's embedding. This is necessary in order to establish our energy estimates in the following sections.

In the last Section 1.4 of this chapter, we establish the following global result purely by the a priori control via (conserved) energy.

Theorem 1.1.3 ([44], Global solutions)**.** *Let $n = 1, 2$ and $k \geq n + 2$. Further let $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ satisfy $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for $x \in \mathbb{R}^n$ with*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n).$$

Then the unique local solutions $u : \mathbb{R}^n \times [0, T_m) \to N$ of Theorem 1.1.1 exist globally in time, i.e. $T_m = \infty$. Especially (1.1.2) *has global smooth solutions starting from smooth, compactly supported initial data in dimension $n = 1, 2$.*

1.2 Preliminaries and Notation

In this section and in the following we will write C for a generic constant only depending on N, n and k, and often also $\lesssim \ldots$ instead of $\leq C\,(\cdots)$.
The projectors P_p are derivatives of the metric distance (with respect to N) in $\mathbb{R}^L$, i.e.,

$$p = \Pi(p) + \frac{1}{2}\nabla_p(\text{dist}^2(p, N)), \quad P_p = d_p\Pi(p), \quad \text{dist}(p, N) < \delta_0. \tag{1.2.1}$$

Moreover, since Π maps to the nearest point on N there holds $\Pi^2 = \Pi$ and hence

$$d\Pi_{|p} = d\Pi(p) = d(\Pi^2(p)) = d\Pi_{|\Pi(p)} d\Pi_{|p},$$

by which the projector maps $P_p : \mathbb{R}^L \to T_{\Pi(p)}N$ are well-defined. Using cut-off functions we extend the identity (1.2.1), and thus also the equation $P_p = d_p\Pi(p)$, to all of $\mathbb{R}^L$. Moreover, all derivatives of P_p are bounded on $\mathbb{R}^n$. In this way one can investigate (1.1.2) without restricting the coefficients a priori. Further, for $l \in \mathbb{N}_0$ we denote by $d^l P_p$ the derivative of order l of the map P_p, which is a $(l+1)$-linear form on $\mathbb{R}^L$. For the coefficients in the standard coordinates in $\mathbb{R}^L$ we write

$$(d^j P_u)^k_{l_0,\ldots,l_j} = \frac{\partial}{\partial p_{l_1}} \cdots \frac{\partial}{\partial p_{l_j}} (P_p)^k_{l_0}(u).$$

We derive the exact coordinate expansion of (1.1.2) in the standard coordinates of $\mathbb{R}^L$ in the Appendix 1.A.4. How to sum the derivatives in (1.1.2) is explained in the introduction above.

In the following, we briefly recall well known results on, e.g. Sobolev embeddings and interpolation inequlities.

Lemma 1.2.1 (Gagliardo-Nirenberg-Sobolev, Morrey)**.** *(i) Let* $s \in \mathbb{R}, 1 \leq p \leq \infty,\ s - \frac{n}{p} < 0$*, then* $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),\ q \in [p, p*],\ p* = \frac{np}{n-sp}$*. In fact we have*

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \lesssim \||\nabla|^s u\|_{L^p(\mathbb{R}^n)} \tag{1.2.2}$$

(ii) Let $s \in \mathbb{R}, 1 \leq p \leq \infty,\ s - \frac{n}{p} > 0$*. Then*

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow C_0^{\lfloor s-\frac{n}{p}\rfloor, s-\frac{n}{p}-\lfloor s-\frac{n}{p}\rfloor}(\mathbb{R}^n).$$

(iii) Let $s, \tilde{s} \in \mathbb{R}$ *and* $1 \leq p \leq q \leq \infty,\ n \in \mathbb{N}$ *such that* $s - \frac{n}{p} \geq \tilde{s} - \frac{n}{q}$*. Then we have*

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{\tilde{s},q}(\mathbb{R}^n).$$

Remark 1.2.2. (i) Here in (i) we use $|\nabla|^s u = \mathcal{F}^{-1}(|\xi|^s \hat{u}(\xi))$ and (1.2.2) is due to Sobolev and Gagliardo-Nirenberg. The full embedding is then due to the interpolation, see also the inequality below .

(ii) If we replace $W^{s,p}$ in part $(i),\ (ii)$ by $W^{s,p}_{loc}$, then the embeddings

$$W^{s,p}_{loc}(\mathbb{R}^n) \hookrightarrow L^q_{loc}(\mathbb{R}^n), \quad W^{s,p}_{loc}(\mathbb{R}^n) \hookrightarrow C_{loc}^{\lfloor s-\frac{n}{p}\rfloor, s-\frac{n}{p}-\lfloor s-\frac{n}{p}\rfloor}(\mathbb{R}^n)$$

are compact for $q \in [p, p*)$.

The following is the well-known Gagliardo-Nirenberg interpolation, for which we refer to any classical book on Sobolev spaces and e.g. to [6] concerning optimality.

Lemma 1.2.3 (Gagliardo-Nirenberg)**.** *(i) Let* $j, m \in \mathbb{N}$ *with* $j \leq m,\ \alpha \in (0,1)$ *and* $1 \leq p, q, r \leq \infty$ *such that*

$$\frac{1}{r} = \frac{j}{n} + (\frac{1}{p} - \frac{m}{n})\alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leq \alpha.$$

Then

$$\left\|D^j u\right\|_{L^r} \lesssim \|D^m u\|^{\alpha}_{L^p} \|u\|^{1-\alpha}_{L^q}.$$

The next Lemma states the sharp improvement of the Sobolev inequality due to Brezis-Galluet $(n = 2,\ k = 1)$ in [5] and Brezis-Wainger in [7]. In our case, this estimate will be necessary in order to obtain global energy control for biharmonic wave maps in the energy subcritical dimension $n \in \{1, 2\}$ in Section 1.4.

Lemma 1.2.4 (Brezis-Gallouet, Brezis-Wainger)**.** *Let* $l \in \mathbb{N},\ 1 \leq q \leq \infty,\ u \in W^{l,q}(\mathbb{R}^n)$ *with* $u \neq 0$, $l - \frac{n}{q} > 0$. *Then there holds for* $k \in \mathbb{N}$ *with* $1 \leq k < \min\{l, n+1\}$

$$\|u\|_{L^\infty} \lesssim \|u\|_{W^{k,\frac{n}{k}}} \left(1 + \log^{\frac{(n-k)}{n}} \left(1 + \frac{\|u\|_{W^{l,q}}}{\|u\|_{W^{k,\frac{n}{k}}}}\right)\right). \tag{1.2.3}$$

Let $N \subset \mathbb{R}^L$ be a Riemmanian submanifold of euclidean space $(\mathbb{R}^L, \delta_{euc})$ with induced metric tensor. We say that N is of class C^k for $k \in \mathbb{N}$ if N is parametrized by an atlas of C^k chart maps. The following is a well known fact, see e.g. [47]

Lemma 1.2.5 (Nearest point map)**.** *Let* $N \subset \mathbb{R}^L$ *be a compact* C^k, $k \geq 2$ *(resp* C^ω*) submanifold of dimension* $d \leq L$. *Then there exists* $\delta = \delta(N) > 0$ *and a* C^{k-1} *(rep.* C^ω*) map*

$$\Pi : \mathcal{V}_\delta(N) := \{x \in \mathbb{R}^L \mid \operatorname{dist}(x, N) < \delta\} \to \mathbb{R}^L,$$

such that for all $x \in \mathcal{V}_\delta(N)$ *there holds*

$$\Pi(x) \in N,\ \ (I - \Pi)(x) \in T^\perp_{\Pi(x)}N,\ \ |x - \Pi(x)| = \operatorname{dist}(x, N).$$

Further we have $\Pi(x + z) = x$ *for* $x \in N$ *and* $z \in T_x^\perp N \cap \mathcal{V}_\delta N$ *and*

$$P_x := d\Pi_{|x} : \mathbb{R}^L \to \mathbb{R}^L, \quad x \in \mathcal{V}_\delta N$$

is a C^{k-2} *(rep.* C^ω*) map to* $T_{\Pi(x)}N$.

Remark 1.2.6. It is easy to verify, see [47], that

$$d^2\Pi_{|x} = dP_x \in T_x^*(N) \otimes T_x^*N \otimes \mathbb{R}^L$$

is the *second fundamental form* denoted by $\mathcal{A}(x)$ of the embedded manifold $N \subset \mathbb{R}^L$. In particular

$$dP_x(X, Y) \perp T_xN, \quad X, Y \in T_xN,\ x \in N.$$

Definition 1.2.7 (Star-Notation)**.** We use the shorthand $\nabla^{k_1}u \star \nabla^{k_2}u$ for (linear combinations of) products of partial derivatives of the components u^l of u for $l = 1, \dots, L$. Here the partial derivatives are of order $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$, respectively.

With this notation we can rewrite equation (1.1.2) as

$$\begin{aligned}\partial_t^2 u + \Delta^2 u = dP_u(u_t, u_t) + dP_u(\nabla^2 u \star \nabla^2 u) + dP_u(\nabla^3 u \star \nabla u)\\ + d^2P_u(\nabla u \star \nabla u \star \nabla^2 u) + d^3P_u(\nabla u \star \nabla u \star \nabla u \star \nabla u).\end{aligned}$$

This notation is also useful in light of the classical Leibniz formula, which implies the following identity.

Lemma 1.2.8. *For $m \in \mathbb{N}$ and $l \in \mathbb{N}_0$ we have*

$$\nabla^m(d^l P_u) = \sum_{j=1}^{m} \sum_{\sum_{k=1}^{j} m_k = m-j} d^{j+l} P_u(\nabla^{m_1+1} u \star \cdots \star \nabla^{m_j+1} u). \tag{1.2.4}$$

In order to include the case $m = 0$ in the Lemma, we will use $\sum_{j=\min\{1,m\}}^{m}$ for the sum in (1.2.4) or similar formulas.

The calculation of derivatives $\nabla^m(\mathcal{N}(u))$ and $\nabla^m(\mathcal{N}(u) - \mathcal{N}(v))$ for sufficiently regular $u, v : \mathbb{R}^n \times [0,T] \to \mathbb{R}^L$ and $m \in \mathbb{N}_0$ has been included in Appendix 1.A.1, employing the $\star$-convention. The results from Appendix 1.A.1 will be used frequently throughout the chapter. In the following sections, we also need a version of the following (Moser-type) estimate, see e.g. [58, Chapter 13].

Lemma 1.2.9. *Let $l, k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ satisfy $\sum_{i=1}^{l} |\alpha_i| = k$. There exists $C > 0$ such that for all $f_1, \ldots, f_l \in C_0(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$ we have*

$$\|D^{\alpha_1} f_1 \cdot \cdots \cdot D^{\alpha_l} f_l\|_{L^2} \leq C \prod_{i=1}^{l} \|f_i\|_{L^\infty}^{1-\frac{|\alpha_i|}{k}} \|f_i\|_{H^k}^{\frac{|\alpha_i|}{k}}. \tag{1.2.5}$$

In particular,

$$\|D^{\alpha_1} f_1 \cdot \cdots \cdot D^{\alpha_l} f_l\|_{L^2} \leq C \sum_{j=1}^{l} \prod_{i \neq j}^{l} \|f_i\|_{L^\infty} \left(\|f_1\|_{H^k} + \cdots + \|f_l\|_{H^k}\right). \tag{1.2.6}$$

Now we state the standard energy estimate for the Cauchy problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u = F \\ u(0) = u_0,\ u_t(0) = u_1 \end{cases} \tag{1.2.7}$$

Lemma 1.2.10. *Let $F \in L^1([0,T), H^{m-2})$, $\nabla u_0 \in H^{m-1}$, $u_1 \in H^{m-2}$, $m \in \mathbb{N}, m \geq 2$. Then the solution u of (1.2.7) satisfies*

$$\sup_{t \leq T}(\|\nabla u(t)\|_{H^{m-1}} + \|\partial_t u(t)\|_{H^{m-2}}) \lesssim (\sqrt{1+T})(\int_0^T \|F(s)\|_{H^{m-2}}\, ds + \|\nabla u_0\|_{H^{m-1}} + \|u_1\|_{H^{m-2}}). \tag{1.2.8}$$

Further for $0 \leq t_0 \leq t < T$ we have

$$\|\nabla^m u(t)\|_{L^2}^2 + \left\|\nabla^{m-2} u_t(t)\right\|_{L^2}^2 = 2\int_{t_0}^{t} \int_{\mathbb{R}^n} \nabla^{m-2}(F(s)) \cdot \nabla^{m-2} u_t(s)\, dx\, ds + \|\nabla^m u(t_0)\|_{L^2}^2 + \left\|\nabla^{m-2} u_t(t_0)\right\|_{L^2}^2. \tag{1.2.9}$$

Remark 1.2.11. Especially, if

$$s \mapsto \int_{\mathbb{R}^n} \nabla^{m-2}(F(s)) \cdot \nabla^{m-2} u_t(s)\, dx$$

is continuous on $[0,T)$, then the map

$$t \mapsto \|\nabla^m u(t)\|_{L^2}^2 + \left\|\nabla^{m-2} u_t(t)\right\|_{L^2}^2, \quad t \in (0,T)$$

is differentiable. A similar identity as (1.2.9) holds for the Cauchy problem of the approximation

$$\partial_t^2 u + \Delta^2 u - \varepsilon \Delta \partial_t u = F, \;\; \varepsilon \in (0,1),$$

which will have relevance below.

Proof. The identity (1.2.9) and hence the estimate (1.2.8) follow in the case of smooth data by differentiating (1.2.7) of order ∇^{m-2} and testing the resulting equation by $\nabla^{m-2} u_t$. For general data as required in the Lemma we regularize

$$F^\delta \to F \;\text{ in }\; L^1([0,T), H^{m-2}),\; \nabla u_0^\delta \to \nabla u_0 \;\text{ in }\; H^{m-1},\; u_1^\delta \to u_1 \;\text{ in }\; H^{m-2}.$$

Then we apply as usual (1.2.8) to $F^\delta - F^{\delta'}$, $\nabla u_0^\delta - \nabla u_0^{\delta'}$, $u_1^\delta - u_1^{\delta'}$ and obtain the convergence of the solution u^δ. Hence (1.2.8) and (1.2.9) hold for F, u_0, u_1. □

1.3 Local wellposedness in high regularity

In this section, we present a proof of Theorem 1.1.1 and Theorem 1.1.2 by a vanishing viscosity approach in a structurally damped approximation. However, the result hinges in a crucial way on a the exploitation of the geometric nonlinearity in order to obtain a uniform existence time in the viscosity parameter and uniqueness of the weak limit. Further, energy estimates that make use of the geometric structre lead to a proof of continuous dependence on the inital map with the Bona-Smith argument. The following four sections are the content of [19] and have largely been taken from this publication.

1.3.1 The parabolic approximation

Since $\mathcal{N}(u) = \mathcal{N}(u, u_t, \nabla u, \nabla^2 u, \nabla^3 u)$, energy estimates for the operator $\partial_t^2 + \Delta^2$ are not sufficient to show the existence of a solution of (1.1.2). Instead, we use the damped plate operator

$$\partial_t^2 + \Delta^2 - \varepsilon \Delta \partial_t,$$

with $\varepsilon \in (0,1]$ fixed, as a regularization. More precisely, we prove the existence of a solution $u^\varepsilon : \mathbb{R}^n \times [0,T_\varepsilon] \to N$ of the Cauchy problem

$$\begin{cases} \partial_t^2 u^\varepsilon(x,t) + \Delta^2 u^\varepsilon(x,t) - \varepsilon\Delta\partial_t u^\varepsilon(x,t) \perp T_{u^\varepsilon(x,t)}N, & (x,t) \in \mathbb{R}^n \times [0,T_\varepsilon], \\ u^\varepsilon(x,0) = u_0(x),\ u_t^\varepsilon(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3.1}$$

where $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ satisfy $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for a.e. $x \in \mathbb{R}^n$ as well as

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. In the following we mostly drop the super-/subscript ε and write (u,T) instead of $(u^\varepsilon, T_\varepsilon)$. We note that the condition in (1.3.1) reads as

$$\partial_t^2 u + \Delta^2 u - \varepsilon\Delta\partial_t u = \mathcal{N}(u) - \varepsilon(I - P_u)(\Delta\partial_t u). \tag{1.3.2}$$

Using $u(t,x) \in N$, we can expand

$$\varepsilon(I - P_u)(\Delta\partial_t u) = \varepsilon d^2P_u(u_t, \nabla u, \nabla u) + \varepsilon 2dP_u(\nabla u_t, \nabla u) + \varepsilon dP_u(u_t, \Delta u). \tag{1.3.3}$$

We thus study the regularized problem

$$\begin{aligned} \partial_t^2 u + \Delta^2 u - \varepsilon\Delta\partial_t u = \mathcal{N}(u) - \varepsilon d^2P_u(u_t, \nabla u, \nabla u) - \varepsilon 2dP_u(\nabla u_t, \nabla u) \\ - \varepsilon dP_u(u_t, \Delta u) =: \mathcal{N}_\varepsilon(u). \end{aligned} \tag{1.3.4}$$

We next solve (1.3.4) without the geometric constraint, recalling that only $u(t) - u_0 \in L^2(\mathbb{R}^n)$.

Lemma 1.3.1. *Let $\varepsilon \in (0,1)$ and take $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for a.e. $x \in \mathbb{R}^n$ such that*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. Then (1.3.4) has a unique local solution $u : \mathbb{R}^n \times [0,T_\varepsilon] \to \mathbb{R}^L$ satisfying $u(0) = u_0$, $u_t(0) = u_1$, and

$$u - u_0 \in C^0([0,T_\varepsilon], H^k(\mathbb{R}^n)) \cap C^1([0,T_\varepsilon], H^{k-2}(\mathbb{R}^n)) \cap H^1(0,T_\varepsilon; H^{k-1}(\mathbb{R}^n)). \tag{1.3.5}$$

In addition,

$$\nabla u \in L^2(0,T_\varepsilon; H^k(\mathbb{R}^n)) \tag{1.3.6}$$

and there exists a constant $C < \infty$ such that for $0 \le t \le T_\varepsilon$

$$\begin{aligned} &\left\|\nabla^{k-2} u_t(t)\right\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \varepsilon \int_0^t \left\|\nabla^{k-1} u_t(s)\right\|_{L^2}^2 ds + \varepsilon \int_0^t \left\|\nabla^{k+1} u(s)\right\|_{L^2}^2 ds \\ &\le C\Big(\int_0^t \int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}_\varepsilon(u)) \cdot \nabla^{k-2} u_t \, dx\, ds + \|\nabla u_0\|_{H^{k-1}}^2 + \|u_1\|_{H^{k-2}}^2 \Big). \end{aligned} \tag{1.3.7}$$

Before we prove Lemma 1.3.1, we reduce the problem to functions in L^2 by setting $v(x,t) = u(x,t) - u_0(x)$. We thus rewrite (1.3.4) as

$$\partial_t U + \mathcal{A}_k U = \begin{pmatrix} 0 \\ f_\varepsilon(U) \end{pmatrix}, \qquad U(0) = \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \tag{1.3.8}$$

where $U = \begin{pmatrix} v \\ v_t \end{pmatrix}$ and $f_\varepsilon(U)$ is defined through

$$\begin{aligned} f_\varepsilon(U) :&= \mathcal{N}(v+u_0) - \varepsilon d^2 P_{v+u_0}(v_t, \nabla(v+u_0), \nabla(v+u_0)) \\ &\quad - \varepsilon 2 d P_{v+u_0}(\nabla v_t, \nabla(v+u_0)) - \varepsilon d P_{v+u_0}(v_t, \Delta(v+u_0)) - \Delta^2 u_0. \end{aligned} \tag{1.3.9}$$

Further the operator $\mathcal{A}_k : H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n) \supseteq \mathcal{D}(\mathcal{A}) \to H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$ is given by

$$\mathcal{A}_k = \begin{pmatrix} 0 & -I \\ \Delta^2 & -\varepsilon\Delta \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = H^{k+2}(\mathbb{R}^n) \times H^k(\mathbb{R}^n). \tag{1.3.10}$$

Since the operators $\mathcal{A}_k$ extend each other we drop the subscript k. It is well known that $-\mathcal{A}$ generates an analytic C^0-semigroup $\{S_\varepsilon(t)\}_{t\geq 0}$, see e.g. [10, Prop. 2.3] for the case $k = 2$. Using also standard parabolic theory, see e.g. [35, Prop. 0.1] and [40, Prop. 1.13], we obtain a first linear existence result with some extra regularity.

Lemma 1.3.2. *Let $r \in \mathbb{N}_0$, $u_1 \in H^{r+1}(\mathbb{R}^n)$, and $g \in C^0([0,T], H^r(\mathbb{R}^n))$. Then there exists a unique solution U of the linear equation*

$$\partial_t U + \mathcal{A} U = \begin{pmatrix} 0 \\ g \end{pmatrix}, \qquad U(0) = \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \tag{1.3.11}$$

satisfying

$$U \in L^2(0,T; H^{r+4} \times H^{r+2}(\mathbb{R}^n)) \cap C^0(0,T; H^{r+3} \times H^{r+1}(\mathbb{R}^n)) \cap H^1(0,T; H^{r+2} \times H^r(\mathbb{R}^n)).$$

We remark that the solution of (1.3.11) is given by

$$U(t) = S_\varepsilon(t) \begin{pmatrix} 0 \\ u_1 \end{pmatrix} + \int_0^t S_\varepsilon(t-s) \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds. \tag{1.3.12}$$

We quantify the above result by the following higher-order energy estimates.

Lemma 1.3.3. *Let $r \in \mathbb{N}_0$, $g \in C^0([0,T], H^r(\mathbb{R}^n))$, $u_1 \in H^{r+1}(\mathbb{R}^n)$, and $u_0 : \mathbb{R}^n \to \mathbb{R}^L$ with $\nabla u_0 \in H^{r+3}(\mathbb{R}^n)$. Then v from Lemma 1.3.2 satisfies*

$$\begin{aligned} &\|v_t(t)\|_{H^{r+1}}^2 + \|v(t)\|_{H^{r+3}}^2 + \varepsilon \int_0^T \|\nabla v_t(s)\|_{H^{r+1}}^2 \, ds + \varepsilon \int_0^T \|\nabla(v+u_0)(s)\|_{H^{r+3}}^2 \, ds \\ &\leq C(1+T) \left(\frac{1}{\varepsilon} \int_0^T \left\|g(s) + \Delta^2 u_0\right\|_{H^r}^2 \, ds + \|u_1\|_{H^{r+1}}^2 + \|\nabla u_0\|_{H^{r+2}}^2 \right) \end{aligned} \tag{1.3.13}$$

for $0 \leq t \leq T$*, and*

$$\left\|\nabla^{r+1}v_t(t)\right\|_{L^2}^2 + \left\|\nabla^{r+3}v(t)\right\|_{L^2}^2 + \varepsilon\int_0^T \left\|\nabla^{r+2}v_t(s)\right\|_{L^2}^2\, ds \tag{1.3.14}$$
$$\leq C\Big(-\int_0^t \int_{\mathbb{R}^n} \nabla^r \left(g(s) + \Delta^2 u_0\right)\cdot \nabla^r \Delta v_t \, dx\, ds + \|u_1\|_{H^{r+1}}^2 + \|\nabla u_0\|_{H^{r+2}}^2 \Big).$$

Proof. Writing $U = (v, v_t)$ in Lemma 1.3.2, the function $u = v + u_0$ fulfills

$$\partial_t^2 u + \Delta^2 u - \varepsilon\Delta\partial_t u = g + \Delta^2 u_0 \tag{1.3.15}$$

in $L^2(0,T;H^r(\mathbb{R}^n))$. We first differentiate (1.3.15) of order ∇^l with $l \in \{0,\dots,r\}$. Testing with $-\nabla^l\Delta u_t \in L^2_{t,x}$ and integrating by parts in x, we derive

$$\frac{d}{dt}\left\|\nabla^{l+1}u_t(t)\right\|_{L^2}^2 + \frac{d}{dt}\left\|\nabla^{l+3}u(t)\right\|_{L^2}^2 + \varepsilon\left\|\nabla^{l+2}u_t(t)\right\|_{L^2}^2 \tag{1.3.16}$$
$$\leq \frac{C}{\varepsilon}\left\|\nabla^l(g + \Delta^2 u_0)\right\|_{L^2}^2 + \frac{\varepsilon}{2}\left\|\nabla^{l+2}u_t(t)\right\|_{L^2}^2,$$

which makes sense for a.e. t. (Here and below we use the duality (H^1, H^{-1}) in intermediate steps.) We then absorb the last term by the left-hand side and integrate the inequality in t.
To control the second summand with ε in (1.3.13), we test the differentiated version of (1.3.15) by $\varepsilon\nabla^l\Delta^2 u$. Here we proceed similarly as before, where we integrate the term

$$\varepsilon\int_0^T\int_{\mathbb{R}^n}\nabla^l\partial_t^2 u\cdot\nabla^l\Delta^2 u\, dx\, ds$$

by parts in t and x before aborbing it.
It remains to estimate the L^2-norm of $v_t(t)$ and the H^2-norm of $v(t)$. These inequalities follow by testing the equation with u_t and using the fact that

$$\|u - u_0\|_{L_t^\infty L^2} \leq T\,\|u_t\|_{L_t^\infty L^2}\,. \qquad \square$$

Before we give a prove of Lemma 1.3.1, we state the following estimates for the nonlinearity.

Lemma 1.3.4. *Let* $u, v : [0,T]\times\mathbb{R}^n \to \mathbb{R}^l,\ k > \lfloor\frac{n}{2}\rfloor + 2$ *be such that*

$$(\nabla u, \partial_t u),\ (\nabla v, \partial_t v) \in L^\infty([0,T], H^{k-1}\times H^{k-2}),$$

and also $u - u_0, v - u_0 \in L^\infty([0,T], L^2_x)$ *for some* $u_0 \in \mathbb{R}^n \to \mathbb{R}^L$*. Then for* $0 \leq s \leq t < T$

$$\|\nabla^{k-3}(\mathcal{N}(u(t)) - \mathcal{N}(v(s)))\|_{L^2} \tag{1.3.17}$$
$$\lesssim (1 + \|\nabla u\|_{L_t^\infty H^{k-1}}^k + \|u_t\|_{L_t^\infty H^{k-2}}^k + \|\nabla v\|_{L_t^\infty H^{k-1}}^k + \|v_t\|_{L_t^\infty H^{k-2}}^k)$$
$$\cdot (\|u(t) - v(s)\|_{H^k} + \|u_t(t) - v_t(s)\|_{H^{k-2}}),$$

$$\|\nabla^{k-3}(\mathcal{N}(u(t)))\|_{L^2} \lesssim (1 + \|\nabla u\|_{L_t^\infty H^{k-1}}^k + \|u_t\|_{L_t^\infty H^{k-2}}^k) \tag{1.3.18}$$
$$\cdot (\|\nabla u(t)\|_{H^{k-1}} + \|u_t(t)\|_{H^{k-2}}).$$

Here we note $u(t) - v(s) = u(t) - u_0 + u_0 - v(s) \in L^2_x$. Further, we note that the estimates above hold for $\mathcal{N}_\varepsilon$, where the constants then depend on $\varepsilon \in (0,1)$. By interpolation with a similar (simpler) estimate for $\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^2},\ \|\mathcal{N}(u)\|_{L^2}$, we obtain the full H^{k-3} norm on the LHS.

Proof. The prove follows by means of the calculations in the Appendix, to be more precise Lemma 1.A.1 and Corollary 1.A.4 combined with a careful application of the Moser estimate in Lemma 1.2.9. We give the relevant details below in Section 1.3.2 in the proof of the a priori estimate and in Section 1.3.3 for the energy bound, respectively the uniqueness. The arguments used there are the very same and in fact require more thought for the a priori estimate in Section 1.3.2. □

Proof of Lemma 1.3.1. We aim at constructing a solution $U \in C^0([0,T], H^k \times H^{k-2})$, but due to $\Delta^2 u_0 \in H^{k-4}$ we have $f_\varepsilon(U) \in C^0([0,T], H^{k-4})$, which is insufficient for an application of Lemmas 1.3.2 and 1.3.3 in a fixed point argument for v.

▷ Step 1: We thus approximate u_0 by $u_0^\delta \in C^\infty(\mathbb{R}^n, \mathbb{R}^L)$ for $\delta > 0$ such that $\operatorname{supp}(\nabla u_0^\delta) \subset \mathbb{R}^n$ is compact with

$$u_0^\delta \to u_0 \text{ a.e. } \text{ and } \nabla u_0^\delta \to \nabla u_0 \text{ in } H^{k-1}(\mathbb{R}^n) \text{ as } \delta \to 0^+. \tag{1.3.19}$$

Defining $f_{\varepsilon,\delta}$ as above with u_0^δ instead of u_0, we obtain $f_{\varepsilon,\delta}(U) \in C^0([0,T], H^{k-3}(\mathbb{R}^n))$. For the data (u_0^δ, u_1) we now prove the existence of a fixed point for the operator $v \mapsto \mathcal{S}(v)$ defined through

$$\begin{pmatrix} \mathcal{S}(v) \\ \partial_t \mathcal{S}(v) \end{pmatrix} = S_\varepsilon(t) \begin{pmatrix} 0 \\ u_1 \end{pmatrix} + \int_0^t S_\varepsilon(t-s) \begin{pmatrix} 0 \\ f_{\varepsilon,\delta}(v) \end{pmatrix} ds, \tag{1.3.20}$$

which acts on the space

$$\begin{aligned} \mathcal{B}_R(T) := \big\{ v \in C^0([0,T], H^k) \cap C^1([0,T], H^{k-2}) \mid v(0) = 0,\ v_t(0) = u_1, \\ \|v\|_{\mathcal{B}} := \|v_t\|_{L^\infty H^{k-2}} + \|v\|_{L^\infty L^2} + \big\|\nabla(v + u_0^\delta)\big\|_{L^\infty H^{k-1}} \le R \big\}, \end{aligned}$$

for parameters $R > 0$ and $T \in (0,1)$ fixed below and the metric given by

$$\|v_1 - v_2\|_{\mathcal{B}(T)} = \|v_1 - v_2\|_{L^\infty H^k} + \|\partial_t v_1 - \partial_t v_2\|_{L^\infty H^{k-2}}, \quad v_1, v_2 \in \mathcal{B}_R(T).$$

Let $\varepsilon \in (0,1)$ be fixed. We will show that the map

$$\mathcal{S} : \mathcal{B}_R(T) \to \mathcal{B}_R(T)$$

is strictly contractive with respect to $\|\cdot\|_{\mathcal{B}(T)}$ if we choose $R = R_\delta$ and $T = T_\delta$ with

$$\begin{aligned} R_\delta^k &= 3(\big\|\nabla u_0^\delta\big\|_{H^{k-1}} + \|u_1\|_{H^{k-2}})^k =: 3R_{0,\delta}^k, \\ T_\delta &= \frac{1}{2}\min\left\{ \left(\frac{\sqrt[k]{3} - 1}{\sqrt[k]{3}}\right)^2 \frac{\varepsilon}{\hat{C}^2(1 + 3R_{0,\delta}^k)^2}, \frac{\varepsilon}{\hat{C}^2(1 + 6R_{0,\delta}^k)^2} \right\} \end{aligned} \tag{1.3.21}$$

for a constant $\hat{C}$ depending only on N, n, and k. To show this statement, we have to prove the estimates

$$\|\mathcal{S}(v)\|_{\mathcal{B}} \le \frac{\hat{C}}{\varepsilon^{\frac{1}{2}}} T^{\frac{1}{2}}(1 + \|v\|_{\mathcal{B}}^k) \|v\|_{\mathcal{B}} + \left\|\nabla u_0^\delta\right\|_{H^{k-1}} + \|u_1\|_{H^{k-2}}, \tag{1.3.22}$$

$$\|\mathcal{S}(v) - \mathcal{S}(\tilde{v})\|_{\mathcal{B}(T)} \le \frac{\hat{C}}{\varepsilon^{\frac{1}{2}}} T^{\frac{1}{2}}(1 + \|v\|_{\mathcal{B}}^k + \|\tilde{v}\|_{\mathcal{B}}^k) \|v - \tilde{v}\|_{\mathcal{B}(T)} \tag{1.3.23}$$

for $v, \tilde{v} \in \mathcal{B}_R(T)$. To employ the inequality (1.3.13) for $r = k-3$, we need to bound the norms

$$\left\|\mathcal{N}_\varepsilon(v(t) + u_0^\delta)\right\|_{H^{k-3}}^2 \quad \text{and} \quad \left\|\mathcal{N}_\varepsilon(v(t) + u_0^\delta) - \mathcal{N}_\varepsilon(\tilde{v}(t) + u_0^\delta)\right\|_{H^{k-3}}^2$$

by $C(1 + \|v\|_{\mathcal{B}}^{2k}) \|v\|_{\mathcal{B}}^2$ and $C(1 + \|v\|_{\mathcal{B}}^{2k} + \|\tilde{v}\|_{\mathcal{B}}^{2k}) \|v - \tilde{v}\|_{\mathcal{B}(T)}^2$, respectively. This is provided by Lemma 1.3.4 and this way we obtain in the fixed point $v^\delta = \mathcal{S}(v^\delta)$ satisyfying

$$\left\|v_t^\delta\right\|_{L^\infty H^{k-2}}^2 + \left\|v^\delta\right\|_{L^\infty H^k}^2 + \varepsilon \int_0^{T_\delta} \left\|v_t^\delta(s)\right\|_{H^{k-1}}^2 ds + \varepsilon \int_0^{T_\delta} \left\|\nabla(v^\delta + u_0^\delta)\right\|_{H^k}^2 ds \lesssim R_\delta^2. \tag{1.3.24}$$

In particular, $v^\delta \in L^2(0, T_\delta; H^{k+1}) \cap H^1(0, T_\delta; H^{k-1})$.

▷ *Step 2*: We next define R_0, R and $\tilde{T} > 0$ in the same way as $R_{0,\delta}$, R_δ and T_δ using u_0 instead of u_0^δ and the R_0 instead of R_0^δ. Thus,

$$R_{0,\delta} \to R_0, \quad R_\delta \to R, \quad T_\delta \to \tilde{T} \quad \text{as } \delta \to 0^+.$$

For sufficiently small $\delta > 0$ we have $T_\delta > \frac{1}{2}\tilde{T} =: T$ and $|R_{0,\delta} - R_0| \le R_0$. Hence $v^\delta : \mathbb{R}^n \times [0, T] \to \mathbb{R}^L$ is well defined and $\left\|v^\delta\right\|_{\mathcal{B}(T)} \le CR$ for a constant $C > 0$. Observe that for sufficiently small $\delta, \delta' > 0$, the differences $v^\delta - v^{\delta'}$+ and $\partial_t v^\delta - \partial_t v^{\delta'}$ solve (1.3.11) with the nonlinearity

$$\mathcal{N}_\varepsilon(v^\delta + u_0^\delta) - \mathcal{N}_\varepsilon(v^{\delta'} + u_0^{\delta'}) + \Delta^2(u_0^\delta - u_0^{\delta'}) \in C^0([0, T], H^{k-3}).$$

Similar to the proof of the Lipschitz estimate (1.3.23), Lemma 1.3.3 then yields the bound

$$\begin{aligned} \left\|v^\delta - v^{\delta'}\right\|_{\mathcal{B}(T)}^2 &+ \varepsilon \int_0^{T_\delta} \left\|v_t^\delta(s) - v_t^{\delta'}(s)\right\|_{H^{k-1}}^2 ds + \varepsilon \int_0^{T_\delta} \left\|\nabla(v^\delta - v^{\delta'}) + \nabla(u_0^\delta - u_0^{\delta'})\right\|_{H^k}^2 ds \\ &\le C\frac{T}{\varepsilon}(1 + R^{2k}) \left\|v^\delta - v^{\delta'}\right\|_{\mathcal{B}(T)}^2 + \tilde{C}_{\varepsilon,R} \left\|\nabla u_0^\delta - \nabla u_0^{\delta'}\right\|_{H^{k-1}}^2. \end{aligned}$$

Hence, if $T = T(\varepsilon)$ is sufficiently small, as $\delta \to 0$ the functions v^δ tend to a function

$$v \in C^0([0, T], H^k) \cap C^1([0, T], H^{k-2}) \cap H^1(0, T; H^{k-1})$$

with $\nabla(v+u_0) \in L^2(0,T;H^k)$, where the limits exist in these spaces. In particular, (v,v_t) is a solution of (1.3.8) and $u = v+u_0$ solves (1.3.4). Moreover, by (1.3.13) the function $u^\delta = v^\delta + u_0^\delta$ satisfies inequality (1.3.7), and therefore this estimate also holds for u since $u_t^\delta \to u_t$ strongly in $C^0([0,T],H^{k-2})$ and $\mathcal{N}_\varepsilon(u^\delta) \to \mathcal{N}_\varepsilon(u)$ strongly in $L^2(0,T;H^{k-2})$ because of Corollary 1.A.4 and Lemma 1.2.9.

▷ *Step 3*: For the uniqueness of v, we note that, for a second solution $\tilde{v}$, the functions $w = v - \tilde{v}$ and $w_t = v_t - \tilde{v}_t$ solve (1.3.11) with the nonlinearity $\mathcal{N}_\varepsilon(v+u_0) - \mathcal{N}_\varepsilon(\tilde{v}+u_0) \in C^0([0,T],H^{k-3})$. Lemma 1.3.3 then yields the estimate

$$\|v-\tilde{v}\|^2_{\mathcal{B}(T)} \leq C\frac{T}{\varepsilon}(1+R^{2k})\,\|v-\tilde{v}\|^2_{\mathcal{B}(T)}\,. \tag{1.3.25}$$

(Note that u_0 from the Lemma is different, namely $u_0 = 0$.) Hence, if T is sufficiently small, we obtain $v = \tilde{v}$ and thus $u = v + u_0$ is unique. □

Next, we show that $u(t) \in N$ for all $t \in (0,T)$ if $u_0 \in N$ and $u_1 \in T_{u_0}N$.

Proposition 1.3.5. *Let $\varepsilon \in (0,1)$ and take $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for a.e. $x \in \mathbb{R}^n$ satisfying*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. Then there exists a maximal existence time $T_{\varepsilon,m} \in (0,\infty]$ and a unique solution $u \in \mathbb{R}^n \times [0,T_{\varepsilon,m}) \to N$ of (1.3.1) *with $u(0) = u_0$, $\partial_t u(0) = u_1$,*

$$u - u_0 \in C^0([0,T_{\varepsilon,m}),H^k) \cap C^1([0,T_{\varepsilon,m}),H^{k-2}) \cap H^1_{loc}([0,T_{\varepsilon,m}),H^{k-1}(\mathbb{R}^n))$$

and $\nabla u \in L^2_{loc}([0,T_{\varepsilon,m}),H^k(\mathbb{R}^n))$ which satisfies (1.3.7) *for $t \in [0,T_{\varepsilon,m})$.*

Proof. Fix $\varepsilon \in (0,1)$. Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^L$ be the solution of (1.3.1) constructed in Lemma 1.3.1. We first show that $u(x,t) \in N$ for $x \in \mathbb{R}^n$ and $t > 0$ small enough. Since

$$C^0([0,T],H^k) \hookrightarrow C^0(\mathbb{R}^n \times [0,T])$$

and $u_0 \in N$ a.e. on $\mathbb{R}^n$, there exists a time $\tilde{T} \in (0,T]$ such that for $t \in [0,\tilde{T}]$ the distance

$$\|\mathrm{dist}(u(t),N)\|_{L^\infty} \leq \sup_{x\in\mathbb{R}^n} |u(x,t) - u_0(x)| \lesssim \|u(t) - u_0\|_{H^k}$$

is so small that $\bar{u} = \Pi(u)$ is well-defined. We then let $w = \bar{u} - u$ and we note that $w(0) = \partial_t w(0) = 0$. Calculating

$$\begin{aligned}
\partial_t^2 \bar{u} &= d\Pi_u \partial_t^2 u + d^2\Pi_u(u_t,u_t), \\
\Delta \bar{u}_t &= d\Pi_u \Delta u_t + d^2\Pi_u(\Delta u, u_t) + 2d^2\Pi_u(\nabla u_t, \nabla u) + d^3\Pi_u(\nabla u, \nabla u, u_t), \\
\Delta^2 \bar{u} &= d\Pi_u \Delta^2 u + d^2\Pi_u(\Delta u, \Delta u) + 4d^2\Pi_u(\nabla u, \nabla \Delta u) + 2d^2\Pi_u(\nabla^2 u, \nabla^2 u) \\
&\quad + 2d^3\Pi_u(\nabla u, \nabla u, \Delta u) + 4d^3\Pi_u(\nabla u, \nabla u, \nabla^2 u) \\
&\quad + d^4\Pi_u(\nabla u, \nabla u, \nabla u, \nabla u),
\end{aligned}$$

we conclude that

$$
\begin{aligned}
(\partial_t^2 + \Delta^2 - \varepsilon \Delta \partial_t) w &= d\Pi_u \Big((\partial_t^2 + \Delta^2 - \varepsilon \Delta \partial_t) u \Big) + \mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(u) \\
&= d\Pi_u (\mathcal{N}_\varepsilon(u)) \in T_{\bar{u}} N.
\end{aligned}
$$

Next, we note that since $\bar{u} = \Pi(u) \in N$, we have $\mathcal{N}_\varepsilon(\bar{u}) \perp T_{\bar{u}} N$ and from $Im(d\Pi_u) \subset T_{\bar{u}} N$, it follows $d\Pi_u(\mathcal{N}_\varepsilon(\bar{u})) = 0$. Hence

$$
(\partial_t^2 + \Delta^2 - \varepsilon \Delta \partial_t) w = d\Pi_u (\mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(\bar{u})).
$$

Now there also holds

$$
\begin{aligned}
\|\nabla \bar{u}\|_{H^{k-1}} + \|\bar{u}_t\|_{H^{k-2}} &\lesssim (1 + \|\nabla u\|_{H^{k-1}}^{k-1} + \|u_t\|_{H^{k-2}}^{k-1}) \qquad (1.3.26) \\
&\qquad \cdot (\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}}), \\
\|d\Pi_u(\mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(\bar{u}))\|_{H^{k-3}} &\lesssim (1 + \|\nabla u\|_{H^{k-1}}^{k-3}) \, \|\mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(\bar{u})\|_{H^{k-3}} . \qquad (1.3.27)
\end{aligned}
$$

We thus employ the energy bound (1.3.13) in Lemma 1.3.3 (with $u_0 = u_1 = 0$ in the Lemma) and it therefore suffices to estimate $\|\mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(\bar{u})\|_{H^{k-3}}$, which is provided by Lemma 1.3.4. Especially, we obtain

$$
\|w(t)\|_{H^k}^2 + \|w_t(t)\|_{H^{k-2}}^2 \leq C\tilde{T}(\|w(t)\|_{H^k}^2 + \|w_t(t)\|_{H^{k-2}}^2) \quad , t \in [0, \tilde{T}],
$$

where $C = C(N, k, \varepsilon, \|\nabla u\|_{L^\infty([0,\tilde{T}], H^{k-1})}, \|u_t\|_{L^\infty([0,\tilde{T}], H^{k-2})}) > 0$. In particular, if $\tilde{T} > 0$ is small enough, there holds $w = 0$, i.e. $u = \Pi(u) \in N$ on $[0, \tilde{T}]$.

In order to see that any $\tilde{T} \in [0, T)$ has this property, we apply a bootstrap argument. Let

$$
J := \{ t \in [0, T) \mid u_{|[0,t]} \text{ maps to } N \}.
$$

Then obviously $0 \in J$ by assumption and J is closed since u is continuous and N is compact. The fact that J is indeed open follows by the above argument starting at $t_0 \in J$, i.e. $u(t_0) \in N$, for which we replace $\tilde{T}$ as above by $\tilde{t} - t_0 > 0$ small enough. The maximal existence time $T_{\varepsilon,m} > 0$, is then defined through the fixed point argument in Lemma 1.3.1. □

1.3.2 The a priori estimate and taking a limit

In the previous section $\varepsilon \in (0, 1)$ was fixed. The constants in the upper bound in estimates such as (1.3.24), however, are of order $O(\varepsilon^{-1})$.

We now have to prove ε independent estimates, which leads to a lower bound of the (maximal) existence times $T_{\varepsilon,m}$ as $\varepsilon \searrow 0$ and the possibility to take a limit by compactness arguments. We then prove the existence of a solution stated in Theorem (1.1.1) and the blow up condition from Theorem 1.1.2. This section is taken from [19] with modifications at the end of the section.

The a priori estimate

We now prove an a priori estimate for the solution $u^\varepsilon : \mathbb{R}^n \times [0, T_{\varepsilon,m}) \to N$ of the equation

$$\partial_t^2 u^\varepsilon + \Delta^2 u^\varepsilon - \varepsilon \Delta \partial_t u^\varepsilon \perp T_{u^\varepsilon} N \quad \text{on } \mathbb{R}^n \times [0, T_{\varepsilon,m}) \tag{1.3.28}$$

given by Proposition 1.3.5 with $\varepsilon \in (0,1)$ and initial data $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ such that $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)} N$ for a.e. $x \in \mathbb{R}^n$ as well as

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. As before we write u instead of u^ε, and we fix a number $T < T_{\varepsilon,m}$. Moreover, (1.3.7) says that

$$\begin{aligned} &\left\|\nabla^{k-2} u_t(t)\right\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \varepsilon \int_0^t \left\|\nabla^{k-1} u_t(s)\right\|_{L^2}^2 ds \\ &\quad \lesssim \int_0^t \int_{\mathbb{R}^n} \nabla^{k-2} \left[\mathcal{N}(u) - \varepsilon (I - P_u)(\Delta u_t)\right] \cdot \nabla^{k-2} u_t \, dx \, ds + \left\|\nabla^{k-2} u_1\right\|_{L^2}^2 + \left\|\nabla^k u_0\right\|_{L^2}^2 \end{aligned} \tag{1.3.29}$$

for $t \in [0, T]$. We recall that the summand with ε on the right-hand side is well defined because of (1.3.3).

In the following, we often make use of the relations $\mathcal{N}(u) \perp T_u N$ and $u_t \in T_u N$ which hold since $u(x,t) \in N$ for a.e. $(x,t) \in \mathbb{R}^n \times [0,T]$. In particular, $\mathcal{N}(u) = (I - P_u)\mathcal{N}(u)$. Using this fact, we first write

$$\begin{aligned} \nabla^{k-2}(\mathcal{N}(u)) \nabla^{k-2} u_t &= \sum_{\substack{m_1+m_2=k-2 \\ m_1>0}} \nabla^{m_1}(I - P_u) \star \nabla^{m_2}(\mathcal{N}(u)) \nabla^{k-2} u_t \\ &\quad + \nabla^{k-2}(\mathcal{N}(u))(I - P_u) \nabla^{k-2} u_t \\ &= \sum_{\substack{m_1+m_2=k-2 \\ m_1>0}} \nabla^{m_1}(I - P_u) \star \nabla^{m_2}(\mathcal{N}(u)) \nabla^{k-2} u_t \\ &\quad - \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \nabla^{k-2}(\mathcal{N}(u)) \star \nabla^{l_1}[(I - P_u)] \nabla^{l_2} u_t \\ &=: I_1 + I_2, \end{aligned} \tag{1.3.30}$$

where the second equality follows from the Leibniz formula

$$0 = \nabla^{k-2}\left[(I - P_u) u_t\right] = \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \nabla^{l_1}[(I - P_u)] \star \nabla^{l_2} u_t + (I - P_u) \nabla^{k-2} u_t. \tag{1.3.31}$$

In (1.3.29) we thus split

$$\begin{aligned}\int_{\mathbb{R}^n} &\nabla^{k-2}(\mathcal{N}(u) - \varepsilon(I-P_u)(\Delta u_t)) \cdot \nabla^{k-2} u_t \, dx \\ &= \int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u)) \cdot \nabla^{k-2} u_t \, dx - \varepsilon \int_{\mathbb{R}^n} \nabla^{k-2}((I-P_u)(\Delta u_t)) \cdot \nabla^{k-2} u_t \, dx \\ &= \int_{\mathbb{R}^n} I_1 \, dx + \int_{\mathbb{R}^n} I_2 \, dx - \varepsilon \int_{\mathbb{R}^n} \nabla^{k-2}((I-P_u)(\Delta u_t)) \cdot \nabla^{k-2} u_t \, dx, \qquad (1.3.32)\end{aligned}$$

We start by estimating

$$\int_{\mathbb{R}^n} I_1 \, dx \leq \sum_{\substack{m_1+m_2=k-2 \\ m_1>0}} \| \nabla^{m_1}(I-P_u) \star \nabla^{m_2}(\mathcal{N}(u)) \|_{L^2} \left\| \nabla^{k-2} u_t \right\|_{L^2} .$$

Lemma 1.2.8 yields the identity

$$\nabla^{m_1}(I-P_u) = -\sum_{j=1}^{m_1} \sum_{\sum_{i=1}^{j} \tilde{k}_i = m_1 - j} d^j P_u(\nabla^{\tilde{k}_1+1} u \star \cdots \star \nabla^{\tilde{k}_j+1} u), \qquad (1.3.33)$$

which implies the pointwise inequality

$$|\nabla^{m_1}(I-P_u)| \lesssim \sum_{j=1}^{m_1} \sum_{\sum_{i=1}^{j} \tilde{k}_i = m_1 - j} |\nabla^{\tilde{k}_1+1} u| \cdots |\nabla^{\tilde{k}_j+1} u|. \qquad (1.3.34)$$

On the other hand, Lemma 1.A.1 allows us to bound $|\nabla^{m_2}(\mathcal{N}(u))|$ pointwise (up to a constant) by terms of the form

$$|\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| \left[|\nabla^{k_1} u_t||\nabla^{k_2} u_t| + |\nabla^{k_1+2} u||\nabla^{k_2+2} u| + |\nabla^{k_1+3} u||\nabla^{k_2+1} u| \right], \quad (1.3.35)$$
$$|\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u \left[|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+2} u| \right], \qquad (1.3.36)$$
$$|\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| \left[|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+1} u||\nabla^{k_4+1} u| \right], \qquad (1.3.37)$$

where $i = 1, \ldots, m_2$ and $\tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + \cdots = m_2 - i$ are as in Lemma 1.A.1. Moreover, in the case $i = 0$ (where no derivatives fall on the coefficients) the terms are of the form

$$\begin{aligned}&|\nabla^{k_1} u_t||\nabla^{k_2} u_t| + |\nabla^{k_1+2} u||\nabla^{k_2+2} u| + |\nabla^{k_1+3} u||\nabla^{k_2+1} u|, \\ &|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+2} u|, \\ &|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+1} u||\nabla^{k_4+1} u|,\end{aligned}$$

where $k_j \in \mathbb{N}_0$ and $k_1 + k_2 + \cdots = m_2$. Note that $m_2 \leq k-3$ since $m_1 > 0$. In the following we use the notation (1.4.10) - (1.4.12) for all five cases, setting $i = 0$ for the latter three. Combining the above considerations with Lemma 1.2.9, we can now estimate the norm

$$\| \nabla^{m_1}(I-P_u) \nabla^{m_2}(\mathcal{N}(u)) \|_{L^2} ,$$

where we distinguish five cases according to the terms in the brackets in (1.4.10) - (1.4.12).
Case 1: $\nabla^{k_1} u_t \star \nabla^{k_2} u_t$
We use Lemma 1.2.9 with

$$f_1 = \nabla u,\ \dots,\ f_j = \nabla u,\ f_{j+1} = \nabla u,\ \dots,\ f_{j+i} = \nabla u,\ f_{j+i+1} = u_t,\ f_{j+i+2} = u_t,$$

and derivatives of order

$$\tilde{k}_1 + \cdots + \tilde{k}_j + \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 = m_1 + m_2 - i - j = k - 2 - (i+j).$$

Employing also Young's inequality, it follows

$$\begin{aligned} &\Big\| |\nabla^{\tilde{k}_1+1} u| \cdots |\nabla^{\tilde{k}_j+1} u| |\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| |\nabla^{k_1} u_t| |\nabla^{k_2} u_t| \Big\|_{L^2} \\ &\lesssim \big((1 + \|\nabla u\|_{L^\infty}^{k-3}) \|u_t\|_{L^\infty}^2 + (1 + \|\nabla u\|_{L^\infty}^{k-2}) \|u_t\|_{L^\infty} \big) (\|\nabla u\|_{H^{k-2-i-j}} + \|u_t\|_{H^{k-2-i-j}}) \\ &\lesssim (1 + \|\nabla u\|_{L^\infty}^{k-1} + \|u_t\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}}). \end{aligned}$$

The other cases will be treated similarly. Note that here and in the following the L^∞ norms and especially $\|u_t\|_{L^\infty}$ are bounded by our choice of k.
Case 2: $\nabla^{k_1+2} u \star \nabla^{k_2+2} u$
Here it is exploited that $m_1 > 0$ in I_1 due to the cancellation from (1.3.31). This time Lemma 1.2.9 is applied with $f_1 = \cdots = f_{j+i+2} = \nabla u$ and derivatives of order

$$\tilde{k}_1 + \cdots + \tilde{k}_j + \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + 2 = m_1 + m_2 + 2 - i - j = k - (i+j) \le k - 1$$

since $j > 0$ by (1.3.33). We estimate

$$\begin{aligned} &\Big\| |\nabla^{\tilde{k}_1+1} u| \cdots |\nabla^{\tilde{k}_j+1} u| |\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| |\nabla^{k_1+2} u| |\nabla^{k_2+2} u| \Big\|_{L^2} \\ &\lesssim \sum_{i,j} \|\nabla u\|_{L^\infty}^{i+j+1} \|\nabla u\|_{H^{k-i-j}} \lesssim (1 + \|\nabla u\|_{L^\infty}^{k-1}) \|\nabla u\|_{H^{k-1}}. \end{aligned}$$

Case 3: $\nabla^{k_1+3} u \star \nabla^{k_2+1} u$
As in the previous case, $C(1 + \|\nabla u\|_{L^\infty}^{k-1}) \|\nabla u\|_{H^{k-1}}$ dominates

$$\Big\| |\nabla^{\tilde{k}_1+1} u| \cdots |\nabla^{\tilde{k}_j+1} u| |\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| |\nabla^{k_1+3} u| |\nabla^{k_2+1} u| \Big\|_{L^2}.$$

Case 4: $\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+2} u$
We apply Lemma 1.2.9 to the functions $f_1 = \cdots = f_{j+i+3} = \nabla u$ with derivatives of order

$$\tilde{k}_1 + \cdots + \tilde{k}_j + \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + k_3 + 1 = m_1 + m_2 + 1 - i - j = k - 1 - (i+j),$$

leading to the bound

$$\begin{aligned} &\Big\| |\nabla^{\tilde{k}_1+1} u| \cdots |\nabla^{\tilde{k}_j+1} u| |\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| |\nabla^{k_1+1} u| |\nabla^{k_2+1} u| |\nabla^{k_3+2} u| \Big\|_{L^2} \\ &\lesssim \sum_{i,j} \|\nabla u\|_{L^\infty}^{i+j+2} \|\nabla u\|_{H^{k-2-i-j}} \lesssim (1 + \|\nabla u\|_{L^\infty}^{k}) \|\nabla u\|_{H^{k-1}}. \end{aligned}$$

<u>*Case 5:*</u> $\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u$
We now use Lemma 1.2.9 with $f_1 = \cdots = f_{j+i+4} = \nabla u$ and derivatives of order

$$\tilde{k}_1 + \cdots + \tilde{k}_j + \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + k_3 + k_4 = m_1 + m_2 - i - j = k - 2 - (i + j).$$

Hence, we have

$$\begin{aligned} &\left\| |\nabla^{\tilde{k}_1+1}u| \cdots |\nabla^{\tilde{k}_j+1}u||\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u||\nabla^{k_1+1}u||\nabla^{k_2+1}u||\nabla^{k_3+1}u||\nabla^{k_4+1}u| \right\|_{L^2} \\ &\quad \lesssim \sum_{i,j} \|\nabla u\|_{L^\infty}^{i+j+3} \|\nabla u\|_{H^{k-2-i-j}} \lesssim (1 + \|\nabla u\|_{L^\infty}^{k+1}) \|\nabla u\|_{H^{k-1}} . \end{aligned}$$

Summing up the five cases, we infer

$$\|I_1\|_{L^1} \lesssim (1 + \|\nabla u\|_{L^\infty}^{k-1} + \|u_t\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}}). \tag{1.3.38}$$

Next, in I_2 from (1.3.32) we integrate by parts in order to conclude

$$\begin{aligned} \int_{\mathbb{R}^n} I_2 \, dx = &\sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}(\mathcal{N}(u)) \star [\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t] \, dx \\ &+ \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}(\mathcal{N}(u)) \star [\nabla^{l_1}(I - P_u)\nabla^{l_2+1} u_t] \, dx \\ =&: I_2^1 + I_2^2. \end{aligned}$$

These terms are estimated by

$$|I_2^1| \lesssim \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2} \left\|\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t\right\|_{L^2}, \tag{1.3.39}$$

$$|I_2^2| \lesssim \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2} \left\|\nabla^{l_1}(I - P_u)\nabla^{l_2+1} u_t\right\|_{L^2}. \tag{1.3.40}$$

We control $\left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2}$ by terms of the form (1.4.10) - (1.4.12) in the L^2 norm, obtaining as above

$$\left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2} \lesssim (1 + \|\nabla u\|_{L^\infty}^{k} + \|u_t\|_{L^\infty}^{k-2})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}}).$$

Equation (1.3.33) and Lemma 1.2.9 further imply

$$\begin{aligned} \left\|\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t\right\|_{L^2} &\lesssim \sum_{j=1}^{l_1+1} \sum_{\sum_{i=1}^j \tilde{m}_i = l_1+1-j} \left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u||\nabla^{l_2} u_t| \right\|_{L^2} \\ &\lesssim (1 + \|\nabla u\|_{L^\infty}^{k-1} + \|u_t\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}}) \end{aligned}$$

where $\tilde{m}_1 + \cdots + \tilde{m}_i + l_2 = k - 1 - i \le k - 2$. Similarly, we have

$$\begin{aligned}\left\|\nabla^{l_1}(I - P_u)\nabla^{l_2+1}u_t\right\|_{L^2} &\lesssim \sum_{j=1}^{l_1} \sum_{\sum_{i=1}^{j} \tilde{m}_i = l_1 - j} \left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u| |\nabla^{l_2+1}u_t| \right\|_{L^2} \\ &\lesssim (1 + \|\nabla u\|_{L^\infty}^{k-2} + \|u_t\|_{L^\infty}^{k-2})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}})\end{aligned}$$

by Lemma 1.2.9 with $\tilde{m}_1 + \cdots + \tilde{m}_i + l_2 + 1 = k - 1 - i \le k - 2$, since $l_1 > 0$. The above three inequalities yield

$$\|I_2\|_{L^1} \lesssim (1 + \|\nabla u\|_{L^\infty}^{2k-1} + \|u_t\|_{L^\infty}^{2k-1})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2). \tag{1.3.41}$$

Finally, for the regularization term, we observe

$$\begin{aligned}-\varepsilon \int_{\mathbb{R}^n} \nabla^{k-2}[(I - P_u)(\Delta u_t)]\nabla^{k-2}u_t\,dx = &\ \varepsilon \int_{\mathbb{R}^n} \nabla^{k-3}[(I - P_u)(\Delta u_t)]\nabla^{k-1}u_t\,dx \\ &\le C\left\|\nabla^{k-3}[(I - P_u)(\Delta u_t)]\right\|_{L^2}^2 + \frac{\varepsilon}{2}\left\|\nabla^{k-1}u_t\right\|_{L^2}^2.\end{aligned}$$

In view of (1.3.3), to bound $\left\|\nabla^{k-3}[(I - P_u)(\Delta u_t)]\right\|_{L^2}^2$ it suffices to estimate

$$\left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u| \left[|\nabla^{k_1+1}u_t||\nabla^{k_2+1}u| + |\nabla^{k_1}u_t||\nabla^{k_2+2}u| \right] \right\|_{L^2}^2, \tag{1.3.42}$$

$$\left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u| |\nabla^{k_1}u_t||\nabla^{k_2+1}u||\nabla^{k_3+1}u| \right\|_{L^2}^2, \tag{1.3.43}$$

where $\tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + 1 = k - 2 - i$ and $\tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + k_3 = k - 3 - i$, respectively. As before, Lemma 1.2.9 implies the inequalities

$$\begin{aligned}&\left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u| \left[|\nabla^{k_1+1}u_t||\nabla^{k_2+1}u| + |\nabla^{k_1}u_t||\nabla^{k_2+2}u| \right] \right\|_{L^2}^2 \\ &\quad \lesssim (1 + \|\nabla u\|_{L^\infty}^{2(k-2)} + \|u_t\|_{L^\infty}^{2(k-2)})(\|u_t\|_{H^{k-2}}^2 + \|\nabla u\|_{H^{k-2}}^2,\end{aligned} \tag{1.3.44}$$

$$\begin{aligned}&\left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u| |\nabla^{k_1}u_t||\nabla^{k_2+1}u||\nabla^{k_3+1}u| \right\|_{L^2}^2 \\ &\quad \lesssim (1 + \|\nabla u\|_{L^\infty}^{2(k-1)} + \|u_t\|_{L^\infty}^{2(k-1)})(\|u_t\|_{H^{k-2}}^2 + \|\nabla u\|_{H^{k-2}}^2).\end{aligned} \tag{1.3.45}$$

Putting together (1.3.38), (1.3.41), (1.3.44) and (1.3.45), we arrive at the inequality

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u) - \varepsilon(I - P_u)(\Delta u_t)) \cdot \nabla^{k-2}u_t\,dx \right| \\ &\quad \lesssim (1 + \|\nabla u\|_{L^\infty}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2) + \frac{\varepsilon}{2}\left\|\nabla^{k-1}u_t\right\|_{L^2}^2.\end{aligned}$$

Subtracting the last term on both sides of (1.3.29), for $t \in [0, T]$ we conclude

$$\begin{aligned}&\left\|\nabla^{k-2}u_t(t)\right\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \frac{\varepsilon}{2}\int_0^t \left\|\nabla^{k-1}u_t(s)\right\|_{L^2}^2\,ds \\ &\quad \lesssim \int_0^t \left[(1 + \|\nabla u\|_{L^\infty}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2)\right]ds + \left\|\nabla^{k-2}u_1\right\|_{L^2}^2 + \left\|\nabla^k u_0\right\|_{L^2}^2.\end{aligned} \tag{1.3.46}$$

It remains to bound the lower order terms. Testing (1.3.28) by $u_t \in T_uN$, we infer

$$\|u_t(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla u_t(s)\|_{L^2}^2 \, ds = \|u_1\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2 . \tag{1.3.47}$$

Since also

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \le \int_{\mathbb{R}^n} |u_t|^2 \, dx + \int_{\mathbb{R}^n} |\Delta u|^2 \, dx,$$

it follows

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 &\le \|\nabla u_0\|_{L^2}^2 + \int_0^t \|\Delta u(s)\|_{L^2}^2 + \|u_t(s)\|_{L^2}^2 \, ds \\ &\le \|\nabla u_0\|_{L^2}^2 + t(\|u_1\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2) \end{aligned} \tag{1.3.48}$$

for $t \in [0, T]$. The other derivatives are treated via interpolation, more precisely

$$\begin{aligned} \left\|\nabla^l u_t\right\|_{L^2}^2 &\lesssim \left\|\nabla^{k-1} u_t\right\|_{L^2}^{\frac{2(l-1)}{k-2}} \|\nabla u_t\|_{L^2}^{\frac{2(k-1-l)}{k-2}}, \quad l = 2, \dots, k-2, \\ \left\|\nabla^l u_t\right\|_{L^2}^2 &\lesssim \left\|\nabla^{k-2} u_t\right\|_{L^2}^{\frac{2l}{k-2}} \|u_t\|_{L^2}^{\frac{2(k-2-l)}{k-2}}, \quad l = 1, \dots, k-3, \\ \left\|\nabla^l u\right\|_{L^2}^2 &\lesssim \left\|\nabla^k u\right\|_{L^2}^{\frac{2(l-2)}{k-2}} \|\Delta u\|_{L^2}^{\frac{2(k-l)}{k-2}}, \quad l = 3, \dots, k-1. \end{aligned}$$

Estimate (1.3.46) and the above inequalities lead to the core estimate

$$\begin{aligned} &\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2 + \frac{\varepsilon}{2} \int_0^t \|\nabla u_t(s)\|_{H^{k-2}}^2 \, ds \\ &\lesssim \int_0^t \Big[(1 + \|\nabla u\|_{L^\infty}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2) \Big] \, ds \\ &\quad + (1+T)(\|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2), \qquad t \in [0, T]. \end{aligned} \tag{1.3.49}$$

for solutions of (1.3.1) and $T < T_{\epsilon,m}$. Using Gronwall's lemma we also obtain

$$\begin{aligned} &\sup_{t \in [0,T]} \left(\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2 \right) \\ &\le C(1+T) \left(\|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2 \right) \exp\left(\int_0^T (1 + \|\nabla u\|_{L^\infty}^{2k} + \|u_t\|_{L^\infty}^{2k}) \, ds \right). \end{aligned} \tag{1.3.50}$$

At least for small times we want to remove the dependence on u on the right-hand side of (1.3.49) and thus we introduce the quantity

$$\alpha(t) = \|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \|u_t(t)\|_{L^2}^2 + \left\|\nabla^{k-2} u_t(t)\right\|_{L^2}^2$$

for $t \in [0, T_{\epsilon,m})$. We observe that $\alpha(t)$ is equivalent to the square of the Sobolev norms appearing in (1.3.49). Since the solutions to (1.3.1) are (locally) unique, our reasoning is also valid for any initial time $t_0 \in (0, T_{\epsilon,m})$. The estimates (1.3.46), (1.3.47) and (1.3.48) thus imply

$$\alpha(t) - \alpha(t_0) \le C \int_{t_0}^t (1 + \alpha(s)^k) \alpha(s) \, ds.$$

By the above arguments, the function α is differentiable a.e. so that

$$\frac{d}{dt}\mathcal{E}(t) \leq C(1+\mathcal{E}(t)^k)\mathcal{E}(t) \tag{1.3.51}$$

for a.e. $0 \leq t_0 \leq t < T_{\varepsilon,m}$. We now proceed similarly to [27], where regularization by the (intrinsic) biharmonic energy has been applied in order to obtain the existence of local Schrödinger maps.

Lemma 1.3.6. *Let $\varepsilon \in (0,1)$ and take data $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for a.e. $x \in \mathbb{R}^n$ satisfying*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n) \quad \text{for some } k \in \mathbb{N} \text{ with } k > \lfloor \tfrac{n}{2} \rfloor + 2.$$

Let $T_{\varepsilon,m} > 0$ be the maximal existence time of the solution $u^\varepsilon : \mathbb{R}^n \times [0, T_{\varepsilon,m}) \to N$ of (1.3.1) with $u^\varepsilon(0) = u_0$ and $\partial_t u^\varepsilon(0) = u_1$ from Proposition 1.3.5. Then there is a time $T_0 = T_0(\|\nabla u_0\|_{H^{k-1}}, \|u_1\|_{H^{k-2}}) > 0$ such that $T_{\varepsilon,m} > T_0$ for all $\varepsilon \in (0,1)$.

Proof. Let $\varepsilon \in (0,1)$ and $t \in [0, T_{\varepsilon,m})$. We write $u = u^\varepsilon$. From (1.3.51) we infer

$$\frac{d}{dt}\log\left(\frac{\mathcal{E}}{(1+\mathcal{E}^k)^{\frac{1}{k}}}\right) = \frac{\mathcal{E}'}{(1+\mathcal{E}^k)\mathcal{E}} \leq C, \tag{1.3.52}$$

With $\alpha_0 = \alpha(0)$ it follows

$$\begin{gathered}\frac{\mathcal{E}(t)^k}{(1+\mathcal{E}(t)^k)} \leq e^{Ctk}\frac{\mathcal{E}_0^k}{(1+\mathcal{E}_0^k)} \leq (1+4Ctk)\frac{\mathcal{E}_0^k}{(1+\mathcal{E}_0^k)},\\ \mathcal{E}(t)^k \leq (1+4Ctk)\mathcal{E}_0^k + 4Ctk\mathcal{E}_0^k\mathcal{E}^k\end{gathered}$$

for $0 \leq t \leq \frac{1}{8Ck}$, and hence

$$\mathcal{E}(t)^k \leq 2(1+4Ctk)\mathcal{E}_0^k \leq 3\mathcal{E}_0^k$$

for $0 \leq t \leq \frac{1}{8Ck}\min\{1, \frac{1}{\mathcal{E}_0^k}\} =: T_0$. Since α and the Sobolev norms are equivalent, we infer

$$\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2 \leq c_0(\|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2) \tag{1.3.53}$$

for $t \in [0, \min\{T_{\varepsilon,m}, T_0\})$ and some constant $c_0 = c_0(k,n) > 0$.
We now assume by contradiction that $T_{\varepsilon,m} \leq T_0$ for some (fixed) $\varepsilon \in (0,1)$. We apply the contraction argument in the proof of Lemma 1.3.1 for the initial time $t_0 \in [0, T_{\varepsilon,m})$ and data $(u(t_0), u_t(t_0))$ in the fixed-point space $\mathcal{B}_r(T)$ with radius

$$r^k = 3r(t_0)^k := 3\Big(\|\nabla u(t_0)\|_{H^{k-1}} + \|u_t(t_0)\|_{H^{k-2}}\Big)^k.$$

Since $t_0 < T_0$, estimate (1.3.53) yields the uniform bound

$$r(t_0) \leq \sqrt{2c_0}(\|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2)^{1/2} =: \hat{c}_0.$$

As a result, the time

$$T := \frac{1}{4}\min\left\{\left(\frac{\sqrt[k]{3}-1}{\sqrt[k]{3}}\right)^2 \frac{\varepsilon}{\hat{C}^2(1+3\hat{c}_0^k)^2}, \frac{\varepsilon}{\hat{C}^2(1+6\hat{c}_0^k)^2}\right\}.$$

is less or equal than the time T_δ for $\mathcal{B}_r(T)$ in (1.3.21). Therefore, the solution can be uniquely extended to $[0, t_0 + T]$ in the regularity class of Proposition 1.3.5. For $t_0 > T_{\varepsilon,m} - T$ this fact contradicts the maximality of $T_{\varepsilon,m}$, showing the result. □

Proof of local existence

We now combine the existence result from Proposition 1.3.5 with Lemma 1.3.6. Thus, there exists a solution $u^\varepsilon : \mathbb{R}^n \times [0, T_0] \to N$ of (1.3.1) for each $\varepsilon \in (0, 1)$, where $T_0 > 0$ only depends on $\|\nabla u_0\|_{H^{k-1}}$ and $\|u_1\|_{H^{k-2}}$. From (1.3.53) and the inequality

$$\|u^\varepsilon - u_0\|_{L_t^\infty L_x^2} \leq T_0 \|u_t^\varepsilon\|_{L_t^\infty L_x^2},$$

we extract a limit $u : \mathbb{R}^n \times [0, T_0] \to \mathbb{R}^L$ as $\varepsilon \to 0^+$ of the solutions $u^\varepsilon_{|[0,T_0]}$ in the sense

$$\nabla^{l_1} u^\varepsilon \overset{*}{\rightharpoonup} \nabla^{l_1} u, \quad u^\varepsilon - u_0 \overset{*}{\rightharpoonup} u - u_0, \quad \text{and} \quad \nabla^{l_2-2} u_t^\varepsilon \overset{*}{\rightharpoonup} \nabla^{l_2-2} u_t \quad \text{in} \;\; L^\infty(0, T_0; L^2),$$

where $1 \leq l_1 \leq k$ and $0 \leq l_2 \leq k$. (Here and below we do not indicate that we pass to subsequences.) In particular,

$$u - u_0 \in L^\infty(0, T_0; H^k) \cap W^{1,\infty}(0, T_0; H^{k-2})$$

and $(\nabla u, \partial_t u)$ is weakly continuous in $H^{k-1} \times H^{k-2}$. We further note that (1.3.53) holds for u by weak* lower semicontinuity of the norm. We first assume $k \geq 4$ (which is no restriction if $n \geq 2$). Estimating the nonlinearity similarly to Section 1.3.2, we also deduce from (1.3.3) and (1.3.53) that $\partial_t^2 u^\varepsilon \in C^0([0, T_0], H^{k-4})$ is uniformly bounded as $\varepsilon \to 0^+$. Compactness and Sobolev's embedding further yield

$$\begin{aligned} &\nabla^3 u^\varepsilon \to \nabla^3 u \quad \text{in} \;\; C^0([0, T_0], L^2_{loc}(\mathbb{R}^n)), \\ &\partial_t u^\varepsilon \to \partial_t u, \;\; u^\varepsilon \to u, \;\; \nabla u^\varepsilon \to \nabla u, \;\; \nabla^2 u^\varepsilon \to \nabla^2 u, \end{aligned} \tag{1.3.54}$$

where the latter holds locally uniformly on $\mathbb{R}^n \times [0, T_0]$. More precisely for $\alpha \in (0, 1)$ and $v^\varepsilon = u^\varepsilon - u_0$, our a priori estimates and [39, Prop. 1.1.4] imply uniform bounds (in ε) in the spaces

$$v^\varepsilon \in C^\alpha H^{k-2\alpha}, \;\; \nabla v^\varepsilon \in C^\alpha H^{k-1-2\alpha}, \;\; \nabla^2 v^\varepsilon \in C^\alpha H^{k-2-2\alpha}, \;\; \partial_t v^\varepsilon \in C^\alpha H^{k-2-2\alpha}. \tag{1.3.55}$$

As a result, u takes values in N. Moreover, since (1.3.49) and (1.3.53) give

$$\int_0^T \left\|\sqrt{\varepsilon}\nabla u_t^\varepsilon(s)\right\|_{H^{k-2}}^2 \, ds \tag{1.3.56}$$
$$\lesssim \big(T_0(1 + \|u_1\|_{H^{k-2}}^{2k} + \|\nabla u_0\|_{H^{k-1}}^{2k}) + 1\big)(\|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2)$$

and $k \geq 3$, we infer that $\varepsilon\Delta\partial_t u^\varepsilon \to 0$ in $L^2_{t,x}$. Combining this fact with (1.3.54) and recalling (1.3.4), we conclude

$$\mathcal{N}_\varepsilon(u^\varepsilon) \to \mathcal{N}(u) \quad \text{in } L^2_{loc}(\mathbb{R}^n \times [0, T_0]).$$

In the case $n = 1$ and $k = 3$ we obtain the convergence $\mathcal{N}_\varepsilon(u^\varepsilon) \to \mathcal{N}(u)$ in the sense of the duality (H^1, H^{-1}) because we still have

$$\nabla u^\varepsilon \to \nabla u, \quad \nabla^2 u^\varepsilon \to \nabla^2 u, \quad \partial_t u^\varepsilon \to \partial_t u$$

locally uniformly, as well as $\nabla^3 u^\varepsilon \to \nabla^3 u$ and $\nabla\partial_t u^\varepsilon \to \nabla\partial_t u$ in $C^0([0, T_0], H^{-1}_{loc})$ as $\varepsilon \to 0^+$.

Summing up, we have constructed a local solution $u : [0, T_0] \times \mathbb{R}^n \to N$ of (1.1.2) with $u(0) = u_0$ and $\partial_t u(0) = u_1$ such that $(\nabla u, \partial_t u)$ is bounded and weakly continuous in $H^{k-1} \times H^{k-2}$. In Lemma 1.3.10 it will be shown that such a solution is locally unique. We recall from the proof of Proposition 1.3.6 that the solution $u : \mathbb{R}^n \times [0, T) \to N$ for some $T > 0$ can be extended if $\limsup_{t \to T^-}(\|\nabla u(t)\|_{H^{k-1}} + \|u_t(t)\|_{H^{k-2}}) < \infty$. There thus exists a maximal time of existence $T_m \in (T_0, \infty]$ of u with

$$\limsup_{t \to T_m^-}(\|\nabla u(t)\|_{H^{k-1}} + \|u_t(t)\|_{H^{k-2}}) = \infty \qquad \text{if } T_m < \infty.$$

Arguing as in Section 1.3.2, we establish the energy equality

$$\left\|\nabla^k u\right\|_{L^2}^2 + \left\|\nabla^{k-2} u_t\right\|_{L^2}^2 = 2\int_0^t \int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u)) \cdot \nabla^{k-2} u_t \, dx \, ds \tag{1.3.57}$$
$$+ \left\|\nabla^k u_0\right\|_{L^2}^2 + \left\|\nabla^{k-2} u_1\right\|_{L^2}^2$$

for $t \in [0, T_m)$. (The integral is well-defined in view of the cancellation of one derivative in (1.3.30).) However, in contrast to the approximations u^ε, the solution u has only k weak spatial derivatives (and $\partial_t u$ has $k - 2$). For this reason, when deriving (1.3.57) we have to replace one spatial derivative by a difference quotient. The details are outlined in Appendix 1.A.3.

We conclude that the highest derivatives $\nabla^{k-2} u_t, \nabla^k u : [0, T_m) \to L^2$ are continuous, employing their weak continuity and that the right-hand side of (1.3.57) is continuous in t. For the continuity of lower order derivatives, we can employ the same argument using the identity (1.2.9) (for lower order derivatives). This follows from the standard energy argument in Lemma

1.2.10 since $\mathcal{N}(u) \in L_t^\infty H^m$ for $0 \leq m \leq k-3$ as Lemma 1.3.4 (by the estimate for the full H^{k-3} norm) shows. In particular

$$u - u_0 \in C^0([0,T_m), H^k) \cap C^1([0,T_m), H^{k-2})$$

as asserted.

Finally, following the proof of the a priori estimate in Section 1.3.2 we can derive the blow-up criterion (1.1.4), cf. Appendix 1.A.3.

In order to show the full statement of Theorem 1.1.1 and Theorem 1.1.2, it thus remains to establish the uniqueness statement and the continuous dependence on the initial data, which is done in the next Sections 1.3.3 and 1.3.4.

Remark 1.3.7. In addition, we apply Lemma 1.3.4 (for the difference with $u(t)$ and $v(s) = u(s)$) in order to see that $\mathcal{N}(u) \in C_t H^m$ for $0 \leq m \leq k-3$. Especially, by the Remark below Lemma 1.2.10, we have that the maps

$$t \mapsto \|\nabla u(t)\|_{H^{m+1}}^2 + \|u_t(t)\|_{H^m}, \quad 0 \leq m \leq k-3,$$

are differentiable with the corresponding identity (1.2.9). A similar statement using (1.2.9) and Lemma 1.3.4 is given for the difference of two solutions, wich will be used in Section 1.3.3. Also, the calculation in the Appendix 1.A.3 shows (combined with Lemma 1.3.4) that the above map is differentiable with $m = k-2$. A similar statement is proven for the difference of solutions at regularity $m = k-2$ if one of the solutions has *higher regularity*. This will be used in Section 1.3.4 and we advise the reader to follow the details in this section.

1.3.3 Energy bounds and uniqueness

In the last Section 1.3.2 (respectively from the Appendix (1.3.57)), we have seen that the solution $u : [0,T_m) \times \mathbb{R}^n \to N$ with regularity

$$(\nabla u, \partial_t u) \in C([0,T], H^{k-1} \times H^{k-2})$$

for some $T \in (0,T_m)$ satisfies

$$\sup_{t \leq T}(\|\nabla u(t)\|_{H^{k-1}}^2 + \|u_t(t)\|_{H^{k-2}}^2) \leq C_1(\|\nabla u_0\|_{H^{k-1}}^2 + \|u_1\|_{H^{k-2}}^2), \tag{1.3.58}$$

where $C_1 = C(T, N, k, \|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^n)}, \|u_t\|_{L^\infty([0,T]\times\mathbb{R}^n)}) > 0$.

For the uniqueness of the limit in Section 1.3.2 and in order to prove that solutions depend continuously on the inital data, we need additional energy bounds for the difference $w = u - v$ of two solutions. These are obtained similarly as in Section 1.3.2. The section is slightly modiefied compared to [19].

Lemma 1.3.8 (Energy bounds)**.** *Let* $T > 0,\ k > \lfloor \frac{n}{2} \rfloor + 2$ *and* u, v *be two solutions of* (1.1.2) *with*

$$(\nabla u, \partial_t u),\ (\nabla v, \partial_t v) \in C([0,T], H^{k-1} \times H^{k-2})$$

and $u(0) = u_0, v(0) = v_0, u_t(0) = u_1$ *and* $v_t(0) = v_1$ *such that* $u_0 - v_0 \in L^2$*. Then there holds*

$$\sup_{t \le T}(\|u(t) - v(t)\|^2_{H^{k-1}} + \|u_t(t) - v_t(t)\|^2_{H^{k-3}}) \tag{1.3.59}$$
$$\le C_2(\|u_0 - v_0\|^2_{H^{k-1}} + \|u_1 - v_1\|^2_{H^{k-3}}),$$

where $C_2 = C(T, N, k) > 0$ *further depends on the norm of* $\nabla u, \nabla v$ *in* $L^\infty(0,T; H^{k-1})$ *and* u_t, v_t *in* $L^\infty(0,T; H^{k-2})$.

Remark 1.3.9. The estimates in Lemma 1.3.8 depend as above on the cancellation introduced by the identities

$$\mathcal{N}(u) = (I - P_u)(\mathcal{N}(u)),\ \ \mathcal{N}(v) = (I - P_v)(\mathcal{N}(v)).$$

However, this effect is weaker for the Lipschitz estimate (1.3.59) and can not be extended to the level of the initial regularity of u, v.

Proof of Lemma 1.3.8. We derive (1.3.59) from a Gronwall argument based on the equality

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n} |\nabla^m w_t|^2 + |\nabla^{m+2} w|^2\, dx = \int_{\mathbb{R}^n} \nabla^m(\mathcal{N}(u) - \mathcal{N}(v)) \cdot \nabla^m w_t\, dx, \tag{1.3.60}$$

for $w = u - v$, $m \in \{0, \dots, k-3\}$ and $t \in [0,T]$, which is a consequence of (1.1.2) and the remarks in the proof of local existence to why this identity holds. Setting

$$\mathcal{E}(t) = \|w(t)\|^2_{H^{k-1}} + \|w_t(t)\|^2_{H^{k-3}},$$

we want to prove

$$\frac{d}{dt}\mathcal{E}(t) \le C(1 + \|\nabla u\|^{2k}_{H^{k-1}} + \|u_t\|^{2k}_{H^{k-2}} + \|\nabla v\|^{2k}_{H^{k-1}} + \|v_t\|^{2k}_{H^{k-2}})\mathcal{E}(t) \tag{1.3.61}$$

for $t \in [0,T]$ and thus first consider (1.3.60) for $m = k-3$. Since u and v map into N, we have $\mathcal{N}(u) = (I - P_u)(\mathcal{N}(u))$ and analogously for v. It follows

$$\begin{aligned}\mathcal{N}(u) - \mathcal{N}(v) &= (I - P_u)\mathcal{N}(u) - (I - P_v)\mathcal{N}(v)\\ &= (P_v - P_u)\mathcal{N}(u) + (I - P_v)(\mathcal{N}(u) - \mathcal{N}(v)),\end{aligned}$$

and hence

$$\begin{aligned}\nabla^{k-3}(\mathcal{N}(u) - \mathcal{N}(v)) \cdot \nabla^{k-3} w_t = \nabla^{k-3}&[(P_v - P_u)\mathcal{N}(u)] \cdot \nabla^{k-3} w_t\\ &+ \nabla^{k-3}[(I - P_v)(\mathcal{N}(u) - \mathcal{N}(v))] \cdot \nabla^{k-3} w_t.\end{aligned}$$

In this way, we can avoid that all derivatives fall on $\nabla^3 w$. We next write

$$\nabla^{k-3}[(P_v - P_u)\mathcal{N}(u)] \cdot \nabla^{k-3} w_t = (P_v - P_u)\nabla^{k-3}[\mathcal{N}(u)] \cdot \nabla^{k-3} w_t + \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \nabla^{l_1}[(P_v - P_u)] \star \nabla^{l_2}[\mathcal{N}(u)] \cdot \nabla^{k-3} w_t =: I_1 + I_2.$$

Observe that

$$\int_{\mathbb{R}^n} I_1 \, dx \lesssim \|w\|_{L^\infty} \left\|\nabla^{k-3}\mathcal{N}(u)\right\|_{L^2} \left\|\nabla^{k-3} w_t\right\|_{L^2}.$$

We then control $\left\|\nabla^{k-3}\mathcal{N}(u)\right\|_{L^2}$ using Lemma 1.2.9 as above for the a priori estimate (1.3.49). Further, Lemma 1.A.2 implies that $\int_{\mathbb{R}^n} I_2 \, dx$ is bounded by terms of the form

$$\|w\|_{L^\infty} \left\| |\nabla^{m_1+1} u| \cdots |\nabla^{m_j+1} u| |\nabla^{l_2}\mathcal{N}(u)| \right\|_{L^2} \left\|\nabla^{k-3} w_t\right\|_{L^2}, \tag{1.3.62}$$

$$\left\|\nabla^{k-3} w_t\right\|_{L^2} \left\| |\nabla^{m_1+1} w| |\nabla^{m_2+1} h_1| \cdots |\nabla^{m_j+1} h_{j-1}| |\nabla^{l_2}\mathcal{N}(u)| \right\|_{L^2}, \tag{1.3.63}$$

where $m_1, \dots, m_j$ and $h_1, \dots, h_{j-1}$ are as in Lemma 1.A.2. In (1.3.62) we then estimate as above in the a priori estimate. For (1.3.63), it suffices to control terms of the form

$$|\nabla^{m_1+1} w| |\nabla^{m_2+1} h_1| \cdots |\nabla^{m_j+1} h_{j-1}| |\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i+1} u| \left[|\nabla^{k_1} u_t| |\nabla^{k_2} u_t| \cdots \right], \tag{1.3.64}$$

where $\left[|\nabla^{k_1} u_t| |\nabla^{k_2} u_t| \cdots \right]$ is given as in the nonlinearity $\mathcal{N}(u)$ and the orders $m_1 \dots, m_j$, $\tilde{m}_1, \dots, \tilde{m}_i$, and $k_1, k_2 \dots$ are as used before. To apply Lemma 1.2.9, as above we choose

$$f_1 = w, \; f_2 = \nabla h_1, \; \dots, f_j = \nabla h_{j-1}, f_{j+1} = \nabla u, \dots, f_{i+j} = \nabla u,$$

and $f_{i+j+1}, \; f_{i+j+2}, \dots,$ according to the respective terms in $\mathcal{N}(u)$. We can thus estimate (1.3.64) in L^2 by

$$\begin{aligned}
&\|w\|_{L^\infty}^{1-\frac{m_1}{k-2-i-j}} \|w\|_{H^{k-2-i-j}}^{\frac{m_1}{k-2-i-j}} (1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}) \\
&+ \|w\|_{L^\infty}^{1-\frac{m_1}{k-1-i-j}} \|w\|_{H^{k-1-i-j}}^{\frac{m_1}{k-1-i-j}} (1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}) \\
&\lesssim \|w\|_{H^{k-1}} (1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}),
\end{aligned}$$

noting that $l_1 > 0$, $j \geq 1$ and $i + j < k - 2$. We continue by computing

$$\begin{aligned}
&\nabla^{k-3}[(I - P_v)(\mathcal{N}(u) - \mathcal{N}(v))] \cdot \nabla^{k-3} w_t \\
&= \nabla^{k-3}(\mathcal{N}(u) - \mathcal{N}(v))(I - P_v)\nabla^{k-3} w_t + \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \nabla^{l_1}(I - P_v) \star \nabla^{l_2}(\mathcal{N}(u) - \mathcal{N}(v)) \cdot \nabla^{k-3} w_t \\
&= \nabla^{k-3}(\mathcal{N}(u) - \mathcal{N}(v))\nabla^{k-3}[(P_u - P_v)u_t] - \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \nabla^{k-3}(\mathcal{N}(u) - \mathcal{N}(v)) \cdot \nabla^{l_1}[(I - P_v)] \star \nabla^{l_2} w_t \\
&\quad + \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \nabla^{l_1}(I - P_v) \star \nabla^{l_2}(\mathcal{N}(u) - \mathcal{N}(v)) \cdot \nabla^{k-3} w_t =: J_1 + J_2 + J_3.
\end{aligned}$$

where the second equality is a consequence of

$$(I - P_v)w_t = (I - P_v)u_t = [(I - P_v) - (I - P_u)]u_t = (P_u - P_v)u_t.$$

We use integration by parts to treat $\int J_1\,dx$ and $\int J_2\,dx$. Here we assume that $k \geq 4$. (If $k = 3$ the estimate becomes easier and we only employ integration by parts for $dP_v(\nabla^3 w \star \nabla u)$ in the difference $\mathcal{N}(u) - \mathcal{N}(v)$.) It follows

$$\begin{aligned}
\int_{\mathbb{R}^n} J_1\,dx &= -\int_{\mathbb{R}^n} \nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)] \cdot \nabla^{k-2}[(P_u - P_v)u_t]\,dx, \\
\int_{\mathbb{R}^n} J_2\,dx &= \sum_{\substack{l_1+l_2=k-3 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)] \cdot [\nabla^{l_1+1}(I - P_v) \star \nabla^{l_2} w_t \\
&\qquad + \nabla^{l_1}(I - P_v) \star \nabla^{l_2+1} w_t]\,dx.
\end{aligned}$$

We first bound

$$\int_{\mathbb{R}^n} J_1\,dx \lesssim \left\|\nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2} \left\|\nabla^{k-2}[(P_u - P_v)u_t]\right\|_{L^2}.$$

Corollary 1.A.3, Lemma 1.A.2 and Lemma 1.2.9 yield

$$\begin{aligned}
\left\|\nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2} &\lesssim (\|w\|_{H^{k-1}} + \|w_t\|_{H^{k-3}}) \\
&\qquad \cdot (1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}), \\
\left\|\nabla^{k-2}[(P_u - P_v)u_t]\right\|_{L^2} &\lesssim \|w\|_{H^{k-1}} (1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).
\end{aligned}$$

The integrals of J_2 and J_3 are treated similary. Summing up, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla^{k-3} w_t|^2 + |\nabla^{k-1} w|^2\,dx \lesssim \mathcal{E}(t)(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).$$

We can similarly derive the estimate (integrating $dP_v(\nabla^3 w \star \nabla u)$ by parts)

$$\frac{d}{dt}\int_{\mathbb{R}^n} |w_t|^2 + |\Delta w|^2\,dx \lesssim \mathcal{E}(t)(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).$$

Interpolation on the left-hand side then yields

$$\frac{d}{dt}\mathcal{E}(t) \lesssim \mathcal{E}(t)(1 + \|\nabla u(t)\|_{H^{k-1}}^{2k} + \|u_t(t)\|_{H^{k-2}}^{2k} + \|\nabla v(t)\|_{H^{k-1}}^{2k} + \|v_t(t)\|_{H^{k-2}}^{2k}),$$

for $t \in [0, T]$. By Gronwall we thus obtain the claimed estimate (1.3.59). $\square$

A direct consequence of (1.3.59) in Lemma 1.3.8 is the following uniqueness statement.

Lemma 1.3.10 (Uniqueness). *Let* $u, v : \mathbb{R}^n \times [0,T] \to N$ *be two solutions of* (1.1.1) *with initial data* $u_0 : \mathbb{R}^n \to N$ *and* $u_1 : \mathbb{R}^n \to \mathbb{R}^L$ *such that* $u_1 \in T_{u_0}N$ *on* $\mathbb{R}^n$ *and*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ *with* $k > \lfloor \frac{n}{2} \rfloor + 2$. *Also let*

$$u - u_0,\ v - u_0 \in L^\infty(0,T; H^k(\mathbb{R}^n)) \cap W^{1,\infty}(0,T; H^{k-2}(\mathbb{R}^n)).$$

Then $u_{|[0,T]} = v_{|[0,T]}$.

1.3.4 Continuity of the flow map

We now prove that the solutions of the Cauchy problem for (1.1.1) depend continuously on the initial data. As mentioned before in section 1.3.3 the difference $u - v$ of two solutions u and v satisfies the Lipschitz estimate with the *loss of one order of derivatives* compared the a priori bounds such as (1.3.49) (or (1.3.58) for the solution of (1.1.2)).

To deal with this problem, we apply the Bona–Smith argument, which is outlined e.g. in [59] (for the Burgers equation) and in [12] (for the KdV equation). The following section is taken from [19] with minor changes.
Let T_m be the maximal existence time of the solution u with initial data (u_0, u_1) from Theorem 1.1.1. Fix $T_0 \in (0, T_m)$. Take data (v_0, v_1) as in the theorem satisfying

$$\|(u_0, u_1) - (v_0, v_1)\|_{H^k \times H^{k-2}} \leq R \tag{1.3.65}$$

for some $R > 0$. (We note that we have to assume $u_0 - v_0 \in L^2$ in order to establish the a priori estimate for the difference of the solutions as in the Section 1.3.3.) We use regularized data (u_0^δ, u_1^δ) and (v_0^δ, v_1^δ) in the sense of Lemma 1.A.6 from Appendix 1.A.2, where $\delta \in (0, \delta^*]$ for some $\delta^* > 0$ depending on N. The corresponding solutions are denoted by u^δ and v^δ. They satisfy the regularity assertions of part a) of Theorem 1.1.1 for *all* $k > \lfloor \frac{n}{2} \rfloor + 2$. It is crucial that the a priori estimates for u^δ and v^δ are uniform in δ. We split $u - v$ into

$$u - v = u - u^\delta + u^\delta - v^\delta + v^\delta - v$$

and bound each of the differences in $H^k \times H^{k-2}$.
In order to estimate $u^\delta - u$ and $v^\delta - v$, we use the geometric structure (as before in Section 1.3.3). It allows us to fix a (small) parameter $\delta > 0$ for which the differences are small in $H^k \times H^{k-2}$. This can be done uniformly for (v_0, v_1) in a certain ball around (u_0, u_1). For fixed δ, one can then estimate $u^\delta - v^\delta$ employing their extra regularity, but paying the price of a large constant (arising from the small parameter δ). We can control this constant, however, by choosing a small radius $R > 0$ in (1.3.65).
We start with some preparations concerning the cancellations caused by the geometric constraints. As in Section 1.3.3, we have

$$\begin{aligned} \mathcal{N}(u^\delta) - \mathcal{N}(u) &= (P_u - P_{u^\delta})(\mathcal{N}(u^\delta)) + (I - P_u)(\mathcal{N}(u^\delta) - \mathcal{N}(u)), \\ (I - P_u)(u^\delta - u)_t &= (P_{u^\delta} - P_u)u_t^\delta. \end{aligned} \tag{1.3.66}$$

We then calculate (again similar to Section 1.3.3)

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u^\delta) - \mathcal{N}(u)) \cdot \nabla^{k-2}(u^\delta - u)_t \, dx \qquad (1.3.67)\\
&\quad = \int_{\mathbb{R}^n} (P_u - P_{u^\delta}) \nabla^{k-2}[\mathcal{N}(u^\delta)] \cdot \nabla^{k-2}(u^\delta - u)_t \, dx \\
&\qquad + \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{l_1}[(P_{u^\delta} - P_u)] \star \nabla^{l_2}\mathcal{N}(u^\delta) \cdot \nabla^{k-2}(u^\delta - u)_t \, dx \\
&\qquad + \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{l_1}(I - P_u) \star \nabla^{l_2}[\mathcal{N}(u^\delta) - \mathcal{N}(u)] \cdot \nabla^{k-2}(u^\delta - u)_t \, dx \\
&\qquad + \int_{\mathbb{R}^n} \nabla^{k-2}[\mathcal{N}(u^\delta) - \mathcal{N}(u)] \cdot (I - P_u) \nabla^{k-2}(u^\delta - u)_t \, dx.
\end{aligned}$$

Using integration by parts and (1.3.66), the last term is rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla^{k-2}[\mathcal{N}(u^\delta) - \mathcal{N}(u)] \cdot (I - P_u) \nabla^{k-2}(u^\delta - u)_t \, dx \qquad (1.3.68)\\
&\quad = \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}[\mathcal{N}(u^\delta) - \mathcal{N}(u)] \cdot \nabla(\nabla^{l_1}(I - P_u) \star \nabla^{l_2}(u^\delta - u)_t) \, dx \\
&\qquad - \sum_{\substack{l_1+l_2=k-1 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}[\mathcal{N}(u^\delta) - \mathcal{N}(u)] \cdot \nabla^{l_1}[(P_{u^\delta} - P_u)] \star \nabla^{l_2} u_t^\delta \, dx \\
&\qquad - \int_{\mathbb{R}^n} \nabla^{k-3}[\mathcal{N}(u^\delta) - \mathcal{N}(u)] \cdot (P_{u^\delta} - P_u) \nabla^{k-1} u_t^\delta \, dx,
\end{aligned}$$

which is well defined by the higher regularity of u^δ. Technically this has to be established by difference quotients as in Appendix 1.A.3, however we omit the details here. The advantage of estimating $u^\delta - u$ is that the *bad terms* (with respect to the regularity of u)

$$\left\| \nabla^{k-2} \mathcal{N}(u^\delta) \right\|_{L^2} \quad \text{and} \quad \left\| \nabla^{k-1} u_t^\delta \right\|_{L^2} \qquad (1.3.69)$$

will be bounded by the regularized initial data from Lemma 1.A.6. Their norm will grow as $\delta \to 0^+$ in a controlled way. Moreover, when estimating (1.3.67) and (1.3.68), these bad terms only appear in the products

$$\begin{aligned}
&\left\| u^\delta - u \right\|_{L^\infty} \left\| \nabla^{k-2} \mathcal{N}(u^\delta) \right\|_{L^2} \left\| \nabla^{k-2}(u^\delta - u)_t \right\|_{L^2}, \\
&\left\| u^\delta - u \right\|_{L^\infty} \left\| \nabla^{k-3}(\mathcal{N}(u^\delta) - \mathcal{N}(u)) \right\|_{L^2} \left\| \nabla^{k-1} u_t^\delta \right\|_{L^2}.
\end{aligned}$$

Here the decay of $\left\| u^\delta - u \right\|_{L^\infty}$ as $\delta \to 0^+$ will compensate the growth in (1.3.69). We now carry out the details in several steps.

▷ *Step 1.* Since $T_0 < T_m$, we have the bound

$$\sup_{t\in[0,T_0]} (\|\nabla u(t)\|_{H^{k-1}} + \|u_t(t)\|_{H^{k-2}}) =: \overline{C} < \infty.$$

Lemma 1.A.6 allows us to fix a parameter $\delta_1' \in (0, \delta^*]$ depending on (u_0, u_1) such that

$$\left\|(\nabla u_0^\delta, u_1^\delta)\right\|_{H^{k-1}\times H^{k-2}} \leq 3\overline{C}/2 \tag{1.3.70}$$

for all $\delta \in (0, \delta_1']$. We let $\delta \in (0, \delta_1']$ and also $R \leq \overline{C}/2$ in (1.3.65). Hence

$$\|(\nabla v_0, v_1)\|_{H^{k-1}\times H^{k-2}} \leq \|(\nabla u_0, u_1)\|_{H^{k-1}\times H^{k-2}} + R \leq 3\overline{C}/2, \tag{1.3.71}$$

$$\left\|(\nabla v_0^\delta, v_1^\delta)\right\|_{H^{k-1}\times H^{k-2}} \leq \left\|(\nabla u_0^\delta, u_1^\delta)\right\|_{H^{k-1}\times H^{k-2}} + R \leq 2\overline{C}. \tag{1.3.72}$$

We define a time $\tilde{T}_0 > 0$ as in Lemma 1.3.6, replacing $\alpha(0)$ there by a multiple of $\overline{C}$. We then combine the uniform a priori bound (1.3.53) for the approximate solution to the ε–problem for v on $[0, \tilde{T}_0]$ with (1.3.71). Likewise one treats u^δ and v^δ using (1.3.70) and (1.3.72), respectively.

Following the existence proof in Section 1.3.2, we then see that the solutions $u_{[0,\tilde{T}_0]}$, $v_{[0,\tilde{T}_0]}$, $u^\delta_{[0,\tilde{T}_0]}$, and $v^\delta_{[0,\tilde{T}_0]}$ exist on $[0, \tilde{T}_0]$. Proceeding as in Section 1.3.2, we further obtain a constant $\tilde{C} = \tilde{C}(N, k, \tilde{T}_0) > 0$ such that

$$\|\nabla u\|^2_{H^{k-1}} + \|u_t\|^2_{H^{k-2}} \leq \tilde{C}(\|\nabla u_0\|^2_{H^{k-1}} + \|u_1\|^2_{H^{k-2}}), \tag{1.3.73}$$

$$\|\nabla v\|^2_{H^{k-1}} + \|v_t\|^2_{H^{k-2}} \leq \tilde{C}(\|\nabla v_0\|^2_{H^{k-1}} + \|v_1\|^2_{H^{k-2}}), \tag{1.3.74}$$

on $[0, \tilde{T}_0]$. Analogously, u^δ and v^δ satisfy the estimates (1.3.73) respectively (1.3.74) with the same constant $\tilde{C} > 0$ independent of $\delta \in (0, \delta^*]$. Further from Lemma 1.3.8 in the previous section combined with (1.3.73), (1.3.74), we have

$$\|u - v\|^2_{H^{k-1}} + \|u_t - v_t\|^2_{H^{k-3}} \leq C(\|u_0 - v_0\|^2_{H^{k-1}} + \|u_1 - v_1\|^2_{H^{k-3}}). \tag{1.3.75}$$

on $[0, \tilde{T}_0]$, where $C = C(N, k, \tilde{T}_0, \overline{C}) > 0$. Analogously, $u - u^\delta$, $v - v^\delta$ and $u^\delta - v^\delta$ fufill (1.3.75) with the same constant $C > 0$. For the regularized data we can replace here k by $k+1$, deriving

$$\left\|\nabla u^\delta\right\|^2_{H^k} + \left\|u_t^\delta\right\|^2_{H^{k-1}} \leq C(\left\|\nabla u_0^\delta\right\|^2_{H^k} + \left\|u_1^\delta\right\|^2_{H^{k-1}}), \tag{1.3.76}$$

$$\left\|\nabla v^\delta\right\|^2_{H^k} + \left\|v_t^\delta\right\|^2_{H^{k-1}} \leq C(\left\|\nabla v_0^\delta\right\|^2_{H^k} + \left\|v_1^\delta\right\|^2_{H^{k-1}}),$$

which follows by (1.3.58) mentioned in the previous section and the fact that $\|\nabla u\|_{L^\infty}$, $\|u_t\|_{L^\infty}$ are bounded by the norm of H^{k-1}, respectively H^{k-2}.

▷ *Step 2.* Estimating (1.3.67) and (1.3.68) as in Section 1.3.3, we derive

$$\begin{aligned}\frac{d}{dt}\left(\left\|u - u^\delta\right\|^2_{H^k} + \left\|u_t - u_t^\delta\right\|^2_{H^{k-2}} \right) &\leq C \left\|u - u^\delta\right\|_{L^\infty} \left\|\nabla^{k-2}\mathcal{N}(u^\delta)\right\|_{L^2} \left\|\nabla^{k-2}(u_t - u_t^\delta)\right\|_{L^2} \\ &\quad + C \left\|u - u^\delta\right\|_{L^\infty} \left\|\nabla^{k-3}(\mathcal{N}(u^\delta) - \mathcal{N}(u))\right\|_{L^2} \left\|\nabla^{k-1}u_t^\delta\right\|_{L^2} \\ &\quad + C(\left\|u - u^\delta\right\|^2_{H^k} + \left\|u_t - u_t^\delta\right\|^2_{H^{k-2}})\end{aligned}$$

for some $C = C(N, \overline{C}, \tilde{C}) > 0$. The nonlinearities are treated as in Sections 1.3.2 and 1.3.3. Using also (1.3.73), (1.3.75) and (1.3.76), we then conclude

$$\begin{aligned}
&\frac{d}{dt}\big(\|u - u^\delta\|_{H^k}^2 + \|u_t - u_t^\delta\|_{H^{k-2}}^2 \big) \\
&\quad \leq C \|u - u^\delta\|_{H^{k-1}} (1 + \|\nabla u^\delta\|_{H^k} + \|u_t^\delta\|_{H^{k-2}})(\|u_t\|_{H^{k-2}} + \|u_t^\delta\|_{H^{k-2}}) \\
&\qquad + C \|u - u^\delta\|_{H^{k-1}} (1 + \|\nabla u\|_{H^{k-1}} + \|\nabla u^\delta\|_{H^{k-1}} + \|u_t\|_{H^{k-3}} + \|u_t^\delta\|_{H^{k-3}}) \|u_t^\delta\|_{H^{k-1}} \\
&\qquad + C(\|u - u^\delta\|_{H^k}^2 + \|u_t - u_t^\delta\|_{H^{k-2}}^2) \\
&\quad \leq C(\|u_0 - u_0^\delta\|_{H^{k-1}} + \|u_t - u_t^\delta\|_{H^{k-3}})(1 + \|\nabla u_0^\delta\|_{H^k} + \|u_1^\delta\|_{H^{k-1}}) \\
&\qquad + C(\|u - u^\delta\|_{H^k}^2 + \|u_t - u_t^\delta\|_{H^{k-2}}^2)
\end{aligned}$$

on $[0, \tilde{T}_0]$. Gronwall's inequality and Lemma 1.A.6 thus yield

$$\begin{aligned}
&\sup_{t \in [0, \tilde{T}_0]} \big(\|u - u^\delta\|_{H^k}^2 + \|u_t - u_t^\delta\|_{H^{k-2}}^2 \big) \\
&\quad \leq \frac{C\tilde{T}_0}{\sqrt{\delta}} \big(\|u_0 - u_0^\delta\|_{H^{k-1}} + \|u_1 - u_1^\delta\|_{H^{k-3}} \big) + C\big(\|u_0 - u_0^\delta\|_{H^k}^2 + \|u_1 - u_1^\delta\|_{H^{k-2}}^2 \big) = o(1)
\end{aligned}$$

as $\delta \to 0^+$. In view of our a priori bounds, we can estimate $v - v^\delta$ in the same way. Here we have to split the initial values, obtaining

$$\begin{aligned}
&\sup_{t \in [0, \tilde{T}_0]} \big(\|v - v^\delta\|_{H^k}^2 + \|v_t - v_t^\delta\|_{H^{k-2}}^2 \big) \\
&\qquad \leq \frac{C\tilde{T}_0}{\sqrt{\delta}} (\|v_0 - v_0^\delta\|_{H^{k-1}} + \|v_1 - v_1^\delta\|_{H^{k-3}}) + C(\|v_0 - v_0^\delta\|_{H^k}^2 + \|v_1 - v_1^\delta\|_{H^{k-2}}^2) \\
&\qquad \leq \frac{C\tilde{T}_0}{\sqrt{\delta}} (\|u_0 - u_0^\delta\|_{H^{k-1}} + \|u_1 - u_1^\delta\|_{H^{k-3}}) + C(\|u_0 - u_0^\delta\|_{H^k}^2 + \|u_1 - u_1^\delta\|_{H^{k-2}}^2) \\
&\qquad\quad + \frac{C\tilde{T}_0}{\sqrt{\delta}} (\|u_0 - v_0\|_{H^{k-1}} + \|u_1 - v_1\|_{H^{k-3}} + \|u_0^\delta - v_0^\delta\|_{H^{k-1}} + \|u_1^\delta - v_1^\delta\|_{H^{k-3}}) \\
&\qquad\quad + C(\|u_0 - v_0\|_{H^k}^2 + \|u_1 - v_1\|_{H^{k-2}}^2 + \|u_0^\delta - v_0^\delta\|_{H^k}^2 + \|u_1^\delta - v_1^\delta\|_{H^{k-2}}^2).
\end{aligned}$$

Lemma 1.A.6 now implies that

$$\begin{aligned}
&\sup_{t \in [0, \tilde{T}_0]} \big(\|v - v^\delta\|_{H^k}^2 + \|v_t - v_t^\delta\|_{H^{k-2}}^2 \big) \\
&\qquad \leq \frac{C\tilde{T}_0}{\sqrt{\delta}} (\|u_0 - u_0^\delta\|_{H^{k-1}} + \|u_1 - u_1^\delta\|_{H^{k-3}}) + C(\|u_0 - u_0^\delta\|_{H^k}^2 + \|u_1 - u_1^\delta\|_{H^{k-2}}^2) \\
&\qquad\quad + \frac{C\tilde{T}_0}{\sqrt{\delta}} R + CR^2.
\end{aligned}$$

On the regularized level, we use the coarse estimate

$$\begin{aligned}\sup_{t\in[0,\tilde T_0]}\big(\left\|u^\delta - v^\delta\right\|_{H^k}^2 + \left\|u_t^\delta - v_t^\delta\right\|_{H^{k-2}}^2\big) &\le \frac{C}{\sqrt{\delta}}\tilde T_0(\left\|v_0^\delta - u_0^\delta\right\|_{H^k} + \left\|v_1^\delta - u_1^\delta\right\|_{H^{k-2}})\\ &\quad + C(\left\|u_0^\delta - v_0^\delta\right\|_{H^k}^2 + \left\|u_1^\delta - v_1^\delta\right\|_{H^{k-2}}^2)\\ &\le \frac{C\tilde T_0}{\sqrt{\delta}}R + CR^2.\end{aligned}$$

Since $u - v = u - u^\delta + u^\delta - v^\delta + v^\delta - v$, it follows

$$\begin{aligned}\sup_{t\in[0,\tilde T_0]}\big(\left\|u - v\right\|_{H^k}^2 + \left\|u_t - v_t\right\|_{H^{k-2}}^2\big) &\le \frac{C\tilde T_0}{\sqrt{\delta}}(\left\|u_0 - u_0^\delta\right\|_{H^{k-1}} + \left\|u_1 - u_1^\delta\right\|_{H^{k-3}})\\ &\quad + C(\left\|u_0 - u_0^\delta\right\|_{H^k}^2 + \left\|u_1 - u_1^\delta\right\|_{H^{k-2}}^2)\\ &\quad + \frac{C\tilde T_0}{\sqrt{\delta}}R + CR^2.\end{aligned} \tag{1.3.77}$$

Now take $\eta \in (0, \overline{C}/2]$ and $r_1 \in (0, \eta]$. We first fix $\delta = \delta_1 = \delta_1(r_1) \in (0, \delta_1']$ and then choose $R_1 = R_1(\delta_1) \in (0, \overline{C}/2]$ such that for all $R \in (0, R_1]$ we have

$$\sup_{t\in[0,\tilde T_0]}\big(\left\|u - v\right\|_{H^k}^2 + \left\|u_t - v_t\right\|_{H^{k-2}}^2\big) \le r_1 \le \eta. \tag{1.3.78}$$

In the above reasoning we now replace (u_0, u_1) with corresponding solution u by data $(\hat u_0, \hat u_1)$ with solution $\hat u$ that satisfy the same assumptions as (v_0, v_1). The function $\hat u$ thus fulfills the same a priori estimates as v and also (1.3.78). Moreover, we assume that

$$\|(\hat u_0, \hat u_1) - (v_0, v_1)\|_{H^k\times H^{k-2}} \le \hat R \tag{1.3.79}$$

for some radius $\hat R > 0$. We can then repeat the above arguments replacing u by $\hat u$. The resulting regularization parameter $\hat\delta_1$ depends on $\hat u$, and thus also the upper bound $\hat R_1 = \hat R_1(\delta_1)$ for the radii in (1.3.79). For given $0 \le \hat r_1 \le \hat\eta$, we infer

$$\sup_{t\in[0,\tilde T_0]}\big(\left\|\hat u - v\right\|_{H^k}^2 + \left\|\hat u_t - v_t\right\|_{H^{k-2}}^2\big) \le \hat r_1 \le \hat\eta \tag{1.3.80}$$

provided that $0 < \hat R \le \hat R_1$ in (1.3.79).

▷ <u>*Step 3.*</u> In the case $\tilde T_0 \ge T_0$ the proof is complete. Otherwise we repeat the same argument starting from

$$(u_0^{(1)}, u_1^{(1)}) = (u(\tilde T_0), u_t(\tilde T_0)) \qquad \text{and} \qquad (v_0^{(1)}, v_1^{(1)}) = (v(\tilde T_0), v_t(\tilde T_0)).$$

Observe that (1.3.78) yields

$$\left\|(\nabla v_0^{(1)}, v_1^{(1)})\right\|_{H^{k-1}\times H^{k-2}} \le \eta + \left\|(\nabla u_0^{(1)}, u_1^{(1)})\right\|_{H^{k-1}\times H^{k-2}} \le 3\overline{C}/2.$$

For a sufficiently small $\delta_2' \in (0,\delta^*]$ and all $\delta \in (0,\delta_2']$, we derive

$$\Big\|(\nabla (u_0^{(1)})^\delta, (u_1^{(1)})^\delta)\Big\|_{H^{k-1}\times H^{k-2}}, \Big\|(\nabla (v_0^{(1)})^\delta, (v_1^{(1)})^\delta)\Big\|_{H^{k-1}\times H^{k-2}} \leq 2\overline{C}$$

as in (1.3.70) and (1.3.72). Based on these bounds we can repeat the arguments of Steps 1 and 2 on the interval $[\tilde{T}_0, \min\{2\tilde{T}_0, T_0\}] =: J_1$. However we have to replace the bound (1.3.65) involving R by (1.3.78) which yields

$$\|(u_0^{(1)}, u_1^{(1)}) - (v_0^{(1)}, v_1^{(1)})\|_{H^k\times H^{k-2}} \leq r_1.$$

Let $r_2 \in (0,\eta]$. Lemma 1.A.6 allows us to fix a parameter $\delta = \delta_2 = \delta_2(r_2) \in (0,\delta_2']$ such that

$$\begin{aligned} &\frac{C\tilde{T}_0}{\sqrt{\delta}}\big(\|u_0^{(1)} - (u_0^{(1)})^\delta\|_{H^{k-1}} + \|u_0^{(1)} - (u_0^{(1)})^\delta\|_{H^{k-3}}\big) \\ &\qquad + C\big(\|u_0^{(1)} - (u_0^{(1)})^\delta\|_{H^{k}} + \|u_0^{(1)} - (u_0^{(1)})^\delta\|_{H^{k-2}}\big) \leq r_2/4. \end{aligned}$$

As in (1.3.77) we then obtain

$$\sup_{t\in J_1}\big(\,\|u-v\|_{H^k}^2 + \|u_t - v_t\|_{H^{k-2}}^2\,\big) \leq r_2/4 + r_2/4 + \frac{C\tilde{T}_0}{\sqrt{\delta_2}} r_1 + C r_1^2 \leq r_2 \leq \eta$$

if we choose r_1, and hence R, small enough.
Again we can argue in the same way for $\hat{u}$ instead of u, replacing r_i, δ_i and R by $\hat{r}_i$, $\hat{\delta}_i$ and $\hat{R}$. For given $0 < \hat{r}_2 \leq \hat{\eta}$, we thus obtain

$$\sup_{t\in J_1}\big(\,\|\hat{u}-v\|_{H^k}^2 + \|\hat{u}_t - v_t\|_{H^{k-2}}^2\,\big) \leq \hat{r}_2/4 + \hat{r}_2/4 + \frac{C\tilde{T}_0}{\sqrt{\hat{\delta}_2}} \hat{r}_1 + C \hat{r}_1^2 \leq \hat{r}_2 \leq \hat{\eta}$$

if $\hat{r}_1$ and $\hat{R}$ are small enough.

▷ *Step 4.*

The previous step can be repeated m times until $m\tilde{T}_0 \geq T_0$. We set $R_0 = R(\overline{C}/2)$ (with $\eta = \overline{C}/2$) and use the resulting radius $\hat{R} = \hat{R}(\hat{\eta})$ for the continuity at $\hat{u}$, concluding the proof of the continuous dependence in Theorem 1.1.2.

1.4 Global regularity for subcritical biharmonic wave maps in low dimension

This section is taken (with modifications) from [44], which appeared as a prepublication.

We start the section by considering the case where N is parallelizable, for which we can work with normal vectors.

Since for solutions u of (1.1.1), resp. the Cauchy problem for (1.1.2), the term $\partial_t^2 u + \Delta^2 u$ is a section over the normal bundle of $u^*(TN)$, we let codim$(N) = L - l$ for $l \in \mathbb{N},\ l \leq L$ and first assume the normalbundle $T^\perp N$ of $N \subset \mathbb{R}^L$ is parallelizable. This means there exists a frame of (smooth) orthogonal vectorfields $\{\nu_1(p), \dots, \nu_{L-l}(p)\} \subset \mathbb{R}^L,\ p \in N$ with $\nu_i(p) \perp T_pN$ for every $p \in N$.

In this case, for any local solution u, we have an explicit representation for the nonlinearity in terms of $\nu_i(u)$.

$$\partial_t^2 u + \Delta^2 u =: \sum_{i=1}^{L-l} G^i(u)\nu_i(u) =: G^i(u)\nu_i(u), \tag{1.4.1}$$

where $G_i(u) = \langle \partial_t^2 u + \Delta^2 u, \nu_i(u)\rangle$. We thus calculate

$$\begin{aligned}
&\langle \partial_t^2 u, \nu_i(u)\rangle = -\langle u_t, d\nu_i(u)u_t\rangle,\\
&\langle \Delta^2 u, \nu_i(u)\rangle = -3\langle \nabla\Delta u, d\nu_i(u)\nabla u\rangle - \langle \nabla u, d\nu_i(u)\nabla\Delta u\rangle\\
&\quad - \langle \nabla u, d^3\nu_i(u)(\nabla u)^3 + 2d^2\nu_i(u)(\nabla u, \nabla^2 u) + d^2\nu_i(u)(\nabla u, \Delta u)\rangle\\
&\quad - 2\langle \nabla^2 u, d^2\nu_i(u)(\nabla u)^2 + d\nu_i(\nabla^2 u)\rangle - \langle \Delta u, d^2\nu_i(u)(\nabla u)^2 + d\nu_i(\Delta u)\rangle,
\end{aligned}$$

where we denote by $d^k\nu_i$ the kth order differential of ν_i on N and write $(\nabla u)^2,\ (\nabla u)^3$ for products of first order derivatives of u with eiter two or three factors, respectively. The precise product, e.g. $\partial_{x_j} u \cdot \partial^{x_j} u$ or $\partial_{x_i} u \cdot \partial^{x_j} u \cdot \partial_{x_j} u$ will become clear in the terms of the expansion. The result in Theorem 1.1.3 is known for $N = \mathbb{S}^{L-1}$ and $n \leq 2$ thanks to [13].

1.4.1 The case $n = 2$

We apply $\Delta = \partial_i\partial^i$ on both sides of (1.4.1). Then, testing the differentiated equation by Δu_t, we infer

$$\frac{d}{2dt}\int_{\mathbb{R}^n} (|\Delta u_t|^2 + |\Delta^2 u|^2)dx = \int_{\mathbb{R}^n} \Delta(G^i(u)\nu_i(u))\Delta u_t dx. \tag{1.4.2}$$

Since $G^i(u)$ contains derivatives of order three, we can not proceed by the Hölder inequality. Instead, we follow [13], where the authors showed that the highest order derivative cancel in the case $N = \mathbb{S}^{L-1},\ \nu(u) = u$. Since

$$\Delta(G^i(u)\nu_i(u))\Delta u_t = \Delta(G^i(u))\nu_i(u)\Delta u_t + 2\nabla(G^i(u)) \cdot \nabla(\nu_i(u))\Delta u_t + G^i(u)\Delta\nu_i(u)\Delta u_t,$$

and

$$0 = \Delta(\nu_i(u)u_t) = 2d\nu_i(u)(\nabla u) \cdot \nabla u_t + \nu_i(u)\Delta u_t + d^2\nu_i(u)(\nabla u)^2 u_t + d\nu_i(u)(\Delta u)u_t,$$

it follows

$$\begin{aligned}
\Delta(G^i(u)\nu_i(u))\Delta u_t = &- \Delta G^i(u)\left(2d\nu_i(u)(\nabla u) \cdot \nabla u_t + d^2\nu_i(u)(\nabla u)^2 u_t + d\nu_i(u)(\Delta u)u_t\right)\\
&+ 2\nabla G^i(u) \cdot d\nu_i(u)(\nabla u)\Delta u_t\\
&+ G^i(u)\left(d^2\nu_i(u)(\nabla u)^2 + d\nu_i(u)\Delta u\right)\Delta u_t.
\end{aligned}$$

Hence we observe, by integration by parts for the first summand,

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta(G^i(u)\nu_i(u))\Delta u_t\, dx = &\int_{\mathbb{R}^n} \nabla G^i(u) \cdot [3d^2\nu_i(u)(\nabla u)^2\nabla u_t + 3d\nu_i(u)(\Delta u)\nabla u_t]dx\\ &+ \int_{\mathbb{R}^n} \nabla G^i(u) \cdot [4d\nu_i(u)(\nabla u)\Delta u_t + d^3\nu_i(u)(\nabla u)^3 u_t]dx\\ &+ \int_{\mathbb{R}^n} \nabla G^i(u) \cdot [3d^2\nu_i(u)(\Delta u, \nabla u)u_t + d\nu_i(u)(\nabla\Delta u)u_t]dx\\ &+ \int_{\mathbb{R}^n} G^i(u)(d^2\nu_i(u)(\nabla u)^2 u_t + d\nu_i(u)(\Delta u))\Delta u_t\, dx.\end{aligned}$$

Instead of deducing bounds for this terms that depend on the normal frame $\{\nu_1, \dots \nu_{L-l}\}$, we turn to the general case and use the normal projector $I - P_u : \mathbb{R}^L \to (T_uN)^\perp$ along the map $u : \mathbb{R}^n \times [0,T) \to N$ in order to represent the nonlinearity in (1.4.1) as

$$\partial_t^2 u + \Delta^2 u = (I - P_u)(\partial_t^2 u + \Delta^2 u). \tag{1.4.3}$$

Here, we proceed similarly, ie. we use

$$\Delta((I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t = \Delta((I - P_u)^2(\partial_t^2 u + \Delta^2 u))\Delta u_t, \tag{1.4.4}$$

and hence

$$\begin{aligned}\Delta((I - P_u)^2(\partial_t^2 u + \Delta^2 u))\Delta u_t =&\Delta[(I - P_u)]((I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t\\ &+ 2\nabla(I - P_u) \cdot \nabla((I - P_u)(\partial_t^2 + \Delta^2 u))\Delta u_t\\ &+ (\Delta[(I - P_u)(\partial_t^2 u + \Delta^2 u)])(I - P_u)\Delta u_t.\end{aligned}$$

In order to treat the last summand, we expand

$$0 = \Delta((I - P_u)u_t) = (I - P_u)\Delta u_t - d^2P_u((\nabla u)^2, u_t) - dP_u(\Delta u, u_t) - 2dP_u(\nabla u, \nabla u_t).$$

Hence, as before, integration by parts yields

$$\begin{aligned}&\int_{\mathbb{R}^n} \Delta((I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t\\ &\quad= - \int_{\mathbb{R}^n} d^2P_u((\nabla u)^2, (I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t\, dx\\ &\quad\quad - \int_{\mathbb{R}^n} dP_u(\Delta u, (I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t\, dx\\ &\quad\quad - 2\int_{\mathbb{R}^n} dP_u(\nabla u, \nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)])\Delta u_t\, dx\\ &\quad\quad - \int_{\mathbb{R}^n} \nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)] \cdot \nabla[dP_u(\Delta u, u_t) + 2dP_u(\nabla u, \nabla u_t) + d^2P_u((\nabla u)^2, u_t)]\, dx.\end{aligned}$$

We first note the pointwise bounds

$$|(I-P_u)(\partial_t^2 u+\Delta^2 u)| \lesssim \quad |u_t|^2+|\nabla^2 u|^2+|\nabla^2 u||\nabla u|^2+|\nabla^3 u||\nabla u|+|\nabla u|^4 \tag{1.4.5}$$

$$\begin{aligned}|\nabla[(I-P_u)(\partial_t^2 u+\Delta^2 u)]| \lesssim\ & |\nabla u_t||u_t|+|\nabla u||u_t|^2+|\Delta^2 u||\nabla u| \\ &+|\nabla^3 u|(|\nabla^2 u|+|\nabla u|^2)+|\nabla u||\nabla^2 u|^2+|\nabla u|^3|\nabla^2 u|+|\nabla u|^5,\end{aligned} \tag{1.4.6}$$

where the constants only depend on the supremum norm

$$\|dP\|_{C_b^3}=\|dP\|_{C_b(N)}+\left\|d^2P\right\|_{C_b(N)}+\left\|d^3P\right\|_{C_b(N)}+\left\|d^4P\right\|_{C_b(N)}.$$

We now estimate, using (1.4.5) and (1.4.6),

$$\begin{aligned}&\|d^2P_u((\nabla u)^2,(I-P_u)(\partial_t^2 u+\Delta^2 u))\Delta u_t\|_{L^1}\\ &\quad\lesssim \|\Delta u_t\|_{L^2}\|\nabla u\|_{L^\infty}^2\Big[\|u_t\|_{L^\infty}\|u_t\|_{L^2}+\left\|\nabla^2 u\right\|_{L^\infty}\|\Delta u\|_{L^2}+\|\nabla u\|_{L^\infty}^2\|\Delta u\|_{L^2}\\ &\quad+\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla u\|_{L^4}^2\|\nabla u\|_{L^\infty}^2\Big],\end{aligned}$$

$$\begin{aligned}&\|dP_u(\Delta u,(I-P_u)(\partial_t^2 u+\Delta^2 u))\Delta u_t\|_{L^1}\\ &\quad\lesssim \|\Delta u_t\|_{L^2}\|\Delta u\|_{L^\infty}\Big[\|u_t\|_{L^\infty}\|u_t\|_{L^2}+\left\|\nabla^2 u\right\|_{L^\infty}\|\Delta u\|_{L^2}+\|\nabla u\|_{L^\infty}^2\|\Delta u\|_{L^2}\\ &\quad+\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla u\|_{L^4}^2\|\nabla u\|_{L^\infty}^2\Big]\\ &\quad=\|\Delta u_t\|_{L^2}\|\Delta u\|_{L^\infty}\left[\,\|u_t\|_{L^\infty}\|u_t\|_{L^2}+\left\|\nabla^2 u\right\|_{L^\infty}\|\Delta u\|_{L^2}\,\right]\\ &\qquad+h(t)^2\|\Delta u_t\|_{L^2}\|\Delta u\|_{L^\infty}\left[\,\|\Delta u\|_{L^2}+\|\nabla u\|_{L^4}^2\,\right]\\ &\qquad+h(t)\|\Delta u_t\|_{L^2}\|\Delta u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}.\end{aligned}$$

where we set $h(t):=\|\nabla u(t)\|_{L^\infty}$. We note further that the equality is up to the constant from the estimate. We hence proceed by estimating

$$\begin{aligned}&\|dP_u(\nabla u,\nabla[(I-P_u)(\partial_t^2 u+\Delta^2 u)])\Delta u_t\|_{L^1}\\ &\quad\lesssim \|\Delta u_t\|_{L^2}\|\nabla u\|_{L^\infty}\Big[\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}\|u_t\|_{L^\infty}^2+\left\|\Delta^2 u\right\|_{L^2}\|\nabla u\|_{L^\infty}\\ &\quad+\|\nabla\Delta u\|_{L^2}(\left\|\nabla^2 u\right\|_{L^\infty}+\|\nabla u\|_{L^\infty}^2)+\|\nabla u\|_{L^\infty}\|\Delta u\|_{L^2}\left\|\nabla^2 u\right\|_{L^\infty}+\|\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}^3\\ &\quad+\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}^4\Big].\end{aligned}$$

The latter upper bound equals the sum of

$$h(t)\|\Delta u_t\|_{L^2}\left[\,\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}\|u_t\|_{L^\infty}^2+\|\nabla\Delta u\|_{L^2}\left\|\nabla^2 u\right\|_{L^\infty}+\|\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}^3\,\right],$$

and

$$h^2(t)\,\|\Delta u_t\|_{L^2}\,\big[\,\|\Delta^2 u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}+\|\Delta u\|_{L^2}\|\nabla^2 u\|_{L^\infty}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}^3\,\big].$$

We calculate

$$\begin{aligned}
&\nabla[dP_u(\Delta u,u_t)+2dP_u(\nabla u,\nabla u_t)+d^2P_u((\nabla u)^2,u_t)]\\
&=d^2P_u(\nabla u,\Delta u,u_t)+dP_u(\nabla\Delta u,u_t)+dP_u(\Delta u,\nabla u_t)+2d^2P_u((\nabla u)^2,\nabla u_t)\\
&\quad+2dP_u(\nabla^2 u,\nabla u_t)+2dP_u(\nabla u,\nabla^2 u_t)+d^3P_u((\nabla u)^3,u_t)\\
&\quad+2d^2P_u(\nabla u,\nabla^2 u,u_t)+d^2P_u((\nabla u)^2,\nabla u_t),
\end{aligned}$$

and hence

$$\begin{aligned}
&\|\nabla[dP_u(\Delta u,u_t)+2dP_u(\nabla u,\nabla u_t)+d^2P_u((\nabla u)^2,u_t)]\cdot\nabla[(I-P_u)(\partial_t^2 u+\Delta^2 u)]\|_{L^1}\\
&\lesssim\big(\,\|\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}\|u_t\|_{L^\infty}+\|\nabla\Delta u\|_{L^2}\|u_t\|_{L^\infty}+(\|\Delta u\|_{L^\infty}+\|\nabla u\|_{L^\infty}^2)\,\|\nabla u_t\|_{L^2}\\
&\quad+\|\Delta u_t\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla u\|_{L^\infty}^3\|u_t\|_{L^2}\,\big)\big[\,\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}\|u_t\|_{L^\infty}^2\\
&\quad+\|\Delta^2 u\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla\Delta u\|_{L^2}\,(\|\nabla^2 u\|_{L^\infty}+\|\nabla u\|_{L^\infty}^2)+\|\nabla u\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla^2 u\|_{L^\infty}\,.\\
&\quad+\|\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}^3+\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}^4\,\big].
\end{aligned}$$

We now collect all terms which are at least quadratic, linear or constant in $h(t)$, i.e. the latter bound equals

$$J_1(u)+h(t)J_2(u)+h(t)J_3(u)+h^2(t)J_4(u),$$

where

$$\begin{aligned}
J_1(u)&=(\|\nabla\Delta u\|_{L^2}\|u_t\|_{L^\infty}+\|\Delta u\|_{L^\infty}\|\nabla u_t\|_{L^2})\big[\,\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}\|u_t\|_{L^\infty}^2\\
&\quad+\|\nabla\Delta u\|_{L^2}\|\nabla^2 u\|_{L^\infty}+\|\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}^3\,\big],\\
J_2(u)&=(\|\nabla\Delta u\|_{L^2}\|u_t\|_{L^\infty}+\|\Delta u\|_{L^\infty}\|\nabla u_t\|_{L^2})\big[\,\|\Delta^2 u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\\
&\quad+\|\Delta u\|_{L^2}\|\nabla^2 u\|_{L^\infty}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}^3\,\big],\\
J_3(u)&=(\|\Delta u\|_{L^2}\|u_t\|_{L^\infty}+\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^\infty}+\|\Delta u_t\|_{L^2}+\|\nabla u\|_{L^\infty}^2\|u_t\|_{L^2})\big[\,\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\,,\\
&\quad+\|\nabla u\|_{L^2}\|u_t\|_{L^\infty}^2+\|\nabla\Delta u\|_{L^2}\|\nabla^2 u\|_{L^\infty}+\|\Delta u\|_{L^2}\|\nabla u\|_{L^\infty}^3\,\big],\\
J_4(u)&=(\|\Delta u\|_{L^2}\|u_t\|_{L^\infty}+\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^\infty}+\|\Delta u_t\|_{L^2}+\|\nabla u\|_{L^\infty}^2\|u_t\|_{L^2})\big[\,\|\Delta^2 u\|_{L^2}\\
&\quad+\|\nabla u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}+\|\Delta u\|_{L^2}\|\nabla^2 u\|_{L^\infty}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}^3\,\big].
\end{aligned}$$

We note that the energy is conserved, ie. for $t\in[0,T)$

$$2E(u(t))=\|\Delta u(t)\|_{L^2}^2+\|\partial_t u(t)\|_{L^2}^2=\|\Delta u_0\|_{L^2}^2+\|u_1\|_{L^2}^2=2E(u_0,u_1),\tag{1.4.7}$$

and further, this implies the bounds

$$\sup_{t\in[0,T)} \|\nabla u(t)\|_{L^2} \lesssim \sqrt{1+T}(\sqrt{E(u_0,u_1)} + \|\nabla u_0\|_{L^2}), \quad \text{and} \tag{1.4.8}$$

$$\sup_{t\in[0,T)} \|u(t)-u_0\|_{L^2} \lesssim T\sqrt{E(u_0,u_1)}. \tag{1.4.9}$$

We recall the following cases of Gagliardo-Nirenberg's interpolation for $n = 2$

$$\|\Delta u\|_{L^\infty} + \|\nabla\Delta u\|_{L^2} \lesssim \left\|\Delta^2 u\right\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}, \quad \|u_t\|_{L^\infty} \lesssim \|\Delta u_t\|_{L^2}^{\frac{1}{2}} \|u_t\|_{L^2}^{\frac{1}{2}}, \tag{1.4.10}$$

$$\|\nabla u\|_{L^\infty} \lesssim \left\|\Delta^2 u\right\|_{L^2}^{\frac{1}{3}} \|\nabla u\|_{L^2}^{\frac{2}{3}}, \quad \|\nabla u\|_{L^4} \lesssim \left\|\Delta^2 u\right\|_{L^2}^{\frac{1}{6}} \|\nabla u\|_{L^2}^{\frac{5}{6}}, \quad \text{and} \tag{1.4.11}$$

$$\|\nabla u_t\|_{L^4} \lesssim \|\Delta u_t\|_{L^2}^{\frac{3}{4}} \|u_t\|_{L^2}^{\frac{1}{4}}. \tag{1.4.12}$$

Setting

$$\mathcal{E}(u(t)) := \|\Delta u_t(t)\|_{L^2} + \left\|\Delta^2 u(t)\right\|_{L^2}, \; t \in [0,T),$$

by (1.4.10), (1.4.11) and the estimates above, there exists a constant $C(T) = C(N, u_0, u_1)(1 + T)^\alpha$ for some $\alpha > 0$, such that $C(N, u_0, u_1)$ only depends on the norm $\|dP\|_{C_b^3}$, the optimal Sobolev constant in Gagliardo-Nirenberg's interpolation and $E(u_0, u_1)$, $\|\nabla u_0\|_{L^2}$ and such that the following holds.

$$\begin{aligned} \frac{d}{dt}\mathcal{E}^2(u(t)) &\leq C(T)(1 + h(t) + h^2(t))(\mathcal{E}(t) + \mathcal{E}^2(t)) \\ &\leq C(T)(1 + h^2(t))(1 + \mathcal{E}^2(t)), \; t \in [0,T). \end{aligned} \tag{1.4.13}$$

Using the idea from [13], we now apply the sharp Sobolev inequality of Brezis-Gallouet-Wainger from [5], [7] in order to bound (we assume u is not a constant)

$$h(t) \leq \tilde{C}\,\|\nabla u(t)\|_{H^1}\left(1 + \log^{\frac{1}{2}}\left(1 + \frac{\|\nabla u(t)\|_{H^2}^2}{\|\nabla u(t)\|_{H^1}^2}\right)\right), \quad t \in [0,T). \tag{1.4.14}$$

Thus, using (1.4.10), (1.4.8) and (1.4.7),

$$h^2(t) \leq C(T)\left(1 + \log\left(1 + \mathcal{E}^2(t)\right)\right), \quad t \in [0,T), \tag{1.4.15}$$

and hence

$$\frac{d}{dt}(e + \mathcal{E}^2(u(t))) \leq C(T)\log\left(e + \mathcal{E}^2(t)\right)(e + \mathcal{E}^2(t)), \; t \in [0,T). \tag{1.4.16}$$

This suffices for a Gronwall-type inequality for $\log(e+\mathcal{E}^2(t))$ and hence by (1.4.7) and (1.4.10), (1.4.11) and (1.4.12), we have

$$\limsup_{t\to T}(\|u_t\|_{H^2}^2 + \|\nabla u\|_{H^3}^2) < \infty,$$

as long as $T < \infty$. Especially, we use that the above norm bounds $\|\nabla u\|_{L^\infty}, \|u_t\|_{L^\infty}$ on compact (time) intervals. By the blow up condition (1.1.4) of Theorem 1.1.2, the solution hence extends to a global solution with $u - u_0 \in C(\mathbb{R}, H^k) \cap C^1(\mathbb{R}, H^{k-2})$.

1.4.2 The case $n = 1$

Here, by Gagliardo-Nirenberg's estimate, we infer the bound

$$\|\nabla u\|_{L^\infty} \lesssim \left\|\nabla^2 u\right\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{1.4.17}$$

Hence, the a priori bound is derived similarly for $(\nabla u(t), u_t(t)) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. We note

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |\nabla u_t|^2 + |\nabla \Delta u|^2 \, dx = -\int_{\mathbb{R}} dP_u(\nabla u, (I - P_u)(\partial_t^2 u + \Delta^2 u)) \cdot \nabla u_t \\ - \int_{\mathbb{R}} (I - P_u)(\partial_t^2 u + \Delta^2 u) \cdot (d^2 P_u((\nabla u)^2, u_t) + dP_u(\nabla^2 u, u_t) + dP_u(\nabla u, \nabla u_t))dx.\end{aligned}$$

Thus we estimate, as before

$$\begin{aligned}&\|dP_u(\nabla u, (I - P_u)(\partial_t^2 u + \Delta^2 u))\nabla u_t\|_{L^1} \\ &\qquad \lesssim \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^\infty} \Big[\|u_t\|_{L^\infty} \|u_t\|_{L^2} + \left\|\nabla^2 u\right\|_{L^\infty} \left\|\nabla^2 u\right\|_{L^2} \\ &\qquad + \|\nabla u\|_{L^\infty}^2 \left\|\nabla^2 u\right\|_{L^2} + \left\|\nabla^3 u\right\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^\infty}^2 \Big], \text{ and}\end{aligned}$$

$$\begin{aligned}&\|(d^2 P_u((\nabla u)^2, u_t) + dP_u(\nabla^2 u, u_t) + dP_u(\nabla u, \nabla u_t))[(I - P_u)(\partial_t^2 u + \Delta^2 u)]\|_{L^1} \\ &\lesssim (\|u_t\|_{L^2} \|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^\infty}) \Big[\|u_t\|_{L^\infty} \|u_t\|_{L^2} \\ &+ \left\|\nabla^2 u\right\|_{L^\infty} \left\|\nabla^2 u\right\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \left\|\nabla^2 u\right\|_{L^2} + \left\|\nabla^3 u\right\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^\infty}^2 \Big].\end{aligned}$$

Hence from the interpolation estimates (1.4.17),

$$\left\|\nabla^2 u\right\|_{L^\infty} \lesssim \left\|\nabla^4 u\right\|_{L^2}^{\frac{1}{4}} \left\|\nabla^2 u\right\|_{L^2}^{\frac{3}{4}}, \quad \|u_t\|_{L^\infty} \lesssim \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|u_t\|_{L^2}^{\frac{1}{2}}, \tag{1.4.18}$$

and (1.4.7), (1.4.8), there holds (for $C(T) > 0$ as before)

$$\frac{d}{dt}(1 + \mathcal{E}(t)) \leq C(T)(1 + \mathcal{E}(t)), \; t \in [0, T) \tag{1.4.19}$$

which suffices to use a Gronwall argument in order to conclude the proof.

Appendix

1.A Auxiliary calculations and approximation of the initial data

In this section, we provide basic calculations in Section 1.A.1 and 1.A.3 that are used throught the chapter, as well as a standard approximation result for the initial data in Section 1.A.2, which is applied for the Bona-Smith argument in Section 1.3.4.

1.A.1 Derivatives of the nonlinearity

In this section we assume $u, v : \mathbb{R}^n \times [0, T] \to \mathbb{R}^L$ are smooth maps. The calculations hold if u and v are sufficiently regular to apply the Leibniz formula (e.g. with weak derivatives in L^2). Lemma 1.2.8 and the Leibniz formula imply the following substitution rule.

Lemma 1.A.1. *Let $l \in \mathbb{N}$. Then we have*

$$\nabla^l(\mathcal{N}(u)) = J_1 + J_2 + J_3,$$

where the terms J_1, J_2, and J_3 are of the form (with $k_i, m_i \in \mathbb{N}_0$)

$$J_1 = \sum_{(*)} d^{j+1}P_u(\nabla^{m_1+1}u \star \ldots \star \nabla^{m_j+1}u)[\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2+1}u]$$

with $(*) : 0 \leq m \leq l,\ \sum_{i=1}^{2} k_i = l - m,\ j = \min\{1, m\}, \ldots, m,\ \sum_{k=1}^{j} m_k = m - j;$

$$J_2 = \sum_{(*)} d^{j+2}P_u(\nabla^{m_1+1}u \star \cdots \star \nabla^{m_j+1}u)[\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u]$$

with $(*) : 0 \leq m \leq l,\ \sum_{i=1}^{3} k_i = l - m,\ j = \min\{1, m\}, \ldots, m,\ \sum_{k=1}^{j} m_k = m - j;$

$$J_3 = \sum_{(*)} d^{j+3}P_u(\nabla^{m_1+1}u \star \cdots \star \nabla^{m_j+1}u)[\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u]$$

with $(*) : 0 \leq m \leq l,\ \sum_{i=1}^{4} k_i = l - m,\ j = \min\{1, m\}, \ldots, m,\ \sum_{k=1}^{j} m_k = m - j.$

The following lemmata are used to prove the existence of a fixed point in Section 1.3.1 and the uniqueness result in Section 1.3.3.

Lemma 1.A.2. *Let* $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, *and* $w = u - v$. *For* $m \geq 2$ *we have*

$$\begin{aligned}
\nabla^m(d^k P_u - d^k P_v) = &\sum_{j=1}^{m} \sum_{m_1+\cdots+m_j=m-j} (d^{j+k}P_u - d^{j+k}P_v)(\nabla^{m_1+1}u, \ldots, \nabla^{m_j+1}u) \\
&+ \sum_{j=2}^{m} \sum_{m_1+\cdots+m_j=m-j} d^{j+k}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}u, \ldots, \nabla^{m_j+1}u) \\
&+ \sum_{j=2}^{m} \sum_{m_1+\cdots+m_j=m-j} d^{j+k}P_v(\nabla^{m_1+1}v, \nabla^{m_2+1}w, \nabla^{m_3+1}u, \ldots, \nabla^{m_j+1}u) \\
&\vdots \\
&+ \sum_{j=2}^{m} \sum_{m_1+\cdots+m_j=m-j} d^{j+k}P_v(\nabla^{m_1+1}v, \ldots, \nabla^{m_{j-1}+1}v, \nabla^{m_j+1}w),
\end{aligned} \tag{1.A.1}$$

and for $m = 1$

$$\nabla(d^k P_u - d^k P_v) = (d^k P_u - d^k P_v)(\nabla u) + d^k P_v(\nabla w). \tag{1.A.2}$$

Proof. The result follows from subtracting the expansion in Lemma 1.2.8 for $d^k P_v$

$$\nabla^m(d^k P_v) = \sum_{j=1}^{m} \sum_{m_1+\cdots+m_j=m-j} d^{j+k}P_v(\nabla^{m_1+1}v \star \cdots \star \nabla^{m_j+1}v),$$

from the same expansion of $\nabla^m(d^k P_u)$. Then subsequently adding and subtracting the intermediate terms in the formula above gives the result. □

Corollary 1.A.3. Let $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $w = u - v$. Then we have

$$\begin{aligned}
&\nabla^m \left[(dP_u - dP_v)(u_t \cdot u_t + \nabla^2 u \star \nabla^2 u + \nabla^3 u \star \nabla u)\right] \\
&= \sum_{(*)} (d^{j+1}P_u - d^{j+1}P_v)(\nabla^{m_1+1}u, \ldots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2}u_t \\
&\qquad + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u) \\
&\quad + \sum_{(**)} d^{j+1}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}u, \ldots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2}u_t \\
&\qquad + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u) \\
&\quad \vdots \\
&\quad + \sum_{(**)} d^{j+1}P_v(\nabla^{m_1+1}v, \ldots, \nabla^{m_{j-1}+1}v, \nabla^{m_j+1}w)(\nabla^{k_1}u_t \star \nabla^{k_2}u_t \\
&\qquad + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u),
\end{aligned}$$

where $(*) : j = 1, \ldots, m$ and $m_1 + \cdots + m_j + k_1 + k_2 = m - j$, and $(**) : j = 2, \ldots, m$ and $m_1 + \cdots + m_j + k_1 + k_2 = m - j$. Likewise we have

$$\begin{aligned}
&\nabla^m \left[(d^2 P_u - d^2 P_v)(\nabla u \star \nabla u \star \nabla^2 u)\right] \\
&\quad = \sum_{(*)} (d^{j+2} P_u - d^{j+2} P_v)(\nabla^{m_1+1} u, \ldots, \nabla^{m_j+1} u)(\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+2} u) \\
&\quad + \sum_{(**)} d^{j+2} P_v(\nabla^{m_1+1} w, \nabla^{m_2+1} u, \ldots, \nabla^{m_j+1} u)(\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+2} u) \\
&\quad \vdots \\
&\quad + \sum_{(**)} d^{j+2} P_v(\nabla^{m_1+1} v, \ldots, \nabla^{m_{j-1}+1} v, \nabla^{m_j+1} w)(\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+2} u)
\end{aligned}$$

where $(*) : j = 1, \ldots, m$ and $m_1 + \cdots + m_j + k_1 + k_2 + k_3 = m - j$, and $(**) : j = 2, \ldots, m$ and $m_1 + \cdots + m_j + k_1 + k_2 + k_3 = m - j$. Further

$$\begin{aligned}
&\nabla^m \left[(d^3 P_u - d^3 P_v)(\nabla u \star \nabla u \star \nabla u \star \nabla u)\right] \\
&= \sum_{(*)} (d^{j+3} P_u - d^{j+3} P_v)(\nabla^{m_1+1} u, \ldots, \nabla^{m_j+1} u)(\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+1} u \star \nabla^{k_4+1} u) \\
&+ \sum_{(**)} d^{j+3} P_v(\nabla^{m_1+1} w, \nabla^{m_2+1} u, \ldots, \nabla^{m_j+1} u)(\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+1} u \star \nabla^{k_4+1} u) \\
&\vdots \\
&+ \sum_{(**)} d^{j+3} P_v(\nabla^{m_1+1} v, \ldots, \nabla^{m_{j-1}+1} v, \nabla^{m_j+1} w)(\nabla^{k_1+1} u \star \nabla^{k_2+1} u \star \nabla^{k_3+1} u \star \nabla^{k_4+1} u)
\end{aligned}$$

where we sum over $(*) : j = 1, \ldots, m$ and $m_1 + \cdots + m_j + k_1 + k_2 + k_3 + k_4 = m - j$, $(**) : j = 2, \ldots, m$ and $m_1 + \cdots + m_j + k_1 + k_2 + k_3 + k_4 = m - j$.

Also, the case $m = 1$ is similar.

Proof. The assertions are consequences of the Leibniz rule and Lemma 1.A.2. □

Corollary 1.A.4. We have for $m \in \mathbb{N},\ m \geq 2$ and $w = u - v$ that

$$\nabla^m (\mathcal{N}(u) - \mathcal{N}(v))$$

is a linear combination of terms of the form

$$(d^{j+1}P_u - d^{j+1}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u),$$

$$d^{j+1}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u),$$

$$(d^{j+2}P_u - d^{j+2}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u),$$

$$d^{j+2}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u),$$

$$(d^{j+3}P_u - d^{j+3}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u),$$

$$d^{j+3}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u), \quad \text{and}$$

$$d^{j+1}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1}w_t \star \nabla^{k_2}h_t + \nabla^{k_1+2}w \star \nabla^{k_2+2}h + \nabla^{k_1+3}w \star \nabla^{k_2}h + \nabla^{k_1+3}h \star \nabla^{k_2}w), \;\; h \in \{u, v\},$$

$$d^{j+2}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1+1}w \star \nabla^{k_2+1}h_1 \star \nabla^{k_3+2}h_2 + \nabla^{k_1+1}h_1 \star \nabla^{k_2+1}h_2 \star \nabla^{k_3+2}w),$$

$$d^{j+3}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1+1}w \star \nabla^{k_2+1}h_1 \star \nabla^{k_3+1}h_2 \star \nabla^{k_4+1}h_3),$$

where $j, k_1, k_2, k_3, k_4, m_1, \dots m_j$ and $h, h_1, \dots, h_{j-1} \in \{u, v\}$ are as above in Corollary 1.A.3. Also, we have a similar (but simpler) statement for $m = 1$.

Proof. We write, according to the definition of $\mathcal{N}(u)$ in (1.1.2),

$$\begin{aligned}
\mathcal{N}(u) - \mathcal{N}(v) &= (dP_u - dP_v)(u_t \cdot u_t + \nabla^2 u \star \nabla^2 u + \nabla^3 u \star \nabla u) \\
&+ (d^2P_u - d^2P_v)(\nabla u \star \nabla u \star \nabla^2 u) + (d^3P_u - d^3P_v)(\nabla u \star \nabla u \star \nabla u \star \nabla u) \\
&+ dP_v(w_t \cdot u_t + v_t \cdot w_t + \nabla w \star \nabla u + \nabla v \star \nabla w + \nabla^3 w \star \nabla u + \nabla^3 v \star \nabla w) \\
&+ d^2P_v(\nabla w \star \nabla u \star \nabla^2 u + \nabla v \star \nabla w \star \nabla^2 u + \nabla v \star \nabla v \star \nabla^2 w) \\
&+ d^3P_v(\nabla w \star \nabla u \star \nabla u \star \nabla u + \nabla v \star \nabla w \star \nabla u \star \nabla u \\
&\quad + \nabla v \star \nabla v \star \nabla w \star \nabla u + \nabla v \star \nabla v \star \nabla v \star \nabla w).
\end{aligned}$$

Then, we use Corollary 1.A.3 for the first three terms in the sum above. For the latter three, we use Lemma 1.2.8 and the Leibniz rule. □

Let $\varepsilon \in (0, 1)$. We recall from (1.3.4) the definition

$$\mathcal{N}_\varepsilon(u) = \mathcal{N}(u) - \varepsilon d^2P_u(u_t, \nabla u, \nabla u) - \varepsilon 2dP_u(\nabla u_t, \nabla u) - \varepsilon dP_u(u_t, \Delta u).$$

Lemma 1.A.5. *For $m \in \mathbb{N}_0$ the derivative $\nabla^m(\mathcal{N}_\varepsilon(u))$ compared to $\nabla^m(\mathcal{N}(u))$ contains the additional terms*

$$d^{j+1}P_u(\nabla^{m_1+1}u \star \cdots \star \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2+2}u + \nabla^{k_1+1}u_t \star \nabla^{k_2+1}u), \quad \textit{and}$$
$$d^{j+2}P_u(\nabla^{m_1+1}u \star \cdots \star \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u),$$

with $j, m_1, \dots, m_j, k_1, k_2, k_3$ similarly to Lemma 1.A.1.
Further $\nabla^m(\mathcal{N}_\varepsilon(u)) - \nabla^m(\mathcal{N}_\varepsilon(v))$ compared to $\nabla^m(\mathcal{N}(u)) - \nabla^m(\mathcal{N}(v))$ contains additional terms of the form

$$(d^{j+1}P_u - d^{j+1}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2+2}u + \nabla^{k_1+1}u_t \star \nabla^{k_2+1}u),$$
$$d^{j+1}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t \star \nabla^{k_2+2}u + \nabla^{k_1+1}u_t \star \nabla^{k_2+1}u),$$

$$(d^{j+2}P_u - d^{j+2}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u),$$
$$d^{j+2}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u), \quad \textit{and}$$

$$d^{j+1}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1}w_t \star \nabla^{k_2+2}h + \nabla^{k_1+1}w_t \star \nabla^{k_2+1}h$$
$$+ \nabla^{k_1}h \star \nabla^{k_2+2}w + \nabla^{k_1+1}h_t \star \nabla^{k_2+1}w), \;\; h \in \{u, v\},$$

$$d^{j+2}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1}w_t \star \nabla^{k_2+1}h_1 \star \nabla^{k_3+1}h_2$$
$$+ \nabla^{k_1}(h_1)_t \star \nabla^{k_2+1}h_2 \star \nabla^{k_3+1}w),$$

with $w = u - v$ and $j, m_1, \dots, m_j, k_1, k_2, k_3, h_1, \dots, h_{j-1}$ similarly to Corollary 1.A.4.

The implicit constants may depend on ε here.

1.A.2 Approximation of the initial data

In this section we construct certain approximations of initial data in order to conclude continuous dependence of the solution on the initial data. As in the previous sections, take functions $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0 \in N$, $u_1 \in T_{u_0}N$ a.e. on $\mathbb{R}^n$, and

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n).$$

for some $k > \lfloor \frac{n}{2} \rfloor + 2$ with $k \in \mathbb{N}$.

Lemma 1.A.6. *Let the functions (u_0, u_1) be as above. Then there is a number $\delta^* = \delta^*(N) > 0$ such that for $\delta \in (0, \delta*]$ there exist maps $u_0^\delta, u_1^\delta \in C^\infty(\mathbb{R}^n, \mathbb{R}^L)$ such that $\nabla u_0^\delta, u_1^\delta \in H^m$ for* *all* $m \in \mathbb{N}$, $u_0^\delta \in N$ *and* $u_1^\delta \in T_{u_0^\delta} N$ *on* $\mathbb{R}^n$ *which satisfy*

$$u_0 - u_0^\delta \in L^2 \quad \text{and} \quad \left\|u_0 - u_0^\delta\right\|_{L^2} \leq C_0 \delta, \tag{1.A.3}$$

$$\left\|(\nabla u_0^\delta, u_1^\delta) - (\nabla u_0, u_1)\right\|_{H^{k-2} \times H^{k-3}} = o(\sqrt{\delta}) \quad \text{as } \delta \to 0^+, \tag{1.A.4}$$

$$\left\|(\nabla u_0^\delta, u_1^\delta) - (\nabla u_0, u_1)\right\|_{H^{k-1} \times H^{k-2}} = o(1) \quad \text{as } \delta \to 0^+, \tag{1.A.5}$$

$$\left\|(\nabla u_0^\delta, u_1^\delta)\right\|_{H^k \times H^{k-1}} \leq C_0 \frac{1}{\sqrt{\delta}} \tag{1.A.6}$$

for a constant $C_0 = C_0(\|P_p\|_{C_b^k}, \|\nabla u_0\|_{H^{k-1}}, \|u_1\|_{H^{k-2}}) > 0$. *Further let* (v_0, v_1) *be as above with* $u_0 - v_0 \in H^k(\mathbb{R}^n)$ *and*

$$\|(u_0, u_1) - (v_0, v_1)\|_{H^k \times H^{k-2}} \leq R$$

for some $R > 0$. *Then for* $\delta \in (0, \delta^*]$ *we have*

$$\left\|(\nabla v_0^\delta, v_1^\delta)\right\|_{H^k \times H^{k-1}} \leq C_0(1 + R^k) \frac{1}{\sqrt{\delta}}, \tag{1.A.7}$$

$$\left\|(u_0^\delta, u_1^\delta) - (v_0^\delta, v_1^\delta)\right\|_{H^k \times H^{k-2}} \leq C_0(1 + R^k) \|(u_0, u_1) - (v_0, v_1)\|_{H^k \times H^{k-2}}. \tag{1.A.8}$$

Proof. We choose the caloric extension for regularization, i.e., we consider $\eta_\delta * u_0$ and $\eta_\delta * u_1$ where

$$\eta_\delta(x) = (4\pi\delta)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\delta}}, \quad \delta > 0, \ x \in \mathbb{R}^n,$$

and $T(\delta)f = \eta_\delta * f$ is the heat semigroup. Since $u_1 \in C_b^0(\mathbb{R}^n)$ and $u_0 \in C_b^2(\mathbb{R}^n)$ by assumption, the convolution is well defined for u_0 and u_1. Moreover, $\eta_\delta * u_0$ tends to u_0 and $\eta_\delta * u_1$ to u_1 uniformly as $\delta \to 0^+$, as well as

$$\nabla(\eta_\delta * u_0) \to \nabla u_0 \ \text{ in } H^{k-1}(\mathbb{R}^n), \qquad \eta_\delta * u_1 \to u_1 \ \text{ in } H^{k-2}(\mathbb{R}^n) \qquad \text{as } \delta \to 0^+.$$

The uniform convergence yields

$$\operatorname{dist}(u_0 * \eta_\delta(x), N) \leq |u_0 * \eta_\delta(x) - u_0(x)| \to 0 \qquad \text{as } \delta \to 0^+ \tag{1.A.9}$$

uniformly in $x \in \mathbb{R}^n$. Hence, if $\delta > 0$ is small enough we can define

$$u_0^\delta := \pi(u_0 * \eta_\delta) \quad \text{and} \quad u_1^\delta := P_{u_0 * \eta_\delta}(u_1 * \eta_\delta).$$

Recall that π is the nearest point map and that $P_{u_0 * \eta_\delta}(u_1 * \eta_\delta) \in T_{u_0^\delta} N$ by definition of the projector P and u_0^δ. Especially we have

$$|u_0^\delta(x) - u_0 * \eta_\delta(x)| = \operatorname{dist}(u_0 * \eta_\delta(x), N) \leq |u_0(x) - u_0 * \eta_\delta(x)|,$$
$$|u_0^\delta(x) - u_0(x)| \leq 2|u_0(x) - u_0 * \eta_\delta(x)|$$

for $x \in \mathbb{R}^n$. We further note that u_0^δ and u_1^δ are smooth maps and that we have the uniform convergence

$$u_0^\delta \to u_0, \qquad u_1^\delta \to u_1$$

as $\delta \to 0^+$ by construction of u_0^δ (and the mean value theorem for u_1^δ). Assertion (1.A.3) follows from

$$\left\| \delta^{-1}(u_0 * \eta_\delta - u_0) \right\|_{L^2} = \left\| \frac{1}{\delta} \int_0^\delta (\Delta u_0) * \eta_s \, ds \right\|_{L^2} \lesssim \|\Delta u_0\|_{L^2},$$

by Young's inequality for the convolution. Since $\nabla u_0^\delta = P_{u_0 * \eta_\delta}((\nabla u_0) * \eta_\delta)$, we further have to treat the terms

$$\begin{aligned} P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta) - \nabla u_0 &= P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta - \nabla u_0) + (P_{u_0*\eta_\delta} - P_{u_0})\nabla u_0, \\ P_{u_0*\eta_\delta}(u_1 * \eta_\delta) - u_1 &= P_{u_0*\eta_\delta}(u_1 * \eta_\delta - u_1) + (P_{u_0*\eta_\delta} - P_{u_0})u_1. \end{aligned}$$

We start by estimating (by means of the mean value theorem for P)

$$\begin{aligned} \|P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta) - \nabla u_0\|_{L^2} &\leq \|P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta - \nabla u_0)\|_{L^2} + \|(P_{u_0*\eta_\delta} - P_{u_0})\nabla u_0\|_{L^2} \\ &\lesssim \delta \left(O(1) + \|\nabla u_0\|_{L^2} \left\| \frac{1}{\delta}(u_0 * \eta_\delta - u_0) \right\|_{L^\infty} \right), \end{aligned}$$

where $\frac{1}{\delta}(u_0 * \eta_\delta - u_0) \to \Delta u_0$ uniformly as $\delta \to 0^+$ since $u_0 \in C_b^2(\mathbb{R}^n)$. Similarly, employing Lemmas 1.2.8, 1.2.9 and 1.A.2 as before, we see

$$\begin{aligned} &\left\| \nabla^{k-2}(P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta) - \nabla u_0) \right\|_{L^2} \\ &\lesssim \sum_{l_1+l_2=k-2} \Big[\left\| \nabla^{l_1}(P_{u_0*\eta_\delta}) \cdot \nabla^{l_2}((\nabla u_0) * \eta_\delta - \nabla u_0) \right\|_{L^2} + \left\| \nabla^{l_1}(P_{u_0*\eta_\delta} - P_{u_0}) \cdot \nabla^{l_2+1} u_0 \right\|_{L^2} \Big] \\ &\lesssim (1 + \|\nabla u_0\|_{H^{k-2}}^k + \|(\nabla u_0) * \eta_\delta\|_{H^{k-2}}^k) \, \|(\nabla u_0) * \eta_\delta - \nabla u_0\|_{H^{k-2}} \\ &\quad + \delta \left\| \nabla^{k-1} u_0 \right\|_{L^2}^k \left\| \delta^{-1}(u_0 * \eta_\delta - u_0) \right\|_{L^\infty} \\ &\lesssim o(\sqrt{\delta}) \qquad \text{as } \delta \to 0^+. \end{aligned}$$

Here we also use [39, Prop. 2.2.4]. Interpolation and an analogous argument for u_1^δ in H^{k-3} then allows us to conclude (1.A.4). Assertion (1.A.5) is shown in the same way, with $o(1)$ instead of $o(\sqrt{\delta})$ in the upper bound. For (1.A.6), we compute

$$\begin{aligned} &\left\| \nabla^k (P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta)) \right\|_{L^2} \\ &\quad \lesssim \sum_{\substack{l_1+l_2=k \\ l_1>0}} \left\| \nabla^{l_1}(P_{u_0*\eta_\delta}) \cdot (\nabla^{l_2+1} u_0 * \eta_\delta) \right\|_{L^2} + \left\| P_{u_0*\eta_\delta} \nabla(\nabla^k u_0 * \eta_\delta) \right\|_{L^2} \\ &\quad \lesssim (1 + \|\nabla u_0\|_{H^{k-1}}^k) \, \|\nabla u_0\|_{H^{k-1}} + \left\| P_{u_0*\eta_\delta} \nabla(\nabla^k u_0 * \eta_\delta) \right\|_{L^2} \end{aligned}$$

as before. The last term is bounded via

$$\left\|P_{u_0*\eta_\delta}\nabla(\nabla^k u_0 * \eta_\delta)\right\|_{L^2} \lesssim \left\|(\nabla^k u_0) * \nabla(\eta_\delta)\right\|_{L^2} \lesssim \frac{1}{\sqrt{\delta}}\left\|\nabla u_0\right\|_{H^{k-1}}$$

again by Young's inequality. Similarly, the term $\nabla^{k-1}u_1^\delta$ is estimated in $L^2(\mathbb{R}^n)$. The above reasoning also shows (1.A.7) if we choose the constant $C_0 > 0$ suitably. In order to prove (1.A.8), similarly as above we compute

$$\left\|u_0^\delta - v_0^\delta\right\|_{L^2} \lesssim \left\|\eta_\delta * (u_0 - v_0)\right\|_{L^2} \lesssim \left\|u_0 - v_0\right\|_{L^2}.$$

by the mean value theorem and Young's inequality. Writing

$$\begin{aligned} &P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta) - P_{v_0*\eta_\delta}((\nabla v_0) * \eta_\delta) \\ &\qquad = P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta - (\nabla v_0) * \eta_\delta) + (P_{u_0*\eta_\delta} - P_{v_0*\eta_\delta})((\nabla v_0) * \eta_\delta), \end{aligned}$$

we deduce

$$\begin{aligned} &\|\nabla^{k-1}(P_{u_0*\eta_\delta}((\nabla u_0) * \eta_\delta) - P_{v_0*\eta_\delta}((\nabla v_0) * \eta_\delta))\|_{L^2} \\ &\quad\lesssim \sum_{l_1+l_2=k-1} \left\|\nabla^{l_1}(P_{u_0*\eta_\delta}) \cdot \nabla^{l_2}((\nabla u_0) * \eta_\delta - (\nabla v_0) * \eta_\delta)\right\|_{L^2} \\ &\qquad + \sum_{l_1+l_2=k-1} \left\|\nabla^{l_1}(P_{u_0*\eta_\delta} - P_{v_0*\eta_\delta}) \cdot (\nabla^{l_2+1}v_0) * \eta_\delta\right\|_{L^2} \\ &\quad\lesssim (1 + \|\nabla u_0\|_{H^{k-1}}^k + \|\nabla v_0\|_{H^{k-1}}^k)\left\|\nabla u_0 - \nabla v_0\right\|_{H^{k-1}} + \left\|\nabla^k v_0\right\|_{L^2}^k \left\|u_0 - v_0\right\|_{L^\infty} \\ &\quad\lesssim (1 + \|\nabla u_0\|_{H^{k-1}}^k + R^k)\left\|\nabla u_0 - \nabla v_0\right\|_{H^{k-1}} + \left\|\nabla^k v_0\right\|_{L^2}^k \left\|u_0 - v_0\right\|_{H^k} \\ &\quad\lesssim (1 + \|\nabla u_0\|_{H^{k-1}}^k + R^k)\left\|\nabla u_0 - \nabla v_0\right\|_{H^{k-1}}, \end{aligned}$$

The claim (1.A.8) then follows by interpolation and a proper choice of $C_0 > 0$. Finally the estimate for

$$u_1^\delta - v_1^\delta = P_{u_0*\eta_\delta}(u_1 * \eta_\delta - v_1 * \eta_\delta) + (P_{u_0*\eta_\delta} - P_{v_0*\eta_\delta})(v_1 * \eta_\delta)$$

works similarly. □

1.A.3 Establishing the identity (1.3.57)

For $f, g \in H^1(\mathbb{R}^n)$, $h \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$ we set

$$D_h^i f(x) = \frac{1}{h}(f(x + e_i h) - f(x)).$$

Observe that $D_h^i(fg)(x) = (D_h^i f)(x)g(x+e_ih) + f(x)(D_h^i g)(x)$. Since we only use the product rule integrated over $x \in \mathbb{R}^n$ and $g(\cdot + he_i) \to g$ strongly in H^1 as $h \to 0$, we drop the h-dependence in $g(\cdot + e_ih)$ in the following calculation.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\big\|D_h^i\nabla^{k-3}u_t\big\|_{L^2}^2 + \big\|D_h^i\nabla^{k-1}u\big\|_{L^2}^2\Big) = \int_{\mathbb{R}^n} D_h^i\nabla^{k-3}\Big((I-P_u)\mathcal{N}(u)\Big)\cdot D_h^i\nabla^{k-3}u_t\,dx \\
&= \sum_{l=1}^{k-3}\int_{\mathbb{R}^n} D_h^i(\nabla^l(I-P_u)\star\nabla^{k-3-l}\mathcal{N}(u))\cdot D_h^i\nabla^{k-3}u_t\,dx \\
&\quad + \int_{\mathbb{R}^n} D_h^i(I-P_u)\nabla^{k-3}\mathcal{N}(u)\cdot D_h^i\nabla^{k-3}u_t\,dx + \int_{\mathbb{R}^n} D_h^i\nabla^{k-3}\mathcal{N}(u)\cdot(I-P_u)D_h^i\nabla^{k-3}u_t\,dx \\
&= \sum_{l=1}^{k-3}\int_{\mathbb{R}^n} D_h^i(\nabla^l(I-P_u)\star\nabla^{k-3-l}\mathcal{N}(u))\cdot D_h^i\nabla^{k-3}u_t\,dx \\
&\quad + \int_{\mathbb{R}^n} D_h^i(I-P_u)\nabla^{k-3}\mathcal{N}(u)\cdot D_h^i\nabla^{k-3}u_t\,dx + \int_{\mathbb{R}^n} D_h^i(\nabla^{k-3}\mathcal{N}(u)\cdot(I-P_u)D_h^i\nabla^{k-3}u_t)\,dx \\
&\quad + \int_{\mathbb{R}^n} \nabla^{k-3}\mathcal{N}(u)\cdot D_h^i(D_h^i(I-P_u)\nabla^{k-3}u_t)\,dx \\
&\quad + \sum_{l=1}^{k-3}\int_{\mathbb{R}^n} \nabla^{k-3}\mathcal{N}(u)\cdot(D_h^i)^2(\nabla^l(I-P_u)\star\nabla^{k-3-l}u_t)\,dx =: \int_{\mathbb{R}^n} T_h^i(u)\,dx,
\end{aligned}
$$

where the second identity follows from $(I-P_u)u_t = 0$. For a fixed time $t \in [0, T_m)$, the regularity of u yields the limit

$$
\begin{aligned}
\lim_{h\to 0}\int_{\mathbb{R}^n} T_h^i(u(t))\,dx &= \sum_{l=1}^{k-3}\int_{\mathbb{R}^n} \partial_{x_i}(\nabla^l(I-P_u)\star\nabla^{k-3-l}\mathcal{N}(u))\cdot\nabla^{k-3}\partial_{x_i}u_t\,dx \\
&\quad - \int_{\mathbb{R}^n} dP_u(\partial_{x_i}u, \nabla^{k-3}\mathcal{N}(u))\cdot\nabla^{k-3}\partial_{x_i}u_t\,dx \\
&\quad - \int_{\mathbb{R}^n} \nabla^{k-3}\mathcal{N}(u)\cdot\partial_{x_i}\big(dP_u(\partial_{x_i}u, \nabla^{k-3}u_t)\big)\,dx \\
&\quad + \sum_{l=1}^{k-3}\int_{\mathbb{R}^n} \nabla^{k-3}\mathcal{N}(u)\cdot\partial_{x_i}^2(\nabla^l(I-P_u)\star\nabla^{k-3-l}u_t)\,dx \\
&=: \int_{\mathbb{R}^n} T^i(u(t))\,dx.
\end{aligned}
$$

Here we also used that

$$
\int_{\mathbb{R}^n} D_h^i(\nabla^{k-3}\mathcal{N}(u)\cdot(I-P_u)D_h^i\nabla^{k-3}u_t)\,dx \to 0 \quad \text{as } h \to 0
$$

by Gauss' Theorem. Estimating as in Section 1.3.2, we derive

$$\left|\int_{\mathbb{R}^n} T^i(u(t))\,dx\right| \lesssim \sup_{s\in[0,T]} (1 + \|\nabla u(s)\|_{H^{k-1}}^{2k} + \|u_t(s)\|_{H^{k-2}}^{2k})(\|\nabla u(s)\|_{H^{k-1}}^2 + \|u_t(s)\|_{H^{k-2}}^2).$$

for $t \in [0,T]$ and $T < T_m$. In the limit $h \to 0$ it follows

$$\left\|\nabla^{k-3}\partial_{x_i} u_t\right\|_{L^2}^2 + \left\|\nabla^{k-1}\partial_{x_i} u\right\|_{L^2}^2 = 2\int_0^t\int_{\mathbb{R}^n} T^i(u(s))\,dx\,ds + \left\|\nabla^{k-3}\partial_{x_i} u_1\right\|_{L^2}^2 + \left\|\nabla^{k-1}\partial_{x_i} u_0\right\|_{L^2}^2$$

by dominated convergence. The right-hand side is continuous in t, and hence the highest derivatives $\nabla^k u_t, \nabla^{k-2} u : [0, T_m) \to L^2$ are continuous, since we already know their weak continuity. Finally, summing over $i = 1, \ldots, n$ and estimating $T^i(u)$ as in Section 1.3.2, we conclude the blow-up criterion from (1.1.4) for the solution u.

1.A.4 Coordinate expansion of the mapping equation

We now derive (1.1.2) from the condition (1.1.1) for smooth solutions $u : \mathbb{R}^m \times [0,T) \to N$. Note that we use the sum convention, i.e. the same indices in super-/subscript means summation. Since $\partial_t u \in T_u N$, we infer the identity

$$\begin{aligned}[(I - P_u)(\partial_t^2 u)]^k &= (\delta_l^k - (P_u)_l^k)(\partial_t^2 u^l) = \partial_t(\delta_l^k - (P_u)_l^k)(\partial_t u^l) + (\partial_m P_u)_l^k \partial_t u^l \partial_t u^m \\ &= (dP_u)_{m,l}^k \partial_t u^l \partial_t u^m\end{aligned}$$

for $k = 1, \ldots, L$. Because of $\nabla u \in T_u N$, we also obtain

$$\begin{aligned}[(I - P_u)(\Delta u)]^k &= \partial^{x_\alpha}(\delta_l^k - (P_u)_l^k)(\partial_{x_\alpha} u^l) + (\partial_m P_u)_l^k \partial^{x_\alpha} u^l \partial_{x_\alpha} u^m \\ &= (dP_u)_{m,l}^k \partial^{x_\alpha} u^l \partial_{x_\alpha} u^m,\end{aligned}$$

and hence

$$\begin{aligned}[(I - P_u)(\Delta^2 u)]^k &= \Delta((dP_u)_{m,l}^k \partial^{x_\alpha} u^l \partial_{x_\alpha} u^m) + \partial^{x_\alpha}((dP_u)_{m,l}^k \Delta u^l \partial_{x_\alpha} u^m) \\ &\quad + (dP_u)_{m,l}^k (\partial^{x_\alpha}\Delta u^l)\partial_{x_\alpha} u^m.\end{aligned}$$

The symmetry of the indices then implies

$$\begin{aligned}[(I - P_u)(\Delta^2 u)]^k &= (d^3 P_u)_{l_0,l_1,l_2,l_3}^k \partial^{x_\alpha} u^{l_0} \partial_{x_\alpha} u^{l_1} \partial^{x_\beta} u^{l_2} \partial_{x_\beta} u^{l_3} \\ &\quad + 2(dP_u)_{l_0,l_1}^k \partial_{x_\alpha}\partial^{x_\beta} u^{l_0} \partial^{x_\alpha}\partial_{x_\beta} u^{l_1} + (dP_u)_{l_0,l_1}^k \Delta u^{l_0} \Delta u^{l_1} \\ &\quad + 2(d^2 P_u)_{l_0,l_1,l_2}^k \partial^{x_\alpha} u^{l_0} \partial_{x_\alpha} u^{l_1} \Delta u^{l_2} + 4(dP_u)_{l_0,l_1}^k \partial^{x_\alpha}\Delta u^{l_0} \partial_{x_\alpha} u^{l_1} \\ &\quad + 4(d^2 P_u)_{l_0,l_1,l_2}^k \partial^{x_\alpha} u^{l_0} \partial_{x_\alpha}\partial^{x_\beta} u^{l_1} \partial_{x_\beta} u^{l_2}.\end{aligned}$$

We briefly state the expressions from (1.1.3) in coordinates, i.e.,

$$\left[P_u(dP_u(\nabla u, \nabla u) \cdot d^2 P_u(\nabla u, \nabla u, \cdot))\right]^l = \sum\nolimits_j (P_u)_j^l dP_u(\nabla u, \nabla u) \cdot (d^2 P_u)_{k,m,j} \partial_{x_\alpha} u^k \partial^{x_\alpha} u^m,$$

$$\left[P_u(\mathrm{div}(dP_u(\nabla u, \nabla u) \cdot dP_u(\nabla u, \cdot)))\right]^l = \sum\nolimits_j (P_u)_j^l \partial^{x_\alpha}(dP_u(\nabla u, \nabla u) \cdot (dP_u)_{kj}\, \partial_{x_\alpha} u^k)$$

for $l = 1, \ldots, L$.

CHAPTER 2

Further global results

In this chapter, we study two Cauchy problems that are motivated by critical points of the *extrinsic (rigid)* action functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^d} |\partial_t u|^2 - |\Delta u|^2 \, dx \, dt,$$

which satisfy the Euler Lagrange condition (1.1.1), i.e.

$$\partial_t^2 u + \Delta^2 u \perp T_u N, \quad \text{on } \mathbb{R} \times \mathbb{R}^d,$$

respectively the biharmonic wave equation (19) for a Riemannian submanifold N. We recall that for maps into the round sphere $N = \mathbb{S}^L$, the condition (1.1.1) leads to equation (2) from the beginning of the introduction, i.e. u satisfies

$$\begin{aligned} \partial_t^2 u + \Delta^2 u &= -|\partial_t u|^2 u - \Delta(|\nabla u|^2)u \\ &\quad - (\nabla \cdot \langle \Delta u, \nabla u\rangle)u - \langle \nabla\Delta u, \nabla u\rangle u \\ &= -(|\partial_t u|^2 + |\Delta u|^2 + 4\langle \nabla u, \nabla \Delta u\rangle + 2\langle \nabla^2 u, \nabla^2 u\rangle)u, \end{aligned} \tag{2.0.1}$$

where Δ^2 denotes the bi-Laplacien and $\langle \nabla^2 u, \nabla^2 u\rangle = \langle \partial_i \partial^j u, \partial_j \partial^i u\rangle$. As mentioned in the introduction, see the surveys [56], [32], the *null condition* of the *wave maps equation*

$$\Box u = \Gamma(u)(\partial_\alpha u, \partial^\alpha u) = \tilde{\Gamma}(u)(\Box(u \cdot u) - 2u \cdot \Box u),$$

leads to improved local and global wellposedness (or regularity) results compared to a generic nonlinearity $\mathcal{N}(u, Du)$. We refer to [29], [31], [38], [11] and [15] for a general overview.

In this chapter, we prove the analogue of the *division problem* for equations of type (2.0.1), i.e. we prove global regularity with small data in $\dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d) \times \dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d)$. For a short overview over the corresponding results for wave maps, we refer to the introductory chapter of this thesis. For (2.0.1), we achieve to solve the division problem in dimension $d \geq 3$ using spaces $Z,\ W = L(Z)$ which are the analogues of Tataru's $F, \Box F$ spaces in [54]. Especially,

$$L : Z \to W$$

is a continuous operator.

2.1 Introduction

We consider the following generalized Cauchy problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u = Q_u(u_t, u_t) + Q_u(\Delta u, \Delta u) + 2Q_u(\nabla u, \nabla \Delta u) \\ \qquad\qquad\qquad +2Q_u(\nabla \Delta u, \nabla u) + 2Q_u(\nabla^2 u, \nabla^2 u) =: \mathcal{Q}(u), \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases} \tag{2.1.1}$$

where

$$\mathcal{Q}^J(u) = [Q_u]^J_{K,M}(\partial_t u^K \partial_t u^M) + [Q_u]^J_{K,M}(\Delta u^K \Delta u^M) + 2[Q_u]^J_{K,M}(\partial_i u^K \partial^i \Delta u^M)$$
$$+ 2[Q_u]^J_{K,M}(\partial_i \Delta u^K \partial^i u^M) + 2[Q_u]^J_{K,M}(\partial_i \partial^j u^K \partial_j \partial^i u^M),$$

and $\{Q_x \mid x \in \mathbb{R}^L\}$ is a *smooth family of bilinear forms* $Q_x(\cdot,\cdot) : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}^L$ (in fact required to be analytic at the origin $x_0 = 0$). Here we contract the derivatives over $i = 1, \dots, d$ and the components of u over $K, M, J \in \{1, \dots, L\}$. The bilinear term $\mathcal{Q}(u)$ in (2.1.1) is *non-generic* for our results, in the sense that for bilinear interactions, the set of *resonances*

$$\big\{((\tau_1, \xi_1), (\tau_2, \xi_2)) \mid (\tau_1 + \tau_2)^2 - |\xi_1 + \xi_2|^4 = \tau_1^2 + \tau_2^2 - |\xi_1|^4 - |\xi_2|^4\big\},$$

is canceled by $\mathcal{Q}(u)$. We use this fact in the form of the following commutator identity for the operator $L = \partial_t^2 + \Delta^2$

$$\mathcal{Q}(u) = \frac{1}{2} Q_u(L(u \cdot u) - u \cdot Lu - Lu \cdot u) \tag{2.1.2}$$
$$= \frac{1}{2}[Q_u]_{K,M}(L(u^K \cdot u^M) - u^K \cdot Lu^M - u^M \cdot Lu^K).$$

This will then be exploited following the work of Tataru in [54], [55] for wave maps. To be precise, the idea used in Tataru's F, $\Box F$ spaces from [54] allow to treat $\mathcal{Q}(u)$ by continuity of L. As a consequence, we find a simple way to solve the disvision problem for (2.1.1) even in *low dimensions* compared to the energy scaling (of (7)) for biharmonic wave maps (2.0.1), see e.g. the introduction and the remark below. However, we do not obtain scattering at $t \to \pm\infty$

from this approach.

The *main difference* to [54] is that we have to use the control of a lateral Strichartz space and a maximal function bound in order to exploit a smoothing effect for the Schrödinger group. More details are given below.

The second Cauchy problem will be solved with the same approach (presented in the following Sections) and further (in Section 2.5) applies to solve (19).

The Euler Lagrange condition (1.1.1) is equivalent to

$$Lu = (I - P_u)(Lu) = (I - d\Pi_u)(Lu), \tag{2.1.3}$$

via the smooth family of orthogonal tangent projector $P_u : \mathbb{R}^L \to T_uN$. Since $u \in N$ we have $\Pi(u) = u$ and $P_u = d\Pi_u$, where Π is nearest point projector

$$\Pi : \mathcal{V}_\varepsilon(N) \to N, \quad |\Pi(p) - p| = \inf_{q \in N} |q - p|.$$

Hence (2.1.3) can be written as

$$\begin{aligned} Lu = L(\Pi(u)) - d\Pi_u(Lu) =& \mathcal{Q}(u) + 2d^3\Pi_u(\partial_j u, \partial^j u, \Delta u) + 4d^3\Pi_u(\partial_j u, \partial^i u, \partial_i \partial^j u) \\ &+ d^4\Pi_u(\partial_i u, \partial^i u, \partial_j u, \partial^j u), \end{aligned}$$

where $\mathcal{Q}(u)$ is as above with $Q_u(\cdot, \cdot) = d^2\Pi_u(\cdot, \cdot)$.

We now generalize this equation and take a smooth vector field

$$\Pi : \mathbb{R}^L \to \mathbb{R}^L,$$

such that Π is *real analytic* at $x_0 = 0$. We consider the Cauchy problem

$$\begin{cases} Lu = \mathcal{Q}(u) + 2d^3\Pi_u(\partial_j u, \partial^j u, \Delta u) + 4d^3\Pi_u(\partial_j u, \partial^i u, \partial_i \partial^j u) \\ \qquad\qquad +d^4\Pi_u(\partial_i u, \partial^i u, \partial_j u, \partial^j u), \\ (u(0), \partial_t u(0)) = (u_0, u_1), \end{cases} \tag{2.1.4}$$

with $\mathcal{Q}$ defined over $d^2\Pi_x(\cdot, \cdot)$. For constructing a solution, we use that the RHS equals

$$L(\Pi(u)) - d\Pi_u(Lu), \tag{2.1.5}$$

from which we infer the formal series expansion for the RHS of (2.1.4),

$$L(\Pi(u)) - d\Pi_u(Lu) = \sum_{k \geq 2} C_k d^k \Pi_0(L(u^k) - ku^{k-1}Lu).$$

This wil be made precise later and we observe that the RHS thus reduces the same non-resonant form (2.1.2). This expression, and in particular the ability to commute L with the series expansion of Π, is justified by the spaces we use.

At least formally, we find a Duhamel representation

$$\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = S(t) \cdot \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t S(t-s) \cdot \begin{pmatrix} 0 \\ Lu(s) \end{pmatrix} \, ds, \tag{2.1.6}$$

where

$$S(t) = \begin{pmatrix} \cos((-\Delta)t) & (-\Delta)^{-1} \sin((-\Delta)t) \\ \Delta \sin((-\Delta)t) & \cos((-\Delta)t) \end{pmatrix} = \frac{1}{2} Q^{-1} \begin{pmatrix} e^{-it\Delta} & 0 \\ 0 & e^{it\Delta} \end{pmatrix} Q$$

with

$$Q = \begin{pmatrix} -i\Delta & 1 \\ i\Delta & 1 \end{pmatrix}. \tag{2.1.7}$$

Thus, in the analysis for *biharmonic wave maps* (19), it is in principle possible to exploit methods developed for derivative Schrödinger equations, which will become apparent below.

Results on the division problem for Schrödinger maps, see e.g. [2], [22], involve versions of lateral Strichartz estimates in the norm ($x \mapsto x_e e + x_{e^\perp},\ e \in \mathbb{S}^{d-1}$)

$$L_e^p L_{t,e^\perp}^q,$$

in order to exploit smoothing effects for Schrödinger equations, see Section 2.3 and the Appendix 2.A below. Especially, we likewise rely on factoring

$$L_e^\infty L_{t,e^\perp}^2 \cdot L_e^2 L_{t,e^\perp}^\infty \subset L_{t,x}^2,$$

where the (lateral) energy $L_e^\infty L_{t,e^\perp}^2$ gives additional regularity of order $|\nabla|^{\frac{1}{2}}$ and the maximal function bound $L_e^2 L_{t,e^\perp}^\infty$ is controlled uniform in $e \in \mathbb{S}^{d-1}$. Apart from the usual Strichartz space S_λ, this will be essential (in one particular frequency interaction) in Section 2.4.2.

Outline of the chapter

In Section 2.3, we provide Strichartz estimates and the lateral version in $L_e^p L_{t,e^\perp}^q$ (including an $L_e^2 L_{t,e^\perp}^\infty$ estimate) for the linear Cauchy problem of the operator $L = \partial_t^2 + \Delta^2$. This is a consequence of the corresponding estimates for $e^{\pm it\Delta}$ which orginally appeared in [21], [22] and [2]. In the Appendix 2.A, we briefly outline proofs of the Strichartz estimates we need for $e^{\pm it\Delta}$ based on the calculation by Bejenaru in [2].

In Section 2.4.1, we construct spaces $Z^{\frac{d}{2}}, W^{\frac{d}{2}}$ such that

$$Z^{\frac{d}{2}} \subset C(\mathbb{R}, \dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)) \cap \dot{C}^1(\mathbb{R}, \dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d)), \tag{2.1.8}$$

$$\|u\|_{Z^{\frac{d}{2}}} \lesssim \|(u_0, u_1)\|_{\dot{B}^{2,1}_{\frac{d}{2}} \times \dot{B}^{2,1}_{\frac{d}{2}-2}} + \|Lu\|_{W^{\frac{d}{2}}}, \tag{2.1.9}$$

and similar Z^s, W^s for $s > \frac{d}{2}$ with data in $\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-2}(\mathbb{R}^d)$.

Further, we prove the algebra properties

$$Z^{\frac{d}{2}} \cdot Z^{\frac{d}{2}} \subset Z^{\frac{d}{2}}, \tag{2.1.10}$$

$$W^{\frac{d}{2}} \cdot Z^{\frac{d}{2}} \subset W^{\frac{d}{2}}, \tag{2.1.11}$$

in Section 2.4.2.

For the higher regularity, we need to provide the following embeddings

$$(Z^{\frac{d}{2}} \cap Z^s) \cdot (Z^{\frac{d}{2}} \cap Z^s) \subset Z^{\frac{d}{2}} \cap Z^s, \tag{2.1.12}$$

$$(W^{\frac{d}{2}} \cap W^s) \cdot (Z^{\frac{d}{2}} \cap Z^s) \subset W^{\frac{d}{2}} \cap W^s. \tag{2.1.13}$$

To be more precise, as in [54] and [2], from the dyadic estimates in Section 2.4.2, we infer for $s > \frac{d}{2}$,

$$\|uv\|_{Z^s} \lesssim \|u\|_{Z^s} \|v\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^s} \|u\|_{Z^{\frac{d}{2}}}, \;\; u, v \in Z^{\frac{d}{2}} \cap Z^s, \tag{2.1.14}$$

$$\|uv\|_{W^s} \lesssim \|u\|_{W^s} \|v\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^s} \|u\|_{W^{\frac{d}{2}}}, \;\; u \in W^{\frac{d}{2}} \cap W^s, \; v \in Z^{\frac{d}{2}} \cap Z^s, \tag{2.1.15}$$

in Section 2.4.2.

Finally, we provide the fixed point argument from [54] in Section 2.4.4. Especially, we state a few details of how to estimate the coefficients in the Cauchy problems (2.1.1) and (2.1.4). We then apply Theorem 2.1.1 to biharmonic wave maps in Corollary 2.1.2 .

We emphasize that the construction of the dyadic blocks Z_λ, W_λ are the analogues of Tataru's $F_\lambda, \; \Box F_\lambda$ spaces in [54], since we globally bound Lu in the spaces $L^1_t L^2_x$. In particular, the operator

$$L : Z_\lambda \to W_\lambda$$

is *continuous by construction* of Z_λ and W_λ. Combining this with (2.1.10) and (2.1.11), it suffices to estimate $\mathcal{Q}(u)$ in (2.1.1) with the identity (2.1.2).

As mentioned above, we can not fully rely on the usual Strichartz norm and have to use the control of the *lateral* Strichartz norm, which exploits additional smoothing in the proof of (2.1.10). This idea has been used in the similar context of the Schrödinger maps flow by Ionescu-Kenig [21], [22], Bejenaru [2] and Bejenaru-Ionescu-Kenig [3].

2.1.1 The main results

The system (2.1.1) is largely motivated by biharmonic wave maps, however the results for (2.1.1) are based on the structural extension of evolution equtions with a nonlinearity that, due to (2.1.2), can be considered *non-generic*.

We turn to general systems (2.1.1) and (2.1.4) for functions $u^1, \ldots, u^L$ with $L \in \mathbb{N}$, where we assume that $x \mapsto Q_x$, as well as $x \mapsto \Pi(x)$ are *real analytic* in the point $x_0 = 0$. That means we require the Taylor series at $x_0 = 0$ to have a positive radius of convergence and to coincide with Q, resp Π on a neighborhood of $x_0 = 0$ (where the series converges uniformly).

Theorem 2.1.1. *For $d \geq 3$ there exists $\delta > 0$ sufficiently small such that the following holds. Let $(u_0, u_1) \in \dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d) \times \dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d)$ such that*

$$\|u_0\|_{\dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)} + \|u_1\|_{\dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d)} \leq \delta. \tag{2.1.16}$$

Then (2.1.1) *and* (2.1.4) *have a global solution $u \in C(\mathbb{R}, \dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)) \cap \dot{C}^1(\mathbb{R}, \dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d))$ with*

$$\sup_{t \geq 0} \big(\|u(t)\|_{\dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)} + \|\partial_t u(t)\|_{\dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d)} \big) \leq C\delta, \tag{2.1.17}$$

for some $C > 0$. Further, the solution depends Lipschitz on the initial data.

If additionally $(u_0, u_1) \in \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-2}(\mathbb{R}^d)$ for some $s > \frac{d}{2}$, then also

$$(u(t), \partial_t u(t)) \in \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-2}(\mathbb{R}^d)$$

for all $t \in \mathbb{R}$ and in fact

$$\sup_{t \geq 0} \big(\|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} + \|\partial_t u(t)\|_{\dot{H}^{s-2}(\mathbb{R}^d)} \big) \leq C(\|u_0\|_{\dot{H}^s(\mathbb{R}^d)} + \|u_1\|_{\dot{H}^{s-2}(\mathbb{R}^d)}).$$

The theorem applies to (2.0.1), however it is not clear if the solution maps to $\mathbb{S}^L$ for all times. This is proven within the following (slightly more general) setup.

Let $N \subset \mathbb{R}^L$ be an embedded manifold and such that the nearest point projector $\Pi : \mathcal{V}_\varepsilon(N) \to N$ is analytic on N with a uniform lower bound on the radius of convergence.

Example: The above class includes the standard space forms $\mathbb{S}^L$, $\mathbb{H}^L$ and $\mathbb{T}^L = \mathbb{R}^L / \mathbb{Z}^L$. In general we take e.g. non-degenerate level sets of uniformly (real) analytic functions.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is uniformly analytic on $S \subset \mathbb{R}^d$, if there exists $C > 0$ such that for all $k \in \mathbb{N}$

$$\sup_{|\alpha| = k} \sup_{x \in S} |D^\alpha f(x)| \leq C^{k+1} k!.$$

We take a uniformly analytic pertubation of the round sphere $\mathbb{S}^{L-1}$. That means we take $\tilde{\eta} \in C^\omega(\mathbb{S}^{L-1})$, uniformly analytic with $\|\tilde{\eta}\|_{L^\infty} < 1$ and $\eta(x) = \tilde{\eta}(x/|x|)$ for $x \in \mathbb{R}^L \setminus \{0\}$. Then for

$$f_\eta(x) = |x|^2 - 1 - \eta(x), \;\; x \in \mathbb{R}^L \setminus \{0\}, \tag{2.1.18}$$

the manifold $N = f_\eta^{-1}(\{0\})$ will have the required property. Note that $\nabla_x \eta(x) \perp x$ and hence $\nabla_x f(x) \neq 0$ for $x \in \mathbb{R}^L \setminus \{0\}$.

The following Corollary holds

Corollary 2.1.2. *Let* $(u_0, u_1) : \mathbb{R}^d \to TN$*, i.e.* $u_0 \in N,\ \ u_1 \in T_{u_0}N$*, be a smooth map such that supp*$(\nabla u_0, u_1)$ *is compact,* $d \geq 3$*. Then if*

$$\|u_0\|_{\dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)} + \|u_1\|_{\dot{B}^{2,1}_{\frac{d}{2}-2}(\mathbb{R}^d)} \leq \delta,$$

where $\delta = \delta(d, N) > 0$ *is sufficienty small, then* (2.1.3)*, respectively the biharmonic wave maps equation* (19)*, has a global smooth solution* $u : \mathbb{R} \times \mathbb{R}^d \to N$ *with* $(u(0), \partial_t u(0)) = (u_0, u_1)$*.*

The statement of Corollary 2.1.2 has to be rigorously corrected to $u - p \in \dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)$ for $p = \lim_{x\to\infty} u_0(x)$ since $u_0 : \mathbb{R}^d \to N$ has no decay. Further, as mentioned in the introduction, equation (2.0.1) has parabolic scaling

$$u_\lambda(t, x) = u(\lambda^2 t, \lambda x),\ x \in \mathbb{R}^d, t \in \mathbb{R}\,.$$

Thus it holds

$$\lambda^{4-d} E(u(\lambda^2 t)) = E(u_\lambda(t))$$

and $d = 4$, the (energy) critical dimension, is included in our results Theorem 2.1.1 and Corollary 2.1.2. This is due to the larger Strichartz range for the dispersion rate $d/2$, whereas the low dimensional case for wave maps is more involved than [54] and has first been solved by Tataru in [55].

2.2 Preliminaries and Notation

The notation in this chapter is mostly common and largely consistent with [15]. However, we have to adapt the Littlewood-Paley projector for frequency and modulation to parabolic scaling (respectively the symbol of L), which is similar to e.g. [2]. Any special notation or definitions will be explained in the following.

Notation

For real $A, B \geq 0$ we write $A \lesssim B$ short for $A \leq cB$, where $c > 0$ is a constant. Likewise we write $A \sim B$ if there holds $A \lesssim B$ and $B \lesssim A$. The space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ will be as usual the Fréchet space

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^n) \mid \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta_x f(x)| < \infty,\ \alpha, \beta \in \mathbb{N}^d_0\},$$

where $x^\alpha = \Pi_{j=1}^d x_j^{\alpha_j},\ D^\beta_x f(x) = \frac{\partial^{|\beta|}}{\partial^{\beta_1}_{x_1} \cdots \partial^{\beta_d}_{x_d}} f(x)$. The semi-norms for the Fréchet property are given by

$$\|f\|_N = \sup_{|\alpha|,|\beta|\leq N} \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta_x f(x)|,\ \ N \in \mathbb{N},$$

and we dentote by $\mathcal{S}'(\mathbb{R}^d)$ its (dual) space of *tempered distributions* with the weak* topology. The Fourier transform is

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\, dx, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{2.2.1}$$

which extends well-defined to $\mathcal{S}'(\mathbb{R}^d)$ by duality

$$\langle \mathcal{F}(v), u\rangle = \langle v, \mathcal{F}(u)\rangle, \ \ u \in \mathcal{S}(\mathbb{R}^d), \ v \in \mathcal{S}'(\mathbb{R}^d).$$

We write $\hat{u}(\xi) = \mathcal{F}(u)(\xi)$ for short and indicate by $\mathcal{F}_{x'}(\xi)$ that the Fourier transform is taken over x' where $x = (x', \tilde{x})$ if necessary. The well known inversion formula for $\mathcal{F}$

$$u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \ \ u \in \mathcal{S}(\mathbb{R}^d),$$

defines the inverse of the isometry $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ through

$$\mathcal{F}^{-1}(u)(x) = (2\pi)^{-d}\mathcal{F}(u)(-x), \ x \in \mathbb{R}^d, u \in \mathcal{S}(\mathbb{R}^d),$$

and extends by duality to $\mathcal{S}'(\mathbb{R}^d)$.

Littlewood-Paley and multiplier

We now let $\varphi \in C^\infty(\mathbb{R})$ be a Littlewood-Paley function, i.e. such that

$$\text{supp}(\varphi) \subset (\frac{1}{2}, 2), \ \ \varphi \in [0,1], \quad \text{and} \ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}s) = 1, \ \ \text{for } s > 0. \tag{2.2.2}$$

We define the multiplier P, Q for $u \in \mathcal{S}'(\mathbb{R}^d), \ v \in \mathcal{S}'(\mathbb{R}^{1+d})$ and dyadic numbers $\lambda, \mu \in 2^{\mathbb{Z}}$ by

$$\widehat{P_\lambda(\nabla)u}(\xi) = \varphi(|\xi|/\lambda)\hat{u}(\xi), \quad \widehat{P_\lambda(D)v}(\tau,\xi) = \varphi((\tau^2 + |\xi|^4)^{\frac{1}{4}}/\lambda)\hat{v}(\tau,\xi),$$

$$\widehat{Q_\mu(D)v}(\tau,\xi) = \varphi(w(\tau,\xi)/\mu)\hat{v}(\tau,\xi),$$

$$P_{\leq\lambda} = \sum_{\tilde{\lambda}\leq\lambda} P_{\tilde{\lambda}}, \ \ Q_{\leq\mu} = \sum_{\tilde{\mu}\leq\mu} Q_{\tilde{\mu}},$$

$$P_{>\lambda} = I - P_{\leq\lambda}, \ \ Q_{>\mu} = I - Q_{\leq\mu}.$$

where

$$w(\tau,\xi) = \frac{|\tau^2 - |\xi|^4|}{(\tau^2 + |\xi|^4)^{\frac{1}{2}}} \sim ||\tau| - \xi^2|, \qquad (\tau^2 + |\xi|^4)^{\frac{1}{4}} \sim (|\tau| + \xi^2)^{\frac{1}{2}}.$$

Further, we write $v_\lambda = P_\lambda v = P_\lambda(D)v, \ P_{\lambda,\leq\mu} = P_\lambda Q_{\leq\mu}(D)$ for short and define

$$A_\lambda = \{(\tau,\xi) \mid \lambda/2 \leq (\tau^2 + \xi^4)^{\frac{1}{4}} \leq 2\lambda\},$$

$$A_\lambda^d = \{\xi \mid \lambda/2 \leq |\xi| \leq 2\lambda\}.$$

For a distribution $f \in \mathcal{S}'(\mathbb{R}^{d+1})$ we say that f is *localized at frequency* $\lambda \in 2^{\mathbb{Z}}$ if

$$\text{supp}(\hat{f}) \subset A_{\lambda/2} \cup A_\lambda \cup A_{2\lambda}$$

and a similar notation is used for $g \in \mathcal{S}'(\mathbb{R}^d)$ and A^d_λ. In addition, we need to localize in the sets

$$A_e := \left\{ \xi \mid \xi \cdot e \geq \frac{|\xi|}{\sqrt{2}} \right\}, \quad e \in \mathbb{S}^{d-1},$$

in order to exploit the smoothing effect for the linear equation. Thus, as in [2], we choose $\mathcal{M} \subset \mathbb{S}^{d-1}$ with $e \in \mathcal{M} \Rightarrow -e \in \mathcal{M}$ such that

$$\mathbb{R}^n = \bigcup_{e \in \mathcal{M}} A_e, \;\; \forall e \in \mathcal{M} \;:\; \#\{ \tilde{e} \in \mathcal{M} \mid A_e \cap A_{\tilde{e}} \neq \emptyset \} \leq K, \tag{2.2.3}$$

with a constant $K = K_d > 0$. Further we require a smooth partition of unity $\{h_e\}_{e \in \mathcal{M}}$ subordinate to $\{A_e\}_{e \in \mathcal{M}}$, i.e.

$$h_e \in C^\infty(\mathbb{R}^d), \;\; \text{supp}(h_e) \subset A_e, \;\; h_e \in [0,1] \tag{2.2.4}$$

$$\sum_{e \in \mathcal{M}} h_e(\xi) = 1, \;\; \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.2.5}$$

We note that this is possible since in particular for $x \in \mathbb{R}^n \setminus \{0\}$ we have $x \in A_e$ if and only if $\measuredangle(x, e) \leq \frac{\pi}{4}$.

Remark 2.2.1. An explicit example would be the set

$$\mathcal{M} = \big\{ x/|x|_2 \mid x \in \{0, 1, -1\}^d \setminus \{0_{\mathbb{R}^d}\} \big\}, \;\; |x|_2^2 = \sum_{i=1}^d x_i^2.$$

Here we take e.g. a non-negative $\psi \in C^\infty(\mathbb{R})$ with $\psi(x) = 1$ if $x \leq \pi/5$ and $\psi(x) = 0$ if $x > \pi/4$. Then we set

$$h_e(\xi) = \frac{\psi(\measuredangle(\xi, e))}{\sum_{\tilde{e} \in \mathcal{M}} \psi(\measuredangle(\xi, \tilde{e}))}, \;\; \measuredangle(v_1, v_2) = \arccos(\langle v_1, v_2 \rangle), \; v_1, v_2 \in \mathbb{S}^{d-1}.$$

We define the respective Fourier multiplier by

$$\widehat{P_e(\nabla)} v(\tau, \xi) = h_e(\xi) \hat{v}(\tau, \xi), \;\; v \in \mathcal{S}'(\mathbb{R}^{d+1}). \tag{2.2.6}$$

Finally, we choose $\chi \in C^\infty(\mathbb{R}^{d+1})$ such that

$$\chi(\tau, \xi) = \begin{cases} 1 & |\tau^2 - |\xi|^4| < \frac{\tau^2 + |\xi|^4}{100}, \\ 0 & |\tau^2 - |\xi|^4| > \frac{\tau^2 + |\xi|^4}{10}. \end{cases}$$

In order to have χ invariant under parabolic scaling, we choose $\chi(\tau,\xi) = \eta(|\frac{\tau^2-|\xi|^4}{\tau^2+|\xi|^4}|)$, where $\eta \in C^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$ and such hat $\eta(x) = 1$ if $|x| < 1/100$ and $\eta(x) = 0$ if $|x| > 1/10$. We then define

$$\widehat{P_0 v}(\tau,\xi) = \chi(\tau,\xi)\hat{v}(\tau,\xi), \quad (\widehat{1-P_0})v(\tau,\xi) = (1-\chi(\tau,\xi))\hat{v}(\tau,\xi). \tag{2.2.7}$$

Thus, we have

$$\text{supp}(\widehat{P_0 v}) \subset \left\{(\tau,\xi) \mid ||\tau| - \xi^2| \leq \frac{|\tau| + \xi^2}{10}\right\}, \tag{2.2.8}$$

$$\text{supp}((\widehat{1-P_0})v) \subset \left\{(\tau,\xi) \mid ||\tau| - \xi^2| \geq \frac{|\tau| + \xi^2}{100}\right\}. \tag{2.2.9}$$

Especially, measuring the distance to the characteristic surface P,

$$\text{dist}((\tau,\xi), P) \sim \frac{||\tau| - \xi^2|}{(|\tau| + \xi^2)^{\frac{1}{2}}}, \quad P = \left\{(\tau,\xi) \mid \tau^2 = \xi^4\right\},$$

we infer that $(1-P_0)v$ (with v being localized at frequency λ) is localized where

$$\text{dist}((\tau,\xi), P) \sim \lambda,$$

such that frequency $(\tau^2 + |\xi|^4)^{\frac{1}{4}} \sim \lambda$ and modulation $||\tau| - \xi^2| \sim \mu$ are of comparable size $\mu \sim \lambda$. For $P_0 v$ we have localization where

$$\text{dist}((\tau,\xi), P) = \mathcal{O}(\lambda),$$

with a small constant that suffices to obtain additional smoothing in the linear estimates of the following sections.

Function spaces

We define the spaces $\dot{H}^s(\mathbb{R}^d),\ H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ are defined as the closure of

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)} = \||\nabla|^s u\|_{L^2(dx)}$$
$$\|u\|_{H^s(\mathbb{R}^d)} = \|\langle\nabla\rangle^s u\|_{L^2(dx)},$$

in $\mathcal{S}(\mathbb{R}^d)$, where in the homogeneous case we restrict to $s > -d/2$. Here

$$\langle\nabla\rangle^s u = \mathcal{F}^{-1}((1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)), \quad |\nabla|^s u = \mathcal{F}^{-1}(|\xi|^s \hat{u}(\xi)).$$

The homogeneous Besov spaces $\dot{B}^{q,p}_s(\mathbb{R}^d),\ 1 \leq q,p < \infty$ are given by the closure of

$$\|u\|^p_{\dot{B}^{q,p}_s(\mathbb{R}^d)} = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{sp} \|P_\lambda u\|^p_{L^q_x},\ u \in \mathcal{S}(\mathbb{R}^d).$$

The semi-norms $\|u\|_{\dot{B}^{q,p}_s(\mathbb{R}^d)}$ degenerate if and only if $\mathrm{supp}(\hat{u}) \subset \{0\}$, i.e. if and only if u is a polynomial. We further have $\dot{H}^s(\mathbb{R}^d) \sim \dot{B}^{2,2}_s(\mathbb{R}^d),\ s > \frac{d}{2}$ and $\dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ is a well-defined Banach space.

Especially, we mention that in the *higher regularity* statement of Theorem 2.1.1, we observe that the solution $u(t) \in \dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$ and all $t \in \mathbb{R}$, which implies $\mathcal{F}_x(u(t,\cdot)) \in L^1(dx)$ for $t \in \mathbb{R}$.

We further introduce the *Hölder space* $C^{k,\alpha}(\mathbb{R}^d)$ with $k \in \mathbb{N}, \alpha \in (0,1]$, defined as usual by the norm

$$\{f \in C^k(\mathbb{R}^d) \mid \|u\|_{C^{k,\alpha}(\mathbb{R}^d)} = \sup_{|\beta|\leq k} \left\|\partial^\beta u\right\|_{L^\infty} + \sup_{|\beta|=k}\sup_{x\neq y} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha} < \infty\}.$$

Perturbative argument

We use the following (straight forward) wellposedness argument (which is similarly stated in e.g. [15, chapter 3.2] and [25]). The scaling argument in the second part of the Lemma was used in [54].

We first state the abstract Cauchy problem

$$\begin{cases} Lu = \mathcal{N}(u) & \text{in } (-T,T) \times \mathbb{R}^n \\ u[0] = f & \text{in } \ \mathbb{R}^n, \end{cases} \tag{2.2.10}$$

where L is an evolution operator of (time) order $d \in \mathbb{N}$, $\mathcal{N}(0) = 0,\ 0 \leq T \leq \infty,\ f = (f_0, \dots, f_{d-1}) \in D$ the initial data. Here we let $D, \tilde{D}$ to be Banach spaces and $Z, \tilde{Z}$ be Banach spaces such that test functions are dense in $Z, \tilde{Z}$.

We choose a function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(t) = 1$ for $t \in (-T,T)$. By Duhamel's formula we write the solution of (2.2.10) as

$$u(t) = \varphi(t)(S(t)f + V(\mathcal{N}(u(t)))),$$

where $L(Sf) = 0,\ S(0)f = f$ and $v = VF$ solves $Lv = F$ with initial data $v(0) = 0$. Then we have the following.

Lemma 2.2.2 (Wellposedness & persistence)**.** *Let Z, D, φ be as above, such that*

$$\|\varphi(Sf)\|_Z \leq C_1 \|f\|_D\,,\ \ f \in D \tag{2.2.11}$$

$$\sup_{t\in(-T,T)} (\|u(t)\|_D) \leq C_2 \|u\|_Z\,,\ u \in Z, \tag{2.2.12}$$

where $C_1, C_2 > 0$. Assume further that for $u, v \in B^Z(0,\delta)$ for some $\delta > 0$ there holds

$$\|\varphi(V(\mathcal{N}(u) - \mathcal{N}(u)))\|_Z \leq C_3(\|u\|_Z + \|v\|_Z)\,\|u-v\|_Z\,. \tag{2.2.13}$$

Then, provided

$$\|f\|_D \le \frac{\delta}{4C_1} < \frac{1}{24C_1C_3}, \tag{2.2.14}$$

the Cauchy problem (2.2.10) *has a unique solution in* $C(-T,T;D) \cap B^Z(0,\delta)$ *such that the map* $f \mapsto u(f)$ *is Lipschitz from* $B^D(0,(3C_1)^{-1}\delta)$ *into* $C(-T,T;D)$.
Now we additionally assume that $\tilde{Z}, \tilde{D}$ *satisfy* (2.2.11), (2.2.12) *and further there holds for* $u, v \in B^Z(0,\delta) \cap \tilde{Z}$

$$\begin{aligned} \|\varphi(V(\mathcal{N}(u) - \mathcal{N}(u)))\|_{\tilde{Z}} \le\ & C_3(\|u\|_Z + \|v\|_Z)\,\|u-v\|_{\tilde{Z}} \\ & + C_3(\|u\|_{\tilde{Z}} + \|v\|_{\tilde{Z}})\,\|u-v\|_Z\,. \end{aligned} \tag{2.2.15}$$

Then, provided also $f \in \tilde{D}$*, the solution from above satisfies* $u \in C(-T,T;\tilde{D})$ *and*

$$\sup_{t\in(-T,T)} (\|u(t)\|_{\tilde{D}}) \lesssim \|f\|_{\tilde{D}}\,.$$

Proof. We define the fixed point map via

$$T(u)(t) = \varphi(t)(S(t)f + V(\mathcal{N}(u))_{|t}), \quad u \in B(0,\delta).$$

Then it is easily seen form (2.2.11) and (2.2.13) with $v = 0$

$$\|Tu\|_Z \le C_1\|f\|_D + C_3\|u\|_Z^2 \le \frac{\delta}{4} + C_3\delta^2 < \delta,$$

by (2.2.14). Similarly it follows by (2.2.13)

$$\|T(u) - T(v)\|_Z \le C_3(2\delta)\,\|u-v\|_Z\,, \quad u, v \in B(0,\delta),$$

which gives the desired fixed point. The continuity of $t \mapsto \|u(t)\|_D$ is obtained by smooth approximation of $u \in Z$ and (2.2.12). The Lipschitz estimate against the initial data follows similarly by

$$\|T(u_1) - T(u_2)\|_Z \le C_1\|f_1 - f_2\|_D + 2C_3\delta\,\|u_1 - u_2\|_Z\,,$$

and hence

$$\|T(u_1) - T(u_2)\|_Z \lesssim \|f_1 - f_2\|_D\,.$$

Now for the second part, we define (as in [54]) the norm

$$\|u\|_{Z_\delta} = \frac{1}{\delta}\|u\|_Z + \frac{1}{M}\|u\|_{\tilde{Z}}\,, \quad M > 0,$$

and we consider $u \mapsto \varphi(V(\mathcal{N}(u)))$. Thus both (2.2.13) and (2.2.15) are defined on $B^{Z_\delta}(0,1) \subset Z_\delta$ and (again for $v = 0$)

$$\begin{aligned} \|\varphi(V(\mathcal{N}(u)))\|_Z &\le C_3 \|u\|_Z^2 \le C_3\delta \|u\|_Z\,, \\ \|\varphi(V(\mathcal{N}(u)))\|_{\tilde{Z}} &\le 2C_3 \|u\|_Z \|u\|_{\tilde{Z}} \le 2C_3\delta \|u\|_{\tilde{Z}}\,. \end{aligned}$$

In particular, we have for $u \in B^{Z_\delta}(0,1)$

$$\|\varphi(V(\mathcal{N}(u)))\|_{Z_\delta} \le \frac{2C_3}{M}\delta \|u\|_{\tilde{Z}} + C_3 \|u\|_Z \le 3C_3\delta \tag{2.2.16}$$

where $3C_3\delta < \frac{1}{2}$ by assumption. For the contraction, we have similarly

$$\begin{aligned} \|\varphi(V(\mathcal{N}(u_1) - \mathcal{N}(u_2)))\|_Z &\le 2C_3\delta \|u_1 - u_2\|_Z\,, \\ \|\varphi(V(\mathcal{N}(u_1) - \mathcal{N}(u_2)))\|_{\tilde{Z}} &\le 2C_3\delta \|u_1 - u_2\|_{\tilde{Z}} + 2C_3 M \|u_1 - u_2\|_Z\,. \end{aligned}$$

Thus we conclude

$$\|\varphi(V(\mathcal{N}(u_1) - \mathcal{N}(u_2)))\|_{Z_\delta} \le 6C_3\delta \|u_1 - u_2\|_{Z_\delta}$$

where $\delta < \frac{1}{6C_3}$. We now fix $M = 4C_1 \|f\|_{\tilde{D}}$. For the fixed point map, we obtain from (2.2.16)

$$\|T(u)\|_{Z_\delta} < C_1 \|f\|_D \frac{1}{\delta} + C_1 \|f\|_{\tilde{D}} \frac{1}{M} + \frac{1}{2} \le C_1 \|f\|_D \frac{1}{\delta} + \frac{3}{4} \le 1.$$

Hence we obtain a solution u in $B^{Z_\delta}(0,1) \subset B^Z(0,\delta)$ (which is unique in Z) and further

$$\|u\|_{\tilde{Z}} \le 4C_1 \|f\|_{\tilde{D}}\,.$$

□

2.2.1 Some preliminary results from harmonic analysis

In this section, we recall statements from Fourier Analysis, for which we refer to [15, chapter 2].

Lemma 2.2.3 (Bernstein)**.** *For $s \ge 0$, $u_\lambda = P_\lambda(D)$, $1 \le p \le q \le \infty$ we have*

$$\left\| |D|^{\pm s} u_\lambda \right\|_{L^p_{t,x}} \sim \lambda^{\pm s} \|u_\lambda\|_{L^p_{t,x}}\,, \quad \mathcal{F}(|D|^s u)(\tau,\xi) = (\tau^2 + |\xi|^4)^{\frac{s}{4}} \hat{u}(\tau,\xi) \tag{2.2.17}$$

$$\left\| |\partial_t|^{\frac{s}{2}} u_\lambda \right\|_{L^p_{t,x}} + \| |\nabla|^s u_\lambda \|_{L^p_{t,x}} \lesssim \lambda^s \|u_\lambda\|_{L^p_{t,x}}\,, \tag{2.2.18}$$

where $\mathcal{F}_t(|\partial_t|^s u)(\tau) = |\tau|^s \hat{u}(\tau)$, and $\mathcal{F}_x(|\nabla|^s u)(\xi) = |\xi|^s \hat{u}(\xi)$. Further

$$\|u_\lambda\|_{L^q_{t,x}} \lesssim \lambda^{d(\frac{1}{p} - \frac{1}{q})} \|u_\lambda\|_{L^p_{t,x}}\,. \tag{2.2.19}$$

Remark 2.2.4. Estimate (2.2.19) is the (bounded frequency) version of the Sobolev embedding

$$\dot{W}^{s,p}(\mathbb{R}^d) \subset L^{p*}(\mathbb{R}^d),\ \ p^* = q,\ s = d\big(\frac{1}{p} - \frac{1}{q}\big),$$

which is why we refer to it as Sobolev embedding in the following. Further Lemma 2.2.3 is stated in [15] for the Littlewood-Paley projector $P_\lambda(\nabla)$ in ξ only.

For the regularity statement of Theorem 2.1.1, we use the following

Lemma 2.2.5 (Embedding into $C^{k,\alpha}$). *Let $s > \frac{d}{2}$ such that $s - \frac{d}{2} \notin \mathbb{N}$. Then there holds*

$$\dot{H}^s(\mathbb{R}^d) \cap \dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d) \subset C^{\lfloor s - \frac{d}{2}\rfloor, s - \frac{d}{2} - \lfloor s - \frac{d}{2}\rfloor}(\mathbb{R}^d).$$

The Lemma is stated in [1, chapter 1.3.4]. However, for $f \in \dot{H}^s(\mathbb{R}^d)$ as defined in [1], it is required that $\hat{f} \in L^1_{loc}$. If we intersect as above, we especially embedd in the homogeneous Sobolev spaces defined in [1].

Lemma 2.2.6 (Littlewood-Paley). *For* $1 < p < \infty$

$$\Big\| \Big(\sum_\lambda |u_\lambda|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \sim \|u\|_{L^p}\,. \tag{2.2.20}$$

$$\sum_\lambda \|u_\lambda\|^2_{L^p} \lesssim \|u\|^2_{L^p}\,,\ \ 1 < p \leq 2 \tag{2.2.21}$$

$$\sum_\lambda \|u_\lambda\|^2_{L^p} \gtrsim \|u\|^2_{L^p}\,,\ \ 2 \leq p < \infty. \tag{2.2.22}$$

Remark 2.2.7. The statement is posed in [15, chapter 2] on $\mathbb{R}^d$ for $u_\lambda = P_\lambda(\nabla)u$. In the case where we localize in $\mathbb{R}^{d+1}$ by $P_\lambda(D)u = u_\lambda$, we define the dyadic regions by $(\tau, \xi) \mapsto (|\tau| + \xi^2)^{\frac{1}{2}}$, which is however equally correct. We refer to [41, chapter 8.2].

We now state the classical convolution inequalities.

Lemma 2.2.8 (Young). *Let $1 \leq r, p, q \leq \infty$ and further*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Then there holds

$$\|f * g\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}\,. \tag{2.2.23}$$

Lemma 2.2.9 (Hardy-Littlewood-Sobolev). *Let $0 < \alpha < d$ and $1 < p < q < \infty$ with*

$$1 - \frac{\alpha}{d} = \frac{1}{p} - \frac{1}{q}.$$

Then there holds

$$\big\| |x|^{-\alpha} * f \big\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}\,. \tag{2.2.24}$$

We further need the following statement on the decay of surface carried measures, for which we also refer to [49, chapter 8]

Lemma 2.2.10. *We let $\mathcal{S} \subset \mathbb{R}^{d+1}$ be a hypersurface with k non-vanishing principle curvature functions. Then for any $\Psi \in C_c^\infty(S)$ we have*

$$\left| \int_S e^{-i\xi\cdot x} \Psi(x) \, d\sigma(x) \right| \lesssim \frac{1}{(1+|\xi|)^{\frac{k}{2}}}, \tag{2.2.25}$$

where σ is the surface measure of $S \subset \mathbb{R}^{d+1}$.

In order to proof Strichartz estimates, we need the next Lemma for wich we denote the space of continuous linear operator between Banach spaces B_1 and B_2 by $\mathcal{L}(B_1, B_2)$.

Lemma 2.2.11 (TT^*)**.** *For a Hilbert space H and a Banach space B we consider a linear operator $T : H \to B$ and its adjoint $T^* : B^* \to H$ defined by*

$$\langle T^* u, v \rangle_H = (u, Tv), \;\; u \in B^*, \; v \in H.$$

Here $(\cdot,\cdot)$ is the duality of (B^*, B). Then we have eqauivalence of

(i) $T \in \mathcal{L}(H, B)$

(ii) $T^* \in \mathcal{L}(B^*, H)$

(iii) $TT^* \in \mathcal{L}(B^*, B)$

(iv) The form $\langle T^*u, T^*v\rangle_H, \;\; u, v \in B^*$ is bounded.

In any case

$$\|TT^*\|_{\mathcal{L}(B^*,B)} = \|T^*\|^2_{\mathcal{L}(B^*,H)} = \|T\|^2_{\mathcal{L}(H,B)} \, .$$

The heuristic argument for our (non-endpoint) Strichartz estimate, see e.g. [24] and [48], is as follows. The operator T in Lemma 2.2.11 is useful for the frequency localized linear propagator, such as

$$T(t)f = e^{\pm it\sqrt{-\Delta}} f, \quad \text{supp}(\hat{f}) \subset A^d_\lambda$$

which has a group structure. In praticular the TT^* bound in (iii) of Lemma 2.2.11 is accessible to Lemma 2.2.8 or Lemma 2.2.9 in $d = 1$ via $T(t)T^*(s) = T(t-s)$, which is controlled by a dispersive factor. In order to prove a dispersive estimate, we e.g. apply Lemma 2.2.10 (or a stationary phase method) to the convolution kernel of $T(t)$.

The T bound in (i) of Lemma 2.2.11 gives the homogeneous Strichartz estimates for the wave equation in this case. The inhomogeneous estimate follows by an application of (i) to the inhomogeneous part in the integral representation of a solution.

Then we would like to use (ii), however one needs to pass to a retarted integral opertor, which is clearified in the following Lemma by Christ-Kiselev.

Lemma 2.2.12 (Christ-Kiselev). *Let B_1, B_2 be Banach spaces and $K : J \times J \to \mathcal{L}(B_1, B_2)$, where $J \subset \mathbb{R}$ is an interval. Then we assume that the operator*

$$Tu(t) = \int_J K(t,s)u(s)\,ds,\ u \in B_1,$$

satisfies

$$\|Tu\|_{L^q(J,B_2)} \le C\,\|u\|_{L^p(J,B_1)},\ \ 1 \le p < q \le \infty.$$

Then

$$\tilde{T}u(t) = \int_J \chi\{s \le t\}K(t,s)u(s)\,ds,\ u \in B_1,$$

satisfies

$$\|\tilde{T}u\|_{L^q(J,B_2)} \lesssim_{p,q} C\,\|u\|_{L^p(J,B_1)}.$$

The next Lemma was proven by Keel-Tao in [24] and is suitable for the *endpoint* Strichartz estimate. Any pair (p, q) with $1 \le p, q \le \infty$ is said to be σ-admissible for $\sigma > 0$, if $(p, q, \sigma) \ne (2, \infty, 1)$ and

$$\frac{1}{p} + \frac{\sigma}{q} \le \sigma.$$

Lemma 2.2.13 (Keel-Tao). *Let H be a Hilbert space and $U(t) : H \to L^2_x$ be a familiy of operator such that*

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_H,$$
$$\|U(t)U^*(s)f\|_{L^\infty_x} \lesssim (1 + |t-s|)^{-\sigma}\,\|f\|_{L^1_x},$$

where $U^*(s)$ is the adjoint of $U(s)$. Then

$$\|U(t)f\|_{L^p_tL^q_x} \lesssim \|f\|_H, \tag{2.2.26}$$

$$\left\|\int U(s)^*F(s)\,ds\right\|_H \lesssim \|F\|_{L^{p'}_tL^{q'}_x}, \tag{2.2.27}$$

$$\left\|\int_{s<t} U(t)U(s)^*F(s)\,ds\right\|_{L^p_tL^q_x} \lesssim \|F\|_{L^{\tilde{p}'}_tL^{\tilde{q}'}_x}, \tag{2.2.28}$$

hold for all σ-admissible pairs $(p, q),\ (\tilde{p}, \tilde{q})$.

In case that $(p, q) \ne (2, \frac{2\sigma}{\sigma-1})$, the proof follows as outlined above by interpolating the energy estimate and the dispersive estimate in Lemma 2.2.13. The three estimates are then concluded from the TT^* argument Lemma 2.2.11 applying Lemma 2.2.8 (for the continuity of the form (iv)) and Christ-Kiselev's Lemma 2.2.12. The endpoint is slightly more delicate and requires untruncated dispersion for scaling reasons. We recommend the reader to consult [24] for further details.

The last Lemma in this section was used by Tataru in [54] and is stated in [15, chapter 2.4, Lemma 2.8]. The Lemma in particular gives a sufficient condition for a multiplier

$$Mf = \mathcal{F}^{-1}(m(\tau,\xi)\hat{u}(\tau,\xi))$$

to define a bounded operator $M : L^1_t L^2_x \to L^1_t L^2_x$. We note that even though $(\tau,\xi) \mapsto m(\tau,\xi)$ is smooth in our case, the classical Mikhlin theorem gives $L^{1,\infty}$ bounds.

Lemma 2.2.14. *Let $C > 0$ and $M = \mathcal{F}^{-1}(m(\tau,\xi)\mathcal{F}(\cdot))$ be a Fourier multiplier such that the following holds.*

(i) For any ξ, there holds supp$(\tau \mapsto m(\tau,\xi)) \subset A_\xi$, where A_ξ has measure $\leq C$.

(ii) For $N \geq 2$ there exists $C_N > 0$ such that

$$\|m\|_{L^\infty_{\tau,\xi}} + C^N \left\|\partial_\tau^N m(\tau,\xi)\right\|_{L^\infty_{\tau,\xi}} \leq C_N.$$

Then the operator

$$M : L^p_t L^2_x \to L^p_t L^2_x, \;\; 1 \leq p \leq \infty, \tag{2.2.29}$$

is continuous and $\|M\| \lesssim C_N$.

Proof. By Plancherel (in ξ) and Young's inequality (in t), it suffices to proof $K \in L^1_t L^\infty_\xi$, where

$$K(t,\xi) = \int e^{it\tau} m(\tau,\xi)\, d\tau.$$

However by $(i),\ (ii)$ it follows $\|K\|_{L^1_t L^\infty_\xi} \lesssim C_N C$ and by (ii) and integration by parts

$$|K(t,\xi)| = \left| \frac{(-1)^N}{|t|^N i^N} \int e^{it\tau} \partial_\tau^N m(\tau,\xi)\, d\tau \right| \lesssim \frac{C_N C}{C^N |t|^N},$$

by which

$$|K(t,\xi)| \lesssim \frac{C_N C}{(1 + C|t|)^N}.$$

□

2.2.2 $X^{s,b}$ spaces and their properties

We define the classical space $X^{s,b}_{\tau=h(\xi)}(\mathbb{R}^{d+1})$ for $s, b \in \mathbb{R}$ adapted to the equation

$$\partial_t u = ih(\nabla)u, \;\; h(\nabla)u = \mathcal{F}^{-1}(h(\xi)\hat{u}(\xi)), \tag{2.2.30}$$

where $\xi \mapsto h(\xi)$ is a real continuous phase, by the closure of the norm

$$\|u\|_{X^{s,b}(\mathbb{R}^{d+1})} = \big\|\langle\xi\rangle^s \langle|\tau - h(\xi)|\rangle^b \hat{u}(\tau,\xi)\big\|_{L^2_{\tau,\xi}}, \quad u \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$$

and similar for the homogeneous version $\dot{X}^{s,b}(\mathbb{R}^{d+1})$ replacing the brackets $\langle\cdot\rangle^s$ by $|\xi|^s$ and $\langle\cdot\rangle^b$ by $|\tau - h(\xi)|^b$.

We state the following Lemma, for which we refer to [53, chapter 2.6]

Lemma 2.2.15. *Let h be a (real) polynomial, $s, b \in \mathbb{R}$. Then*

$$\begin{aligned} &\big\|e^{ith(\nabla)} f\big\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim \|f\|_{H^s_x}, \\ &(X^{s,b}_{\tau=h(\xi)})' = X^{-s,-b}_{\tau=-h(-\xi)}, \quad \bar{X}^{s,b}_{\tau=-h(-\xi)} = X^{s,b}_{\tau=h(\xi)}, \end{aligned}$$

where $u \in \bar{X}^{s,b}$ if and only if $\bar{u} \in X^{s,b}$.
If $s \in \mathbb{R}, b > \frac{1}{2}$ then there holds

$$\|u\|_Y \lesssim \|u\|_{X^{s,b}_{\tau=h(\xi)}},$$

if for all $\tau \in \mathbb{R}$ and $f \in H^s_x(\mathbb{R}^d)$ there holds

$$\big\|e^{it(\tau + h(\nabla))} f\big\|_Y \lesssim \|f\|_{H^s_x(\mathbb{R}^d)}.$$

Further for $s \in \mathbb{R}, b > \frac{1}{2}$ and $\eta(t)$ a smooth compact cut-off function, we have

$$\|\eta(t)u(t)\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim \|u(0)\|_{H^s_x(\mathbb{R}^d)} + \|(\partial_t - ih(\nabla))u\|_{X^{s,b-1}_{\tau=h(\xi)}}.$$

The results that require $b > \frac{1}{2}$ are in particular useful to work with e.g. Strichartz estimates or energy estimates in the $X^{s,b}_{\tau=h(\xi)}$ space and to use perturbative methods directly in $X^{s,b}$.

There are different (similar) ways to define $X^{s,b}$, see e.g. [15] for wave equations. Most notably we can use (non-degenerate) space-time weights $a(\tau,\xi)^s$ instead of $\langle\xi\rangle^s$. For higher order (time) evolution operator, we can also replace $\tau - h(\xi)$ in the definition by a normalized symbol. For our purpose, we set

$$\|u\|_{\dot{X}^{s,b}} = \big\|(\tau^2 + |\xi|^4)^{\frac{s}{4}} |w(\tau,\xi)|^b \hat{u}(\tau,\xi)\big\|_{L^2_{\tau,\xi}}, \tag{2.2.31}$$

where

$$w(\tau,\xi) = \frac{|\tau^2 - |\xi|^4|}{(\tau^2 + |\xi|^4)^{\frac{1}{2}}} \sim ||\tau| - \xi^2|, \quad (\tau,\xi) \in \mathbb{R}^{1+d},$$

as defined above. In the context of wave equtions $h(\xi) = |\xi|$ these spaces have first been used (implicitly) by Klainerman-Machedon in [29] and for dispersive equations by Bourgain, we refer to [53] for a good introduction.

With the regularity imposed in Theorem 2.1.1, we have to consider the limit case $b = \frac{1}{2}$, for which the corresponding estimates in Lemma 2.2.15 fail. Further, it suffices for us to restrict to functions with a fixed frequency $\lambda \in 2^{\mathbb{Z}}$.

We define the Besov-type modification of (2.2.31). Let λ be a fixed dyadic number. Then for $b \leq \frac{1}{2}$ and $p \in [1, \infty)$ we set

$$\|f\|^p_{X^{b,p}_\lambda} = \sum_{\mu \leq 4\lambda^2} \mu^{pb} \, \|Q_\mu(D)f\|^p_{L^2_{t,x}}\,, \tag{2.2.32}$$

and denote by $X^{b,p}_\lambda$ the closure of the (semi-)norm in $\mathcal{S}$ restricted to functions f localized at frequency λ. This definition is extended as usual to the case $p = \infty$. We observe that $f \in X^{b,p}_\lambda$ has the representation

$$f = \sum_{\mu \leq 4\lambda^2} h_\mu + h, \tag{2.2.33}$$

where h is a solution of $Lh = 0$ (with initial data localized at frequency λ). Thus f is only well-defined up to homogeneous solutions $Lh = 0$. In the following, we will correct (2.2.32) by $\|h\|_{L^\infty_t L^2_x} + \|\partial_t h\|_{L^\infty_t \dot{H}^{-2}_x}$ as a limiting dyadic block ($\mu \searrow 0$), where $Lh = 0$ and $h(0) = f(0)$ and $\partial_t h(0) = \partial_t f(0)$.

More precisely, the atomic decomposition (2.2.33) has the form

$$h_\mu(t,x) = \int_{-\infty}^{\infty} \frac{e^{its}}{|s|^b} h^\mu(s,x)\, ds, \tag{2.2.34}$$

$$\|f\|^p_{\dot{X}^{b,p}_\lambda} \sim \sum_{\mu \lesssim \lambda^2} \left(\int_{-\infty}^{\infty} \|h^\mu(s,x)\|^2_{L^2_x}\; ds \right)^{\frac{p}{2}}. \tag{2.2.35}$$

Here the $h^\mu(s,\cdot)$ solves $Lh^\mu(s,\cdot) = 0$ for some $L^2 \times \dot{H}^{-2}$ initial data and is localized where $s \sim \mu$. Further (2.2.35) only holds up to $\mu = 0$ as mentioned above. We infer (2.2.34) and (2.2.35) by foliation, which also shows that the sum in (2.2.33) is well-defined distributionally for the cases $b < \frac{1}{2}$ and $p \geq 1$ or $b = \frac{1}{2}$ and $p = 1$.

We will give the foliation explicitly in the proof of Lemma 2.3.7. The analogue statements from Lemma 2.2.15 are proven in the following sections.

2.3 Strichartz estimates and (local) smoothing for the linear Cauchy problem

The goal of this section is to develope estimates for the linear equation

$$\begin{cases} \partial_t^2 u(t,x) + \Delta^2 u(t,x) = F(t,x) & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u[0] = (u(0), \partial_t u(0)) = (u_0, u_1), & \text{on } \mathbb{R}^d, \end{cases} \tag{2.3.1}$$

with data F, u_0, u_1. In the following we provide *Strichartz estimates* and a maximal function estimate for the Cauchy problem (2.3.1) in case $F \in L^1_t L^2_x$. The main results of this section summarize all necessary homogeneous bounds in Lemma 2.3.5 and the inhomogeneous bounds in Lemma 2.3.6. Further we give a proof of the trace estimate in Lemma 2.3.7.

We start with the classical Strichartz estimate, which follows directly from the results in Section 2.2.1, respectively the framework in Lemma 2.2.13 (especially for the endpoint we rely on the work of Keel-Tao).

Definition 2.3.1. We say that a pair (p, q) with $1 \leq p, q \leq \infty$ is *admissible* in dimension $d \in \mathbb{N},\ d \geq 2$ if there holds

$$\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}. \tag{2.3.2}$$

and $(p, q) \neq (2, \infty)$ in the case of $d = 2$.

Lemma 2.3.2 (Strichartz)**.** *Let u be a weak solution of (2.3.1) for data u_0, u_1, F. Then there holds*

$$\|u\|_{C(\mathbb{R},\dot{H}^\gamma)} + \|u\|_{L^p_t L^q_x} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-2}} + \|F\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}_x}, \tag{2.3.3}$$

where $(p, q), (\tilde{p}, \tilde{q})$ are admissible pairs with $q, \tilde{q} < \infty$ and $\gamma \in [0, 2]$ satisfies

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - \gamma = \frac{2}{\tilde{p}'} + \frac{d}{\tilde{q}'} - 4 \tag{2.3.4}$$

Proof. We prove the inequality for $P_\lambda(\nabla)u,\ P_\lambda(\nabla)F$, where λ is a dyadic number. Then (2.3.3) follows by Lemma 2.2.6 since $q, \tilde{q} < \infty$. Further, (2.3.3) is invariant under scaling

$$u_\lambda(t, x) = u(\lambda^2 t, \lambda x),\ F_\lambda = \lambda^4 F(\lambda^2 t, \lambda x),$$

which follows from (2.3.4). Especially, since $(P_\lambda u)_{\lambda^{-1}} = P_1 u_{\lambda^{-1}}$, we assume $\lambda \sim 1$. By Duhamels formula we obtain

$$u(t) = \cos(-t\Delta)u_0 + \frac{\sin(-t\Delta)}{(-\Delta)}u_1 + \int_0^t \frac{\sin(-(t-s)\Delta)}{(-\Delta)}F(s)\, ds.$$

Therefore, as used above already, we decompose

$$\sin(-t\Delta)f = \frac{1}{2i}(e^{-it\Delta}f - e^{it\Delta}f),\ \ \cos(-t\Delta)f = \frac{1}{2}(e^{-it\Delta}f + e^{it\Delta}f),$$

and by $\lambda \sim 1$ this can hence be estimated via

$$\widehat{U_\pm(t)f}(\xi) = \chi\{t \geq 0\}e^{\mp it\xi^2}\psi(|\xi|)\hat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\psi \in C_c^\infty((0,\infty))$ with $\psi(x) = 1$ for $x \in \operatorname{supp}(\varphi)$ and φ is the Littlewood-Paley function from Section 2.2. Clearly $U_\pm(t)$ extends to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ satisfying the energy bound in Lemma 2.2.13. For the dispersive estimate, we use

$$U_\pm(t)f = K_\pm(t,\cdot) *_d f, \;\; K_\pm(t,x) = \chi\{t \geq 0\}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi \mp it|\xi|^2}\psi(|\xi|)\,d\xi.$$

The kernel then applies to Lemma 2.2.10 with the characteristic $P_\pm = \{(\tau,\xi) \mid \pm\tau + \xi^2 = 0\}$, for which all principle curvature functions are non-vanishing ($(\tau,\xi) \neq (0,0)$). Thus, we conclude

$$\|U_\pm(t)f\|_{L^\infty(\mathbb{R}^d)} \lesssim (1+|t|)^{-\frac{d}{2}}\|f\|_{L^1(\mathbb{R}^d)},$$

and Lemma 2.2.13 applies. In particular (2.2.26) implies the homogeneous estimate, whereas since $U(s)^*$ is the adjoint of $U(s)$ on L^2_x, we obtain the inhomogeneous estimate from (2.2.28). This in particular implies (2.3.3) on $[0,\infty)$ and we apply this inequality to $u_-(t,x) = u(-t,x)$, $F_-(t,x) = F(-t,x)$, which in turn implies the full estimate. It remains to prove $u \in C_tL^2_x$, for which we refer to the argument (for the wave equation) in [24]. □

Corollary 2.3.3. Let u have Fourier support in A_λ. Then

$$\|u\|_{S_\lambda} \lesssim \|u(0)\|_{L^2} + \|\partial_t u(0)\|_{\dot{H}^{-2}} + \lambda^{-2}\|Lu\|_{L^1_tL^2_x}, \tag{2.3.5}$$

where

$$S_\lambda = \left\{ f \in C_tL^2_x \mid \operatorname{supp}(\hat{f}) \subset A_\lambda,\; \|f\|_{S_\lambda} = \sup_{(p,q)}\left(\lambda^{\frac{2}{p}+\frac{d}{q}-\frac{d}{2}}\|f\|_{L^p_tL^q_x}\right) < \infty \right\}$$

and the supremum is taken over admissible pairs (p,q).

Proof. We obtain by Lemma 2.2.3

$$\|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-2}} \lesssim \lambda^\gamma(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-2}}),$$

which by Lemma 2.3.3 and the gap (2.3.4) implies the desired estimate for all admissible pairs (p,q) with $q < \infty$. For the case $q = \infty$ in $d \geq 3$ we estimate by Soblev embedding in Lemma 2.2.3 for any $q \geq \frac{2d}{d-2}$

$$\begin{aligned}\lambda^{-\frac{d-2}{2}}\|u\|_{L^2_tL^\infty_x} &\lesssim \lambda^{-\frac{d}{2}+1+\frac{d}{q}}\|u\|_{L^2_tL^q_x}\\ &\lesssim \|u(0)\|_{L^2} + \|\partial_t u(0)\|_{\dot{H}^{-2}} + \lambda^{-2}\|Lu\|_{L^1_tL^2_x}.\end{aligned}$$

□

The Corollary 2.3.3 is not sufficient for our proof of bilinear estimates in Section 2.4.2 and we additionally need to apply a well known smoothing estimate for the Schrödinger group.
For this reason, we define the following norm (see also 2.A)

$$\|u\|^p_{L^p_eL^q_{t,e^\perp}} = \int_{-\infty}^{\infty}\left(\int_{[e]^\perp}\int_{-\infty}^{\infty}|u(t,re+x)|^q dt\,dx\right)^{\frac{p}{q}}dr, \;\; e \in \mathbb{S}^{d-1}. \tag{2.3.6}$$

In order to introduce the necessary notation, we recall Bejenaru's calculus from [2] (see also the Appendix 2.A) for the Cauchy problem (2.3.1). In the case $F = 0$ we have

$$\text{supp}(\hat{u}) \subset P = \{(\tau, \xi) \mid \tau^2 - |\xi|^4 = 0\},$$

which is a paraboloid in the variables (τ, ξ). More precisely, denoting by $\Xi = (\tau, \xi)$ the Fourier variables, we split the symbol (in case of general F)

$$\widehat{F}(\tau, \xi) = L(\Xi)\hat{u}(\tau, \xi) = -(\tau - \xi^2)(\tau + \xi^2)\hat{u}(\tau, \xi). \tag{2.3.7}$$

Hence, we further split in the Fourier space into

$$-(\tau + \xi^2)^{-1}\widehat{F}(\tau, \xi)\chi\{\tau > 0\} = (\tau - \xi^2)\hat{u}(\tau, \xi)\chi\{\tau > 0\}, \tag{2.3.8}$$

$$(-\tau + \xi^2)^{-1}\widehat{F}(\tau, \xi)\chi\{\tau \leq 0\} = (\tau + \xi^2)\hat{u}(\tau, \xi)\chi\{\tau \leq 0\}, \;\; |\xi| \neq 0, \tag{2.3.9}$$

and introduce coordinates adapted to a characteristic unit normal $e \in \mathbb{S}^{d-1}$. That means we use the change of coordinates

$$\Xi \mapsto (\tau, \xi \cdot e, \xi - (\xi \cdot e)e) =: (\tau, \xi_e, \xi_{e^\perp}) =: \Xi_e,$$

and the sets

$$A_e = \left\{\xi \mid \xi_e \geq \frac{|\xi|}{\sqrt{2}}\right\}, \quad B_e := \left\{(\tau, \xi) \mid ||\tau| - \xi^2| \leq \frac{|\tau| + \xi^2}{10}, \; \xi \in A_e\right\} \tag{2.3.10}$$

$$B_e^{\pm} := \left\{(\tau, \xi) \mid | \pm \tau - \xi^2| \leq \frac{|\tau| + \xi^2}{10}, \; \xi \in A_e\right\} = B_e \cap \{\pm\tau > 0\} \cup \{(0, 0)\}. \tag{2.3.11}$$

Then for any $(\tau, \xi) \in B_e$, we clearly have

$$|\tau| - \xi_{e^\perp}^2 \geq 0, \;\; \xi_e \sim (|\tau| + \xi^2)^{\frac{1}{2}}, \;\; \xi_e + \sqrt{|\tau| - \xi_{e^\perp}^2} \sim (|\tau| + \xi^2)^{\frac{1}{2}}, \tag{2.3.12}$$

and similar for $\pm\tau$ on $B_e^{\pm}$.

Especially, the latter two quantities in (2.3.12) are controlled by frequency. Also, if we assume that $\text{supp}(\hat{u}) \subset B_e$, then for $|\tau| + \xi^2 > 0$, we have from (2.3.8) and (2.3.9)

$$-(|\tau| + \xi^2)^{-1}\left(\xi_e + \sqrt{|\tau| - \xi_{e^\perp}^2}\right)^{-1}\widehat{F}(\tau, \xi) = \left(\sqrt{|\tau| - \xi_{e^\perp}^2} - \xi_e\right)\hat{u}(\tau, \xi), \tag{2.3.13}$$

Now, taking the FT in the variable Ξ_e, we obtain that (2.3.1) is equivalent to

$$\big(i\partial_{x_e} + D_{t,e^\perp}\big)\tilde{u}(t, x_e, x_{e^\perp}) = \tilde{F}(t, x_e, x_{e^\perp}), \tag{2.3.14}$$

where

$$\widehat{D_{t,e^\perp}u}(\tau, \xi_e, \xi_{e^\perp}) = \left(\sqrt{|\tau| - \xi_{e^\perp}^2}\right)\hat{u}(\tau, \xi), \tag{2.3.15}$$

$$\mathcal{F}(\tilde{F})(\tau, \xi_e, \xi_{e^\perp}) = -(|\tau| + \xi^2)^{-1}\left(\xi_e + \sqrt{|\tau| - \xi_{e^\perp}^2}\right)^{-1}\widehat{F}(\tau, e\xi_e + \xi_{e^\perp}) \tag{2.3.16}$$

$$\mathcal{F}(\tilde{u})(\tau, \xi_e, \xi_{e^\perp}) = \hat{u}(\tau, \xi_e e + \xi_{e^\perp}), \tag{2.3.17}$$

Remark 2.3.4. The calculations above apply to prove inhomogeneous linear estimates for (2.3.1) with $F \in L^1_e L^2_{t,e^\perp}$ that are based on on the reduction to (2.3.14). However, using the above notation for the sets B_e and A_e, we only need estimates for $F \in L^1_t L^2_x$ localized on $B_e \cap A_\lambda$. These estimates follow directly from Corollary 2.A.3 (a) and Lemma 2.A.5 (a) in the Appendix 2.A.

We now state the homogeneous estimates which follow from the Appendix 2.A.

Lemma 2.3.5 (Linear estimates I)**.** *Let $u_0, u_1 \in L^2(\mathbb{R}^d)$, $e \in \mathcal{M}$, $\lambda > 0$ dyadic with supp$(\hat{u}_0)$, supp$(\hat{u}_1) \subset A^d_\lambda \cap A_e$. Then the solution u of* (2.3.1) *with $F = 0$ satisfies*

$$\|u\|_{L^p_e L^q_{t,e^\perp}} \leq C\lambda^{\frac{d}{2}-\frac{1}{p}-\frac{(d+1)}{q}} \left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-2}}\right), \tag{2.3.18}$$

where (p, q) is an admissible pair. Further if $d \geq 3$ and $\hat{u}_0$, $\hat{u}_1$ have Fourier support in A^d_λ, then the solution u of (2.3.1) *with $F = 0$ satisfies*

$$\sup_{e\in\mathcal{M}} \|u\|_{L^2_e L^\infty_{t,e^\perp}} \leq C\lambda^{\frac{d-1}{2}}(\|u_0\|_{L^2_x} + \|u_1\|_{\dot{H}^{-2}_x}). \tag{2.3.19}$$

$$\|u\|_{L^p_t L^q_x} \leq C\lambda^{\frac{d}{2}-\frac{2}{p}-\frac{d}{q}} \left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-2}}\right). \tag{2.3.20}$$

Proof. By (2.1.6), we note

$$u(t) = \frac{1}{2}e^{-it\Delta}(u_0 - i(-\Delta)^{-1}u_1) + \frac{1}{2}e^{it\Delta}(u_0 + i(-\Delta)^{-1}u_1),$$

hence (2.3.18) follows from Corollary 2.A.3 and (2.3.19) follows from Lemma 2.A.5. Estimate (2.3.20) is the classical Strichartz estimate for the Schrödinger group, for which we refer to Corollary 2.3.3. □

Lemma 2.3.6 (Linear estimates II)**.** *For $e \in \mathcal{M}$ and $\lambda > 0$ a dyadic number let $F \in L^1_t L^2_x$ be localized in $A_\lambda \cap B_e$. Then the solution u of* (2.3.1) *with $u_0 = u_1 = 0$ satisfies*

$$\|u\|_{L^p_e L^q_{t,e^\perp}} \lesssim \lambda^{(d+1)(\frac{1}{2}-\frac{1}{q})-\frac{1}{p}-\frac{5}{2}} \|F\|_{L^1_t L^2_x}, \tag{2.3.21}$$

$$\sup_{\tilde{e}\in\mathcal{M}} \left(\|u\|_{L^2_{\tilde{e}} L^\infty_{t,\tilde{e}^\perp}} \right) \lesssim \lambda^{\frac{d}{2}-\frac{5}{2}} \|F\|_{L^1_t L^2_x}, \tag{2.3.22}$$

where (p, q) is an admissible pair. If $\hat{F}$ has support in A_λ, then the solution u of (2.3.1) *with $u_0 = u_1 = 0$ satisfies*

$$\|u\|_{L^p_t L^q_x} \lesssim \lambda^{d(\frac{1}{2}-\frac{1}{q})-\frac{2}{p}-2} \|F\|_{L^1_t L^2_x}, \tag{2.3.23}$$

where (p, q) is an admissible pair.

Proof. The estimate (2.3.23) is the classical Strichartz estimate, which is stated in Corollary (2.3.3). For the remaining bounds (2.3.21), (2.3.22), we decompose the solution

$$u(t) = \int_0^t \frac{\sin(-(t-s)\Delta)}{(-\Delta)} F(s)\,ds = \frac{1}{2i}\int_0^t e^{-i(t-s)\Delta}(-\Delta)^{-1}F(s)\,ds + \frac{1}{2i}\int_0^t e^{i(t-s)\Delta}(-\Delta)^{-1}F(s)\,ds.$$

Especially, we have the pointwise bound

$$\left|\int_0^t e^{\pm i(t-s)\Delta}(-\Delta)^{-1}F(s)\,ds\right| \leq \int_0^\infty |e^{\pm i(t-s)\Delta}(-\Delta)^{-1}F(s)|\,ds,$$

and observe (2.3.21) and (2.3.22) by Corollary 2.A.3 (a), Lemma 2.A.5 (a). If X denotes either one of the spaces on the LHS of (2.3.23) and (2.3.22), we estimate

$$\left\|\int_{-\infty}^t e^{\pm i(t-s)\Delta}(-\Delta)^{-1}F(s)\,ds\right\|_X \leq \int_{-\infty}^\infty \left\|e^{-\pm i(t-s)\Delta}(-\Delta)^{-1}F(s)\right\|_X ds \lesssim \int_{-\infty}^\infty \|(-\Delta)^{-1}F(s)\|_{L^2_x}\,ds.$$

Here we note that in order to use the Corollary and the Lemma, we verify that $e^{\mp is\Delta}(-\Delta)^{-1}F(s)$ has Fourier support (in ξ) in $(A^d_\lambda \cup A^d_{\lambda/2}) \cap A_e$ for all $s \in \mathbb{R}$. This follows since F is localized on $B_e \cap A_\lambda$ and hence also implies for normalized frequencies $\lambda \sim 1$

$$\left\|(-\Delta)^{-1}F\right\|_{L^1_tL^2_x} \lesssim \|F\|_{L^1_tL^2_x}\,.$$

□

The next lemma follows from the homogeneous estimates in Lemma 2.3.5, resp. Corollary 2.A.3 and Lemma 2.A.5.

Lemma 2.3.7 (Trace estimate)**.** *Let $F \in X_\lambda^{\frac{1}{2},1}$ for a dyadic number λ.*

(a) There holds

$$\sup_{e\in\mathcal{M}} \left(\|F\|_{L^2_eL^\infty_{te^\perp}} \right) \lesssim \lambda^{\frac{d-1}{2}} \|F\|_{X_\lambda^{\frac{1}{2},1}} \tag{2.3.24}$$

$$\|F\|_{L^p_tL^q_x} \lesssim \lambda^{\frac{d}{2}-\frac{d}{q}-\frac{2}{p}} \|F\|_{X_\lambda^{\frac{1}{2},1}}\,, \tag{2.3.25}$$

for any admissible pair (p,q).

(b) We additionally assume $\hat{F}(\tau,\cdot)$ has support in A_e for some $e \in \mathcal{M}$ and all $\tau \in \mathbb{R}$. Then there holds

$$\|F\|_{L^p_eL^q_{t,e^\perp}} \lesssim \lambda^{\frac{d}{2}-\frac{d+1}{q}-\frac{1}{p}} \|F\|_{X_\lambda^{\frac{1}{2},1}}\,, \tag{2.3.26}$$

where (p,q) is an admissible pair, $p \geq 2$.

Remark 2.3.8. In the following, we often use the dual estimates of (2.3.25) - (2.3.26), i.e.

$$\|F\|_{X_\lambda^{-\frac{1}{2},\infty}} \lesssim \lambda^{\frac{d}{2}-\frac{d}{q}-\frac{2}{p}} \|F\|_{L_t^{p'} L_x^{q'}},$$
$$\|F\|_{X_\lambda^{-\frac{1}{2},\infty}} \lesssim \lambda^{\frac{d}{2}-\frac{d+1}{q}-\frac{1}{p}} \|F\|_{L_e^{p'} L_{t,e^\perp}^{q'}},$$

Proof of Lemma 2.3.7. For $F \in X_\lambda^{\frac{1}{2},1}$, we have the representation

$$F = \sum_{\mu \leq 4\lambda^2} Q_\mu F + h,$$

where $Lh = 0$ as mentioned in the previous section. We want to use (2.2.34) and (2.2.35). However, here we split over $sign(\tau)$ and write

$$\begin{aligned}
\sum_{\mu \in 2^{\mathbb{Z}}} Q_\mu F &= \sum_{\mu \in 2^{\mathbb{Z}}} \int\int e^{ix\cdot\xi + it\tau} \varphi(w(\tau,\xi)/\mu)\hat{F}(\tau,\xi)\, d\tau\, d\xi \\
&= \sum_{\mu \in 2^{\mathbb{Z}}} \int\int \chi(s+\xi^2 > 0) e^{ix\cdot\xi + it(s+\xi^2)} \varphi(w(s+\xi^2,\xi)/\mu)\hat{F}(s+\xi^2,\xi)\, ds\, d\xi \\
&\quad + \sum_{\mu \in 2^{\mathbb{Z}}} \int\int \chi(-s+\xi^2 > 0) e^{ix\cdot\xi + it(s-\xi^2)} \varphi(w(s-\xi^2,\xi)/\mu)\hat{F}(s-\xi^2,\xi)\, ds\, d\xi \\
&= \sum_{\mu \in 2^{\mathbb{Z}}} \int e^{it(s-\Delta)} h_\mu^+(s)\, ds + \sum_{\mu \in 2^{\mathbb{Z}}} \int e^{it(s+\Delta)} h_\mu^-(s)\, ds,
\end{aligned}$$

where

$$h_\mu^\pm(s) = \chi\{\mu/2^{\frac{3}{2}} \leq |s| \leq 2\mu\} \int e^{ix\cdot\xi} \chi(\pm s + \xi^2 > 0)\varphi(w(s\pm\xi^2,\xi)/\mu)\hat{F}(s\pm\xi^2,\xi)\, d\xi,$$

and we used

$$\mu/2 \leq w(\tau,\xi) = ||\tau| - \xi^2| \frac{|\tau| + \xi^2}{(\tau^2 + |\xi|^4)^{\frac{1}{2}}} \leq \sqrt{2}||\tau| - \xi^2| \leq \sqrt{2} w(\tau,\xi) \leq 2\sqrt{2}\mu.$$

Now we assume there holds $\left\|e^{i\theta} e^{\pm it\Delta} f\right\|_X \lesssim \|f\|_{L_x^2}$ for some space X and all $\theta, t \in \mathbb{R}$, then

$$\begin{aligned}
\sum_\mu \sum_\pm \left\| \int e^{it(s\mp\Delta)} h_\mu^\pm(s)\, ds \right\|_X &\lesssim \sum_\mu \sum_\pm \mu^{\frac{1}{2}} \left(\int \left\| h_\mu^\pm(s) \right\|_{L_x^2}^2\, ds \right)^{\frac{1}{2}} \\
&\sim \sum_\mu \sum_\pm \mu^{\frac{1}{2}} \left\| \chi(\pm\tau > 0)\varphi(w(\pm\tau,\xi)/\mu)\hat{F}(\tau,\xi) \right\|_{L_{\xi,\tau}^2} \\
&\lesssim \sum_\mu \mu^{\frac{1}{2}} \|Q_\mu F\|_{L_{t,x}^2}.
\end{aligned}$$

Hence (2.3.25) follows from the Strichartz estimate for Schrödinger groups and Lemma 2.A.5, since (for the limiting dyadic block $\mu = 0$ with $Lh = 0$) we have (see Lemma 2.3.3, resp. Lemma 2.3.5)

$$\|h\|_X \lesssim \|h(0)\|_{L^2} + \|\partial_t h(0)\|_{\dot{H}^{-2}}\,, \tag{2.3.27}$$

where $X = \lambda^{\frac{d}{2}-\frac{2}{p}-\frac{d}{q}} L_t^p L_x^q$. For (2.3.24), (2.3.26), we use the decomposition

$$F = Q_{\leq\frac{\lambda^2}{16}} F + (1 - Q_{\leq\frac{\lambda^2}{16}})F.$$

Then we check that calculating $h_\mu^\pm(s)$ in the above argument for $Q_{\leq\frac{\lambda^2}{16}} F$, the function

$$\xi \mapsto \widehat{h_\mu^\pm(s)}(\xi)$$

has support in $A_{\lambda/2}^d \cup A_\lambda^d$ for all $s \in \mathbb{R}$. Hence, following the argument with

$$X = \lambda^{\frac{d}{2}-\frac{1}{p}-\frac{(d+1)}{q}} L_e^p L_{t,e^\perp}^q, \quad X = \bigcap_e \lambda^{\frac{d-1}{2}} L_e^2 L_{t,e^\perp}^\infty,$$

we obtain (2.3.24), (2.3.26) by Corollary 2.A.3 and Lemma 2.A.5 for $Q_{\leq\frac{\lambda^2}{16}} F$ on the LHS. For (2.3.26), we further note that by assumption $h_\mu^\pm(s)$ localizes in A_e for all $s \in \mathbb{R}$. The remaining estimates for $(1 - Q_{\leq\frac{\lambda^2}{16}})F$ are equivalent to

$$\|(1 - Q_{\leq\frac{\lambda^2}{16}})F\|_X \lesssim \lambda\|(1 - Q_{\leq\frac{\lambda^2}{16}})F\|_{L^2_{t,x}},$$

which follow from Sobelev embedding (thus the restriction to $p \geq 2$). As above, we obtain the estimates for the $Lh = 0$ part of the limiting dyadic block $\mu = 0$ by Lemma 2.3.5. □

2.4 Global wellposedness in the scaling critical Besov space

In this section, we define the space $Z^{\frac{d}{2}}$ that solves the division problem for the Cauchy problems (2.1.1) and (2.1.4) (as stated above in Theorem 2.1.1). Further, we establish the bilinear estimates in Section 2.4.2 that are necessary to prove the Lipschitz bound in Lemma 2.2.2 and finally state the proof of Theorem 2.1.1. The proof of Corollary 2.1.2 is a consequence of Theorem 2.1.1 and will be stated in the next section.

2.4.1 Function spaces

We now define the dyadic building blocks of the function spaces $Z^{\frac{d}{2}}$, $W^{\frac{d}{2}}$ and use the convention

$$\|\cdot\|_{\lambda B_\lambda} = \lambda^{-1}\|\cdot\|_{B_\lambda}\,.$$

We set

$$Z_\lambda = X_\lambda^{\frac{1}{2},1} + Y_\lambda, \tag{2.4.1}$$

where Y_λ is the closure of

$$\{ f \in \mathcal{S} \mid \operatorname{supp}(\hat{f}) \subset A_\lambda,\ \|f\|_{Y_\lambda} < \infty \},$$
$$\|f\|_{Y_\lambda} = \lambda^{-2} \|Lf\|_{L_t^1 L_x^2} + \|f\|_{L_t^\infty L_x^2},$$

and the norm of Z_λ is given by

$$\|u\|_{Z_\lambda} = \inf_{u_1+u_2=u} \big(\|u_1\|_{X_\lambda^{\frac{1}{2},1}} + \|u_2\|_{Y_\lambda} \big).$$

For the nonlinearity, we construct $W_\lambda = L(Z_\lambda)$, i.e.

$$W_\lambda = \lambda^2 \big(X_\lambda^{-\frac{1}{2},1} + (L_t^1 L_x^2)_\lambda \big) \tag{2.4.2}$$

where $(L_t^1 L_x^2)_\lambda$ is the closure of

$$\{ F \in \mathcal{S} \mid \operatorname{supp}(\hat{F}) \subset A_\lambda,\ \|F\|_{L_t^1 L_x^2} < \infty \},$$

and

$$\|F\|_{W_\lambda} = \lambda^{-2} \inf_{F_1+F_2=F} \big(\|F_1\|_{X_\lambda^{-\frac{1}{2},1}} + \|F_2\|_{L_t^1 L_x^2} \big).$$

Then, we define

$$\|u\|_Z = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{\frac{d}{2}} \|P_\lambda(D)u\|_{Z_\lambda}, \tag{2.4.3}$$

$$\|F\|_W = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{\frac{d}{2}} \|P_\lambda(D)F\|_{W_\lambda}, \tag{2.4.4}$$

and

$$\|u\|_{Z^s}^2 = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{2s} \|P_\lambda(D)u\|_{Z_\lambda}^2 \quad \text{for } s > \frac{d}{2}. \tag{2.4.5}$$

$$\|F\|_{W^s}^2 = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{2s} \|P_\lambda(D)F\|_{W_\lambda}^2 \quad \text{for } s > \frac{d}{2}. \tag{2.4.6}$$

Embeddings, linear estimates and continuous operator

In this section, we provide some useful embeddings and multiplier theorems concerning Z_λ and W_λ. We also show that the solution of (2.3.1) satisfies $u \in Z^s$ if $Lu \in W^s$ with the correct initial regularity $\dot{B}_{\frac{d}{2}}^{2,1} \times \dot{B}_{\frac{d}{2}-2}^{2,1}$ or $\dot{H}^s \times \dot{H}^{s-2}$, respectively for $s > \frac{d}{2}$.

At the end of this section, we show that Z_λ bounds

$$\lambda^{-\frac{1}{2}} L_e^\infty L_{t,e^\perp}^2, \quad \bigcap_e \lambda^{\frac{d-1}{2}} L_e^2 L_{t,e^\perp}^\infty,$$

in a suitable sense. Therefore, we apply the following heurstic argument, used similarly in [2] for Schrödinger maps. For solving (2.3.1) by $u = V(F)$ with $u_0 = u_1 = 0$, we rely on the inhomogeneous Strichartz estimate Lemma 2.3.6

$$\|V(F)\|_{\lambda^{-\frac{1}{2}} L_e^\infty L_{t,e^\perp}^2} \lesssim \lambda^{-2} \|F\|_{L_t^1 L_x^2}, \quad (\tau, \xi) \in \text{supp}(\widehat{P_0 F})$$

and otherwise on *inverting the symbol* of L

$$V(F) = \mathcal{F}^{-1}\Big(\frac{\widehat{F}(\tau,\xi)}{\tau^2 - |\xi|^4}\Big), \quad (\tau, \xi) \in \text{supp}(\widehat{(1 - P_0)F}).$$

We first consider the following Lemma, which clearifies how the the spaces Z_λ, W_λ behave under modulation cut-off and is essentially from [54] (adapted to the paraboloid $\tau^2 = |\xi|^4$).

Lemma 2.4.1. *The following operator are continuous for $1 \leq p < \infty$ with norms that are uniformly bounded in $\mu \leq 4\lambda^2$.*

$$(a) \;\; P_{\lambda, \leq \mu}, \; P_\lambda P_0 : \; L_t^p L_x^2 \to L_t^p L_x^2, \;\; \mu \leq 4\lambda^2$$

$$(b) \;\; (1 - Q_{\leq \mu}) P_\lambda : \; Y_\lambda \to \mu^{-1} L_t^1 L_x^2, \;\; \mu \leq 4\lambda^2.$$

Proof. We follow Tataru's argument in [54], which we stated in Lemma 2.2.14 in Section 2.2.1. This argument applies, similar to [54], to $P_{\lambda,\leq\mu}$ with multiplier

$$m_{\mu,\lambda}(\tau,\xi) = \sum_{\tilde{\mu} \leq \mu} \varphi((\tau^2 + \xi^4)^{\frac{1}{4}}/\lambda) \varphi(w(\tau,\xi)/\tilde{\mu}),$$

since there holds for $N \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$ fixed

$$|\partial_\tau^N m_{\mu,\lambda}(\tau,\xi)| \lesssim_N \mu^{-N}, \;\; \text{supp}(m_{\mu,\lambda}) \subset \{(\tau,\xi) \mid ||\tau| - \xi^2| \leq 2\mu\}. \tag{2.4.7}$$

For the second operator

$$P_\lambda P_0 u = \mathcal{F}^{-1}(\varphi((\tau^2 + \xi^4)^{\frac{1}{4}}/\lambda)) \chi(\tau,\xi) \hat{u}(\tau,\xi))$$

in (a), we note that χ is invariant under scaling and hence the claim reduces to continuity of $P_1 P_0 : L_t^p L_x^2 \to L_t^p L_x^2$. This follows directly from the above argument.

Now for part (b), we write

$$\begin{aligned}
\mathcal{F}(1 - Q_{\leq\mu}) P_\lambda u)(\tau,\xi) &= \big(1 - \sum_{\tilde{\mu} \leq \mu} \varphi(w(\tau,\xi)/\tilde{\mu})\big) \varphi((\tau^2 + \xi^4)^{\frac{1}{4}}/\lambda) \hat{u}(\tau,\xi) \\
&= \mu^{-1}\lambda^{-2}\big(1 - \sum_{\tilde{\mu} \leq \mu} \varphi(w(\tau,\xi)/\tilde{\mu})\big) \frac{\varphi((\tau^2 + \xi^4)^{\frac{1}{4}}/\lambda)\mu\lambda^2}{w(\tau,\xi)(\tau^2 + \xi^4)^{\frac{1}{2}}} \widehat{Lu}(\tau,\xi) \\
&=: \mu^{-1}\lambda^{-2} \tilde{m}_{\mu,\lambda}(\tau,\xi) \widehat{Lu}(\tau,\xi).
\end{aligned}$$

It hence suffices to prove continuity of the operator $\mathcal{F}^{-1}(\tilde{m}_{\mu,\lambda}\mathcal{F}(\cdot))$ on $L^1_tL^2_x$. As in Lemma 2.2.14, by Young's inequality and Plancherel, this reduces to a proof of $K \in L^1_tL^\infty_\xi$, where

$$K(t,\xi) = \int e^{it\tau}\tilde{m}_{\mu,\lambda}(\tau,\xi)\, d\tau,$$

is the convolution kernel (in t). This follows similarly as in the proof for the cone in [54]. We sketch the argument following the proof in [15, chapter 2.4]. There holds

$$||\tau| - \xi^2|\tilde{m}_{\mu,\lambda}(\tau,\xi) + ||\tau| - \xi^2|^3\partial_\tau^2\tilde{m}_{\mu,\lambda}(\tau,\xi) \lesssim \mu. \tag{2.4.8}$$

Hence, considering the support

$$\{(\tau,\xi) \mid ||\tau| - \xi^2| \geq \mu/\sqrt{2}, \quad |\tau| + \xi^2 \leq 4\sqrt{2}\lambda^2\ \},$$

we infer

$$\left|\int e^{it\tau}\tilde{m}_{\mu,\lambda}(\tau,\xi)\, d\tau\right| \lesssim \mu\log(\lambda^2/\mu), \qquad \left|t^2\int e^{it\tau}\tilde{m}_{\mu,\lambda}(\tau,\xi)\, d\tau\right| \lesssim \mu^{-1}.$$

Integration gives boundedness of the following terms (uniform in μ, λ)

$$\|K\|_{L^1_tL^2_\xi} \lesssim \int_{|t|\leq\frac{1}{4\sqrt{2}\lambda^2}} \|K(t,\cdot)\|_{L^\infty}\, dt + \int_{|t|\geq\frac{\sqrt{2}}{\mu}} \|K(t,\cdot)\|_{L^\infty}\, dt + \int_{\frac{1}{4\sqrt{2}\lambda^2}\leq|t|\leq\frac{\sqrt{2}}{\mu}} \|K(t,\cdot)\|_{L^\infty}\, dt.$$

For the last term, we estimate

$$\|K(t,\cdot)\|_{L^\infty} \lesssim \int_{\frac{\mu}{\sqrt{2}}\leq||\tau|-\xi^2|\leq\frac{1}{|t|}} \frac{\mu}{||\tau|-\xi^2|}\, d\tau + \frac{1}{t^2}\int_{||\tau|-\xi^2|\geq\frac{1}{|t|}} \frac{\mu}{||\tau|-\xi^2|^3}\, d\tau \lesssim \mu(1-\log(|t|\mu)),$$

and hence

$$\int_{\frac{1}{4\sqrt{2}\lambda^2}\leq|t|\leq\frac{\sqrt{2}}{\mu}} \|K(t,\cdot)\|_{L^\infty}\, dt \lesssim 1.$$

□

Lemma 2.4.2. *We have*

$$W_\lambda \subset \lambda^3 L^2_{t,x}, \quad Z_\lambda \subset \lambda^{\frac{d}{2}} L^\infty_{t,x} \tag{2.4.9}$$

$$X_\lambda^{\frac{1}{2},1} \subset Z_\lambda \subset X_\lambda^{\frac{1}{2},\infty}. \tag{2.4.10}$$

Proof. For (2.4.10), we note that $X_\lambda^{\frac{1}{2},1} \subset Z_\lambda$ follows by definiton and $Z_\lambda \subset X_\lambda^{\frac{1}{2},\infty}$ is proven as follows.

The norm of $X_\lambda^{\frac{1}{2},\infty}$ is estimated against the norm of the $X_\lambda^{\frac{1}{2},1}$ part and further, for the $L^1_tL^2_x$ part in Y_λ, we deduce from Lemma 2.3.7

$$\begin{aligned}\|u_\lambda\|_{X_\lambda^{\frac{1}{2},\infty}} &\lesssim \lambda^{-2}\|Lu_\lambda\|_{X_\lambda^{-\frac{1}{2},\infty}} + \|u(0)\|_{L^2_x} + \|\partial_t u(0)\|_{H^{-2}_x} \\ &\lesssim \lambda^{-2}\|Lu_\lambda\|_{L^1_tL^2_x} + \|u_\lambda\|_{L^\infty_tL^2_x},\end{aligned}$$

which reads as

$$\|u_\lambda\|_{X_\lambda^{\frac{1}{2},\infty}} \lesssim \|u_\lambda\|_{Y_\lambda}, \;\; u_\lambda \in Y_\lambda.$$

Concerning (2.4.9) in the Lemma, we note

$$\|u_\lambda\|_{L^2_{t,x}} \lesssim \lambda \|u_\lambda\|_{L^1_t L^2_x} \sim \lambda^3 \|u_\lambda\|_{\lambda^2 L^1_t L^2_x}, \;\; u_\lambda \in L^1_t L^2_x,$$

where we used that $\hat{u}_\lambda(\cdot, \xi)$ is localized (in τ) on an interval on length $\sim \lambda^2$. Hence, since also,

$$\|u_\lambda\|_{L^2_{t,x}} \lesssim \lambda \sum_{\mu \lesssim \lambda^2} \mu^{-\frac{1}{2}} \|Q_\mu(u_\lambda)\|_{L^2_{t,x}}, \quad u_\lambda \in X_\lambda^{-\frac{1}{2},1},$$

we obtain the first claim. For the $L^\infty_{t,x}$ embedding, we estimate similarly by Lemma 2.3.7

$$\|u_\lambda\|_{L^\infty_{t,x}} \lesssim \lambda^{\frac{d}{2}} \|u_\lambda\|_{X_\lambda^{\frac{1}{2},1}}.$$

For the Y_λ part, we obtain by a direct application of the classical Strichartz estimate

$$\|u_\lambda\|_{L^\infty_{t,x}} \lesssim \|u(0)\|_{\dot{H}^{\frac{d}{2}}} + \|\partial_t u(0)\|_{\dot{H}^{\frac{d}{2}-2}} + \lambda^{\frac{d}{2}-2} \|Lu\|_{L^1_t L^2_x} \lesssim \lambda^{\frac{d}{2}} \|u_\lambda\|_{Y_\lambda}.$$

from Lemma 2.3.5 and Lemma 2.3.6 for $p = q = \infty$. □

Proposition 2.4.3. There holds

$$Z^{\frac{d}{2}} \subset C(\mathbb{R}, \dot{B}^{2,1}_{\frac{d}{2}}) \cap \dot{C}^1(\mathbb{R}, \dot{B}^{2,1}_{\frac{d}{2}-2}) \tag{2.4.11}$$

$$Z^s \subset C(\mathbb{R}, \dot{H}^s) \cap \dot{C}^1(\mathbb{R}, \dot{H}^{s-2}) \tag{2.4.12}$$

Further we have

$$\|u\|_{Z^{\frac{d}{2}}} \lesssim \|(u(0), \partial_t u(0))\|_{\dot{B}^{2,1}_{\frac{d}{2}} \times \dot{B}^{2,1}_{\frac{d}{2}-2}} + \|Lu\|_{W^{\frac{d}{2}}}, \tag{2.4.13}$$

$$\|u\|_{Z^s} \lesssim \|(u(0), \partial_t u(0))\|_{\dot{H}^s \times \dot{H}^{s-2}} + \|Lu\|_{W^s}, \quad s > \frac{d}{2}, \tag{2.4.14}$$

$$\|Lu\|_{W^{\frac{d}{2}}} \lesssim \|u\|_{Z^{\frac{d}{2}}}, \;\; \|Lu\|_{W^s} \lesssim \|u\|_{Z^s}, \quad s > \frac{d}{2}. \tag{2.4.15}$$

Proof. The claim (2.4.15) follows from the definition of Z_λ, W_λ since $\lambda^2 L^1_t L^2_x = LY_\lambda$ and for the $X_\lambda^{\frac{1}{2},p}$ part, we use

$$\|Lu\|_{X_\lambda^{-\frac{1}{2},1}} \lesssim \lambda^2 \|u\|_{X_\lambda^{\frac{1}{2},1}}.$$

For (2.4.11) and (2.4.12), if suffices to show

$$\|P_\lambda(D)u\|_{L^\infty_t \dot{B}^{2,1}_{\frac{d}{2}}} + \|P_\lambda(D)\partial_t u(t)\|_{L^\infty_t \dot{B}^{2,1}_{\frac{d}{2}-2}} \lesssim \lambda^{\frac{d}{2}} \|P_\lambda(D)u\|_{Z_\lambda},$$

where by Bernstein

$$\|P_\lambda(D)\partial_t u(t)\|_{L_t^\infty \dot{B}^{2,1}_{\frac{d}{2}-2}} \lesssim \|P_\lambda(D)u\|_{L_t^\infty \dot{B}^{2,1}_{\frac{d}{2}}}. \tag{2.4.16}$$

Then, since

$$\|P_\lambda(D)u\|_{L_t^\infty \dot{B}^{2,1}_{\frac{d}{2}}} \leq \sum_{\tilde{\lambda}\leq\lambda} \left(\tilde{\lambda}/\lambda\right)^{\frac{d}{2}} \lambda^{\frac{d}{2}} \|P_\lambda(D)P_{\tilde{\lambda}}(\nabla)u\|_{L_t^\infty L_x^2}, \tag{2.4.17}$$

the embedding and the continuity in time follow from $Z_\lambda \subset S_\lambda \subset C_t L_x^2$ and we proceed similarly for the emdedding of Z^s using square sums. Now for (2.4.13) and (2.4.14), we use Duhamel's formula

$$u = S(u(0), \partial_t u(0)) + V(Lu),$$

where $S(u_0, u_1)$ solves (2.3.1) for $F = 0$ and VF solves (2.3.1) for $u_0 = u_1 = 0$. The homogeneous solution is estimated by the Strichartz bound in Lemma 2.3.5 in the energy case $p = \infty, q = 2$. This is also directly verified by

$$\mathcal{F}_x(P_\lambda(D)S(u(0), \partial_t u(0)))(t,\xi) = \varphi(2^{\frac{1}{4}}|\xi|/\lambda)(\cos(|\xi|^2 t)\widehat{u(0)}(\xi) + |\xi|^{-2}\sin(|\xi|^2 t)\widehat{\partial_t u(0)}(\xi)),$$

and hence

$$\|P_\lambda S(u(0), u_t(0))\|_{Z_\lambda} \lesssim \|P_\lambda S(u(0), u_t(0))\|_{L_t^\infty L_x^2} \lesssim \|u_\lambda(0)\|_{L^2} + \|\partial_t u_\lambda(0)\|_{H^{-2}}.$$

For the inhomogeneous solution $V(Lu)$ we estimate the $X_\lambda^{\frac{1}{2},p}$ part by

$$\|V(Lu_\lambda)\|_{X_\lambda^{\frac{1}{2},1}} \lesssim \lambda^{-2}\|Lu_\lambda\|_{X_\lambda^{-\frac{1}{2},1}}, \tag{2.4.18}$$

and for Y_λ, we use Lemma 2.3.6 in order to conclude

$$\|V(Lu_\lambda)\|_{Y_\lambda} = \lambda^{-2}\|Lu_\lambda\|_{L_t^1 L_x^2} + \|u_\lambda\|_{L_t^\infty L_x^2} \lesssim \lambda^{-2}\|Lu_\lambda\|_{L_t^1 L_x^2}.$$

□

We further estimate the lateral Strichartz norm and establish the maximal function estimate.

Proposition 2.4.4. For any dyadic number $\lambda \in 2^{\mathbb{Z}}$ we have

$$Z_\lambda \subset S_\lambda \cap \sum_{e\in\mathcal{M}} S_\lambda^e, \tag{2.4.19}$$

$$Z_\lambda \subset \bigcap_{e\in\mathcal{M}} \lambda^{\frac{n-1}{2}} L_e^2 L_{t,e^\perp}^\infty. \tag{2.4.20}$$

where S_λ^e is the closure of

$$\left\{ f \in \mathcal{S} \mid \operatorname{supp}(\hat{f}) \subset A_\lambda,\ \|f\|_{S_\lambda^e} = \sup_{(p,q)} \left(\lambda^{\frac{1}{p}+\frac{(d+1)}{q}-\frac{d}{2}} \|f\|_{L_e^p L_{t,e^\perp}^q}\right) < \infty \right\}$$

with (p,q) ranging over all admissible pairs with $p \geq 2$.

Proof. For (2.4.19), we first consider the embedding $Z_\lambda \subset S_\lambda$. Thus, the $X_\lambda^{\frac{1}{2},1}$ part satisfies for any admissible pair (p, q)

$$\lambda^{\frac{2}{p}+\frac{d}{q}-\frac{d}{2}} \|u_\lambda\|_{L_t^p L_x^q} \lesssim \|u_\lambda\|_{X_\lambda^{\frac{1}{2},1}},$$

by Lemma 2.3.7. Likewise, we obtain the same bound against the Y_λ part by Lemma 2.3.5 and Lemma 2.3.6. For the S_λ^e embedding, we decompose as follows

$$u_\lambda = \sum_{e \in \mathcal{M}} u_\lambda^e, \quad u_\lambda^e = P_e(\nabla) u_\lambda, \tag{2.4.21}$$

which suffices to obtain (2.4.19) for the $X_\lambda^{\frac{1}{2},1}$ part directly from Lemma 2.3.7. Now, considering the Y_λ part of Z_λ, we further write

$$u_\lambda^e = P_0 u_\lambda^e + (1 - P_0) u_\lambda^e.$$

Then, $P_0 u_\lambda^e$ is localized in B_e and (by definition of $P_0,\ 1 - P_0$)

$$P_0 u_\lambda^e = S(u_\lambda^e(0), \partial_t u_\lambda^e(0)) + V(P_0 L(u_\lambda^e)), \quad (1 - P_0) u_\lambda^e = V((1 - P_0) L(u_\lambda^e)).$$

Hence by Lemma 2.3.5 and Lemma 2.3.6 we have

$$\begin{aligned} \lambda^{\frac{1}{p}+\frac{(d+1)}{q}-\frac{d}{2}} \|P_0 u_\lambda^e\|_{L_e^p L_{t,e^\perp}^q} &\lesssim \lambda^{-2} \|P_0 L u_\lambda^e\|_{L_t^1 L_x^2} + \|u_\lambda^e(0)\|_{L^2} + \|\partial_t u_\lambda^e(0)\|_{H^{-2}} \\ &\lesssim \|u_\lambda\|_{Y_\lambda^e}, \end{aligned} \tag{2.4.22}$$

by Lemma 2.4.1 and continuity of $P_e(\nabla)$ on $L_t^1 L_x^2$. Similarly, by Lemma 2.3.7, we infer

$$\begin{aligned} \lambda^{\frac{1}{p}+\frac{(d+1)}{q}-\frac{d}{2}} \|(1 - P_0) u_\lambda^e\|_{L_e^p L_{t,e^\perp}^q} \lesssim \|V(1 - P_0)(L u_\lambda^e)\|_{X_\lambda^{\frac{1}{2},1}} &\lesssim \lambda^{-2} \|(1 - P_0) L u_\lambda^e\|_{X_\lambda^{-\frac{1}{2},1}} \\ &\lesssim \lambda^{-2} \|(1 - P_0) L u_\lambda^e\|_{X_\lambda^{-\frac{1}{2},\infty}} \\ &\lesssim \|u_\lambda\|_{Y_\lambda^e}, \end{aligned}$$

where we used $(1 - P_0) X_\lambda^{\frac{1}{2},1} \sim (1 - P_0) X_\lambda^{\frac{1}{2},\infty}$ uniform in the frequency $\lambda \in 2^{\mathbb{Z}}$ and the dual trace inequelity from Lemma 2.3.7 in the last step. Hence we sum over $e \in \mathcal{M}$ and take the infimum over $u_\lambda = \sum_e u_\lambda^e$ with $u_\lambda^e \in S_\lambda^e$. The $L_e^2 L_{t,e^\perp}^\infty$ embedding (2.4.20) follows similarly using Lemma 2.3.7, the decomposition (2.4.21) and Lemma 2.3.5, 2.3.6. Especially

$$\sup_{\tilde{e}} \left(\lambda^{\frac{1-d}{2}} \|P_0 u_\lambda^e\|_{L_{\tilde{e}}^2 L_{t,\tilde{e}^\perp}^\infty}\right) \lesssim \|u_\lambda\|_{Y_\lambda}, \tag{2.4.23}$$

$$\sup_{\tilde{e}} \left(\lambda^{\frac{1-d}{2}} \|(1 - P_0) u_\lambda^e\|_{L_{\tilde{e}}^2 L_{t,\tilde{e}^\perp}^\infty}\right) \lesssim \lambda^{-2} \|L u_\lambda\|_{X_\lambda^{-\frac{1}{2},\infty}} \lesssim \|u_\lambda\|_{Y_\lambda} \tag{2.4.24}$$

Again the estimate for the $X_\lambda^{\frac{1}{2},1}$ part follows directly by Lemma 2.3.7. □

2.4.2 Bilinear estimates

For the bilinear interaction, we write

$$
\begin{aligned}
u \cdot v &= \sum_{\lambda_1,\lambda_2,\lambda} (u_{\lambda_1} v_{\lambda_2})_\lambda \\
&= \sum_{\lambda_2 \gg \lambda_1} \big[(u_{\lambda_1} v_{\lambda_2})_{\lambda_2/2} + (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} + (u_{\lambda_1} v_{\lambda_2})_{2\lambda_2} \big] && (2.4.25) \\
&\quad + \sum_{\lambda_1 \gg \lambda_2} \big[(u_{\lambda_1} v_{\lambda_2})_{\lambda_1/2} + (u_{\lambda_1} v_{\lambda_2})_{\lambda_1} + (u_{\lambda_1} v_{\lambda_2})_{2\lambda_1} \big] && (2.4.26) \\
&\quad + \sum_{|\log_2(\lambda_1/\lambda_2)| \sim 1} \ \sum_{\lambda \lesssim \max\{\lambda_1,\lambda_2\}} (u_{\lambda_1} v_{\lambda_2})_\lambda. && (2.4.27)
\end{aligned}
$$

Due to symmetry, we restrict (2.4.25) - (2.4.27) to

$$
\begin{aligned}
&\sum_{\lambda_2 \gg \lambda_1} \big[(u_{\lambda_1} v_{\lambda_2})_{\lambda_2/2} + (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} + (u_{\lambda_1} v_{\lambda_2})_{2\lambda_2} \big] \\
&+ \sum_{\lambda_1 \sim \lambda_2} \sum_{\lambda \lesssim \lambda_2} (u_{\lambda_1} v_{\lambda_2})_\lambda,
\end{aligned}
$$

and thus further reduce to the interactions

$$
\lambda_1 \ll \lambda_2 : \quad (u_{\lambda_1} v_{\lambda_2})_{\lambda_2}, \quad \text{and} \quad \lambda_1 \leq \lambda_2 : \quad (u_{\lambda_2} v_{\lambda_2})_{\lambda_1}.
$$

Lemma 2.4.5.

$$
(a) \quad \| u_{\lambda_1} v_{\lambda_2} \|_{Z_{\lambda_2}} \lesssim \lambda_1^{\frac{d}{2}} \| u_{\lambda_1} \|_{Z_{\lambda_1}} \| v_{\lambda_2} \|_{Z_{\lambda_2}}, \ \ \lambda_1 \ll \lambda_2 \tag{2.4.28}
$$

$$
(b) \quad \| (u_{\lambda_2} v_{\lambda_2})_{\lambda_1} \|_{Z_{\lambda_1}} \lesssim \lambda_2^{\frac{d}{2}} \| u_{\lambda_2} \|_{Z_{\lambda_2}} \| v_{\lambda_2} \|_{Z_{\lambda_2}}, \ \ \lambda_1 \leq \lambda_2. \tag{2.4.29}
$$

Proof. For part (a), we decompose

$$
(u_{\lambda_1} v_{\lambda_2})_{\lambda_2} = Q_{\leq 4\lambda_1\lambda_2} (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} + (1 - Q_{\leq 4\lambda_1\lambda_2})(u_{\lambda_1} v_{\lambda_2})_{\lambda_2}. \tag{2.4.30}
$$

First, we place $Q_{\leq 4\lambda_1\lambda_2} (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} \in X_{\lambda_2}^{\frac{1}{2},1}$ by estimating

$$
\| (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} \|_{L^2_{t,x}} \lesssim \lambda_1^{\frac{d-1}{2}} \lambda_2^{-\frac{1}{2}} \| u_{\lambda_1} \|_{Z_{\lambda_1}} \| v_{\lambda_2} \|_{Z_{\lambda_2}}. \tag{2.4.31}
$$

Then, from $X_{\lambda_2}^{\frac{1}{2},1} \subset Z_{\lambda_2}$, (2.4.31) gives

$$
\begin{aligned}
\| Q_{\leq 4\lambda_1\lambda_2} (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} \|_{Z_{\lambda_2}} &\lesssim \| Q_{\leq 4\lambda_1\lambda_2} (u_{\lambda_1} v_{\lambda_2})_{\lambda_2} \|_{X_{\lambda_2}^{\frac{1}{2},1}} \\
&\lesssim \Big(\sum_{\mu \leq 4\lambda_1\lambda_2} \mu^{\frac{1}{2}} (\lambda_1\lambda_2)^{-\frac{1}{2}} \Big) \lambda_1^{\frac{d}{2}} \| u_{\lambda_1} \|_{Z_{\lambda_1}} \| v_{\lambda_2} \|_{Z_{\lambda_2}}.
\end{aligned}
$$

For (2.4.31), we write $u_{\lambda_1} v_{\lambda_2} = \sum_{e\in\mathcal{M}} u_{\lambda_1} v^e_{\lambda_2}$ where $v^e_{\lambda_2} \in S^e_{\lambda_2}$. Hence

$$\left\|(u_{\lambda_1} v^e_{\lambda_2})_{\lambda_2}\right\|_{L^2_{t,x}} \le \|u_{\lambda_1}\|_{L^2_e L^\infty_{t,e^\perp}} \left\|v^e_{\lambda_2}\right\|_{L^\infty_e L^2_{t,e^\perp}} \tag{2.4.32}$$

$$\le \lambda_1^{\frac{d-1}{2}} \|u_{\lambda_1}\|_{\lambda_1^{\frac{d-1}{2}} \bigcap_{\tilde e} L^2_{\tilde e} L^\infty_{t,\tilde e^\perp}} \lambda_2^{-\frac{1}{2}} \left\|v^e_{\lambda_2}\right\|_{\lambda_2^{-\frac{1}{2}} L^\infty_e L^2_{t,e^\perp}}. \tag{2.4.33}$$

Summing over $e \in \mathcal{M}$, the claim follows from Proposition 2.4.4. Secondly, we note for $\lambda_1^2 \ll \mu$

$$Q_\mu(u_{\lambda_1} v_{\lambda_2})_{\lambda_2} = Q_\mu\big(u_{\lambda_1} \sum_{|j|\le 2} Q_{2^j\mu} v_{\lambda_2}\big) \tag{2.4.34}$$

Hence we write

$$(1 - Q_{\le 4\lambda_1\lambda_2})(u_{\lambda_1} v_{\lambda_2})_{\lambda_2} = (1 - Q_{\le 4\lambda_1\lambda_2})(u_{\lambda_1}(1 - Q_{\le \lambda_1\lambda_2}) v_{\lambda_2})_{\lambda_2}$$

In order to estimate the remaining part in (2.4.30), using Lemma 2.4.1, it thus suffices to prove

$$\|(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2})_{\lambda_2}\|_{X^{\frac{1}{2},1}_{\lambda_2}} \lesssim \|u_{\lambda_1}\|_{Z_{\lambda_1}} \|v_{\lambda_2}\|_{X^{\frac{1}{2},1}_{\lambda_2}} \tag{2.4.35}$$

$$\|(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2})_{\lambda_2}\|_{Y_{\lambda_2}} \lesssim \|u_{\lambda_1}\|_{Z_{\lambda_1}} \|v_{\lambda_2}\|_{Y_{\lambda_2}}. \tag{2.4.36}$$

The estimate (2.4.35) and the $L^\infty_t L^2_x$ summand of (2.4.36) follow from the embedding $Z_{\lambda_1} \subset \lambda_1^{\frac{d}{2}} L^\infty_{t,x}$ by factoring off the $L^\infty_{t,x}$ norm of u_{λ_1}. For the second estimate (2.4.36), we further calculate

$$\begin{aligned} L(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2})_{\lambda_2} &= u_{\lambda_1} L(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2} + \partial_t u_{\lambda_1} \partial_t (1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2} \\ &\quad + \partial_t^2 u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2} + \Delta^2(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}) \\ &\quad - u_{\lambda_1} \Delta^2 (1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}), \end{aligned}$$

hence we estimate

$$\begin{aligned} &\|L(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2})_{\lambda_2}\|_{L^1_t L^2_x} \\ &\quad\lesssim \|u_{\lambda_1} L(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\|_{L^1_t L^2_x} + \|\partial_t u_{\lambda_1} \partial_t (1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\|_{L^1_t L^2_x} \\ &\quad\quad + \left\|\partial_t^2 u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\right\|_{L^1_t L^2_x} \\ &\quad\quad + \left\|\Delta^2(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}) - u_{\lambda_1} \Delta^2 (1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2})\right\|_{L^1_t L^2_x}. \end{aligned}$$

Calculating the expression in the latter norm and factoring off the derivatives of u_{λ_1} in L^∞, we infer (using Bernstein's inequality)

$$\begin{aligned} &\|L(u_{\lambda_1}(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2})_{\lambda_2}\|_{L^1_t L^2_x} \\ &\quad\lesssim \|u_{\lambda_1} L(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\|_{L^1_t L^2_x} + \lambda_1 \|u_{\lambda_1}\|_{L^\infty} \lambda_2^3 \|(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\|_{L^1_t L^2_x} \\ &\quad\approx \|u_{\lambda_1} L(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\|_{L^1_t L^2_x} \\ &\quad\quad + \|u_{\lambda_1}\|_{L^\infty} \lambda_1 \lambda_2^3 (\lambda_1\lambda_2)^{-1} \|(1 - Q_{\le\lambda_1\lambda_2}) v_{\lambda_2}\|_{(\lambda_1\lambda_2)^{-1} L^1_t L^2_x} \end{aligned}$$

where we note $\lambda_1 \ll \lambda_2$. We now proceed by Lemma 2.4.1 (b) (for $\mu = \lambda_1\lambda_2$)

$$\begin{aligned}\lambda_2^{-2}\|L(u_{\lambda_1}(1-Q_{\leq\lambda_1\lambda_2})v_{\lambda_2})_{\lambda_2}\|_{L^1_tL^2_x} \\ \lesssim \lambda_1^{\frac{d}{2}}\|u_{\lambda_1}\|_{Z_{\lambda_1}}(\lambda_2^{-2}\|Lv_{\lambda_2}\|_{L^1_tL^2_x}+\|v_{\lambda_2}\|_{Y_{\lambda_2}}),\end{aligned}$$

which gives the claim. The proof part (b) follows similarly, in fact easier, since we can directly place $(u_{\lambda_2}v_{\lambda_2})_{\lambda_1} \in X^{\frac{1}{2},1}_{\lambda_1}$ by estimating

$$\|u_{\lambda_2}v_{\lambda_2}\|_{L^2_{t,x}} \lesssim \lambda_2^{\frac{d}{2}-1}\|u_{\lambda_2}\|_{Z_{\lambda_2}}\|v_{\lambda_2}\|_{Z_{\lambda_2}}. \tag{2.4.37}$$

Then, from $X^{\frac{1}{2},1}_{\lambda_1} \subset Z_{\lambda_1}$, (2.4.37) gives

$$\begin{aligned}\|(u_{\lambda_2}v_{\lambda_2})_{\lambda_1}\|_{Z_{\lambda_1}} &\lesssim \sum_{\mu\leq 4\lambda_1^2}\left(\frac{\mu}{\lambda_1^2}\right)^{\frac{1}{2}}\lambda_2\|u_{\lambda_2}v_{\lambda_2}\|_{L^2_{t,x}} \\ &\lesssim \lambda_2(\lambda_2^{\frac{d}{2}-1}\|u_{\lambda_2}\|_{Z_{\lambda_2}}\|v_{\lambda_2}\|_{Z_{\lambda_2}}).\end{aligned}$$

For (2.4.37), we write $u_{\lambda_2}v_{\lambda_2} = \sum_{e\in\mathcal{M}} u^e_{\lambda_2}v_{\lambda_2}$ where $u^e_{\lambda_2} \in S^e_{\lambda_2}$. Hence

$$\begin{aligned}\left\|u^e_{\lambda_2}v_{\lambda_2}\right\|_{L^2_{t,x}} &\leq \left\|u^e_{\lambda_2}\right\|_{L^\infty_e L^2_{t,e^\perp}}\|v_{\lambda_2}\|_{L^2_e L^\infty_{t,e^\perp}} \\ &\leq \lambda_2^{-\frac{1}{2}}\left\|u^e_{\lambda_2}\right\|_{\lambda_2^{-\frac{1}{2}}L^\infty_e L^2_{t,e^\perp}}\lambda_2^{\frac{d-1}{2}}\|v_{\lambda_2}\|_{\lambda_2^{\frac{d-1}{2}}\bigcap_{\tilde{e}}L^2_{\tilde{e}}L^\infty_{t,\tilde{e}^\perp}}.\end{aligned}$$

Summing over $e \in \mathcal{M}$, we infer the claim.

□

From Lemma 2.4.5, we obtain (2.1.10) as outlined above by summation according to the definiton of $Z^{\frac{d}{2}}$ and $W^{\frac{d}{2}}$. Note that the estimates for the remaining frequency interactions in (2.4.25) and (2.4.27) follow the same arguments provided in Lemma 2.4.5.

Similarly, for the embedding (2.1.11) we prove the subsequent estimates.

Lemma 2.4.6.

$$\|u_{\lambda_2}v_{\lambda_1}\|_{W^{\frac{d}{2}}} \lesssim \lambda_1^{\frac{d}{2}}\lambda_2^{\frac{d}{2}}\|u_{\lambda_2}\|_{Z_{\lambda_2}}\|v_{\lambda_1}\|_{W_{\lambda_1}}, \quad \lambda_1 \leq \lambda_2 \tag{2.4.38}$$

$$\|u_{\lambda_2}v_{\lambda_1}\|_{W_{\lambda_2}} \lesssim \lambda_1^{\frac{d}{2}}\|u_{\lambda_2}\|_{W_{\lambda_2}}\|v_{\lambda_1}\|_{Z_{\lambda_1}}, \quad \lambda_1 \ll \lambda_2 \tag{2.4.39}$$

Proof. We first estimate by Sobolev embedding

$$\begin{aligned}
\lambda_2^{-2} \|u_{\lambda_2} v_{\lambda_1}\|_{L_t^1 L_x^2} &\lesssim \lambda_2^{-2} \|u_{\lambda_2}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|v_{\lambda_1}\|_{L_t^2 L_x^d} \\
&\lesssim \lambda_2^{-2} \|u_{\lambda_2}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lambda_1^{\frac{d-2}{2}} \|v_{\lambda_1}\|_{L_{t,x}^2} \\
&\lesssim \|u_{\lambda_2}\|_{S_{\lambda_2}} \lambda_1^{\frac{d}{2}-3} \|v_{\lambda_1}\|_{L_{t,x}^2} \\
&\lesssim \|u_{\lambda_2}\|_{Z_{\lambda_2}} \lambda_1^{\frac{d}{2}} \|v_{\lambda_1}\|_{W_{\lambda_1}}
\end{aligned}$$

where we used Lemma 2.4.2 for $W_{\lambda_1} \subset \lambda_1^3 L_{t,x}^2$. Thus from

$$\left\| u_{\lambda_2}^e v_{\lambda_1} \right\|_{W^{\frac{d}{2}}} \lesssim \lambda_2^{\frac{d-4}{2}} \|u_{\lambda_2} v_{\lambda_1}\|_{L_t^1 L_x^2} \lesssim \lambda_2^{\frac{d}{2}} \lambda_1^{\frac{d}{2}} \|u_{\lambda_2}\|_{Z_{\lambda_2}} \|v_{\lambda_1}\|_{W_{\lambda_1}},$$

we obtain the claim (2.4.38). Estimate (2.4.39) is implied by

$$\lambda_1^{-\frac{d}{2}} Z_{\lambda_1} \cdot L_t^1 L_x^2 \subset L_t^1 L_x^2, \tag{2.4.40}$$

$$\lambda_1^{-\frac{d}{2}} Z_{\lambda_1} \cdot X_{\lambda_2}^{-\frac{1}{2},1} \subset W_{\lambda_2} \lambda_2^{-2}, \tag{2.4.41}$$

where the first embedding follows from $Z_{\lambda_1} \subset \lambda_1^{\frac{d}{2}} L_{t,x}^\infty$. For (2.4.41), we note that since we restrict to $\lambda_1 \ll \lambda_2$, we only consider $\lambda_1 \leq \frac{\lambda_2}{C}$ for a large, fixed constant $C > 0$. We thus decompose

$$u_{\lambda_2} = Q_{\leq C^2 \lambda_1^2} u_{\lambda_2} + (1 - Q_{\leq C^2 \lambda_1^2}) u_{\lambda_2}.$$

In particular, each dyadic piece $Q_\mu u_{\lambda_2}$ in $(1 - Q_{\leq C^2 \lambda_1^2}) X_{\lambda_2}^{-\frac{1}{2},1}$ satisfies $\lambda_1^2 \ll \mu \leq \lambda_2^2$. We then estimate (note that we use (2.4.34))

$$\begin{aligned}
\left\| v_{\lambda_1} (1 - Q_{\leq C^2 \lambda_1^2}) u_{\lambda_2} \right\|_{X_{\lambda_2}^{-\frac{1}{2},1}} &\sim \sum_{C^2 \lambda_1^2 \leq \mu \leq 4\lambda_2^2} \mu^{-\frac{1}{2}} \|v_{\lambda_1} Q_\mu u_{\lambda_2}\|_{L_{t,x}^2} \\
&\lesssim \sum_{C^2 \lambda_1^2 \leq \mu \leq 4\lambda_2^2} \mu^{-\frac{1}{2}} \|v_{\lambda_1}\|_{L_{t,x}^\infty} \|Q_\mu u_{\lambda_2}\|_{L_{t,x}^2} \\
&\lesssim \lambda_1^{\frac{d}{2}} \|v_{\lambda_1}\|_{Z_{\lambda_1}} \|u_{\lambda_2}\|_{X_{\lambda_2}^{-\frac{1}{2},1}}.
\end{aligned}$$

Further

$$\begin{aligned}
\lambda_2^2 \left\| v_{\lambda_1} Q_{\leq C^2 \lambda_1^2} u_{\lambda_2} \right\|_{W_{\lambda_2}} &\lesssim \left\| v_{\lambda_1} Q_{\leq C^2 \lambda_1^2} u_{\lambda_2} \right\|_{L_t^1 L_x^2} \\
&\lesssim \|v_{\lambda_1}\|_{L_t^2 L_x^\infty} \left\| Q_{\leq C^2 \lambda_1^2} u_{\lambda_2} \right\|_{L_{t,x}^2} \\
&\lesssim \lambda_1^{\frac{d}{2}} \|v_{\lambda_1}\|_{\lambda_1^{\frac{d-2}{2}} L_t^2 L_x^\infty} \sum_{\mu \leq C^2 \lambda_1^2} \mu^{-\frac{1}{2}} \|Q_\mu u_{\lambda_2}\|_{L_{t,x}^2} \\
&\lesssim \lambda_1^{\frac{d}{2}} \|v_{\lambda_1}\|_{Z_{\lambda_1}} \|u_{\lambda_2}\|_{X_{\lambda_2}^{-\frac{1}{2},1}},
\end{aligned}$$

which follows from $Z_{\lambda_1} \subset S_{\lambda_1}$. □

We now infer (2.1.10) and (2.1.11) by the summation argument provided in the beginning of the section.

2.4.3 Higher regularity

The percisteny of higher regularity of the $\dot{B}^{2,1}_{\frac{d}{2}} \times \dot{B}^{2,1}_{\frac{d}{2}-2}$ solution as stated in Theorem 2.1.1 follows as in [54] and [2] from (2.1.14) and (2.1.15). We will show how to employ these estimates in order to apply Lemma 2.2.2 in the next Section 2.4.4.

For (2.1.14), we rely again on Lemma 2.4.5 and the decomposition

$$uv = \sum_{\lambda_1 \ll \lambda_2} u_{\lambda_2} v_{\lambda_1} + \sum_{\lambda_2 \ll \lambda_1} u_{\lambda_2} v_{\lambda_1} + \sum_{\lambda_1 \sim \lambda_2} u_{\lambda_2} v_{\lambda_1},$$

from the beginning of the Section 2.4.2. However, we now sum as follows

$$\Big(\sum_{\lambda} \lambda^{2s} \|(uv)_\lambda\|^2_{Z_\lambda} \Big)^{\frac{1}{2}} \lesssim \sum_{\lambda_1} \Big(\sum_{\lambda} \lambda^{2s} \Big\| \Big(\sum_{\lambda_1 \ll \lambda_2} u_{\lambda_2} v_{\lambda_1} \Big)_\lambda \Big\|^2_{Z_\lambda} \Big)^{\frac{1}{2}}$$

$$+ \sum_{\lambda_2} \Big(\sum_{\lambda} \lambda^{2s} \Big\| \Big(\sum_{\lambda_2 \ll \lambda_1} u_{\lambda_2} v_{\lambda_1} \Big)_\lambda \Big\|^2_{Z_\lambda} \Big)^{\frac{1}{2}} + \sum_{\lambda_1 \sim \lambda_2} \|u_{\lambda_2} v_{\lambda_1}\|_{Z^s}.$$

Hence, we need to estimate the three terms

$$\sum_{\lambda_1} \Big(\sum_{\lambda_1 \ll \lambda} \lambda^{2s} \|(u_\lambda v_{\lambda_1})_\lambda\|^2_{Z_\lambda} \Big)^{\frac{1}{2}}, \quad \sum_{\lambda_2} \Big(\sum_{\lambda_2 \ll \lambda} \lambda^{2s} \|(u_{\lambda_2} v_\lambda)_\lambda\|^2_{Z_\lambda} \Big)^{\frac{1}{2}}, \quad \sum_{\lambda_1 \sim \lambda_2} \|u_{\lambda_2} v_{\lambda_1}\|_{Z^s},$$

where for $s > \frac{d}{2}$, the latter sum is treated by Lemma 2.4.5 (b) similar as before via (note that we identify λ_1 and λ_2 for simplicity)

$$\sum_{\lambda_2} \sum_{\lambda \lesssim \lambda_2} \lambda^s \|(u_{\lambda_2} v_{\lambda_2})_\lambda\|_{Z_\lambda} \lesssim \sum_{\lambda_2} (\lambda_2^{2s} \|u_{\lambda_2}\|^2_{Z_{\lambda_2}})^{\frac{1}{2}} \lambda_2^{\frac{d}{2}} \|v_{\lambda_2}\|_{Z_{\lambda_2}} \lesssim \|u\|_{Z^s} \|v\|_{Z^{\frac{d}{2}}}.$$

The LHS of this inequality now bounds the $l^2(\mathbb{Z})$ norm (wrt λ) and for the first two sums above we directly estimate the squares via Lemma 2.4.5 (a). For (2.1.15), we sum in the same way and use the following dyadic estimates

$$\sum_{\lambda_1 \lesssim \lambda_2} \lambda_1^s \|(u_{\lambda_2} v_{\lambda_2})_{\lambda_1}\|_{W_{\lambda_1}} \lesssim \lambda_2^{s+\frac{d}{2}} \|u_{\lambda_2}\|_{Z_{\lambda_2}} \|v_{\lambda_2}\|_{W_{\lambda_2}},$$

$$\|(u_{\lambda_2} v_{\lambda_1})_{\lambda_2}\|_{W_{\lambda_2}} \lesssim \lambda_1^{\frac{d}{2}} \|u_{\lambda_2}\|_{W_{\lambda_2}} \|v_{\lambda_1}\|_{Z_{\lambda_1}}, \quad \lambda_1 \ll \lambda_2$$

$$\|(u_{\lambda_2} v_{\lambda_1})_{\lambda_2}\|_{W_{\lambda_2}} \lesssim \lambda_1^{\frac{d}{2}} \|u_{\lambda_2}\|_{Z_{\lambda_2}} \|v_{\lambda_1}\|_{W_{\lambda_1}}, \quad \lambda_1 \ll \lambda_2$$

which are the same as in (or follow from) Lemma 2.4.6.

2.4.4 Proof of the main theorem

The proof of Theorem 2.1.1 follows straight forward pertubatively by convergence of

$$u_{k+1} = Su[0] + V(\mathcal{Q}(u_k)), \;\; k \geq 0, \; u_0(t,x) = 0 \tag{2.4.42}$$

in the space $Z^{\frac{d}{2}}$, where $Su[0] = S(u_0, u_1)$ solves (2.3.1) with $F = 0$ and VF solves (2.3.1) with $u[0] = 0$. To be more precise, we combine Proposition 2.4.3, i.e.

$$\|u_{k+1}\|_{Z^{\frac{d}{2}}} \lesssim \|u_0\|_{\dot{B}^{2,1}_{\frac{d}{2}}} + \|u_1\|_{\dot{B}^{2,1}_{\frac{d}{2}-2}} + \|\mathcal{Q}(u_k)\|_{W^{\frac{d}{2}}}$$

$$\sup_{t \in \mathbb{R}} \big(\|u(t)\|_{\dot{B}^{2,1}_{\frac{d}{2}}} + \|\partial_t u(t)\|_{\dot{B}^{2,1}_{\frac{d}{2}-2}} \big) \lesssim \|u\|_{Z^{\frac{d}{2}}} ,$$

with the Lipschitz estimates

$$\|\mathcal{Q}(u_k) - \mathcal{Q}(u_l)\|_{W^{\frac{n}{2}}} \lesssim C(\|u_k\|_{Z^{\frac{d}{2}}}, \|u_l\|_{Z^{\frac{d}{2}}}) \|u_k - u_l\|_{Z^{\frac{d}{2}}} ,$$

$$\|\mathcal{Q}(u_k) - \mathcal{Q}(u_l)\|_{W^s} \lesssim \|u_k - u_l\|_{Z^s} (\|u_k\|_{Z^{\frac{d}{2}}} + \|u_l\|_{Z^{\frac{d}{2}}}) + \|u_k - u_l\|_{Z^{\frac{d}{2}}} (\|u_k\|_{Z^s} + \|u_l\|_{Z^s}).$$

The first estimate is a direct consequence of (2.1.10), (2.1.11), (2.1.2) and Lemma 2.4.3 provided $\mathcal{Q}$ is analytic (at $x_0 = 0$) and (2.1.16) holds, which is a priori necessary to expand the coefficients of $\mathcal{Q}$. The latter estimate follows similarly using (2.1.14) and (2.1.15) and hence this framework applies to Lemma 2.2.2.

For the interested reader, we now make this precise and establish (2.2.11) and (2.2.12) in the Lemma for $Z = Z^{\frac{d}{2}}$ in Proposition 2.4.3. It hence suffices to prove the Lipschitz estimates (2.2.13) and (2.2.15), which follow from the continuity $V : W_\lambda \to Z_\lambda$ (see Proposition 2.4.3) and for a small $\delta > 0$ and $s > \frac{d}{2}$, as mentioned above,

$$\|\mathcal{Q}(u) - \mathcal{Q}(v)\|_{W^{\frac{n}{2}}} \lesssim (\|u\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^{\frac{d}{2}}}) \|u - v\|_{Z^{\frac{d}{2}}}, \quad u, v \in B^{Z^{\frac{d}{2}}}(0, \delta), \tag{2.4.43}$$

$$\begin{aligned} \|\mathcal{Q}(u) - \mathcal{Q}(v)\|_{W^s} &\lesssim \|u - v\|_{Z^s} (\|u\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^{\frac{d}{2}}}) \\ &\quad + \|u - v\|_{Z^{\frac{d}{2}}} (\|u\|_{Z^s} + \|v\|_{Z^s}), \quad u, v \in B^{Z^{\frac{d}{2}}}(0, \delta) \cap Z^s, \end{aligned} \tag{2.4.44}$$

and similar for the nonlinearity in (2.1.4). Then we conclude Theorem 2.1.1 from Lemma 2.2.2 for $\delta > 0$ such that additionally (2.2.14) is satisfied. In order to obtain (2.4.43) and (2.4.44), we first note that from (2.1.14) (combined with (2.1.10)) there holds by induction over $k_j \in \mathbb{N}$ with $j = 1, \dots, m$ for $u_j \in Z^{\frac{d}{2}} \cap Z^s$

$$\Big\| \prod_{j=1}^m u_j^{k_j} \Big\|_{Z^s} \lesssim \sum_{j=1}^m \prod_{i \neq j} k_j \|u_j\|_{Z^s} \|u_j\|_{Z^{\frac{d}{2}}}^{k_j - 1} \|u_i\|_{Z^{\frac{d}{2}}}^{k_i} . \tag{2.4.45}$$

Here we ignore the fact that u_j are vector-valued. In particular, the smallness assumption is only necessary in $Z^{\frac{d}{2}}$ in order to estimate the series expansion of $\mathcal{Q}(u),\ \mathcal{Q}(v)$ in (2.1.1) and Π in (2.1.4). We apply (2.4.45) for power of $w = u - v, u, v$ where $u, v \in Z^{\frac{d}{2}} \cap Z^s$.

We proceed by expanding $x \mapsto Q_x(\cdot,\cdot)$ and $x \mapsto \Pi(x)$ at $x_0 = 0$, i.e. by assumption

$$Q_x = \sum_{k=0}^{\infty} \sum_{k=|\alpha|} \frac{1}{\alpha!} D^\alpha (Q_x)_{|x=0} x^\alpha, \quad \Pi(x) = \sum_{k=0}^{\infty} \sum_{k=|\alpha|} \frac{1}{\alpha!} D^\alpha \Pi(x)_{|x=0} x^\alpha,$$

converge uniformly in $B(0,\varepsilon) \subset \mathbb{R}^L$ for some $\varepsilon > 0$, where $\alpha \in \mathbb{N}_0^L,\ \alpha! = \Pi_{j=1}^L \alpha_j!,\ x^\alpha = \Pi_{j=1}^L x_j^{\alpha_j}$. For convenience, we write

$$Q_x = \sum_{k=0}^{\infty} \frac{1}{k!} (d^k Q_x)_{|x=0}(x^k), \quad \Pi(x) = \sum_{k=0}^{\infty} \frac{1}{k!} d^k \Pi(x)_{|x=0}(x^k), \tag{2.4.46}$$

where $d^k Q_x,\ d^k\Pi(x)$ are $(k+2)$-, resp. k-tensors with the notation for $l = 1, \dots, L$

$$d^k Q_0^l(x^k) = d^k Q_0^l(x, \dots, x) = \sum_{l_1 + \cdots + l_L = k} \frac{k!}{l_1! \cdots l_L!} (\partial_{x_1}^{l_1} \cdots \partial_{x_L}^{l_L} Q_0^l) x_1^{l_1} \cdots x_L^{l_L},$$

and similar for $d^k\Pi$. Especially we have for any $v \in \mathbb{R}^L$

$$\begin{aligned} d\Pi_x(v) &= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^{L} d^{k-1} \partial_{x_l} \Pi(x)_{|x=0}(x^{k-1}) v_l \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{l=1}^{L} d^{k-1} \partial_{x_l} \Pi(x)_{|x=0}(x^{k-1}) k v_l. \end{aligned} \tag{2.4.47}$$

Since e.g.

$$\left\| \frac{1}{k!} (d^k Q_x)_{|x=0}(x^k) \right\|_{Z^{\frac{d}{2}}} \lesssim \sum_{|\alpha|=k} \frac{1}{\alpha!} |D^\alpha (Q_x)_{|x=0}|_{\mathbb{R}^L} \delta^k, \ \ u \in B^{Z^{\frac{d}{2}}}(0,\delta),$$

we have that (2.4.46) exist absolutely in $B^{Z^{\frac{d}{2}}}(0,\delta)$ if $\delta > 0$ is small enough. We now consider

$$\begin{aligned} \mathcal{Q}(u) &= \frac{1}{2} Q_u (L(u \cdot u) - u \cdot Lu - Lu \cdot u), \\ \mathcal{N}(u) &= L(\Pi(u)) - d\Pi_u(Lu). \end{aligned}$$

Then we have for $w = u - v$

$$\begin{aligned}
\mathcal{Q}(u) - \mathcal{Q}(v) &= \frac{1}{2}(Q_u - Q_v)(L(u \cdot u) - u \cdot Lu - Lu \cdot u) \\
&\quad + \frac{1}{2} Q_v (L(w \cdot u) + L(v \cdot w) - w \cdot Lu - v \cdot Lw - Lw \cdot u - Lv \cdot w), \\
(Q_u - Q_v) &= \sum_{k \geq 1} \frac{1}{k!} (d^k Q_x)_{|x=0} (u^k - v^k) \\
&= \sum_{k \geq 2} \sum_{l=0}^{k-1} \frac{1}{k!} (d^k Q_x)_{|x=0} (v^l w u^{k-l-1}) + (dQ_x)_{|x=0}(w),
\end{aligned}$$

where in $(d^k Q_x)_{|x=0}(v^l w u^{k-l-1})$, we capture all terms of the form

$$\sum_{\substack{l_1 + \dots l_m = l \\ l_{m+2} \dots + l_L = k-1-l}} C_{l_1, \dots, l_L} (\partial_{x_{i_1}}^{l_1} \cdots \partial_{x_{i_L}}^{l_L} Q_0) v_{i_1}^{l_1} \cdots v_{i_m}^{l_m} w_{i_{m+1}} u_{i_{m+2}}^{l_{m+2}} \cdots u_{i_L}^{l_L}, \qquad i_j \in \{1, \dots, L\}.$$

Especially, this implies (2.4.43) by (2.1.10), (2.1.11) and continuity of $L : Z^{\frac{d}{2}} \to W^{\frac{d}{2}}$ (Proposition 2.4.3).

We show more details for the proof of (2.4.44). Therefore, we use simplyfied notation and estimate using (2.1.15)

$$\|(v^l w u^{k-l-1})(L(u \cdot u) - u \cdot Lu - Lu \cdot u)\|_{W^s} \lesssim J_1 + J_2,$$

where

$$\begin{aligned}
J_1 &\lesssim \left\|(v^l w u^{k-l-1})\right\|_{Z^s} (\left\|u^2\right\|_{Z^{\frac{d}{2}}} + \|uLu\|_{W^{\frac{d}{2}}}) \\
&\lesssim \left\|(v^l w u^{k-l-1})\right\|_{Z^s} \|u\|_{Z^{\frac{d}{2}}}^2 \\
&\lesssim_\delta l \|v\|_{Z^s} \|v\|_{Z^{\frac{d}{2}}}^{l-1} \|w\|_{Z^{\frac{d}{2}}} \|u\|_{Z^{\frac{d}{2}}}^{k-l-1} + \|w\|_{Z^s} \|v\|_{Z^{\frac{d}{2}}}^l \|u\|_{Z^{\frac{d}{2}}}^{k-l-1} \|u\|_{Z^{\frac{d}{2}}}^2 \\
&\qquad + (k-l-1) \|u\|_{Z^s} \|v\|_{Z^{\frac{d}{2}}}^l \|w\|_{Z^{\frac{d}{2}}} \|u\|_{Z^{\frac{d}{2}}}^{k-l-2} \\
&\lesssim_\delta (k-1)\delta^{k-2} (\|u\|_{Z^s} + \|v\|_{Z^s}) \|w\|_{Z^{\frac{d}{2}}} + \delta^k \|w\|_{Z^s} (\|u\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^{\frac{d}{2}}}).
\end{aligned}$$

by (2.4.45) and (2.1.14). Further

$$J_2 \lesssim \left\|(v^l w u^{k-l-1})\right\|_{Z^{\frac{d}{2}}} \|u\|_{Z^s} \|u\|_{Z^{\frac{d}{2}}} \lesssim_\delta \|w\|_{Z^{\frac{d}{2}}} \delta^k \|u\|_{Z^s}.$$

Similarly we estimate

$$\|v^k (L(w \cdot u) + L(v \cdot w) - w \cdot Lu - v \cdot Lw - Lw \cdot u - Lv \cdot w)\|_{W^s} \lesssim I_1 + I_2$$

where

$$
\begin{aligned}
I_1 &\lesssim \left\|v^k\right\|_{Z^s} (\|u\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^{\frac{d}{2}}}) \|w\|_{Z^{\frac{d}{2}}} \lesssim_\delta k\delta^{k-1} \|v\|_{Z^s} \|w\|_{Z^{\frac{d}{2}}} , \\
I_2 &\lesssim \left\|v^k\right\|_{Z^{\frac{d}{2}}} ((\|u\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^{\frac{d}{2}}}) \|w\|_{Z^s} + (\|u\|_{Z^s} + \|v\|_{Z^s}) \|w\|_{Z^{\frac{d}{2}}}) \\
&\lesssim_\delta \delta^k((\|u\|_{Z^{\frac{d}{2}}} + \|v\|_{Z^{\frac{d}{2}}}) \|w\|_{Z^s} + (\|u\|_{Z^s} + \|v\|_{Z^s}) \|w\|_{Z^{\frac{d}{2}}}).
\end{aligned}
$$

Now we turn to the second Cauchy problem (2.1.4), respectively the nonlinearity $\mathcal{N}$. We note that by continuity of $L : Z^{\frac{d}{2}} \to W^{\frac{d}{2}}$, i.e.

$$
\|Lv\|_{W^{\frac{d}{2}}} \lesssim \|v\|_{Z^{\frac{d}{2}}} \lesssim \delta,
$$

and by convergence of (2.4.46) in $B^{Z^{\frac{d}{2}}}(0,\delta)$, we justify to pull L into the series expansion and all terms in the series expression of $\mathcal{N}(u)$ are at least quadratic. More precisely in the above notation

$$
\mathcal{N}(u) = \sum_{k\geq 2} \frac{1}{k!}(d^k\Pi(x))_{|x=0}(L(u^k) - ku^{k-1}Lu),
$$

converges absolutely in $W^{\frac{d}{2}}$ if $u \in B^{Z^{\frac{d}{2}}}(0,\delta)$ and $\delta > 0$ is small enough. Here we use the identity (2.4.47) and write similarly as before

$$
\mathcal{N}(u) - \mathcal{N}(v) = \sum_{k\geq 2}\sum_{l=0}^{k-1} \frac{1}{k!}(d^k\Pi(x))_{|x=0}(L(v^l wu^{k-1-l}) - kv^l wu^{k-2-l}Lu - kv^{k-1}Lw),
$$

where for the middle term, we only sum $l = 0, \dots, k-2$. This form applies to derive (2.4.43) and (2.4.44) with the same arguments used above.

2.5 Application to biharmonic wave maps

We now want to prove Corollary 2.1.2 and construct a global solution of (2.1.3), which reads as

$$
\begin{aligned}
\partial_t^2 u + \Delta^2 u = {} & dP_u(u_t,u_t) + dP_u(\Delta u, \Delta u) + 4dP_u(\nabla u, \nabla\Delta u) + 2dP_u(\nabla^2 u, \nabla^2 u) \\
& + 2d^2P_u(\nabla u, \nabla u, \Delta u) + 4d^2P_u(\nabla u, \nabla u, \nabla^2 u) \\
& + d^3P_u(\nabla u, \nabla u, \nabla u, \nabla u),
\end{aligned}
$$

where

$$
\begin{aligned}
d^2P_u(\nabla u, \nabla u, \nabla^2 u) &= d^2P_u(\partial^i u, \partial_j u, \partial_i\partial^j u), \\
d^3P_u(\nabla u, \nabla u, \nabla u, \nabla u) &= d^3P_u(\partial_i u, \partial^i u, \partial_j u, \partial^j u),
\end{aligned}
$$

and $dP_u,\ d^2P_u,\ d^3P_u$ are derivatives of the orthogonal tangent projector $P_p : \mathbb{R}^L \to T_pN$ for $p \in N$.

We extend this equation via the nearest point map Π ($d\Pi_u = P_u$ for $u \in N$) to functions that only map to the neighborhood $\mathcal{V}_\varepsilon(N)$. By direct calculation or comparison to (2.1.5), it can be verified that (2.1.4) is the canonical extension for this setting. We thus consider

$$Lv = L(\Pi(v+p)) - d\Pi_{v+p}(Lv) =: \mathcal{N}(v),$$

for $v = u - p$ where $p := \lim_{|x|\to\infty} u_0(x)$. Since for $\delta > 0$ small enough, we conclude (note that in $\dot{B}^{2,1}_{\frac{d}{2}}$ we have C_0 data)

$$\sup_{t\in\mathbb{R}} \operatorname{dist}(u, N) \le \|u-p\|_{L^\infty_{t,x}} \lesssim \|v\|_{L^\infty_t B^{2,1}_{\frac{d}{2}}} \lesssim \|v\|_{Z^{\frac{n}{2}}} \lesssim \delta, \tag{2.5.1}$$

the map $v \mapsto \Pi(v+p)$ and thus (2.1.4) is welldefined in a $B(0, C\delta)$ ball in $Z^{\frac{d}{2}}$. In particular, we apply Theorem 2.1.1 to the Cauchy problem (2.1.4) with the nearest point map $v \mapsto \Pi(v+p)$ and obtain a global solution in $\dot{B}^{2,1}_{\frac{d}{2}}(\mathbb{R}^d)$, which belongs to $\dot{H}^s(\mathbb{R}^d)$ for any $s > \frac{d}{2}$. Especially we recover that $u = v + p$ is a smooth map from Lemma 2.2.5.

The only thing left to show is that $u(t) \in N$ for $t \in \mathbb{R}$, such that in particular, (2.1.4) implies (2.1.3).

If $v = u - p \in B(0, C\delta) \subset Z^{\frac{d}{2}}$ for a small $\delta > 0$, then since $\Pi(p) = p$ we have $\Pi(u) - p = \Pi(v+p) - p \in Z^{\frac{d}{2}}$ by the series expansion and $\Pi(u) - u = \Pi(v+p) - p - v \in Z^{\frac{d}{2}}$ with

$$\|\Pi(u) - u\|_{Z^{\frac{d}{2}}} + \|\Pi(u) - p\|_{Z^{\frac{d}{2}}} \lesssim \|v\|_{Z^{\frac{d}{2}}},$$

provided $\delta > 0$ is small. We now have

$$L(u - \Pi(u)) = L(v - \Pi(v+p)) = -d\Pi_{v+p}(Lv) = -d\Pi_{v+p}(\mathcal{N}(v)), \tag{2.5.2}$$

Note that, since we have $\Pi(u)$ on the LHS, the linear part of the expansion of $d\Pi_{v+p}$ on the RHS is present. Since $\Pi(u) \in N$, we have $\mathcal{N}(\Pi(u) - p) \perp T_{\Pi(u)}N$ and from $Im(d\Pi_u) \subset T_{\Pi(u)}N$, $u = v + p$, we obtain

$$d\Pi_{v+p}(\mathcal{N}(\Pi(u) - p)) = 0.$$

Hence (2.5.2) reads as

$$L(u - \Pi(u)) = -d\Pi_{v+p}(\mathcal{N}(v) - \mathcal{N}(\Pi(u) - p)). \tag{2.5.3}$$

At this point, however, we mention that since

$$\mathcal{N}(v) = L(\Pi(v+p)) - d\Pi_{v+p}(Lv),$$

$$\mathcal{N}(\Pi(u) - p) = L(\Pi(\Pi(u))) - d\Pi_{\Pi(u)}(L(\Pi(u))),$$

we cancel the linear part in the series expansion for $L(\Pi(v+p))$ with the linear part of $d\Pi_{v+p}(Lv)$ in $\mathcal{N}(v)$. Likewise we cancel the linear part of $L(\Pi(\Pi(u)))$ with $d\Pi_{\Pi(u)}(L(\Pi(u)))$ in $\mathcal{N}(\Pi(u)-p)$ on the RHS. We proceed using the notation as before and obtain (following the definition of $\mathcal{N}$)

$$\begin{aligned} L(u-\Pi(u)) = -d\Pi_{v+p}\big(& (d\Pi_{(\Pi(u)-p)+p} - d\Pi_{v+p})L(\Pi(u)-p) \\ & + d\Pi_{v+p}(L(\Pi(u)-p-v)) \\ & + L(\Pi(v+p) - \Pi((\Pi(u)-p)+p))\big). \end{aligned} \tag{2.5.4}$$

Note that we don't want to use $\Pi^2 = \Pi$, since technically we want the identity for the series expressions for $\Pi,\ d\Pi$ with missing linear (resp. constant) parts. Especially, all terms appearing on the RHS are at least quadratic in $v,\ \Pi(u)-p$.

This implies (note that $u(0) = \Pi u(0),\ u_t(0) = \partial_t(\Pi u)(0)$ by assumption)

$$\begin{aligned} \|u-\Pi(u)\|_{Z^{\frac{d}{2}}} \lesssim\ & (1+\|v\|_{Z^{\frac{d}{2}}})\left\|(d\Pi_{(\Pi(u)-p)+p} - d\Pi_{v+p})L(\Pi(u)-p)\right\|_{W^{\frac{d}{2}}} \\ & + (1+\|v\|_{Z^{\frac{d}{2}}})\|d\Pi_{v+p}(L(\Pi(u)-p-v))\|_{W^{\frac{d}{2}}} \\ & + (1+\|v\|_{Z^{\frac{d}{2}}})\|L(\Pi(v+p) - \Pi((\Pi(u)-p)+p))\|_{W^{\frac{d}{2}}} \\ \lesssim\ & (1+\|v\|_{Z^{\frac{d}{2}}})\|u-\Pi(u)\|_{Z^{\frac{d}{2}}}\|L(\Pi(u)-p)\|_{W^{\frac{d}{2}}} \\ & + (1+\|v\|_{Z^{\frac{d}{2}}})\|v\|_{Z^{\frac{d}{2}}}\|L(\Pi(u)-p-v)\|_{W^{\frac{d}{2}}} \\ & + (1+\|v\|_{Z^{\frac{d}{2}}})(\|v\|_{Z^{\frac{d}{2}}} + \|\Pi(u)-p\|_{Z^{\frac{d}{2}}})\|u-\Pi(u)\|_{Z^{\frac{d}{2}}} \\ \lesssim\ & (1+\|v\|_{Z^{\frac{d}{2}}})\|v\|_{Z^{\frac{d}{2}}}\|u-\Pi(u)\|_{Z^{\frac{d}{2}}}. \end{aligned}$$

In particular, if $\|v\|_{Z^{\frac{d}{2}}} \le \delta$ is sufficiently small, we have $u = \Pi(u) \in N$.

Appendix

2.A Local smoothing and Strichartz inequalities

In this section, we recall the *local smoothing* effect (i.e. lateral Strichartz estimates with localized data) and a *maximal function estimate* for the linear Cauchy problem

$$\begin{cases} i\partial_t u(t,x) \pm \Delta u(t,x) = f(t,x) & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases} \tag{2.A.1}$$

in the lateral space $L_e^p L_{t,e^\perp}^q$ for $e \in \mathbb{S}^{d-1}$ with norm

$$\|f\|_{L_e^p L_{t,e^\perp}^q}^p = \int_{-\infty}^{\infty} \left(\int_{[e]^\perp} \int_{-\infty}^{\infty} |f(t, re + x)|^q dt\, dx \right)^{\frac{p}{q}} dr. \tag{2.A.2}$$

The norm (2.A.2) was used by Kenig, Ponce, Vega, see e.g. [26], in order to establish local smoothing estimates for nonlinear Schrödinger equations.
The estimates for $L_e^p L_{t,e^\perp}^q$, $L_e^1 L_{t,e^\perp}^2$, $L_e^2 L_{t,e^\perp}^\infty$ in Corollary 2.A.3 and Lemma 2.A.5 below are substantial in the wellposedness theory of Schrödinger maps and were proven by Ionescu, Kenig in [21], [22] (see also the work of Bejenaru in [2] and Bejenaru, Ionescu, Kenig in [3]). Similar ideas (however more involved due to the absence of the $L_e^2 L_{t,e^\perp}^\infty$ estimate in $d = 2$) have been used by Bejenaru, Ionescu, Kenig and Tataru in [4] for global Schrödinger maps into $\mathbb{S}^2$ in dimension $d \geq 2$ with small initial data in $H^{\frac{d}{2}}$.

Here we follow Bejenaru's calculation in [2], which recovers the smoothing effect for (2.A.1) provided the data u_0, f is sufficiently localized in the sets

$$A_e = \{\xi \mid \xi \cdot e \geq \frac{|\xi|}{\sqrt{2}}\},$$

$$B_e^\pm = \left\{(\tau, \xi) \mid | \pm \tau - \xi^2| \leq \frac{|\tau| + \xi^2}{10},\ \xi \in A_e \right\}$$

$$A_\lambda = \{(\tau, \xi) \mid \lambda/2 \leq (\tau^2 + |\xi|^4)^{\frac{1}{4}} \leq 2\lambda\},$$

as defined in Section 2.3. Especially for $(\tau,\xi)\in B_e^{\pm}\cap A_\lambda$, there holds

$$\pm\tau-\xi_{e^\perp}^2\geq 0,\quad \xi_e\sim\lambda,\quad \xi_e+\sqrt{\pm\tau-\xi_{e^\perp}^2}\sim\lambda. \tag{2.A.3}$$

Remark 2.A.1. We note that our definition of $B_e^{\pm}$ slightly differs from [2].

Taking the FT (in t,x) of (2.A.1), with u being localized in $B_e^{\pm}$,

$$\hat f(\tau,\xi)=(\tau\mp|\xi|^2)\hat u(\tau,\xi)=\pm\left(\sqrt{\pm\tau-\xi_{e^\perp}^2}-\xi_e\right)\left(\sqrt{\pm\tau-\xi_{e^\perp}^2}+\xi_e\right)\hat u(\tau,\xi). \tag{2.A.4}$$

Hence, considering (2.A.3), we proceed by taking the (inv.) FT in the coordinates $t,x_{e^\perp}$,

$$\begin{aligned}&\pm\mathcal{F}^{-1}(\hat f(\xi_e,\tau,\xi_{e^\perp})\big(\sqrt{\pm\tau-\xi_{e^\perp}^2}+\xi_e\big)^{-1}) \\ &\qquad=\mathcal{F}^{-1}\left(\sqrt{\pm\tau-\xi_{e^\perp}^2}\,\hat u(\xi_e,\tau,\xi_{e^\perp})\right)-\xi_e\hat u(\xi_e,t,x_{e^\perp}).\end{aligned} \tag{2.A.5}$$

Thus, (2.A.1) is equivalent to an intial value problem of the following type

$$\begin{cases}(i\partial_r+D_{t,x}^{\pm})v(t,r,x)=f,\\ v(t,0,x)=u(t,x),\end{cases} \tag{2.A.6}$$

where $\widehat{D_{t,x}^{\pm}v}(\tau,\xi)=\sqrt{\pm\tau-|\xi|^2}\hat v(\tau,\xi)$. Thus (at least formally) the homogeneous solution of (2.A.6) is represented as

$$v(t,r,x)=e^{irD_{t,x}^{\pm}}u(t,x).$$

In the following, we only consider homogeneous estimates for (2.A.1), wich imply all linear estimates we need in Section 2.3. Inhomogeneous bounds for the biharmonic problem (2.3.1) with $F\in L_e^1L_{t,e^\perp}^2$ can be proven similarly as for the Schrödinger equation using the calculation in Section 2.3.

The equation (2.A.6) has the scaling $v_\lambda(t,r,x)=v(\lambda^2t,\lambda r,\lambda x),\ \lambda>0$ and we now prove the following Strichartz estimate.

Lemma 2.A.2 (Strichartz estimate)**.** *Let* $u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}^{d-1}),\ \ f\in\mathcal{S}'(\mathbb{R}\times(\mathbb{R}\times\mathbb{R}^{d-1}))$ *have Fourier support in*

$$\{\pm\tau\geq\xi^2\}\cap A_\lambda$$

for some dyadic $\lambda\in 2^{\mathbb{Z}}$*. Then there holds*

$$\left\|e^{irD_{t,x}^{\pm}}u(t,x)\right\|_{L_r^pL_{t,x}^q}\lesssim\lambda^{\frac{d+1}{2}-\frac{1}{p}-\frac{d+1}{q}}\|u\|_{L_{t,x}^2}, \tag{2.A.7}$$

$$\left\|\int_{-\infty}^r e^{i(r-s)D_{t,x}^{\pm}}f(s,t,x)\,ds\right\|_{L_r^pL_{t,x}^q}\lesssim\lambda^{\frac{1}{\tilde p'}-\frac{1}{p}+(d+1)(\frac{1}{\tilde q'}-\frac{1}{q})-1}\|f\|_{L_r^{\tilde p'}L_{t,x}^{\tilde q'}}, \tag{2.A.8}$$

where $(p,q),\ (\tilde p,\tilde q)$ *are admissible, i.e.* $1\leq p,q\leq\infty,\ (p,q)\neq(2,\infty)$ *if* $d=2$ *and*

$$\frac{2}{p}+\frac{d}{q}\leq\frac{d}{2}. \tag{2.A.9}$$

Proof. We use the Littlewood-Paley decomposition from Section 2.2

$$\widehat{P_\lambda(u)}(\tau,\xi) = \varphi((\tau^2+|\xi|^4)^{\frac{1}{4}}/\lambda)\hat{u}(\tau,\xi),$$

and by scaling of (2.A.6), we have $P_1(u_{\lambda^{-1}}) = (P_\lambda u)_{\lambda^{-1}}$. Thus we reduce the estimate (2.A.7) to

$$\left\| e^{irD^{\pm}_{t,x}} P_1 u(t,x) \right\|_{L^p_r L^q_{t,x}} \leq C_{d,p,q} \left\| \varphi((\tau^2+|\xi|^4)^{\frac{1}{4}})\hat{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}}. \tag{2.A.10}$$

We further have $e^{irD^{\pm}_{t,x}} P_1 u(t,x) = K *_{t,x} P_1 u$, with kernel

$$K(t,r,x) = \int\int e^{i(x,t,r)\cdot(\xi,\tau,\sqrt{\pm\tau-\xi^2})}\psi(\tau,\xi)\, d\tau d\xi, \tag{2.A.11}$$

where $\psi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^{d-1})$ with $\psi(\tau,\xi) = 1$ for $(\tau,\xi) \in \operatorname{supp}((\tau,\xi) \mapsto \varphi((\tau^2+|\xi|^4)^{\frac{1}{4}}))$. which is the Fourier transform of a (compactly supported) surface carried measure on the hypersurface

$$S = \left\{ \left(\xi,\tau,\sqrt{\pm\tau-\xi^2}\right) \mid \xi \in \mathbb{R}^{d-1}, \tau\in\mathbb{R}, \xi^2 \leq \pm\tau \right\},$$

and S has d non-vanishing principal curvature functions in the relevant coordinate patch. Thus, from Lemma 2.2.10 we observe

$$|K(t,r,x)| \lesssim (1+|r|)^{-\frac{d}{2}}, \;\; t\in\mathbb{R},\; x\in\mathbb{R}^{d-1}, \tag{2.A.12}$$

which gives

$$\left\| e^{irD^{\pm}_{t,x}} P_1 u(t,x) \right\|_{L^\infty_{t,x}} \lesssim (1+|r|)^{-\frac{d}{2}} \|P_1 u\|_{L^1_{t,x}}. \tag{2.A.13}$$

Now in the endpoint case $(p,q) = (2,\frac{2d}{d-2})$, we apply Lemma 2.2.13 and otherwise we can use a direct argument as outlined in Section 2.2. More precisely, combining (2.A.13) and the fact that $e^{irD^{\pm}_{t,x}}$ is a group on $L^2_{t,x}$ with a classical TT^* argument and the Christ-Kiselev Lemma 2.2.12, we deduce (2.A.7) and (2.A.8) for $P_1 u$. □

We remark that Lemma 2.A.2 is valid if u, f are localized at frequency λ as stated in Section 2.2. For a dyadic number $\lambda \in 2^{\mathbb{Z}}$, we recall the definition of $A^d_\lambda = \{\xi \mid \lambda/2 \leq |\xi| \leq 2\lambda\}$. An immediate consequence of the Strichartz estimate is the following Corollary.

Corollary 2.A.3. *Let* $u_0 \in L^2(\mathbb{R}^d),\; e\in\mathbb{S}^{d-1},\; \lambda > 0$ *dyadic with* $\operatorname{supp}(\hat{u}_0) \subset A^d_\lambda \cap A_e$. *Then there holds*

$$\left\| e^{\pm it\Delta} u_0 \right\|_{L^p_e L^q_{t,e^\perp}} \leq C\lambda^{\frac{d}{2}-\frac{1}{p}-\frac{(d+1)}{q}} \|u_0\|_{L^2_x}, \tag{2.A.14}$$

where (p,q) *is an admissible pair. Let* $u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d),\; e\in\mathbb{S}^{d-1},\; \lambda>0$ *dyadic such that*

$$\operatorname{supp}(\hat{u}) \subset \left\{ (\tau,\xi_{e^\perp}) \mid (\tau,\xi) \in B^{\pm}_e \cap A_\lambda \right\}$$

Then there holds

$$\left\| e^{ix_e D^{\pm}_{t,x_{e^\perp}}} u(t,x_{e^\perp}) \right\|_{L^p_t L^q_x} \leq C_{d,p,q} \lambda^{\frac{d+1}{2}-\frac{2}{p}-\frac{d}{q}} \left\| u(t,x_{e^\perp}) \right\|_{L^2_{t,x_{e^\perp}}}, \tag{2.A.15}$$

where (p,q) *is an admissible pair.*

Proof. For the first statement, we identify $\xi = (\xi \cdot e)e + \xi_{e^\perp} \mapsto (\xi_e, \xi_{e^\perp})$ and proceed as follows. By the change of coordinates $\sqrt{\pm\tau - \xi^2_{e^\perp}} = \xi_e$, we have

$$\int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \hat{u}_0(\xi)\, d\xi$$
$$= \int_{[e]^\perp} \int_{\{\pm\tau \geq \xi^2_{e^\perp}\}} e^{\pm it\tau} e^{ix_{e^\perp}\xi_{e^\perp}} e^{ix_e\sqrt{\pm\tau-\xi^2_{e^\perp}}} \hat{u}_0\Big(\sqrt{\pm\tau - \xi^2_{e^\perp}}\, e + \xi_{e^\perp}\Big) \frac{d\tau}{2\sqrt{\pm\tau - \xi^2_{e^\perp}}}\, d\xi_{e^\perp}.$$

Now we set

$$\hat{u}(\tau, \xi_{e^\perp}) = \hat{u}_0\Big(\sqrt{\pm\tau - \xi^2_{e^\perp}}\, e + \xi_{e^\perp}\Big) \left(2\sqrt{\pm\tau - \xi^2_{e^\perp}}\right)^{-1}, \text{ if } \pm\tau \geq |\xi_{e^\perp}|^2,$$

and $\hat{u}(\tau,\xi_{e^\perp}) = 0$ elsewhere. By assumption on $\hat{u}_0$, we have $u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^{d-1})$ (upon the identification of $[e]^\perp = \mathbb{R}^{d-1}$) and for $(\tau,\xi_{e^\perp}) \in \operatorname{supp}(\hat{u})$ there holds $\sqrt{\pm\tau - \xi^2_{e^\perp}} = \xi_e \sim \lambda$ and $\lambda/2 \leq \sqrt{|\tau| + \xi^2_{e^\perp}} \leq 4\lambda$ (since in particular u_0 localizes in $A_e \cap A^d_\lambda$). Thus, we apply Lemma 2.A.2 and conclude

$$\begin{aligned}
\left\| e^{\pm it\Delta} u_0(x) \right\|_{L^p_e L^q_{t,e^\perp}} &= \left\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \hat{u}_0(\xi)\, d\xi \right\|_{L^p_e L^q_{t,e^\perp}} \\
&= \left\| \int_{[e]^\perp} \int_{\{\pm\tau \geq \xi^2_{e^\perp}\}} e^{\pm it\tau} e^{ix_{e^\perp}\xi_{e^\perp}} e^{ix_e\sqrt{\pm\tau-\xi^2_{e^\perp}}} \hat{u}(\tau,\xi_{e^\perp}) d\tau d\xi_{e^\perp} \right\|_{L^p_e L^q_{t,e^\perp}} \\
&\lesssim \lambda^{\frac{d+1}{2}-\frac{1}{p}-\frac{d+1}{q}} \left\| \hat{u}_0\Big(\sqrt{\tau - \xi^2_{e^\perp}}\, e + \xi_{e^\perp}\Big) \left(2\sqrt{\pm\tau - \xi^2_{e^\perp}}\right)^{-1} \chi\{\pm\tau > \xi^2_{e^\perp}\} \right\|_{L^2_{\tau,e^\perp}} \\
&\lesssim \lambda^{\frac{d}{2}-\frac{1}{p}-\frac{d+1}{q}} \left\| u_0 \right\|_{L^2_x},
\end{aligned}$$

where, for the last inequality, we reverse the coordinate change and estimate the Jacobian. For the second statement, the estimate follows from Strichartz estimates for the Schrödinger group and from the above coordinate transform in the backward direction. To be more precise, we estimate

$$\begin{aligned}
\left\| e^{-ix_e D^{\pm}_{t,x_{e^\perp}}} u(t,x_{e^\perp}) \right\|_{L^p_t L^q_x} &= \left\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \hat{u}(\pm|\xi|^2, \xi_{e^\perp})\, 2(\xi\cdot e) d\xi \right\|_{L^p_t L^q_x} \\
&\lesssim \lambda^{\frac{d+1}{2}-\frac{2}{p}-\frac{d}{q}} \left\| (\xi\cdot e)^{\frac{1}{2}} \hat{u}(\pm|\xi|^2, \xi_{e^\perp}) \chi\{\xi\cdot e \geq 0\} \right\|_{L^2_x} \\
&\lesssim \lambda^{\frac{d+1}{2}-\frac{2}{p}-\frac{d}{q}} \left\| u(t,x_{e^\perp}) \right\|_{L^2_{t,x_{e^\perp}}}.
\end{aligned}$$

□

Remark 2.A.4. In the case $\text{supp}(\hat{u}_0) \subset A^d_\lambda$, we obtain from Corollary 2.A.3

$$\sup_{e \in \mathcal{M}} \left\| e^{\pm it\Delta} P_e(\nabla) u_0 \right\|_{L^p_e L^q_{t,e^\perp}} \leq C \lambda^{\frac{d}{2} - \frac{1}{p} - \frac{(d+1)}{q}} \left\| u_0 \right\|_{L^2_x}, \tag{2.A.16}$$

and especially the $L^p_e L^\infty_{t,e^\perp}$ estimate for $q = \infty,\ pd \geq 4,\ d \geq 3$.

The next Lemma (from [2]) shows that the $P_e(\nabla)$ localization on the LHS of (2.A.16) is not necessary in the case $q = \infty$ if $dp > 4$.

Lemma 2.A.5. *Let $u_0 \in L^2(\mathbb{R}^d)$ such that $\text{supp}(\hat{u}_0) \subset A^d_\lambda$ for some dyadic $\lambda \in 2^{\mathbb{Z}}$. Then there holds*

$$\sup_{e \in \mathcal{M}} \left\| e^{\pm it\Delta} u_0 \right\|_{L^p_e L^\infty_{t,e^\perp}} \leq C_{d,p} \lambda^{\frac{d}{2} - \frac{1}{p}} \left\| u_0 \right\|_{L^2_x}, \tag{2.A.17}$$

where $1 \leq p \leq \infty$ and $dp > 4$. Let $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$, such that

$$\textit{supp}(\hat{u}) \subset \{(\tau, \xi_{\tilde{e}^\perp}) \mid (\tau, \xi) \in B^\pm_{\tilde{e}} \cap A_\lambda\}$$

for some $\lambda \in 2^{\mathbb{Z}}$ and $\tilde{e} \in \mathcal{M}$. Then there holds

$$\sup_{e \in \mathcal{M}} \left\| e^{-irD^\pm_{t,x_e}} u(t, x_{e^\perp}) \right\|_{L^p_r L^\infty_{t,x_{e^\perp}}} \leq C_{d,p} \lambda^{\frac{d+1}{2} - \frac{1}{p}} \left\| u(t, x_{\tilde{e}^\perp}) \right\|_{L^2_{t,x_{\tilde{e}^\perp}}}, \tag{2.A.18}$$

where (d, p) are as above.

Proof. By scaling we reduce again to the unit frequency $\lambda = 1$. Then estimate (2.A.17) is a consequence of the TT^* argument for the Schrödinger group in the space $L^p_e L^\infty_{t,e^\perp}$ and Young's inequality. As mentioned before, we obtain the decay

$$\left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \varphi(|\xi|) d\xi \right| \lesssim (1 + |x \cdot e|)^{-\frac{d}{2}},$$

which implies (for $dp > 4$)

$$\left\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \varphi(|\xi|) d\xi \right\|_{L^{\frac{p}{2}}_e L^\infty_{t,e^\perp}} \lesssim 1.$$

Then by Young's inequality

$$\left\| \int e^{\pm i(t-s)\Delta} f(s)\, ds \right\|_{L^p_e L^\infty_{t,e^\perp}} \lesssim \|f\|_{L^{p'}_e L^1_{t,e^\perp}},$$

which implies (2.A.17) by TT^*. For (2.A.18), we use again (note $\hat{u}$ is localized in $B_e^{\pm}$, thus (2.A.3) holds)

$$\int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \varphi(|\xi|) d\xi \tag{2.A.19}$$

$$= \int_{[e]^\perp} \int_{\pm\tau \geq \xi_{e^\perp}^2} e^{ix_e\sqrt{\pm\tau - \xi_{e^\perp}}} e^{i(x_{e^\perp},t)\cdot(\xi_{e^\perp},\pm\tau)} \varphi_0(\sqrt{\pm\tau}) \frac{d\tau}{2\sqrt{\pm\tau - \xi_{e^\perp}}} d\xi_{e^\perp} \tag{2.A.20}$$

and thus we obtain (2.A.18) also from TT^* and Young's inequality for $\exp(-iD_{t,e^\perp}^{\pm})$. □

Remark 2.A.6. The first estimate in Lemma 2.A.5 holds more general by the same argument in the following sense. Let u_0, u as above in Lemma 2.A.5 and further $1 \leq p, q \leq \infty$ such that $q > 4$ and

$$\begin{cases} \frac{4q}{q-4} < dp, & q < \infty \\ \quad 4 < dp, & q = \infty. \end{cases} \tag{2.A.21}$$

Then there holds

$$\sup_{e\in\mathcal{M}} \left\| e^{\pm it\Delta} u_0 \right\|_{L_e^p L_{t,e^\perp}^q} \leq C_{d,p,q} \lambda^{\frac{d}{2} - \frac{d+1}{q} - \frac{1}{p}} \left\| u_0 \right\|_{L_x^2}, \tag{2.A.22}$$

Provided (2.A.21) holds, it is verified that

$$\int_{\infty}^{\infty} \left(\int_0^\infty (1 + |x_e| + r)^{-\frac{dq}{4}} r^{d-1}\, dr \right)^{\frac{p}{q}} dx_e < \infty,$$

which is required by the argument in the proof of Lemma 2.A.5, if we use

$$\left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|^2} \varphi(|\xi|) d\xi \right| \lesssim (1 + |x_e| + |(t, x_{e^\perp})|)^{-\frac{d}{2}}.$$

Under the assumption (2.A.21), we especially infer

$$\frac{2}{p} + \frac{d}{q} < \frac{2d(q-4) + 4d}{4q} = \frac{d(q-2)}{2q} < \frac{d}{2},$$

so that (p, q) is admissible. This is a natural requirement, since typically Strichartz bounds with bounded frequency rely on estimating the truncated dispersion factor via Young's inequality.

Remark 2.A.7. We apply the estimates to Lemma 2.3.5 and Lemma 2.3.6 in Section 2.3. Also, in Section 2.3, we need to use Corollary 2.A.3 and Lemma 2.A.5 for functions on $\mathbb{R}^d$ that are *localized at frequency* λ as stated in Section 2.2. This is observed (for all $t \in \mathbb{R}$) e.g. for functions on $\mathbb{R}^{d+1}$ localized (in (τ, ξ)) in $B_e \cap A_\lambda$, which have Fourier support in $A_\lambda^d \cup A_{\lambda/2}^d$, and poses no problem to the proof.

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