Sensitivity and statistical inference in Markov decision models and collective risk models

Dissertation

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Abstract

The first part of this thesis deals with the sensitivity and statistical estimation of the optimal value of a Markov decision model (MDM) in the transition probability function, i.e. the family of all transition probabilities. Such models are used for modelling stochastic optimization problems with sequential decision making which appear in many application areas. Since in practice, the used MDM is most often less complex than the underlying 'true' MDM, we first discuss the impact of a reduction of the model complexity in the transition probability function on the optimal value of the MDM, i.e. the solution of the underlying stochastic control problem. Besides a statement on the continuity of the optimal value regarded as a real-valued functional on a set of transition probability functions, we will in particular introduce a sort of derivative of this functional which can be used to measure the (first-order) sensitivity of the optimal value w.r.t. deviations in the transition probability function.

In addition, we perform a statistical analysis of the optimal value of a MDM where the underlying transition probability function is unknown, a situation that often occurs in practice. By limiting ourselves to a simple MDM in which the transition probability function is generated only by a single distribution function, we show that the optimal value construed as a real-valued functional defined on a set of distribution functions is continuous and functionally differentiable in a certain sense. By means of these regularity properties, we discuss the asymptotics of suitable estimators for the optimal value of the MDM in nonparametric and parametric statistical models. Our theoretical findings in the first part of this thesis are illustrated by means of optimization problems in inventory control and mathematical finance.

The second part of this thesis is devoted to the nonparametric estimation of risk measures of collective risks in a non-homogeneous individual risk model in connection with the determination of appropriate insurance premiums. We present two nonparametric candidates for the estimator of the exact insurance individual premium and show several asymptotic properties for the estimated premiums, such as strong consistency, asymptotic normality, and qualitative robustness, that are applicable in 'large' insurance collectives.

Zusammenfassung

Der erste Teil dieser Arbeit befasst sich mit der Sensitivität und statistischen Schätzung des optimalen Wertes eines Markov Entscheidungsmodells (MEMs) in der Übergangswahrscheinlichkeitsfunktion, d. h. der Familie aller Übergangswahrscheinlichkeiten. Solche Modelle werden zur Modellierung von stochastischen Optimierungsproblemen mit sequentieller Entscheidungsfindung verwendet, die in vielen Anwendungsbereichen auftreten. Da das verwendete MEM in der Praxis meist weniger komplex ist als das zugrundeliegende „wahre" MEM, diskutieren wir zunächst den Einfluss einer Reduktion der Modellkomplexität in der Übergangswahrscheinlichkeitsfunktion auf den optimalen Wert des MEM, d.h. der Lösung des zugrundeliegenden stochastischen Kontrollproblems. Neben einer Aussage über die Stetigkeit des optimalen Wertes, aufgefasst als ein reellwertiges Funktional definiert auf einer Menge von Übergangswahrscheinlichkeitsfunktionen, werden wir insbesondere eine Art Ableitung dieses Funktionals vorstellen, die zur Messung der Sensitivität (ersten Ordnung) des optimalen Wertes bezüglich Abweichungen in der Übergangswahrscheinlichkeitsfunktion verwendet werden kann.

Darüber hinaus führen wir eine statistische Untersuchung des optimalen Wertes eines MEMs durch, bei dem die zugrundeliegende Übergangswahrscheinlichkeitsfunktion unbekannt ist, eine Situation, die in der Praxis häufig vorkommt. Indem wir uns auf ein einfaches MEMs beschränken, in welchem die Übergangswahrscheinlichkeitsfunktion nur durch eine einzelne Verteilungsfunktion erzeugt wird, zeigen wir, dass der optimale Wert, welcher als ein Funktional auf einer Menge von Verteilungsfunktionen betrachtet wird, stetig und funktional differenzierbar in einem gewissen Sinn ist. Mit Hilfe dieser Regularitätseigenschaften diskutieren wir in nichtparametrischen und parametrischen statistischen Modellen die Asymptotiken geeigneter Schätzer für den optimalen Wert des MEM. Unsere theoretischen Erkenntnisse im ersten Teil dieser Arbeit werden anhand von Optimierungsproblemen in der Lagerbestandskontrolle und der Finanzmathematik veranschaulicht.

Der zweite Teil dieser Arbeit widmet sich der nichtparametrischen Schätzung von Risikomaßen kollektiver Risiken in einem individuellen Risikomodell im Zusammenhang mit der Bestimmung geeigneter Versicherungsprämien. Wir stellen zwei nichtparametrische Kandidaten für den Schätzer der exakten individuellen Versicherungs -prämie vor und zeigen für die geschätzten Prämien mehrere asymptotische Eigenschaften wie starke Konsistenz, asymptotische Normalität und qualitative Robustheit, welche in „großen" Versicherungskollektiven anwendbar sind.

Introduction

Markov decision models (MDMs), whose theoretical foundations can be traced back to the pioneer works of Bellman [9, 10], Shapley [84], and Howard [43], are a common and widely used mathematical framework for modelling stochastic optimization problems with sequential decision making that have a Markovian structure. These stochastic control problems, which may also be referred to as Markov decision optimization problems, appear in a variety of application areas, such as economics (e.g. optimal replacement, inventory control), finance (e.g. terminal wealth optimization), logistics (e.g. dynamic routing problems), engineering (e.g. elevator control), computer science (e.g. robotic control), and medicine (e.g. optimal cadaveric organ acceptance or rejection). The central object of a Markov decision optimization problem is a stochastic system (modelled via a so-called Markov decision process (MDP)) whose random transition mechanism, described by a family of transition probabilities, can be controlled over time by a decision maker through a strategy, i.e. a sequence of actions. The aim of the decision maker is to find a 'good' strategy so that the underlying Markov decision optimization problem admits an optimal solution, the so-called optimal value of the corresponding MDM.

The theory of MDMs has become increasingly important in recent decades, especially for the reason that in practice robust procedures for the computation of the optimal value of a MDM are highly demanded. This is based on the fact that, in contrast to the theory which usually assumes that the model components of a MDM are known precisely, some of these components, such as the transition probabilities, are unknown or difficult to determine in practice, for instance due to the lack of historical observations. On the one hand, the corresponding model components can be estimated by statistical methods to avoid this problem. On the other hand, in many applications, the 'true' model is replaced by an approximate version of the 'true' model or by a variant which is simplified and thus less complex. As a result, the optimal value of a MDM is often calculated in practice on the basis of model components which differ from the underlying 'true' model elements. Therefore, the sensitivity of the optimal value w.r.t. deviations in the model components of the corresponding MDM is of interest and has become an important research field in the theory of MDMs. Exemplary for these investigations, we refer to the works of Chin Hon and Hartman [24], Kolonko [50], Mastin and Jaillet [66], Müller [68], Van Dijk and Puterman [88], and others.

In the first part of this thesis, we first of all deal with the sensitivity of the optimal value of a finite horizon discrete time MDM w.r.t. deviations in the so-called transition probability function, i.e. the family of all transition probabilities. Already in the 1990s, Müller [68] pointed out that the impact of the transition probabilities of a MDP on the optimal value of a corresponding MDM can not be ignored for practical issues. He showed that the optimal value of a time-homogeneous MDM depends continuously on the transition probabilities, and he established bounds for the approximation error. Even earlier, Kolonko [50] obtained analogous bounds in a MDM in which the transition probabilities depend on a parameter. Error bounds for the optimal value of a discrete time MDM with countable state space and action spaces were also specified by Van Dijk and Puterman [88]. Morever, Mastin and Jaillet [66] presented loss bounds for the optimal value of a MDM with unknown transition probabilities.

In Chapter 2, we will focus on the situation where in the MDM the 'true' transition probability function is replaced by a simplified and thus less complex version. We refer to Subsection 3.1.4 for a simple example of this situation. The reduction of model complexity in practical applications is common and performed for several reasons. Apart from computational aspects and the difficulty of considering all relevant factors, one major point is that statistical inference for certain transition probabilities can be costly in terms of both time and money. For this reason, it is obviously of interest with regard to the optimal value to know what kind of model reduction is reasonable and what not. To put it another way, we are interested in how a change from a simplified to a more complex (more realistic) variant of the transition probability function affects the optimal value of a MDM. In Section 2.2, we will show that the so-called value function specifying the optimal value of a MDM is 'Lipschitz continuous' in a certain sense w.r.t. the transition probability function. However, with the help of this result we are not able to quantify the effect of changing the less complex version of a transition probability function to a more realistic version on the optimal value. For this reason, in Section 2.3, we will present a sort of derivative of the value function regarded as a real-valued functional defined on a set of transition probability functions which can be used to measure the (first-order) sensitivity of the optimal value w.r.t. changes in the transition probability function. Compared to the existing theory of MDMs, this approach is new and of interest for many application areas.

Besides this, in the first part of this thesis, we also consider the situation where in a MDM the underlying transition probability function is not known, but can be estimated with statistical methods. The motivation for our investigations results from the fact that, as described above, in many practical applications the transition probabilities (and thus the transition probability function) of a MDM are completely or partially unknown so that, for a computation of the optimal value of the corresponding MDM, the missing transition probabilities must first be determined. We consider a simple finite horizon discrete time MDM in which the transition probability function is generated only by a single distribution function, a situation that occurs in many real applications. Subsections 5.1.1 and 5.2.1 will each describe such a situation by way of example. In Chapter 4, we will present two methods which can be used to estimate the unknown transition probability function (and thus the optimal value) of the corresponding MDM. These two approaches are based on a nonparametric and a parametric estimation of the unknown distribution function and require the knowledge of historical observations. As a consequence, we derive a reasonable estimator for the unknown distribution function and thus the optimal value of the corresponding MDM within each approach. This leads to the following questions: Under which assumptions on the underlying MDM and the estimators for the unknown distribution function can we derive asymptotic properties for the respective estimators of the optimal value? In this case, how does the estimation of the unknown distribution function affects the estimation of the optimal value of the corresponding MDM? What validity and conclusions do such asymptotic properties have? Sections 4.4 and 4.5 are devoted to address these issues.

The existing literature has already dealt with the statistical inference for the optimal value of a MDM in which not only the transition probabilities are unknown on several occasions. Cooper and Rangarajan [26] considered a MDM in which the expected cost functions as well as the transition probabilities are not known, and assumed that the latter expressions are governed by a family of (unknown) distribution functions. By means of a nonparametric approach, they estimated the unknown family of distribution functions based on a sequence of i.i.d. random variables, and provided under some structural assumptions on the corresponding MDM bounds for the probability that the optimal value computed with estimated components is within a prescribed distance of the optimal value with 'true' components. Loss bounds for the expected approximation error of the estimated and the 'true' optimal value of a MDM with transition probabilities depending on an unknown parameter were given by Kolonko [51]. Here the author used a Bayes estimator for the estimation of the unknown parameter and thus the transition probabilities.

The second part of this thesis is concerned with the statistical estimation of an appropriate individual premium for the next insurance period from a non-homogeneous insurance collective consisting of a finite and deterministic number of independent risks. In the context of actuarial practice, insurers are confronted with the task of calculating a premium for (insurable) risks that is both competitive and sufficient to cover future claims. However, the observed single claim amounts (and thus the corresponding distributions) of the individual risks in an insurance collective may differ, sometimes considerably, so that the resulting premium based on each of these claims would be unacceptable to the policyholder. For this reason, insurers group 'similar' risks together in 'large' collectives and take advantage of the effect that in such collectives the random risk is reduced and thus a lower premium can be realised for each individual risk. To model this approach, we will consider a so-called non-homogeneous individual model as a (standard) mathematical setting, in which the individual risks from a non-homogeneous insurance collective are expressed by a sequence of independent but not necessarily identically distributed random variables. Here we stress the fact that such a non-homogeneous risk model can better reflect actual actuarial practice than a homogeneous risk model, where the involved random variables modelling the individual risks in the insurance collective are independent and additionally identically distributed.

Within this theoretical framework, in Chapter 7, we will present a candidate for the exact (collective and) individual premium which is based on the total claim amount of the non-homogeneous insurance collective evaluated at an appropriate risk measure. The choice of such a risk measure describes to a certain extent the risk position from the insurer's point of view and expresses its so-called risk appetite, i.e. what level of risk the insurance company is prepared to accept insurable risks in return for payment of a premium. In Section 6.4, we give examples of risk measures that are frequently used in actuarial practice. For the estimation of the (distribution of the) future total claim amount in the insurance collective we will present, with the normal approximation and a convolution estimation method, two nonparametric approaches which are based on observed historical individual claims. The use of historical data for estimating the distribution of future claims is common in actuarial practice. By inserting the resulting nonparametric estimators for the total claim distribution into a so-called risk functional, which is linked to the risk measure chosen by the insurance company, one obtains two candidates for the estimator of the (collective and) individual premium.

In the existing literature, there are several studies about the statistical estimation of the total claim distribution. On the one hand, Krätschmer and Zähle [54] as well as Lauer and Zähle [61] used the normal approximation with estimated parameters to construct an estimator for the distribution of the total claim in a homogeneous insurance collective with independent identically distributed individual risks. On the other hand, Lauer and Zähle [61] showed that the convolution of the empirical measure of independent and identically distributed individual risks is also a suitable estimator for the total claim distribution.

The main task in the second part of this thesis is to investigate the asymptotics of the nonparametric estimators for the exact individual premium. From the insurer's point of view, it is on the one hand interesting to know how the deviation of the estimated premiums from the exact premium behaves asymptotically depending on, for example, the collective size, the choice of the risk measure or the distributions of the observed single claims. Motivated by the works of [54, 61], Section 7.2 is devoted to these studies. There we will also see that both the estimated individual premiums and the exact individual premium can be approximated on the basis of a premium principle, which corresponds to a standard deviation principle widely used in actuarial practice. On the other hand, insurers are confronted in practice with observed single claims from insurance collectives whose distributions can sometimes differ considerably from each other. For pragmatic reasons, however, insurers generally assume a homogeneous risk model with a hypothetical single claim distribution to calculate the exact individual premium. In practice, it is therefore of interest how a deviation of the observed single claim distributions from the hypothetically assumed single claim distribution affects the (individual premium and the distribution of the) estimated individual premium based on a homogeneous risk model, especially for 'large' insurance collectives. In Section 7.3, we will deal with this issue in context of the convolution based premium estimator.

This thesis is organized as follows. Chapter 1 will provide a theoretical background in the field of finite horizon discrete time MDMs based on the standard literature on MDMs, such as Bäuerle and Rieder [5], Bertsekas and Shreve [11], Herández-Lerma and Lasserre [38], Hinderer [39], and Puterman [73]. Since it is important to have an elaborate notation in order to formulate our main results in Chapters 2 and 4, we are very precise in this chapter. In Sections 1.1–1.2, we will formally introduce our finite horizon discrete time MDM used throughout the first part of this thesis to model a stochastic optimization problem with sequential decision making based on a specific performance criterion, and define the so-called value function specifying the solution and thus the optimal strategy for the considered maximization problem. Afterwards, in Sections 1.3–1.4, we will also discuss under which conditions the value function and an optimal strategy exist, and Section 1.5 is devoted to MDMs in a specific finite setting.

In Chapter 2, we will first introduce an appropriate distance between transition probability functions based on so-called integral probability metrics which have already been discussed, for example, in [68]. By means of this distance, we will introduce a reasonable notion of 'continuity' and 'differentiability' and show that the value function, regarded as a real-valued functional on some set of transition probability functions, is 'continuous' and 'differentiable'. This will be discussed in detail in Sections 2.2 and 2.3. These investigations will justify in a way that the optimal value is sensitive w.r.t. deviations in the transition probability function.

Our theoretical findings in Chapters 1–2 will be illustrated in Chapter 3 by means of classical Markov decision optimization problems in inventory control and mathematical finance. In particular, the numerical example presented in Subsection 3.1.4 shows that the 'derivative' of the optimal value (known from Subsection 3.1.3) can be used to quantify the effect of a change from simplified to a more complex variant of the transition probability function on the optimal value of the corresponding MDM.

Chapter 4 deals with a study of a simple MDM, in which the underlying transition probability function is generated only by an (unknown) single distribution function. There we will use the general terminology and notations introduced in Chapter 1 to formulate our specific setting (see Sections 4.1–4.2). In Section 4.3, we show that the value function construed as a real-valued functional defined on a set of distribution functions is continuous and functionally differentiable in a certain sense. Based on these notations and regularity results, in Sections 4.4–4.5, we will discuss two approaches to estimate the unknown distribution function and thus the optimal value of the MDM in a nonparametric and a parametric statistical model. In both sections, we will present asymptotic properties of the corresponding estimators for the optimal value of the MDM, such as strong consistency, asymptotic error distribution, and bootstrap consistency (in probability).

Exemplary for the theory presented in Chapter 4, in Chapter 5, we will take up the stochastic inventory control problem as well as the terminal wealth problem already considered in Sections 3.1–3.2, and assume that the random transition mechanism of the corresponding Markov decision optimization problems is now described by an unknown distribution function. Within this framework, we will perform a nonparametric and a parametric estimation of the optimal value of both Markov decision optimization problems in the unknown distribution function which will illustrate the results presented in Sections 4.4–4.5.

In Chapter 6, we will first formally introduce the notion of a risk measure in the context of non-life insurance mathematics and discuss, for certain classes of risk measures, regularity properties of the associated risk functionals w.r.t. the so-called Wasserstein metric. Finally, in Section 6.4, we will present examples of risk measures used in practice.

Chapter 7 is devoted to a nonparametric estimation of the individual premium in a non-homogeneous insurance collective consisting of a finite number of independent but not identically distributed risks. After motivating and introducing two nonparametric estimators for the individual premium based on a normal approximation and a convolution approach in Section 7.1, we will show asymptotic properties of these estimators, such as strong consistency and asymptotic normality which is part of Section 7.2. Finally, in Section 7.3, we will investigate the sequence of estimators which are based on the convolution approach for qualitative robustness. This investigation motivates somehow the choice of the latter estimator for the individual premium in a 'slightly' non-homogeneous insurance collective when the insurer assumes a homogeneous individual risk model for the computation of future single premiums.

The majority of the results in the first part of this thesis can also be found in the article jointly with Axel Simroth and Professor Henryk Zähle. The results of Chapters 1–3 are based on [48]:

> Kern, P., Simroth, A. and Zähle, H. (2020). First-order sensitivity of the optimal value in a Markov decision model with respect to deviations in the transition probability function. *Mathematical Methods of Operations Research*, 92(1), 165–197.

The elaborations in Chapters 4–5 result from a joint project with Professor Henryk Zähle:

> Kern, P. and Zähle, H., project on the "Statistical estimation of the optimal value in a specific Markov decision model", *work in progress.*

Finally, the investigations in the second part of this thesis are based on joint work with Professor Henryk Zähle. The results in Chapter 6–7 rely on:

> Kern, P. and Zähle, H., project on the "Statistical inference for risk measures of collective risks in an individual model with independent but not identically distributed observed single claims", *work in progress.*

Part I

Sensitivity analysis and statistical inference for the optimal value in a Markov decision model

Chapter 1

Foundations of finite horizon discrete time Markov decision models

In this chapter we give a detailed introduction into the theory of finite horizon discrete time Markov decision models (MDMs). As already mentioned in the main introduction, these mathematical models are powerful tools that are used to model stochastic optimization problems with sequential decision making which have a Markovian structure.

To explain such a stochastic control problem informally, suppose that we have a system of states whose dynamic can be controlled or regulated at finitely many discrete points of time by a sequence of decisions or actions. Moreover we assume that the transitions between different system states are random and that the process describing the stochastic evolution of the system states is Markovian. The latter means that transitions to future states of the process are not influenced by past states. The evolution of the system can be described as follows:

> Given a system state x at some point of time n, a decision maker (or controller) chooses an (admissible) action a. If action a is applied, then the decision maker receives a reward $r_n(x, a)$ and a random transition of the system occurs according to a probability distribution (or law) $P_n((x, a), \bullet)$ which leads to a new system state x' at time $n + 1$.

For the formulation of a reasonable optimization criterion, we suppose that at *any* point of time the decision maker selects an action and receives a reward. The objective of the decision maker is now to choose a suitable strategy (or policy), i.e. a sequence of actions, which leads to the fact that the system state process perform optimally with respect to some specific predetermined performance criterion based on the rewards. In Section 1.2, we will look at the so-called expected total reward criterion in detail which is one of the most commonly used performance criterion in the classical theory of MDMs.

All these quantities together characterize in an informal way some of the key features of a discrete time MDM with finite time horizon. In Section 1.1, we will precisely introduce all basic model components of a finite horizon discrete time MDM. There we will see that the stochastic evolution of the system states will be modelled by a so-called Markov decision process (MDP). Figure 1.1 below illustrates the general evolution of a MDP.

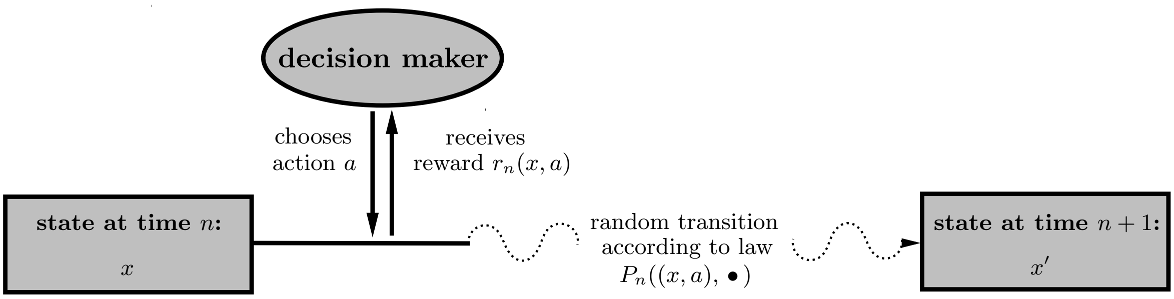

Figure 1.1: Schematic evolution of a MDP.

The first chapter is organized as follows. In Section 1.1, we will formally introduce quite general MDMs in the fashion of the standard monographs [5, 11, 38, 39, 73]. Later on, in Section 1.2 we introduce the value function of a MDM which will be derived from a reward maximization problem, and define subsequently the notion of an optimal strategy which corresponds to a solution of the latter optimization problem. The existence (and computation) of optimal strategies in general MDMs will be carried out in Section 1.3. Afterwards, in Section 1.4 we discuss some conditions under which the value function is well-defined. Finally, Section 1.5 is devoted to the special case of MDMs with finite state space and finite actions spaces.

1.1 Formal definition of a Markov decision model

In this section and elsewhere we will only consider finite horizon discrete time MDMs. Note that discrete time MDMs with infinite time horizon can be approximated by discrete time MDMs with finite but large time horizon; see, for example, [5, Chapter 7] or [12, Chapter 7]. Also note that in the finite horizon case Howard [43, p. 124] showed that discrete time MDMs can be seen as approximations for continuous time MDMs. As already mentioned in the main introduction, we will carefully introduce in this section the required notations and terminologies in order to present our main results in Chapter 2. As a result, this section is a little longer compared to the respective sections in other works on MDMs.

1.1.1 Basic model components

In this subsection we will introduce the basic model components of a finite horizon discrete time MDM which will be formally defined in Subsection 1.1.3.

Now, we let $N \in \mathbb{N}$ be a fixed number of discrete points of time at which the decision maker may choose actions in order to influence the dynamics of the stochastic system. The number N is also called *time* or *planning horizon* in discrete time. Moreover we will assume in the sequel that an action is always applied at the very beginning of the period between time n and $n+1$, $n = 0, \ldots, N-1$. Therefore the set of points of time at which actions may be chosen is given by $\{0, \ldots, N-1\}$.

Since the probabilistic system occupies at each point of time a state, we denote by E the *state space* of the system. Here we may and do assume that the state space E is a non-empty set, and we will equip E with a σ-algebra $\mathcal{E}$. The elements of the state space E represents the information about the system that is available for the decision maker.

For each state $x \in E$ and each time point $n = 0, \dots, N-1$, we let $A_n(x)$ be a non-empty set. The elements of $A_n(x)$ correspond to the *admissible* (or *allowable) actions* which the decision maker may choose at time $n = 0, \dots, N-1$ in state $x \in E$. Moreover, for each $n = 0, \dots, N-1$, we let

$$A_n := \bigcup_{x \in E} A_n(x) \quad \text{and} \quad D_n := \big\{(x, a) \in E \times A_n : a \in A_n(x)\big\}. \tag{1.1}$$

Note that the elements of A_n can be seen as the actions that may basically be selected at time n, whereas the elements of D_n are the possible state-action combinations at time n. For our subsequent analysis, we equip A_n with a σ-algebra $\mathcal{A}_n$, and let $\mathcal{D}_n := (\mathcal{E} \otimes \mathcal{A}_n) \cap D_n$ be the trace of the product σ-algebra $\mathcal{E} \otimes \mathcal{A}_n$ in D_n.

Given some possible state-action combination $(x, a) \in D_n$ at time $n = 0, \dots, N-1$, the system state visited at time $n+1$ will be drawn by the probability measure $P_n((x, a), \bullet\,)$, where P_n refers to a *probability* (or *Markov) kernel* from $(D_n, \mathcal{D}_n)$ to $(E, \mathcal{E})$. By definition a map $P_n : D_n \times \mathcal{E} \to [0, 1]$ is said to be a probability kernel from $(D_n, \mathcal{D}_n)$ to $(E, \mathcal{E})$ if $P_n(\,\cdot\,, B)$ is a $(\mathcal{D}_n, \mathcal{B}([0, 1]))$-measurable map for any $B \in \mathcal{E}$, and $P_n((x, a), \bullet\,) \in \mathcal{M}_1(E)$ for any $(x, a) \in D_n$. Here $\mathcal{M}_1(E)$ stands for the set of all probability measures on $(E, \mathcal{E})$. In this context P_n will be referred to as *one-step transition (probability) kernel at time n* (or *from time n to $n+1$*) and the probability measure $P_n((x, a), \bullet\,)$ is referred to as *one-step transition probability at time n* (or *from time n to $n+1$*) *given state x and action a*. In particular, the transitions of the system at each point of time $n = 0, \dots, N-1$ can be characterized by the N-tuple

$$\boldsymbol{P} = (P_0, \dots, P_{N-1})$$

whose n-th entry P_n is a probability kernel from $(D_n, \mathcal{D}_n)$ to $(E, \mathcal{E})$. Therefore, the N-tuple $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ will be referred to as *(Markov decision) transition (probability) function*. The set of all transition functions will be denoted by $\overline{\mathcal{P}}$.

In the sequel, we will assume that the actions are performed by a so-called N-stage strategy (or N-stage policy). An *(N-stage) strategy* is an N-tuple

$$\pi = (f_0, \dots, f_{N-1})$$

of decision rules at times $n = 0, \dots, N-1$, where a *decision rule at time n* is an $(\mathcal{E}, \mathcal{A}_n)$-measurable map $f_n : E \to A_n$ satisfying $f_n(x) \in A_n(x)$ for all $x \in E$. Note that $f_n(x)$ determines an admissible action which is taken in state x at time n. Also note that a decision rule at time n is (deterministic and) 'Markovian' since it only depends on the current state and is independent of previous states and actions. We denote by $\overline{\mathbb{F}}_n$ the set of *all* decision rules at time n, and assume that $\overline{\mathbb{F}}_n$ is non-empty. Hence a strategy is an element of the set $\overline{\Pi} := \overline{\mathbb{F}}_0 \times \cdots \times \overline{\mathbb{F}}_{N-1}$, and this set can be seen as the set of *all* strategies. Moreover, we fix for any $n = 0, \dots, N-1$ some $\mathbb{F}_n \subseteq \overline{\mathbb{F}}_n$ which can be seen as the set of all *admissible* decision rules at time n. In particular, the set $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$ $(\subseteq \overline{\Pi})$ can be seen as the set of all *admissible* strategies.

1.1.2 Markov decision process

Based on the notation and terminology from Subsection 1.1.1 we will present in this subsection a formal definition of an E-valued (finite horizon discrete time) Markov decision process (MDP) associated with a given initial state $x_0 \in E$, a given transition function $\boldsymbol{P} \in \overline{\mathcal{P}}$ and a given strategy $\pi \in \Pi$. We will see that the MDP describes the stochastic evolution of the system states.

To this end, let us consider in the following the measurable space

$$(\Omega, \mathcal{F}) := (E^{N+1}, \mathcal{E}^{\otimes(N+1)}).$$

Moreover, for any transition function $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, strategy $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and time point $n \in \{0, \ldots, N-1\}$, we can obviously derive from P_n a probability kernel P_n^π from $(E, \mathcal{E})$ to $(E, \mathcal{E})$ through

$$P_n^\pi(x, B) := P_n\big((x, f_n(x)), B\big), \qquad x \in E,\, B \in \mathcal{E}. \tag{1.2}$$

Note that the probability measure $P_n^\pi(x, \bullet)$ can be seen as the one-step transition probability at time n given state x when the transitions and actions are governed by $\boldsymbol{P}$ and π, respectively.

In virtue of (1.2), we may define for any $x_0 \in E$, $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, and $\pi \in \Pi$ a probability measure $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{P}^{x_0,\boldsymbol{P};\pi} := \delta_{x_0} \otimes P_0^\pi \otimes \cdots \otimes P_{N-1}^\pi, \tag{1.3}$$

where x_0 should be seen as the *initial state* of the MDP to be constructed and δ_{x_0} refers to the Dirac measure at point x_0. Note that the right-hand side of (1.3) is the usual product of the probability measure δ_{x_0} and the kernels $P_0^\pi, \ldots, P_{N-1}^\pi$. That is, the precise meaning of the definition of the probability measure $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ in (1.3) is in view of (1.2)

$$\begin{aligned}
\mathbb{P}^{x_0,\boldsymbol{P};\pi}[B] \;:=\; & \int_E \int_E \cdots \int_E \int_E \mathbb{1}_B(y_0, \ldots, y_N)\, P_{N-1}^\pi(y_{N-1}, dy_N) \\
& \qquad P_{N-2}^\pi(y_{N-2}, dy_{N-1}) \cdots P_0^\pi(y_0, dy_1)\, \delta_{x_0}(dy_0) \\
=\; & \int_E \int_E \cdots \int_E \int_E \mathbb{1}_B(y_0, \ldots, y_N)\, P_{N-1}\big((y_{N-1}, f_{N-1}(y_{N-1})), dy_N\big) \\
& \qquad P_{N-2}\big((y_{N-2}, f_{N-2}(y_{N-2})), dy_{N-1}\big) \cdots P_0\big((y_0, f_0(y_0)), dy_1\big)\, \delta_{x_0}(dy_0)
\end{aligned} \tag{1.4}$$

for $B \in \mathcal{F}$, for any given $x_0 \in E$, $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, and $\pi = (f_n)_{n=0}^{N-1} \in \Pi$.

Further let $\boldsymbol{X} = (X_0, \ldots, X_N)$ be the identity on $\Omega = E^{N+1}$, i.e.

$$X_n(x_0, \ldots, x_N) := x_n, \qquad (x_0, \ldots, x_N) \in E^{N+1},\, n = 0, \ldots, N. \tag{1.5}$$

Note that for any $x_0 \in E$, $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, and $\pi \in \Pi$ the map $\boldsymbol{X}$ can be regarded as an $(E^{N+1}, \mathcal{E}^{\otimes(N+1)})$-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^{x_0,\boldsymbol{P};\pi})$ with distribution $\delta_{x_0} \otimes P_0^\pi \otimes \cdots \otimes P_{N-1}^\pi$.

In the following $\mathbb{P}_{X\|Y}^{x_0,\boldsymbol{P};\pi}(\,\cdot\,, \bullet)$ refers to the factorized conditional distribution of X given Y under $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$, where X and Y correspond to any random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^{x_0,\boldsymbol{P};\pi})$. More precisely, by a (regular version of the) factorized conditional distribution of X given Y under $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ we mean a probability kernel $\mathbb{P}_{X\|Y}^{x_0,\boldsymbol{P};\pi}(\,\cdot\,, \bullet)$ for which for every $B \in \mathcal{E}$ the random variable

$\omega \mapsto \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X\|Y}(Y(\omega), B)$ is a conditional probability of $\{X \in B\}$ given Y under $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$. This object is only $\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{Y}$-a.s. unique. Thus the formulation of parts (iii)–(ix) in the following Lemma 1.1.1 is somewhat sloppy. Assertion (vi) in fact means that the probability kernel $P_n((\,\cdot\,, f_n(\,\cdot\,)), \bullet)$ provides a (regular version of the) factorized conditional distribution of X_{n+1} given X_n under $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$, and analogously for parts (iii)–(v) and (vii)–(ix). Note that it is also customary to write $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X \in \bullet\}]$ and $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X \in \bullet\}\|Y = \,\cdot\,]$ instead of $\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X}[\,\bullet\,]$ and $\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X\|Y}(\,\cdot\,,\bullet)$, respectively.

Lemma 1.1.1 *For any $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, $x_0, \widetilde{x}_0, x_1, \ldots, x_n \in E$ and $1 \le n < k \le N$ as well as $x_m \in E$ and $m = 1, \ldots, N$ we have*

(i) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_0 \in \bullet\}] \,=\, \delta_{x_0}[\,\bullet\,]$.

(ii) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{(X_0, \ldots, X_m) \in \bullet\}] \,=\, \delta_{x_0} \otimes P_0^\pi \otimes \cdots \otimes P_{m-1}^\pi[\,\bullet\,]$.

(iii) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_0 \in \bullet\}\|X_0 = \widetilde{x}_0] \,=\, \delta_{x_0}[\,\bullet\,]$.

(iv) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_1 \in \bullet\}\|X_0 = \widetilde{x}_0] \,=\, P_0\big((x_0, f_0(x_0)), \bullet\big)$.

(v) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_{n+1} \in \bullet\}\|(X_0, X_1, \ldots, X_n) = (\widetilde{x}_0, x_1, \ldots, x_n)] \,=\, P_n\big((x_n, f_n(x_n)), \bullet\big)$.

(vi) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_{n+1} \in \bullet\}\|X_n = x_n] \,=\, P_n\big((x_n, f_n(x_n)), \bullet\big)$.

(vii) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_m \in \bullet\}\|X_0 = \widetilde{x}_0] \,=\, \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_m \in \bullet\}] \,=\, \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0} \cdots \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(x_0, \bullet)$.

(viii) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_k \in \bullet\}\|X_n = x_n] \,=\, \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_{n+1}\|X_n} \cdots \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_{k-1}}(x_n, \bullet)$.

(ix) $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_m \in \bullet\}\|X_m = x_m] \,=\, \delta_{x_m}[\bullet]$.

Similarly to (1.4), the right-hand side of part (ii) of the preceding lemma is given by

$$\begin{aligned}
&\delta_{x_0} \otimes P_0^\pi \otimes \cdots \otimes P_{m-1}^\pi[B] \\
&:= \int_E \int_E \cdots \int_E \int_E \mathbb{1}_B(y_0, \ldots, y_m)\, P_{m-1}^\pi(y_{m-1}, dy_m) \\
&\qquad P_{m-2}^\pi(y_{m-2}, dy_{m-1}) \cdots P_0^\pi(y_0, dy_1)\, \delta_{x_0}(dy_0) \\
&= \int_E \int_E \cdots \int_E \int_E \mathbb{1}_B(y_0, \ldots, y_m)\, P_{m-1}\big((y_{m-1}, f_{m-1}(y_{m-1})), dy_m\big) \\
&\qquad P_{m-2}\big((y_{m-2}, f_{m-2}(y_{m-2})), dy_{m-1}\big) \cdots P_0\big((y_0, f_0(y_0)), dy_1\big)\, \delta_{x_0}(dy_0).
\end{aligned} \tag{1.6}$$

Also note that for parts (vii) and (viii) in Lemma 1.1.1 the compositions on the right-hand side are for every $B \in \mathcal{E}$ defined by

$$\begin{aligned}
&\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0} \cdots \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(x_0, B) \\
&:= \int_E \cdots \int_E \mathbb{1}_B(y_m)\, P_{m-1}\big((y_{m-1}, f_{m-1}(y_{m-1})), dy_m\big) \cdots P_0\big((x_0, f_0(x_0)), dy_1\big)
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}
&\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_{n+1}\|X_n} \cdots \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_{k-1}}(x_n, B) \\
&:= \int_E \cdots \int_E \mathbb{1}_B(y_k)\, P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big).
\end{aligned} \tag{1.8}$$

Note that the factorized conditional distributions in parts (iii)–(iv) and (vii) of Lemma 1.1.1 are constant w.r.t. $\widetilde{x}_0 \in E$ and that the probability measure $\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_n}(x_n, \bullet)$ in part (viii) of Lemma

1.1.1 can be seen in view of (1.8) as a $(k-n)$-step transition probability from stages n to k given state x_n.

Now, let us turn to the proof of Lemma 1.1.1. Note that $\mathbb{E}^{x_0,\boldsymbol{P};\pi}$ refers to the expectation w.r.t. the probability measure $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ as defined in (1.3)–(1.4).

Proof of Lemma 1.1.1: First of all it is clear that assertions (i)–(ii) hold. Thus it suffices to show the claims in (iii)–(ix).

(iii): The claim holds true, because in view of (1.4) and part (i)

$$\begin{aligned}
&\mathbb{E}^{x_0,\boldsymbol{P};\pi}\big[\delta_{X_0}[B]\mathbb{1}_{B_1}(X_0)\big]\\
&= \int_\Omega \delta_{X_0(\omega)}[B]\mathbb{1}_{B_1}(X_0(\omega))\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}(d\omega)\\
&= \int_E\int_E\cdots\int_E \delta_{y_0}[B]\mathbb{1}_{B_1}(y_0)\\
&\qquad P_{N-1}\big((y_{N-1},f_n(y_{N-1})),dy_N\big)\cdots P_0\big((y_0,f_0(y_0)),dy_1\big)\,\delta_{x_0}(dy_0)\\
&= \int_E \delta_{y_0}[B]\mathbb{1}_{B_1}(y_0)\,\delta_{x_0}(dy_0) \;=\; \delta_{x_0}[B]\mathbb{1}_{B_1}(x_0) \;=\; \delta_{x_0}[B\cap B_1]\\
&= \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_0\in B\}\cap\{X_0\in B_1\}]
\end{aligned}$$

for any $B\in\mathcal{E}$ and $B_1\in\mathcal{E}$.

(iv): The claim holds true, because in view of (1.4), (1.6), and part (ii)

$$\begin{aligned}
&\mathbb{E}^{x_0,\boldsymbol{P};\pi}\big[P_0\big((X_0,f_0(X_0)),B\big)\mathbb{1}_{B_1}(X_0)\big]\\
&= \int_\Omega P_0\big((X_0(\omega),f_0(X_0(\omega)),B\big)\mathbb{1}_{B_1}(X_0(\omega))\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}(d\omega)\\
&= \int_E\int_E\cdots\int_E P_0\big((y_0,f_0(y_0)),B\big)\mathbb{1}_{B_1}(y_0)\\
&\qquad P_{N-1}\big((y_{N-1},f_{N-1}(y_{N-1})),dy_N\big)\cdots P_0\big((y_0,f_0(y_0)),dy_1\big)\,\delta_{x_0}(dy_0)\\
&= \int_E\int_E \mathbb{1}_B(y_1)\,P_0\big((y_0,f_0(y_0)),dy_1\big)\mathbb{1}_{B_1}(y_0)\,\delta_{x_0}(dy_0)\\
&= \int_E\int_E \mathbb{1}_{B_1\times B}(y_0,y_1)\,P_0\big((y_0,f_0(y_0)),dy_1\big)\,\delta_{x_0}(dy_0)\\
&= \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_1\in B\}\cap\{X_0\in B_1\}]
\end{aligned}$$

for any $B\in\mathcal{E}$ and $B_1\in\mathcal{E}$.

(v): The claim holds true, because in view of (1.4), (1.6), and part (ii)

$$\begin{aligned}
&\mathbb{E}^{x_0,\boldsymbol{P};\pi}\big[P_n\big((X_n,f_n(X_n)),B\big)\mathbb{1}_{B_{n+1}}(X_0,\ldots,X_n)\big]\\
&= \int_\Omega P_n\big((X_n(\omega),f_n(X_n(\omega))),B\big)\mathbb{1}_{B_{n+1}}(X_0(\omega),\ldots,X_n(\omega))\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}(d\omega)\\
&= \int_E\int_E\cdots\int_E P_n\big((y_n,f_n(y_n)),B\big)\mathbb{1}_{B_{n+1}}(y_0,\ldots,y_n)\\
&\qquad P_{N-1}\big((y_{N-1},f_{N-1}(y_{N-1})),dy_N\big)\cdots P_0\big((y_0,f_0(y_0)),dy_1\big)\,\delta_{x_0}(dy_0)\\
&= \int_E\int_E\cdots\int_E\int_E \mathbb{1}_B(y_{n+1})\,P_n\big((y_n,f_n(y_n)),dy_{n+1}\big)\,\mathbb{1}_{B_{n+1}}(y_0,\ldots,y_n)
\end{aligned}$$

$$
\begin{aligned}
&\qquad P_{n-1}\big((y_{n-1}, f_{n-1}(y_{n-1})), dy_n\big) \cdots P_0\big((y_0, f_0(y_0)), dy_1\big)\, \delta_{x_0}(dy_0)\\
&= \int_E\int_E \cdots \int_E\int_E \mathbb{1}_{B_{n+1}\times B}(y_0,\ldots,y_n,y_{n+1})\, P_n\big((y_n, f_n(y_n)), dy_{n+1}\big)\\
&\qquad P_{n-1}\big((y_{n-1}, f_{n-1}(y_{n-1})), dy_n\big) \cdots P_0\big((y_0, f_0(y_0)), dy_1\big)\, \delta_{x_0}(dy_0)\\
&= \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_{n+1}\in B\}\cap\{(X_0,\ldots,X_n)\in B_{n+1}\}]
\end{aligned}
$$

for any $B \in \mathcal{E}$ and $B_{n+1} \in \mathcal{E}^{\otimes(n+1)}$.

(vi): As in the proof of (v) we obtain

$$
\mathbb{E}^{x_0,\boldsymbol{P};\pi}\big[P_n\big((X_n, f_n(X_n)), B\big)\mathbb{1}_{B_1}(X_n)\big] \;=\; \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_{n+1}\in B\}\cap\{X_n\in B_1\}]
$$

for any $B \in \mathcal{E}$ and $B_1 \in \mathcal{E}$.

(vii): First of all, it is known from the Chapman–Kolmogorov relation (see, e.g., [46, p. 143]) that the identity

$$
\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_j}(x_j, \bullet\,) \;=\; \int_E \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_l}(y, \bullet\,)\, \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_l\|X_j}(x_j, dy) \tag{1.9}
$$

holds for any $x_j \in E$ and $0 \le j \le l < m \le N$. Hence, by iterating (1.9), we obtain by means of parts (iv) and (vi) as well as (1.7)

$$
\begin{aligned}
&\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_m\in B\}\|X_0=\widetilde{x}_0] \;=\; \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_0}(\widetilde{x}_0, B)\\
&= \int_E\int_E\cdots\int_E \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(y_{m-1}, B)\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_2\|X_1}(y_1, dy_2)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0}(\widetilde{x}_0, dy_1)\\
&= \int_E\int_E\cdots\int_E P_{m-1}\big((y_{m-1}, f_{m-1}(y_{m-1})), B\big)\cdots P_1\big((y_1, f_1(y_1)), dy_2\big)\, P_0\big((x_0, f_0(x_0)), dy_1\big)\\
&= \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(x_0, B)
\end{aligned} \tag{1.10}
$$

for any $B \in \mathcal{E}$. Moreover, as an immediate consequence of the characterization of the (regular version of the) factorized conditional distribution, we have in view of (1.10) and part (i)

$$
\begin{aligned}
&\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_m\in B\}] \;=\; \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m}[B]\\
&= \int_E \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_0}(y, B)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_0}(dy) \;=\; \int_E \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(y, B)\,\delta_{x_0}(dy)\\
&= \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(x_0, B)
\end{aligned}
$$

for any $B \in \mathcal{E}$.

(viii): As in the proof of (vii) we obtain by iterating (1.9) along with part (vi) and (1.8)

$$
\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_k\in B\}\|X_n=x_n] \;=\; \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_n}(x_n, B) \;=\; \mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_{n+1}\|X_n}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_{k-1}}(x_n, B)
$$

for any $B \in \mathcal{E}$.

(ix): Analogously to the proof of (iii) we obtain by means of (1.7) and part (vii)

$$
\mathbb{E}^{x_0,\boldsymbol{P};\pi}\big[\delta_{X_m}[B]\mathbb{1}_{B_1}(X_m)\big] \;=\; \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\{X_m\in B\}\cap\{X_m\in B_1\}]
$$

for any $B \in \mathcal{E}$ and $B_1 \in \mathcal{E}$. This completes the proof of Lemma 1.1.1. □

Parts (v) and (vi) of Lemma 1.1.1 together imply that the temporal evolution of X_n is Markovian. That is, Lemma 1.1.1 describe the so-called *Markov property* of the map $\boldsymbol{X} = (X_n)_{n=0}^{N-1}$ defined by (1.5). Thus the following definition is justified:

Definition 1.1.2 (MDP) *Under law $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ the random variable $\boldsymbol{X} = (X_n)_{n=0}^{N-1}$ is called (finite horizon discrete time) Markov decision process (MDP) associated with initial state $x_0 \in E$, transition function $\boldsymbol{P} \in \overline{\mathcal{P}}$, and strategy $\pi \in \Pi$.*

1.1.3 Markov decision model

Maintain the notation and terminology introduced in Subsections 1.1.1–1.1.2. In the following we will formally define our finite horizon discrete time Markov decision model (MDM).

For this reason, let for each point of time $n = 0, \ldots, N-1$

$$r_n : D_n \longrightarrow \mathbb{R}$$

be a $(\mathcal{D}_n, \mathcal{B}(\mathbb{R}))$-measurable map, referred to as *one-stage reward function.* Here $r_n(x,a)$ specifies the one-stage reward when action a is taken at time n in state x. Further let

$$r_N : E \longrightarrow \mathbb{R}$$

be an $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable map, referred to as *terminal reward function.* The value $r_N(x)$ specifies the reward of being in state x at terminal time N.

In the sequel, we use $\boldsymbol{A}$ to denote the family of all sets $A_n(x)$, $x \in E$, $n = 0, \ldots, N-1$, and set $\boldsymbol{r} := (r_n)_{n=0}^{N}$. Moreover let $\boldsymbol{X}$ be defined as in (1.5), and recall Definition 1.1.2. Then we define our finite horizon discrete time MDM as follows.

Definition 1.1.3 (MDM) *The sextuple $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ is called (finite horizon discrete time) Markov decision model (MDM) associated with state space E, the family of action spaces $\boldsymbol{A}$, transition function $\boldsymbol{P} \in \overline{\mathcal{P}}$, set of admissible strategies Π, and reward functions $\boldsymbol{r}$.*

Remark 1.1.4 (i) In Definition 1.1.3 we do not impose any assumptions on the state space E and the action spaces A_n. So it is possible to consider E and A_n as finite (or countable) sets or as Borel subsets of a complete, separable and metric space. In the latter case, the corresponding σ-fields $\mathcal{E}$ and $\mathcal{A}_n$ are then given by $\mathcal{B}(E)$ and $\mathcal{B}(A_n)$, respectively. For an example of these situations, see Sections 3.1 and 3.2. At this point we emphasize that our theoretical results in Chapters 2 and 4 will hold for arbitrary state space E and actions spaces A_n.

(ii) In the existing literature the sextuple $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ in Definition 1.1.3 is also referred to as a Markov decision process; see, for instance, [44, 73]. However, the latter expression is in our terminology reserved for the random variable $\boldsymbol{X}$ as defined in (1.5) which satisfies the Markov property; see the discussion in Subsection 1.1.2.

(iii) We allow in a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ that the reward functions $\boldsymbol{r} = (r_n)_{n=0}^{N}$ can take negative values, which are then interpreted as costs. This is be benificial when regarding stochastic optimization problems with sequential decision making with a performance criterion based on cost functions; see, for example, [38].

(iv) If, within the framework of Definition 1.1.3, the action spaces A_n, the components of the transition function $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, and the reward functions $\boldsymbol{r} = (r_n)_{n=0}^{N}$ do not depend on time n, then the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ will be called *stationary*. ◇

1.2 Value function and optimal strategies

In this section we consider a specific sequential decision making optimization problem where the expected total reward over a time horizon of N stages is maximized over all admissible strategies. As motivated in the main introduction, maximization problems of this kind can be modelled via a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ as introduced in Definition 1.1.3.

Now, fix $\boldsymbol{P} \in \overline{\mathcal{P}}$. In the sequel, we will always assume that a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ satisfies the following Assumption 1.2.1. In Section 1.4 we will discuss some conditions on the MDM under which Assumption 1.2.1 holds. Denote by $\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}$ the expectation w.r.t. the factorized conditional distribution $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\,\bullet\,\|X_n = x_n]$. Note that for $n = 0$ we clearly have $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\,\bullet\,\|X_0 = x_0] = \mathbb{P}^{x_0,\boldsymbol{P};\pi}[\,\bullet\,]$ for every $x_0 \in E$; see Lemma 1.1.1. In what follows we will use the convention that the sum over the empty set is zero.

Assumption 1.2.1 $\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}[\sum_{k=n}^{N-1} |r_k(X_k, f_k(X_k))| + |r_N(X_N)|\,] < \infty$ *for any* $x_n \in E$ *and* $n = 0, \ldots, N$.

Under Assumption 1.2.1 we may in particular define in a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for any $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ and $n = 0, \ldots, N$ a map $V_n^{\boldsymbol{P};\pi} : E \to \mathbb{R}$ through

$$V_n^{\boldsymbol{P};\pi}(x_n) := \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\Big[\sum_{k=n}^{N-1} r_k(X_k, f_k(X_k)) + r_N(X_N)\Big]. \tag{1.11}$$

The value $V_n^{\boldsymbol{P};\pi}(x_n)$ specifies the expected total reward from time n to N of $\boldsymbol{X}$ under $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ when strategy π is used and $\boldsymbol{X}$ is in state x_n at time n. Therefore, in the following the map $V_n^{\boldsymbol{P};\pi}$ defined by (1.11) will be referred to as *policy value function (at time n)*.

Remark 1.2.2 (i) It follows from the factorization lemma (see, e.g., [6, p. 62]) that the map $V_n^{\boldsymbol{P};\pi}(\cdot)$ as a factorized conditional expectation is in particular $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable for any $\pi \in \Pi$ and $n = 0, \ldots, N$.

(ii) The policy value function $V_n^{\boldsymbol{P};\pi}$ depends in view of the right-hand side of (1.11) only on the last $N - n$ components $(f_n, \ldots, f_{N-1})$ of a strategy $\pi = (f_n)_{n=0}^{N-1} \in \Pi$.

(iii) Note that for $n = 1, \ldots, N$ the right-hand side of (1.11) does *not* depend on x_0; see Lemma 1.4.4 in Section 1.4. Therefore the map $V_n^{\boldsymbol{P};\pi}(\cdot)$ need *not* be equipped with an index x_0. ◇

Let us turn to our sequential decision making optimization problem. It is natural to ask for those strategies $\pi \in \Pi$ for which the policy value function at time 0 evaluated at x_0 is maximal for all initial states $x_0 \in E$. This results in the following (finite horizon discrete time Markov decision) optimization problem:

$$V_0^{\boldsymbol{P};\pi}(x_0) \;\longrightarrow\; \max\ \ (\text{in } \pi \in \Pi)\,! \tag{1.12}$$

If a solution $\pi^{\boldsymbol{P}}$ to the optimization problem (1.12) (in the sense of Definition 1.2.5 ahead) exists, then the corresponding maximal expected total reward is given by the so-called *value function (at time* 0*)*.

Definition 1.2.3 (Value function) *For a MDM* $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ *the value function at time* $n \in \{0, \ldots, N\}$ *is the map* $V_n^{\boldsymbol{P}} : E \to \mathbb{R}$ *defined by*

$$V_n^{\boldsymbol{P}}(x_n) \;:=\; \sup_{\pi \in \Pi} V_n^{\boldsymbol{P};\pi}(x_n). \tag{1.13}$$

The value $V_n^{\boldsymbol{P}}(x_n)$ specifies the maximal expected total reward from time n to N of $\boldsymbol{X}$ under $\mathbb{P}^{x_0,\boldsymbol{P};\pi}$ when strategy π is used and $\boldsymbol{X}$ is in state x_n at time n. Note that the value function $V_n^{\boldsymbol{P}}$ is well-defined due to Assumption 1.2.1.

Remark 1.2.4 It follows from the right-hand side of (1.13) that the value function $V_n^{\boldsymbol{P}}$ is not necessarily $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable. The measurability holds true, for example, if the sets $\mathbb{F}_n, \ldots, \mathbb{F}_{N-1}$ are at most countable (by the right-hand side of (1.13) along with Remark 1.2.2(ii)) or if conditions (a)–(c) of Theorem 1.3.3 in Section 1.3 are satisfied (see Remark 1.3.4(i) in Section 1.3). $\diamond$

Definition 1.2.5 (Optimal strategy) *In a MDM* $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ *a strategy* $\pi^{\boldsymbol{P}} \in \Pi$ *is called optimal w.r.t.* $\boldsymbol{P}$ *if*

$$V_0^{\boldsymbol{P};\pi^{\boldsymbol{P}}}(x_0) \;=\; V_0^{\boldsymbol{P}}(x_0) \quad \text{for all } x_0 \in E. \tag{1.14}$$

In this case $V_0^{\boldsymbol{P};\pi^{\boldsymbol{P}}}(x_0)$ *is called optimal value (function), and we denote by* $\Pi(\boldsymbol{P})$ *the set of all optimal strategies w.r.t.* $\boldsymbol{P}$*. Further, for any given* $\delta > 0$*, a strategy* $\pi^{\boldsymbol{P};\delta} \in \Pi$ *is called* δ*-optimal w.r.t.* $\boldsymbol{P}$ *in a MDM* $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ *if*

$$V_0^{\boldsymbol{P}}(x_0) - \delta \;\le\; V_0^{\boldsymbol{P};\pi^{\boldsymbol{P};\delta}}(x_0) \quad \text{for all } x_0 \in E, \tag{1.15}$$

and we denote by $\Pi(\boldsymbol{P};\delta)$ *the set of all* δ*-optimal strategies w.r.t.* $\boldsymbol{P}$*.*

Note that condition (1.14) requires that $\pi^{\boldsymbol{P}} \in \Pi$ is an optimal strategy for *all* possible initial states $x_0 \in E$. Though, in some situations it might be sufficient to ensure that $\pi^{\boldsymbol{P}} \in \Pi$ is an optimal strategy only for some fixed initial state x_0. We refer to Section 1.3 for a brief discussion of the existence and computation of optimal strategies.

Remark 1.2.6 (i) In practice, the choice of an (admissible) action can possibly be based on historical observations of states and actions. In particular one could relinquish the Markov property of the decision rules and allow them to depend also on previous states and actions. Then one might hope that the corresponding (deterministic) history-dependent strategies improve the optimal value of a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$. However, it is known that the optimal value of a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ can *not* be enhanced by considering history-dependent strategies; see, e.g., Theorem 18.4 in [39] or Theorem 4.5.1 in [73].

(ii) Instead of considering the reward maximization problem (1.12) one could as well be interested in minimizing expected total costs over the time horizon N. In this case, one can maintain the previous notation and terminology when regarding the functions r_n and r_N as the one-stage costs and the terminal costs, respectively. The only thing one has to do is to replace "sup" by "inf" in the representation (1.13) of the value function. Accordingly, a strategy $\pi^{\boldsymbol{P};\delta} \in \Pi$ will be δ-optimal for a given $\delta > 0$ if in condition (1.15) "$-\delta$" and "$\leq$" are replaced by "$+\delta$" and "$\geq$". ◇

1.3 Existence of optimal strategies

Consider the setting of Sections 1.1 and 1.2, that is, let $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ be a MDM in the sense of Definition 1.1.3 with fixed transition function $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$. In this section we will recall from [5] a statement on the existence of optimal strategies in the sense of Definition 1.2.5; see Theorem 1.3.3 below. Part (i) of the latter theorem will ensure that the maximization problem (1.12) can be solved via *dynamic programming* using the so-called *Bellman equation*. Moreover Proposition 1.3.1 below recalls the so-called *reward iteration* from [5] which is used for the proof of Theorem 1.3.3 (see [5, p. 23]) and in our elaborations in Sections 2.2–2.3 and 3.2.

Recall that we used E to denote the state space of the MDP $\boldsymbol{X}$ and that E was equipped with a σ-algebra $\mathcal{E}$. For any $n = 0, \ldots, N-1$ we used $\overline{\mathbb{F}}_n$ to denote the set of *all* decision rules at time n and we fixed some $\mathbb{F}_n \subseteq \overline{\mathbb{F}}_n$ which was regarded as the set of all *admissible* decision rules at time n. We referred to $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$ as the set of all *admissible* strategies.

In the following we denote by $\mathbb{M}(E)$ the set of all $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable functions in $\mathbb{R}^E$. For any $n = 0, \ldots, N-1$, let $\mathbb{M}_n^{\boldsymbol{P}}(E)$ be the set of all $h \in \mathbb{M}(E)$ satisfying

$$\int_E |h(y)|\, P_n\big((x, f_n(x)), dy\big) < \infty \qquad \text{for all } x \in E \text{ and } f_n \in \mathbb{F}_n. \tag{1.16}$$

Thus, for any $h \in \mathbb{M}_n^{\boldsymbol{P}}(E)$, $n = 0, \ldots, N-1$, and $f_n \in \mathbb{F}_n$, we may define maps $\mathcal{T}_{n,f_n}^{\boldsymbol{P}} h : E \to \mathbb{R}$ and $\mathcal{T}_n^{\boldsymbol{P}} h : E \to (-\infty, \infty]$ by

$$\mathcal{T}_{n,f_n}^{\boldsymbol{P}} h(x) := r_n(x, f_n(x)) + \int_E h(y)\, P_n\big((x, f_n(x)), dy\big) \quad \text{and} \quad \mathcal{T}_n^{\boldsymbol{P}} h(x) := \sup_{f_n \in \mathbb{F}_n} \mathcal{T}_{n,f_n}^{\boldsymbol{P}} h(x). \tag{1.17}$$

Note that $\mathcal{T}_{n,f_n}^{\boldsymbol{P}}$ and $\mathcal{T}_n^{\boldsymbol{P}}$ can be seen as maps from $\mathbb{M}_n^{\boldsymbol{P}}(E)$ to $\mathbb{M}(E)$ and from $\mathbb{M}_n^{\boldsymbol{P}}(E)$ to $(-\infty, \infty]^E$ respectively, and that $\mathcal{T}_n^{\boldsymbol{P}}$ is also called *maximal reward operator at time n*.

Recall from (1.11) the definition of the policy value function $V_n^{\boldsymbol{P};\pi}$. The following proposition, whose statements can be proven with the same arguments as in the proof of Theorem 2.3.4 in [5], shows that the policy value function can be computed via the so-called *reward iteration*.

Proposition 1.3.1 *Let $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ be fixed. If $V_{n+1}^{\boldsymbol{P};\pi}(\cdot) \in \mathbb{M}_n^{\boldsymbol{P}}(E)$ for any $n = 0, \ldots, N-1$, then the following two assertions hold.*

(i) $V_N^{\boldsymbol{P};\pi} = r_N$, *and* $V_n^{\boldsymbol{P};\pi} = \mathcal{T}_{n,f_n}^{\boldsymbol{P}} V_{n+1}^{\boldsymbol{P};\pi}$ *for* $n = 0, \ldots, N-1$.

(ii) $V_n^{\boldsymbol{P};\pi} = \mathcal{T}_{n,f_n}^{\boldsymbol{P}} \mathcal{T}_{n+1,f_{n+1}}^{\boldsymbol{P}} \cdots \mathcal{T}_{N-1,f_{N-1}}^{\boldsymbol{P}} r_N$ *for* $n = 0, \ldots, N-1$.

Note that the assumption $V_{n+1}^{\boldsymbol{P};\pi}(\cdot) \in \mathbb{M}_n^{\boldsymbol{P}}(E)$ (for any $n = 0, \ldots, N-1$) in the preceding proposition is not trivially satisfied. It holds, for example, if the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ possesses a bounding function ψ (in the sense of Definition 1.4.1 in Section 1.4 with $\mathcal{P} := \{\boldsymbol{P}\}$). This is ensured by Proposition 1.4.3 ahead applied to $\mathcal{P} := \{\boldsymbol{P}\}$, taking into account that by condition (c) of Definition 1.4.1 we clearly have $\mathbb{M}_\psi(E) \subseteq \mathbb{M}_n^{\boldsymbol{P}}(E)$ (with $\mathbb{M}_\psi(E)$ as in Section 1.4) for any $n = 0, \ldots, N-1$. In some cases, however, the assumption in Proposition 1.3.1 can also be shown directly; see e.g. the proof of Lemma 3.2.6 in Subsection 3.2.3.

Theorem 1.3.3 below is concerned with the existence of optimal strategies. It invokes the following definition.

Definition 1.3.2 *For any* $n = 0, \ldots, N-1$, *a decision rule* $f_n^{\boldsymbol{P}} \in \mathbb{F}_n$ *is called a* maximizer of $h \in \mathbb{M}_n^{\boldsymbol{P}}(E)$ w.r.t. $\boldsymbol{P}$ *if* $\mathcal{T}_{n,f_n^{\boldsymbol{P}}}^{\boldsymbol{P}} h(x) = \mathcal{T}_n^{\boldsymbol{P}} h(x)$ *for all* $x \in E$.

The following theorem, which is also known as *structure theorem*, provides sufficient conditions for the existence of optimal strategies. Its statements can be proven while using the same arguments as in the proof of Theorem 2.3.8 in [5]. Recall from (1.13) the definition of the value function $V_n^{\boldsymbol{P}}$.

Theorem 1.3.3 (Existence of optimal strategies) *Suppose that there are for any* $n = 0, \ldots, N-1$ *sets* $\mathbb{M}_n^{\boldsymbol{P}} \subseteq \mathbb{M}_n^{\boldsymbol{P}}(E)$ *and* $\mathbb{F}_n' \subseteq \mathbb{F}_n$ *such that the following three conditions hold.*

(a) $r_N \in \mathbb{M}_{N-1}^{\boldsymbol{P}}$.

(b) *For any* $n = 1, \ldots, N-1$ *and* $h \in \mathbb{M}_n^{\boldsymbol{P}}$, *we have* $\mathcal{T}_n^{\boldsymbol{P}} h \in \mathbb{M}_{n-1}^{\boldsymbol{P}}$.

(c) *For any* $n = 0, \ldots, N-1$ *and* $h \in \mathbb{M}_n^{\boldsymbol{P}}$, *there exists a maximizer* $f_n^{\boldsymbol{P}} \in \mathbb{F}_n$ *of* h *w.r.t.* $\boldsymbol{P}$ *with* $f_n^{\boldsymbol{P}} \in \mathbb{F}_n'$.

Then the following three assertions are valid:

(i) $V_0^{\boldsymbol{P}} \in \mathbb{M}(E)$, *and* $V_{n+1}^{\boldsymbol{P}} \in \mathbb{M}_n^{\boldsymbol{P}}$ *for any* $n = 0, \ldots, N-1$. *Moreover* $V_N^{\boldsymbol{P}} = r_N$, *and* $V_n^{\boldsymbol{P}} = \mathcal{T}_n^{\boldsymbol{P}} V_{n+1}^{\boldsymbol{P}}$ *for any* $n = 0, \ldots, N-1$.

(ii) $V_n^{\boldsymbol{P}} = \mathcal{T}_n^{\boldsymbol{P}} \mathcal{T}_{n+1}^{\boldsymbol{P}} \cdots \mathcal{T}_{N-1}^{\boldsymbol{P}} r_N$ *for any* $n = 0, \ldots, N-1$.

(iii) *For any* $n = 0, \ldots, N-1$ *there exists a maximizer* $f_n^{\boldsymbol{P}} \in \mathbb{F}_n$ *of* $V_{n+1}^{\boldsymbol{P}}$ *w.r.t.* $\boldsymbol{P}$ *with* $f_n^{\boldsymbol{P}} \in \mathbb{F}_n'$. *Any such maximizers* $f_0^{\boldsymbol{P}}, \ldots, f_{N-1}^{\boldsymbol{P}}$ *form an optimal strategy* $\pi^{\boldsymbol{P}} := (f_n^{\boldsymbol{P}})_{n=0}^{N-1} \in \Pi$ *w.r.t.* $\boldsymbol{P}$ *in the MDM* $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$. *In particular,* $\pi^{\boldsymbol{P}}$ *is even an element of the subset* $\Pi' := \mathbb{F}_0' \times \cdots \times \mathbb{F}_{N-1}'$ *of* Π.

The iteration scheme in part (i) of Theorem 1.3.3 is known as *Bellman equation.* This backward iteration scheme can be seen as a dynamic programming principle which is a general approach for solving multi-stage Markov decision optimization problems. Therefore, part (i) of the preceding theorem shows that the underlying idea for solving the (Markov decision) optimization problem (1.12) is to reduce the complexity by using iteratively the Bellman equation, that is, solving N (one-stage) optimization problems.

Note that conditions (a)–(c) of Theorem 1.3.3 are not trivially satisfied. It is discussed in Subsection 2.4 of the monograph [5] that these conditions hold in so-called structured MDMs. In some situations, however, these conditions can be verified directly; see Subsection 3.2.3 (proof of Theorem 3.2.5) for an example. For original work on the existence of optimal strategies in MDMs see, for instance, [39, 79].

Remark 1.3.4 (i) Under conditions (a)–(c) of Theorem 1.3.3, part (i) of the latter theorem implies that the value function $V_n^{\boldsymbol{P}}(\cdot)$ is $(\mathcal{E},\mathcal{B}(\mathbb{R}))$-measurable for any $n=0,\dots,N$. The measurability of the value function has been discussed in the literature several times; see, for instance, [39, 79].

(ii) It follows from Theorem 1.3.3 that any N-tuple $(f_n^{\boldsymbol{P}})_{n=0}^{N-1}$ of maximizers provides an optimal strategy $\pi^{\boldsymbol{P}}$ w.r.t. $\boldsymbol{P}$ in the MDM $(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r})$ via $\pi^{\boldsymbol{P}} := (f_n^{\boldsymbol{P}})_{n=0}^{N-1}$. The reverse statement, however, is not true since even under the assumptions of Theorem 1.3.3 optimal strategies are *not* necessarily composed of maximizers; see, e.g., [5, Example 2.3.10]. Hence, Theorem 1.3.3 provides only a sufficient criterion for the existence of optimal strategies.

(iii) In view of the second part of (ii), an optimal strategy in a MDM can in general be non-unique. However, this does not exclude that in specific situations there is *exactly* one optimal strategy. For an example see Theorem 3.2.5 in Subsection 3.2.3.

(iv) In the case where we are interested in minimizing expected total costs in the MDM $(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r})$ (see Remark 1.2.6(ii)), the integral operator $\mathcal{T}_n^{\boldsymbol{P}}$ is given by (1.17) with "sup" replaced by "inf" and in Definition 1.3.2 we have to replace "maximizer" by "minimizer". $\diamond$

1.4 Bounding functions

In the following we will discuss some sufficient conditions under which Assumption 1.2.1 is fulfilled. Throughout this section, we fix the components E, $\boldsymbol{A}$, Π, and $\boldsymbol{r}$ of a MDM $(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r})$ (introduced in Definition 1.1.3).

Recall from Section 1.1 that $\overline{\mathcal{P}}$ stands for the set of all transition functions, i.e. of all N-tuples $\boldsymbol{P}=(P_n)_{n=0}^{N-1}$ of probability kernels P_n from $(D_n,\mathcal{D}_n)$ to $(E,\mathcal{E})$, and we defined $\mathbb{M}(E)$ to be the set of all $(\mathcal{E},\mathcal{B}(\mathbb{R}))$-measurable functions in $\mathbb{R}^E$. Let $\psi: E\to\mathbb{R}_{\geq 1}$ be an $(\mathcal{E},\mathcal{B}(\mathbb{R}_{\geq 1}))$-measurable map, referred to as *gauge function*, where $\mathbb{R}_{\geq 1} := [1,\infty)$. Denote by $\mathbb{M}_\psi(E)$ the set of all $h\in\mathbb{M}(E)$ satisfying

$$\|h\|_\psi := \sup_{x\in E}\frac{|h(x)|}{\psi(x)} < \infty. \tag{1.18}$$

The following definition is adapted from [5, 68, 91]. Conditions (a)–(c) of this definition are sufficient for the well-definiteness of the policy value function $V_n^{\boldsymbol{P};\pi}$ and the value function $V_n^{\boldsymbol{P}}$ introduced in (1.11) and (1.13), respectively; see Proposition 1.4.3 ahead.

Definition 1.4.1 (Bounding function) *Let $\mathcal{P}\subseteq\overline{\mathcal{P}}$. A gauge function $\psi: E\to\mathbb{R}_{\geq 1}$ is said to be a bounding function for the family of MDMs $\{(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r}) : \boldsymbol{P}\in\mathcal{P}\}$ if there exist finite constants $K_1,K_2,K_3>0$ such that the following three conditions hold for any $n=0,\dots,N-1$ and $\boldsymbol{P}=(P_n)_{n=0}^{N-1}\in\mathcal{P}$:*

(a) $|r_n(x,a)|\leq K_1\psi(x)$ *for all* $(x,a)\in D_n$.

(b) $|r_N(x)|\leq K_2\psi(x)$ *for all* $x\in E$.

(c) $\int_E \psi(y)\,P_n\big((x,a),dy\big)\leq K_3\psi(x)$ *for all* $(x,a)\in D_n$.

If $\mathcal{P} = \{\boldsymbol{P}\}$ for some $\boldsymbol{P} \in \overline{\mathcal{P}}$, then ψ is called a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$.

Note that the conditions in Definition 1.4.1 do *not* depend on the set Π. That is, the terminology *bounding function* is independent of the set of all (admissible) strategies. Also note that conditions (a) and (b) can be satisfied by unbounded reward functions.

Remark 1.4.2 (i) It is an immediate consequence of Definition 1.4.1 that $\psi :\equiv 1$ provides a bounding function for the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \mathcal{P}\}$ (with $\mathcal{P} \subseteq \overline{\mathcal{P}}$) if the reward functions $\boldsymbol{r} = (r_n)_{n=0}^N$ are bounded. This is the case, for example, when the state space E as well as the actions spaces A_n at time $n \in \{0, \dots, N-1\}$ are finite. We refer to Section 1.5 for further discussions when in a MDM the state space and the actions spaces are finite.

(ii) If ψ is a bounding function for the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \mathcal{P}\}$ (with $\mathcal{P} \subseteq \overline{\mathcal{P}}$) then, for example, any gauge function $\widetilde{\psi}$ that is a multiple of ψ provides also a bounding function for the same family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \mathcal{P}\}$. This means in particular that, in general, a bounding function can *not* be unique. $\diamond$

The following proposition ensures that Assumption 1.2.1 is satisfied when the underlying MDM possesses a bounding function.

Proposition 1.4.3 *Let $\mathcal{P} \subseteq \overline{\mathcal{P}}$. If the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \mathcal{P}\}$ possesses a bounding function ψ, then Assumption 1.2.1 is satisfied for any $\boldsymbol{P} \in \mathcal{P}$. Moreover, $\sup_{\pi \in \Pi} \|V_n^{\boldsymbol{P};\pi}(\cdot)\|_\psi < \infty$ for every $\boldsymbol{P} \in \mathcal{P}$ and $n = 0, \dots, N$. In particular, $V_n^{\boldsymbol{P};\pi}(\cdot)$ is contained in $\mathbb{M}_\psi(E)$ for any $\boldsymbol{P} \in \mathcal{P}$, $\pi \in \Pi$, and $n = 0, \dots, N$.*

The proof of Proposition 1.4.3 avails the following lemma. Recall that $\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}$ refers to the expectation w.r.t. the factorized conditional distribution $\mathbb{P}^{x_0,\boldsymbol{P};\pi}[\,\bullet\,\|X_n = x_n]$. Finally, let $L^1(\Omega, \mathcal{F}, \mathbb{P}^{x_0,\boldsymbol{P};\pi})$ be the usual L^1-space on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^{x_0,\boldsymbol{P};\pi})$.

Lemma 1.4.4 *Let $x_0 \in E$, $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$, and $\pi = (f_n)_{n=0}^{N-1} \in \Pi$. Moreover let $h \in \mathbb{M}(E)$ such that $h(X_n) \in L^1(\Omega, \mathcal{F}, \mathbb{P}^{x_0,\boldsymbol{P};\pi})$ for all $n = 0, \dots, N$. Then for any $\widetilde{x}_0, x_n \in E$ and $1 \le n < k \le N$ as well as $x_m \in E$ and $m = 1, \dots, N$ we have*

(i) $\mathbb{E}^{x_0,\boldsymbol{P};\pi}[h(X_0)] \;=\; h(x_0)$.

(ii) $\mathbb{E}_{0,\widetilde{x}_0}^{x_0,\boldsymbol{P};\pi}[h(X_0)] \;=\; h(x_0)$.

(iii) $\mathbb{E}_{m,x_m}^{x_0,\boldsymbol{P};\pi}[h(X_m)] \;=\; h(x_m)$.

(iv) $\mathbb{E}_{0,\widetilde{x}_0}^{x_0,\boldsymbol{P};\pi}[h(X_m)] \;=\; \mathbb{E}^{x_0,\boldsymbol{P};\pi}[h(X_m)] \;=\; \int_E h(y_m)\, \mathbb{P}_{X_1\|X_0}^{x_0,\boldsymbol{P};\pi} \cdots \mathbb{P}_{X_m\|X_{m-1}}^{x_0,\boldsymbol{P};\pi}(x_0, dy_m)$.

(v) $\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}[h(X_k)] \;=\; \int_E h(y_k)\, \mathbb{P}_{X_{n+1}\|X_n}^{x_0,\boldsymbol{P};\pi} \cdots \mathbb{P}_{X_k\|X_{k-1}}^{x_0,\boldsymbol{P};\pi}(x_n, dy_k)$.

Moreover the right-hand side of parts (iv) and (v) can be represented as

$$\begin{aligned} &\int_E h(y_m)\, \mathbb{P}_{X_1\|X_0}^{x_0,\boldsymbol{P};\pi} \cdots \mathbb{P}_{X_m\|X_{m-1}}^{x_0,\boldsymbol{P};\pi}(x_0, dy_m) \\ &= \int_E \cdots \int_E h(y_m)\, P_{m-1}\big((y_{m-1}, f_{m-1}(y_{m-1})), dy_m\big) \cdots P_0\big((x_0, f_0(x_0)), dy_1\big) \end{aligned}$$

and

$$\int_E h(y_k)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_{n+1}\|X_n}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_{k-1}}(x_n,dy_k)$$
$$= \int_E\cdots\int_E h(y_k)\,P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big).$$

Proof First of all, it is easily seen that the identities

$$\mathbb{E}^{x_0,\boldsymbol{P};\pi}[h(X_m)] \;=\; \int_E h(y)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m}(dy) \tag{1.19}$$

and

$$\mathbb{E}^{x_0,\boldsymbol{P};\pi}_{j,x_j}[h(X_m)] \;=\; \int_E h(y)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_j}(x_j,dy) \tag{1.20}$$

hold for any $x_j\in E$ and $0\le j\le m\le N$.

(i): The claim is an immediate consequence of (1.19) and part (i) of Lemma 1.1.1.

(ii)–(iii): The assertions follow from (1.20) along with parts (iii) and (ix) of Lemma 1.1.1, respectively.

(iv): For the assertions it suffices in view of (1.19)–(1.20) to show that

$$\int_E h(y_m)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_0}(\widetilde{x}_0,dy_m) = \int_E h(y_m)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(x_0,dy_m) \tag{1.21}$$

and

$$\int_E h(y_m)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m}(dy_m) = \int_E h(y_m)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_1\|X_0}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_m\|X_{m-1}}(x_0,dy_m). \tag{1.22}$$

Clearly, in view of part (vii) of Lemma 1.1.1, the assertions in (1.21) and (1.22) are valid for indicator functions and thus by linearity for simple functions. The latter assertions can be extended by the Monotone Convergence theorem to arbitrary nonnegative maps $h\in\mathbb{M}(E)$. Since the integrals on the left-hand sides of (1.21) and (1.22) exist and are finite (recall that $h(X_n)\in L^1(\Omega,\mathcal{F},\mathbb{P}^{x_0,\boldsymbol{P};\pi})$ for all $n=0,\ldots,N$ by assumption), it follows that the equalities in (1.21) and (1.22) hold even for all $h\in\mathbb{M}(E)$.

(v): Analogously to the proof of (1.21) we obtain by means of (1.20)

$$\mathbb{E}^{x_0,\boldsymbol{P};\pi}_{n,x_n}[h(X_k)] = \int_E h(y_k)\,\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_{n+1}\|X_n}\cdots\mathbb{P}^{x_0,\boldsymbol{P};\pi}_{X_k\|X_{k-1}}(x_n,dy_k).$$

The additional assertions can be verified easily by means of (1.7) and (1.8) with the same arguments as in the proof of (1.21) and (1.22). □

Note that (for any given $x_0\in E$, $\boldsymbol{P}\in\overline{\mathcal{P}}$, and $\pi\in\Pi$) the assumption $h(X_n)\in L^1(\Omega,\mathcal{F},\mathbb{P}^{x_0,\boldsymbol{P};\pi})$ (for some $h\in\mathbb{M}(E)$ and any $n=0,\ldots,N$) in the preceding lemma is not trivially satisfied. It holds, for example, if ψ provides a bounding function for the MDM $(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r})$ (in the sense of Definition 1.4.1 with $\mathcal{P}:=\{\boldsymbol{P}\}$) and if $h\in\mathbb{M}_\psi(E)$. In this case it can be verified easily by means of part (c) of Definition 1.4.1 (with $\mathcal{P}:=\{\boldsymbol{P}\}$) that indeed $h(X_n)\in L^1(\Omega,\mathcal{F},\mathbb{P}^{x_0,\boldsymbol{P};\pi})$ for all $n=0,\ldots,N$.

Now, we are in the position to prove Proposition 1.4.3.

Proof of Proposition 1.4.3: Fix $x_0 \in E$. By assumption there exist finite constants $K_1, K_3 > 0$ such that in view of part (v) of Lemma 1.4.4 as well as parts (a) and (c) of Definition 1.4.1

$$\begin{aligned}
\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[|r_k(X_k, f_k(X_k))|\big] &\le \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}[K_1\psi(X_k)] \\
&= K_1 \int_E \cdots \int_E \int_E \psi(y_k)\, P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad P_{k-2}\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
&\le K_1 K_3^{k-n} \psi(x_n)
\end{aligned}$$

for any $x_n \in E$, $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \mathcal{P}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $1 \le n < k \le N-1$. Moreover, in view of part (iii) of Lemma 1.4.4 and part (a) of Definition 1.4.1, we have

$$\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[|r_n(X_n, f_n(X_n))|\big] = |r_n(x_n, f_n(x_n))| \le K_1\psi(x_n)$$

for any $x_n \in E$, $\boldsymbol{P} \in \mathcal{P}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 1, \dots, N-1$. Similarly, we find by assumption some finite constant $K_2 > 0$ such that in view of parts (iii) and (v) of Lemma 1.4.4 as well as parts (b) and (c) of Definition 1.4.1

$$\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[|r_N(X_N)|\big] \le K_2 K_3^{N-n} \psi(x_n)$$

for any $x_n \in E$, $\boldsymbol{P} \in \mathcal{P}$, $\pi \in \Pi$, and $n = 1, \dots, N$. In the same way we obtain with parts (ii) and (iv) of Lemma 1.4.4 and the characteristic properties of the bounding function ψ

$$\mathbb{E}_{0,x_0}^{x_0,\boldsymbol{P};\pi}\big[|r_k(X_k, f_k(X_k))|\big] \le K_1 K_3^k \psi(x_0)$$

and

$$\mathbb{E}_{0,x_0}^{x_0,\boldsymbol{P};\pi}\big[|r_N(X_N)|\big] \le K_2 K_3^N \psi(x_0)$$

for any $\boldsymbol{P} \in \mathcal{P}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $k = 0, \dots, N-1$. Then Assumption 1.2.1 holds for any $\boldsymbol{P} \in \mathcal{P}$. Moreover by choosing $C_n := K_1 \sum_{k=n}^{N-1} K_3^{k-n} + K_2 K_3^{N-n}$ $(< \infty)$ we have $\sup_{\pi\in\Pi} \|V_n^{\boldsymbol{P};\pi}(\cdot)\|_\psi \le C_n$ for every $\boldsymbol{P} \in \mathcal{P}$, $\pi \in \Pi$, and $n = 0, \dots, N$. In particular the latter implies $V_n^{\boldsymbol{P};\pi}(\cdot) \in \mathbb{M}_\psi(E)$ for every $\boldsymbol{P} \in \mathcal{P}$, $\pi \in \Pi$, and $n = 0, \dots, N$. Take into account that $V_n^{\boldsymbol{P};\pi}(\cdot)$ is $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable for every $\boldsymbol{P} \in \mathcal{P}$, $\pi \in \Pi$, and $n = 0, \dots, N$ by Remark 1.2.2(i). This completes the proof of Proposition 1.4.3. □

In particular, Proposition 1.4.3 shows that in a MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ (for some given $\boldsymbol{P} \in \overline{\mathcal{P}}$) a sufficient condition for the existence of the policy value function $V_n^{\boldsymbol{P};\pi}$ as well as the value function $V_n^{\boldsymbol{P}}$ is that the MDM possesses a bounding function ψ (in the sense of Definition 1.4.1). In some MDMs, however, it is sometimes cumbersome to find a suitable gauge function ψ that satisfies conditions (a)–(c) of Definition 1.4.1.

1.5 Markov decision models with finite state space and finite action spaces

In this section we will briefly discuss the special case when in the setting of Sections 1.1–1.4 the state space E and the set A_n of all actions at time $n \in \{0, \dots, N-1\}$ are finite. The following elaborations will be beneficial for later purposes, in particular in Chapter 2 to present all definitions and our theoretical results in a more intuitive and comprehensible way if both the state space as well as the action spaces are finite. Finite horizon discrete time Markov decision optimization problems in which the state space as well as the action spaces are finite often appear in practice as, for example, for optimal stopping problems, bandit models or discrete time queueing systems. We refer to Section 3 in [73] for further examples. In Section 3.1 we will exemplary discuss a single-product stochastic inventory control problem which will be used throughout the first part of this thesis to illustrate our theoretical results.

Now, let for some fixed $\mathfrak{e} \in \mathbb{N}$ the state space be equal to

$$E := \{x_1, \ldots, x_{\mathfrak{e}}\}, \tag{1.23}$$

and set $\mathcal{E} := \mathfrak{P}(E)$. Moreover, for any $i = 1, \ldots, \mathfrak{e}$ and $n = 0, \ldots, N-1$, let

$$A_n(x_i) := \{a_{n,i;1}, \ldots, a_{n,i;\mathfrak{t}_{n,i}}\} \tag{1.24}$$

be the finite set of all admissible actions that can be performed when the MDP is in state x_i at time n, where $\mathfrak{t}_{n,i} \in \mathbb{N}$ is fixed. Therefore, the sets $A_n = \bigcup_{i=1}^{\mathfrak{e}} A_n(x_i)$ and $D_n = \{(x_i, a) \in E \times A_n : a \in A_n(x_i)\}$ of all actions and possible state-action combinations at time $n \in \{0, \ldots, N-1\}$ are also finite. The set $\overline{\mathbb{F}}_n$ of all decision rules at time $n \in \{0, \ldots, N-1\}$ consists of all maps $f_n : \{x_1, \ldots, x_{\mathfrak{e}}\} \to A_n$ which satisfy $f_n(x_i) \in \{a_{n,i;1}, \ldots, a_{n,i;\mathfrak{t}_{n,i}}\}$ for every $i = 1, \ldots, \mathfrak{e}$. Note that in the finite setting the set $\overline{\mathbb{F}}_n$ is clearly non-empty and finite. Finally, let $\mathbb{F}_n \subseteq \overline{\mathbb{F}}_n$ be a fixed subset, and set as before $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$ for the finite set of all admissible strategies.

For any $i = 1, \ldots, \mathfrak{e}$, $n = 0, \ldots, N-1$, and $a \in A_n(x_i)$, the (one-step transition) probability measure on E from which the state of the MDP at time $n+1$ is drawn, given that the MDP is in state x_i and action a is selected at time n, can be identified with an element

$$p_{n,i;a} := \big(p_{n,i;a}(1), \ldots, p_{n,i;a}(\mathfrak{e})\big)$$

of $\mathbb{R}^{\mathfrak{e}}_{\geq 0,1}$. Here $\mathbb{R}^{\mathfrak{e}}_{\geq 0,1}$ is the set of all vectors from $\mathbb{R}^{\mathfrak{e}}$ whose entries are nonnegative and sum up to 1, and $p_{n,i;a}(j)$ specifies the probability that the MDP will be in state x_j at time $n+1$, given it is in state x_i and action $a \in A_n(x_i)$ is selected at time n. As the state space E as well as the sets $D_0, \ldots, D_{N-1}$ are finite, the set $\overline{\mathcal{P}}$ of all transition functions can be represented in the finite setting as a finite product of the set $\mathcal{M}_1(E)$:

$$\overline{\mathcal{P}} = \times_{n=0}^{N-1} \times_{(x,a) \in D_n} \mathcal{M}_1(E). \tag{1.25}$$

In particular, for any fixed $i_0 \in \{1, \ldots, \mathfrak{e}\}$ (where $x_0 = x_{i_0} \in E$ refers to the corresponding initial state of the MDP), we may and do identify any transition function $\boldsymbol{P}$ from $\overline{\mathcal{P}}$ with a vector $\boldsymbol{p}$ in $\mathbb{R}^{\mathfrak{d}}$ defined by

$$\boldsymbol{p} := \big(\oplus_{k=1}^{\mathfrak{t}_{0,i_0}} p_{0,i_0;a_{0,i_0;k}} \big) \oplus \big(\oplus_{n=1}^{N-1} \oplus_{i=1}^{\mathfrak{e}} \oplus_{k=1}^{\mathfrak{t}_{n,i}} p_{n,i;a_{n,i;k}} \big) \tag{1.26}$$

with $\mathfrak{d} := (\mathfrak{t}_{0,i_0} + \sum_{n=1}^{N-1} \sum_{i=1}^{\mathfrak{e}} \mathfrak{t}_{n,i})\mathfrak{e}$. Here $\oplus$ is the 'clueing operator' defined by $(\alpha_1, \ldots, \alpha_s) \oplus (\beta_1, \ldots, \beta_t) := (\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t)$. In fact one can show that $\boldsymbol{p}$ is even an element of the following subset of $\mathbb{R}^{\mathfrak{d}}$:

$$\widetilde{\mathcal{P}} := \big(\mathbb{R}^{\mathfrak{e}}_{\geq 0,1}\big)^{\times(\mathfrak{d}/\mathfrak{e})}. \tag{1.27}$$

If $V_n^{\boldsymbol{p};\pi}$ corresponds to the policy value function at time $n \in \{0, \ldots, N-1\}$ associated with vector $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ and strategy $\pi \in \Pi$ (introduced in (1.11)), then for any fixed $i_0 \in \{1, \ldots, \mathfrak{e}\}$ and $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ the (Markov decision) optimization problem (1.12) reads as

$$V_0^{\boldsymbol{p};\pi}(x_{i_0}) \longrightarrow \max \ (\text{in } \pi \in \Pi)\,! \tag{1.28}$$

(recall that $x_{i_0} \in E$ refers to the initial state). Here $V_n^{\boldsymbol{p};\pi}(\cdot)$ can be obtained from the usual backward iteration scheme (see, e.g., [39, Lemma 3.5] or [73, p. 80]):

$$\begin{aligned} V_N^{\boldsymbol{p};\pi}(x_i) &:= r_N(x_i), \\ V_n^{\boldsymbol{p};\pi}(x_i) &:= r_n(x_i, f_n(x_i)) + \sum_{j=1}^{\mathfrak{e}} V_{n+1}^{\boldsymbol{p};\pi}(x_j)\, p_{n,i;f_n(x_i)}(j), \quad n = 0, \ldots, N-1, \end{aligned} \tag{1.29}$$

$i = 1, \ldots, \mathfrak{e}$. Since in the finite setting above the gauge function $\psi : E \to \mathbb{R}_{\geq 1}$ defined by

$$\psi :\equiv 1 \tag{1.30}$$

provides a bounding function for the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \overline{\mathcal{P}}\}$ (see part (i) of Remark 1.4.2), we observe by iterating (1.29) from Definition 1.4.1 that the left-hand side in (1.28) (and $V_n^{\boldsymbol{p};\pi}(\cdot)$) is well-defined. Moreover note that it follows from [73, Proposition 4.4.3] that in the finite setting one can always find for any $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ an optimal strategy $\pi^{\boldsymbol{p}} \in \Pi$ w.r.t. $\boldsymbol{p}$ which solves the optimization problem (1.28). Therefore, the set $\Pi(\boldsymbol{p})$ of all optimal strategies w.r.t. $\boldsymbol{p}$ is non-empty and finite.

Chapter 2

‘Continuity’ and ‘differentiability’ of the value function in the transition probability function

In this chapter we use the notation and terminology introduced in Sections 1.1–1.4 to show that the value function of a MDM, regarded as a real-valued functional defined on a set of transition functions, is ‘continuous’ as well as ‘differentiable’ in a certain sense. Here we are particularly interested in reasonable quantifying the effect of changing a less complex version of the transition probability function to a more realistic version on the optimal value.

The motivation for our investigations comes from the field of optimal logistics transportation planning, where ongoing projects like SYNCHRO-NET (`https://www.synchronet.eu/`) aim at stochastic decision models based on transition probabilities estimated from historical route information. Due to the lack of historical data for unlikely events, transition probabilities are often modelled in a simplified way. In fact, events with small probabilities are often ignored in the model. However, the impact of these events on the optimal value (here the minimal expected transportation costs) of the corresponding MDM may nevertheless be significant. The identification of unlikely but potentially cost sensitive events is therefore a major challenge. In logistics planning operations engineers have indeed become increasingly interested in comprehensibly quantifying the sensitivity of the optimal value w.r.t. the incorporation of unlikely events into the model. For background see, for instance, [41, 42]. The assessment of rare but risky events takes on greater importance also in other areas of applications; see, for instance, [52, 92] and references cited therein.

By an incorporation of an unlikely event into the model we mean, for instance, that under performance of an action a at some time n a previously impossible transition from one state x to another state x' gets now assigned small but strictly positive probability ε. Mathematically this means that the transition probability $P_n((x,a),\bullet\,)$ is replaced by a new transition probability

$$P_{n;\varepsilon}\big((x,a),\bullet\,\big) \;:=\; (1-\varepsilon)P_n\big((x,a),\bullet\,\big)+\varepsilon Q_n\big((x,a),\bullet\,\big)$$

with $Q_n((x,a),\bullet\,):=\delta_{x'}[\,\bullet\,]$, where $\delta_{x'}$ is the Dirac measure at point x'. More generally one could consider a change of the whole transition function, i.e. the family of all transition probabilities, $\boldsymbol{P}$ to

$$\boldsymbol{P}_\varepsilon \;:=\; (1-\varepsilon)\boldsymbol{P}+\varepsilon\boldsymbol{Q}$$

with $\varepsilon>0$ small. For operations engineers it is here interesting to know how this change affects the (optimal) value $\mathcal{V}_0^{x_0}(\boldsymbol{P}):=V_0^{\boldsymbol{P}}(x_0)$ (with $V_0^{\boldsymbol{P}}$ introduced in (1.13)) for some fixed initial state $x_0\in E$. If the effect is minor, then an incorporation can be seen as superfluous, at least from a pragmatic point of view. If on the other hand the effect is significant, then the engineer should consider the option to extend the MDM and to make an effort to get access to statistical data for the extended MDM.

At this point it is worth mentioning that a change of the transition function from $\boldsymbol{P}$ to $\boldsymbol{P}_\varepsilon$ with $\varepsilon > 0$ small can also have a different interpretation than an incorporation of an (unlikely) *new event*. It could also be associated with an incorporation of an (unlikely) *divergence from the normal transition rules*. We refer to Subsection 3.2.5 for an example.

In Section 2.2, we will show that the value functional $\mathcal{V}_0^{x_0}$ is in some sense 'continuous' in the transition function $\boldsymbol{P}$. However, with this result we are *not* able to quantify the effect of changing the transition function from $\boldsymbol{P}$ to $\boldsymbol{P}_\varepsilon$, with $\varepsilon > 0$ small, on the (optimal) value $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ of the MDM (with $x_0 \in E$ fixed). For this reason, we will discuss in Section 2.3 an approach to measure this effect. In view of $\boldsymbol{P}_\varepsilon = \boldsymbol{P} + \varepsilon(\boldsymbol{Q} - \boldsymbol{P})$, we feel that it is reasonable to quantify the above effect by a sort of derivative of the value functional $\mathcal{V}_0^{x_0}$ at $\boldsymbol{P}$ evaluated at direction $\boldsymbol{Q} - \boldsymbol{P}$. To some extent the 'derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q} - \boldsymbol{P})$ specifies the first-order sensitivity of $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ w.r.t. a change of $\boldsymbol{P}$ as above. Take into account that

$$\mathcal{V}_0^{x_0}(\boldsymbol{P} + \varepsilon(\boldsymbol{Q} - \boldsymbol{P})) - \mathcal{V}_0^{x_0}(\boldsymbol{P}) \approx \varepsilon \cdot \dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q} - \boldsymbol{P}) \qquad \text{for } \varepsilon > 0 \text{ small.} \tag{2.1}$$

To be able to compare the first-order sensitivity for (infinitely) many different $\boldsymbol{Q}$, it is favourable to know that the approximation in (2.1) is uniform in $\boldsymbol{Q} \in \mathcal{K}$ for preferably large sets $\mathcal{K}$ of transition functions. Moreover, it is not always possible to specify the relevant $\boldsymbol{Q}$ exactly. For that reason it would be also good to have robustness (i.e. some sort of continuity) of the 'derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q} - \boldsymbol{P})$ in $\boldsymbol{Q}$. These two things induced us to focus on a variant of tangential $\mathcal{S}$-differentiability as introduced by Sebastião e Silva [82] and Averbukh and Smolyanov [4] (here $\mathcal{S}$ is a family of sets $\mathcal{K}$ of transition functions). In Subsection 2.3.2, we present a result on '$\mathcal{S}$-differentiability' of the value functional $\mathcal{V}_0^{x_0}$ for the family $\mathcal{S}$ of all *relatively compact* sets of admissible transition functions and a reasonably broad class of MDMs, where we measure the distance between transition functions by means of metrics based on probability metrics as in [68]; see Section 2.1 for further details.

The 'derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q} - \boldsymbol{P})$ of the (optimal) value functional $\mathcal{V}_0^{x_0}$ at $\boldsymbol{P}$ quantifies the effect of a change from $\boldsymbol{P}$ to $\boldsymbol{P}_\varepsilon$, with $\varepsilon > 0$ small, assuming that after the change the strategy π is chosen such that it optimizes the target value $\mathcal{V}_0^{x_0;\pi}(\boldsymbol{P}_\varepsilon) := V_0^{\boldsymbol{P}_\varepsilon;\pi}(x_0)$ (with $V_0^{\boldsymbol{P}_\varepsilon;\pi}$ defined as in (1.11)) in π under the new transition function $\boldsymbol{P}_\varepsilon$. On the other hand, practitioners are also interested in quantifying the impact of a change of $\boldsymbol{P}$ when the optimal strategy (under $\boldsymbol{P}$) is kept after the change. Such a quantification would somehow answers the question: How much different does a strategy derived in a simplified MDM perform in a more complex (more realistic) variant of the MDM? Since the 'derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi}(\boldsymbol{Q} - \boldsymbol{P})$ of the functional $\mathcal{V}_0^{x_0;\pi}$ under a *fixed* strategy π (and initial state $x_0 \in E$) turns out to be a building stone for the derivative $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q} - \boldsymbol{P})$ of the (optimal) value functional $\mathcal{V}_0^{x_0}$ at $\boldsymbol{P}$ (see Displays (2.29)–(2.30) ahead), our elaborations cover both situations anyway. For fixed strategy π (and initial state $x_0 \in E$), we obtain '$\mathcal{S}$-differentiability' of $\mathcal{V}_0^{x_0;\pi}$ even for the broader family $\mathcal{S}$ of all *bounded* sets of admissible transition functions.

The rest of this chapter is organized as follows. Motivated by the works of Müller [68, 69], we will first explain in Section 2.1 how we measure the distance between transition functions. In Section 2.2, we will show, using the distance measure between transition functions introduced in Section

2.1, that (under some structural assumptions) the value function of a MDM regarded as a real-valued functional defined on a set of transition functions is 'Lipschitz continuous' in a certain sense. This statement is in line with a result in Müller [68] in the case of stationary MDMs. Afterwards we carefully introduce in Section 2.3 our notion of 'differentiability' and state our main result concerning the computation of the 'derivative' of the value functional. We stress the fact that this result can be obtained with the same assumptions as the statement about the continuity of the value functional. Throughout this chapter we fix the components E, $\boldsymbol{A}$, Π, and $\boldsymbol{r}$ of a MDM.

2.1 Measuring the distance between transition functions

In this section we will introduce in Display (2.12) below a reasonable (semi-) metric defined on a set of admissible transition functions which can be used to measure the distance between transition functions. The motivation for this (semi-) metric comes from the work of Müller in [68] where the author defines a distance between transition probabilities based on so-called integral probability metrics. The latter concept will be explained in detail in Subsection 2.1.1. After introducing a reasonable distance measure between transition functions in Subsection 2.1.2, we will discuss in Subsection 2.1.3 how to measure the distance between transition functions in the setting of Section 1.5 where both the state space and the action spaces are finite.

2.1.1 Integral probability metrics

In Sections 2.2 and 2.3 we will work with a (semi-) metric (on a set of transition functions) to be defined in (2.12) below. As it is common in the theory of probability metrics (see, e.g., p. 10 ff in [74]), we allow the distance between two probability measures and the distance between two transition functions to be infinite. That is, we adapt the axioms of a (semi-) metric but we allow a (semi-) metric to take values in $\overline{\mathbb{R}}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{\infty\}$ rather than only in $\mathbb{R}_{\geq 0} := [0, \infty)$.

Let ψ be any gauge function, and denote by $\mathcal{M}_1^{\psi}(E)$ the set of all $\mu \in \mathcal{M}_1(E)$ for which $\int_E \psi \, d\mu < \infty$. Note that the integral $\int_E h \, d\mu$ exists and is finite for any $h \in \mathbb{M}_{\psi}(E)$ and $\mu \in \mathcal{M}_1^{\psi}(E)$, where the set $\mathbb{M}_{\psi}(E)$ is introduced in Section 1.4. For any fixed subset $\mathbb{M} \subseteq \mathbb{M}_{\psi}(E)$, the distance between two probability measures $\mu, \nu \in \mathcal{M}_1^{\psi}(E)$ can be measured by

$$d_{\mathbb{M}}(\mu, \nu) \; := \; \sup_{h \in \mathbb{M}} \Big| \int_E h \, d\mu - \int_E h \, d\nu \Big|. \tag{2.2}$$

Note that (2.2) indeed defines a map $d_{\mathbb{M}} : \mathcal{M}_1^{\psi}(E) \times \mathcal{M}_1^{\psi}(E) \to \overline{\mathbb{R}}_{\geq 0}$ which is symmetric and fulfills the triangle inequality, i.e. $d_{\mathbb{M}}$ provides a semi-metric. If $\mathbb{M}$ separates points in $\mathcal{M}_1^{\psi}(E)$ (i.e. if any two $\mu, \nu \in \mathcal{M}_1^{\psi}(E)$ coincide when $\int_E h \, d\mu = \int_E h \, d\nu$ for all $h \in \mathbb{M}$), then $d_{\mathbb{M}}$ is even a metric. It is sometimes called *integral probability metric* or *probability metric with a ζ-structure*; see [69, 95].

In some situations the (semi-) metric $d_{\mathbb{M}}$ (with $\mathbb{M} \subseteq \mathbb{M}_{\psi}(E)$ fixed) can be represented by the right-hand side of (2.2) with $\mathbb{M}$ replaced by a different subset $\mathbb{M}'$ of $\mathbb{M}_{\psi}(E)$. Each such set $\mathbb{M}'$ is said to be a *generator of* $d_{\mathbb{M}}$. The largest generator of $d_{\mathbb{M}}$ is called the *maximal generator of* $d_{\mathbb{M}}$ and denoted by $\overline{\mathbb{M}}$. That is, $\overline{\mathbb{M}}$ is defined according to [69, Definition 3.1] as the set of all $h \in \mathbb{M}_{\psi}(E)$ for which $| \int_E h \, d\mu - \int_E h \, d\nu | \leq d_{\mathbb{M}}(\mu, \nu)$ for all $\mu, \nu \in \mathcal{M}_1^{\psi}(E)$.

Below we will give some examples for the distance $d_{\mathbb{M}}$. The metrics in Examples 2.1.2–2.1.5 were already mentioned in [68, 69]. In Examples 2.1.4–2.1.6 the metric $d_{\mathbb{M}}$ generates the ψ-weak topology $\mathcal{O}_{\mathrm{w}}^{\psi}$. The latter is defined to be the coarsest topology on $\mathcal{M}_1^{\psi}(E)$ for which all mappings $\mu \mapsto \int_E h\,d\mu$, $h \in \mathbb{C}_{\psi}(E)$, are continuous. Here $\mathbb{C}_{\psi}(E)$ is the set of all continuous functions in $\mathbb{M}_{\psi}(E)$. If specifically $\psi :\equiv 1$, then $\mathcal{M}_1^{\psi}(E) = \mathcal{M}_1(E)$ and the ψ-weak topology is nothing but the classical weak topology $\mathcal{O}_{\mathrm{w}}$. In Section 2 in [58] one can find characterizations of those subsets of $\mathcal{M}_1^{\psi}(E)$ on which the relative ψ-weak topology coincides with the relative weak topology.

In the sequel, we will say that a sequence $(\mu_m)_{m\in\mathbb{N}}$ in $\mathcal{M}_1^{\psi}(E)$ *converges ψ-weakly to some* $\mu \in \mathcal{M}_1^{\psi}(E)$ (in symbol $\mu_m \to \mu$ ψ-weakly) if $\int_E h\,d\mu_m \to \int_E h\,d\mu$ for all $h \in \mathbb{C}_{\psi}(E)$. Note that for $\psi :\equiv 1$ we will write $\mu_m \xrightarrow{\mathrm{w}} \mu$ instead of $\mu_m \to \mu$ ψ-weakly.

The following result is a direct consequence of Lemma 2.1 in [58] and characterizes ψ-weak convergence of probability measures.

Lemma 2.1.1 *Assume that (E, d_E) is a complete and separable metric space. Then the set $\mathcal{M}_1^{\psi}(E)$ equipped with the ψ-weak topology is a Polish space. Further the ψ-weak topology is metrizable by the metric d_{ψ} defined by*

$$d_{\psi}(\mu,\nu) \;:=\; d_{\mathrm{w}}(\mu,\nu) + \Big|\int_E \psi\,d\mu - \int_E \psi\,d\nu\Big|, \qquad \mu,\nu \in \mathcal{M}_1^{\psi}(E), \tag{2.3}$$

where d_{w} refers to any metric on $\mathcal{M}_1(E)$ which generates the weak topology. Additionally, for every choice of $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_1^{\psi}(E)$ we have the following equivalent statements.

(i) $\mu_m \longrightarrow \mu$ *ψ-weakly.*

(ii) $\mu_m \xrightarrow{\mathrm{w}} \mu$ *and* $\int_E \psi\,d\mu_m \longrightarrow \int_E \psi\,d\mu$.

The statements in Lemma 2.1.1 are used to prove Lemma 2.1.7 below. Now, let us turn to examples for the (semi-) metric $d_{\mathbb{M}}$.

Example 2.1.2 (Total variation metric) Let $\psi :\equiv 1$ and $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$, where $\mathbb{M}_{\mathrm{TV}} := \{\mathbb{1}_B : B \in \mathcal{E}\} \subseteq \mathbb{M}_{\psi}(E)$. Then $d_{\mathbb{M}}$ equals the *total variation metric* d_{TV} defined by

$$d_{\mathrm{TV}}(\mu,\nu) \;:=\; \sup_{B\in\mathcal{E}} \big|\mu[B] - \nu[B]\big|, \qquad \mu,\nu \in \mathcal{M}_1^{\psi}(E). \tag{2.4}$$

The set $\mathbb{M}_{\mathrm{TV}}$ clearly separates points in $\mathcal{M}_1^{\psi}(E) = \mathcal{M}_1(E)$. The maximal generator of d_{TV} is the set $\overline{\mathbb{M}}_{\mathrm{TV}}$ of all $h \in \mathbb{M}(E)$ with $\mathrm{sp}(h) := \sup_{x\in E} h(x) - \inf_{x\in E} h(x) \le 1$; see Theorem 5.4 in [69]. $\diamond$

Example 2.1.3 (Kolmogorov metric) For $E = \mathbb{R}$, let $\psi :\equiv 1$ and $\mathbb{M} := \mathbb{M}_{\mathrm{Kolm}}$, where $\mathbb{M}_{\mathrm{Kolm}} := \{\mathbb{1}_{(-\infty,t]} : t \in \mathbb{R}\} \subseteq \mathbb{M}_{\psi}(\mathbb{R})$. Then $d_{\mathbb{M}}$ equals the *Kolmogorov metric* d_{Kolm} defined by

$$d_{\mathrm{Kolm}}(\mu,\nu) \;:=\; \sup_{t\in\mathbb{R}} \big|F_{\mu}(t) - F_{\nu}(t)\big|, \qquad \mu,\nu \in \mathcal{M}_1^{\psi}(\mathbb{R}),$$

where F_{μ} and F_{ν} refer to the distribution functions of μ and ν, respectively. The set $\mathbb{M}_{\mathrm{Kolm}}$ clearly separates points in $\mathcal{M}_1^{\psi}(\mathbb{R}) = \mathcal{M}_1(\mathbb{R})$. The maximal generator of d_{Kolm} is the set $\overline{\mathbb{M}}_{\mathrm{Kolm}}$ of all maps $h \in \mathbb{R}^{\mathbb{R}}$ with $\mathbb{V}_h(\mathbb{R}) \le 1$; see Theorem 5.2 in [69]. Recall from Display (B.2) in Section B.1 that $\mathbb{V}_h(\mathbb{R})$ denotes the variation of h on $\mathbb{R}$. $\diamond$

Example 2.1.4 (Bounded Lipschitz metric) Assume that (E, d_E) is a metric space and let $\mathcal{E} := \mathcal{B}(E)$. Let $\psi :\equiv 1$ and $\mathbb{M} := \mathbb{M}_{\mathrm{BL}}$, where $\mathbb{M}_{\mathrm{BL}} := \{h \in \mathbb{R}^E : \|h\|_{\mathrm{BL}} \leq 1\} \subseteq \mathbb{M}_\psi(E)$ with $\|h\|_{\mathrm{BL}} := \max\{\|h\|_\infty, \|h\|_{\mathrm{Lip}}\}$ for $\|h\|_\infty := \sup_{x\in E}|h(x)|$ and $\|h\|_{\mathrm{Lip}} := \sup_{x,y\in E:\, x\neq y}|h(x) - h(y)|/d_E(x,y)$. Then $d_{\mathbb{M}}$ is nothing but the *bounded Lipschitz metric* d_{BL} defined by

$$d_{\mathrm{BL}}(\mu,\nu) := \sup_{h\in\mathbb{M}_{\mathrm{BL}}}\Big|\int_E h\,d\mu - \int_E h\,d\nu\Big|, \qquad \mu,\nu\in\mathcal{M}_1^\psi(E). \tag{2.5}$$

The set $\mathbb{M}_{\mathrm{BL}}$ separates points in $\mathcal{M}_1^\psi(E) = \mathcal{M}_1(E)$; see Lemma 9.3.2 in [32]. Moreover it is known (see, e.g., Theorem 11.3.3 in [32]) that if E is separable then d_{BL} generates the weak topology $\mathcal{O}_{\mathrm{w}}$ on $\mathcal{M}_1^\psi(E) = \mathcal{M}_1(E)$. ◇

Example 2.1.5 (Kantorovich metric) Assume that (E, d_E) is a metric space and let $\mathcal{E} := \mathcal{B}(E)$. For some fixed $x' \in E$, let $\psi(\cdot) := 1 + d_E(\,\cdot\,, x')$ and $\mathbb{M} := \mathbb{M}_{\mathrm{Kant}}$, where $\mathbb{M}_{\mathrm{Kant}} := \{h \in \mathbb{R}^E : \|h\|_{\mathrm{Lip}} \leq 1\} \subseteq \mathbb{M}_\psi(E)$ with $\|\cdot\|_{\mathrm{Lip}}$ as in Example 2.1.4. Then $d_{\mathbb{M}}$ is nothing but the *Kantorovich metric* d_{Kant} defined by

$$d_{\mathrm{Kant}}(\mu,\nu) := \sup_{h\in\mathbb{M}_{\mathrm{Kant}}}\Big|\int_E h\,d\mu - \int_E h\,d\nu\Big|, \qquad \mu,\nu\in\mathcal{M}_1^\psi(E). \tag{2.6}$$

The set $\mathbb{M}_{\mathrm{Kant}}$ separates points in $\mathcal{M}_1^\psi(E)$, because $\mathbb{M}_{\mathrm{BL}}$ ($\subseteq \mathbb{M}_{\mathrm{Kant}}$) does. It is known (see, e.g., Theorem 7.12 in [89]) that if E is complete and separable then d_{Kant} generates the ψ-weak topology $\mathcal{O}_{\mathrm{w}}^\psi$ on $\mathcal{M}_1^\psi(E)$.

Recall from [85] that for $E = \mathbb{R}$ the L^1-*Wasserstein metric* $d_{\mathrm{Wass},1}$ given by

$$d_{\mathrm{Wass},1}(\mu,\nu) := \int_{-\infty}^{\infty}|F_\mu(t) - F_\nu(t)|\,dt, \qquad \mu,\nu\in\mathcal{M}_1^\psi(\mathbb{R})$$

coincides with the Kantorovich metric d_{Kant}. In this case the ψ-weak topology $\mathcal{O}_{\mathrm{w}}^\psi$ is also referred to as L^1-weak topology. Note that the L^1-Wasserstein metric is a conventional metric for measuring the distance between probability distributions; see, for instance, [28, 47, 85] for the general concept and [8, 49, 55, 59] for recent applications. ◇

Although the Kantorovich metric is a popular and well established metric, for the application in Section 3.2 we will need the following generalization from $\alpha = 1$ to $\alpha \in (0, 1]$.

Example 2.1.6 (Hölder-α metric) Assume that (E, d_E) is a metric space and let $\mathcal{E} := \mathcal{B}(E)$. For some fixed $x' \in E$ and $\alpha \in (0, 1]$, let $\psi(\cdot) := 1 + d_E(\,\cdot\,, x')^\alpha$ and $\mathbb{M} := \mathbb{M}_{\mathrm{Höl},\alpha}$, where $\mathbb{M}_{\mathrm{Höl},\alpha} := \{h \in \mathbb{R}^E : \|h\|_{\mathrm{Höl},\alpha} \leq 1\} \subseteq \mathbb{M}_\psi(E)$ for $\|h\|_{\mathrm{Höl},\alpha} := \sup_{x,y\in E:\, x\neq y}|h(x) - h(y)|/d_E(x,y)^\alpha$. The set $\mathbb{M}_{\mathrm{Höl},\alpha}$ separates points in $\mathcal{M}_1^\psi(E)$ (this follows with similar arguments as in the proof of Lemma 9.3.2 in [32]). Then $d_{\mathbb{M}}$ provides a metric on $\mathcal{M}_1^\psi(E)$ which we denote by $d_{\mathrm{Höl},\alpha}$, and refer to as *Hölder-α metric*. That is, the Hölder-α metric $d_{\mathrm{Höl},\alpha}$ is defined by

$$d_{\mathrm{Höl},\alpha}(\mu,\nu) := \sup_{h\in\mathbb{M}_{\mathrm{Höl},\alpha}}\Big|\int_E h\,d\mu - \int_E h\,d\nu\Big|, \qquad \mu,\nu\in\mathcal{M}_1^\psi(E).$$

Especially when dealing with risk averse utility functions (as, e.g., in Section 3.2) this metric can be beneficial. ◇

The following Lemma 2.1.7 shows that if in the setting of Example 2.1.6 the state space E is complete and separable then the Hölder-α metric $d_{\mathrm{Höl},\alpha}$ generates the ψ-weak topology $\mathcal{O}_{\mathrm{w}}^{\psi}$ on $\mathcal{M}_1^{\psi}(E)$.

Lemma 2.1.7 *Assume that (E, d_E) is a complete and separable metric space, and let $\alpha \in (0,1]$ and $x' \in E$ be arbitrary but fixed. Then the Hölder-α metric $d_{\mathrm{Höl},\alpha}$ introduced in Example 2.1.6 generates the ψ-weak topology $\mathcal{O}_{\mathrm{w}}^{\psi}$ on $\mathcal{M}_1^{\psi}(E)$ for $\psi(\cdot) := 1 + d_E(\,\cdot\,, x')^{\alpha}$.*

Proof As the ψ-weak topology is metrizable (see, for example, Corollary A.45 in [35]), it suffices to show that for any choice of $\mu, \mu_1, \mu_2 \ldots \in \mathcal{M}_1^{\psi}(E)$ we have $\mu_m \to \mu$ ψ-weakly if and only if $d_{\mathrm{Höl},\alpha}(\mu_m, \mu) \to 0$.

First assume that $d_{\mathrm{Höl},\alpha}(\mu_m, \mu) \to 0$. As $\mu_m \to \mu$ ψ-weakly if and only if $\mu_m \overset{\mathrm{w}}{\longrightarrow} \mu$ and $\int_E \psi \, d\mu_m \to \int_E \psi \, d\mu$ (by Lemma 2.1.1), it suffices to show that $\mu_m \overset{\mathrm{w}}{\longrightarrow} \mu$ and $\int_E \psi \, d\mu_m \to \int_E \psi \, d\mu$. Any bounded $h \in \mathbb{R}^E$ with $\|h\|_{\mathrm{Lip}} < \infty$ satisfies $\|h\|_{\mathrm{Höl},\alpha} \le C_h := \max\{\|h\|_{\mathrm{Lip}}, 2\|h\|_\infty\}$. Since h/C_h lies in $\mathbb{M}_{\mathrm{Höl},\alpha}$, our assumption implies $\int_E h \, d\mu_m \to \int_E h \, d\mu$. That is, $\int_E h \, d\mu_m \to \int_E h \, d\mu$ for any bounded and Lipschitz continuous $h \in \mathbb{R}^E$. By the portmanteau theorem we can conclude $\mu_m \overset{\mathrm{w}}{\longrightarrow} \mu$. Moreover, as ψ lies in $\mathbb{M}_{\mathrm{Höl},\alpha}$, our assumption also implies $\int_E \psi \, d\mu_m \to \int_E \psi \, d\mu$.

Conversely, assume that $\mu_m \to \mu$ ψ-weakly. We have to show that for every $\varepsilon > 0$ there exists some $m_0 \in \mathbb{N}$ such that

$$\sup_{h \in \mathbb{M}_{\mathrm{Höl},\alpha}} \Big| \int_E h \, d\mu_m - \int_E h \, d\mu \Big| \le \varepsilon \qquad \text{for all } m \ge m_0. \tag{2.7}$$

For any $K > 0$, the left hand side of (2.7) is bounded above by

$$\sup_{h \in \mathbb{M}_{\mathrm{Höl},\alpha}} \Big| \int_E h_K \, d\mu_m - \int_E h_K \, d\mu \Big| + \sup_{h \in \mathbb{M}_{\mathrm{Höl},\alpha}} \Big| \int_E h^K \, d\mu_m - \int_E h^K \, d\mu \Big| \tag{2.8}$$

with $h_K := h\mathbb{1}_{\{|h| \le K\}} + K\mathbb{1}_{\{h > K\}} - K\mathbb{1}_{\{h < -K\}}$, and $h^K := h - h_K$. Without loss of generality we may and do assume that $h(x') = 0$ for all $h \in \mathbb{M}_{\mathrm{Höl},\alpha}$; take into account that $|\int_E h \, d\mu_m - \int_E h \, d\mu|$ remains unchanged when a constant is added to h. Then $|h(x)| = |h(x) - h(x')| \le d_E(x, x')^{\alpha} \le \psi(x)$ for all $h \in \mathbb{M}_{\mathrm{Höl},\alpha}$. In particular, $|h^K| \le |h|\mathbb{1}_{\{|h| > K\}} \le \psi \mathbb{1}_{\{\psi > K\}}$. Thus the second summand in (2.8) is bounded above by

$$\int_E \psi \mathbb{1}_{\{\psi > K\}} \, d\mu_m + \int_E \psi \mathbb{1}_{\{\psi > K\}} \, d\mu \tag{2.9}$$

Now we can choose $K > 0$ so large that the second summand in (2.9) is at most $\varepsilon/5$. The first summand in (2.9) is bounded above by

$$\Big| \int_E \psi \mathbb{1}_{\{\psi > K\}} \, d\mu_m - \int_E \psi \mathbb{1}_{\{\psi > K\}} \, d\mu \Big| + \int_E \psi \mathbb{1}_{\{\psi > K\}} \, d\mu \tag{2.10}$$

The second summand in (2.10) is at most $\varepsilon/5$ (see above) and the first summand in (2.10) is bounded above by

$$\Big| \int_E \psi \, d\mu_m - \int_E \psi \, d\mu \Big| + \Big| \int_E \psi \mathbb{1}_{\{\psi \le K\}} \, d\mu_m - \int_E \psi \mathbb{1}_{\{\psi \le K\}} \, d\mu \Big|. \tag{2.11}$$

The first summand in (2.11) converges to 0 as $m \to \infty$, because $\mu_m \to \mu$ ψ-weakly. Thus we can find $m_0 \in \mathbb{N}$ such that it is bounded above by $\varepsilon/5$ for every $m \ge m_0$. Since $\mu \circ \psi^{-1}$ as a probability measure on the real line has at most countably many atom, we may and do assume that $K > 0$ is chosen such that $\mu[\{\psi = K\}] = 0$. Since $\mu_m \to \mu$ ψ-weakly and thus $\mu_m \xrightarrow{\mathrm{w}} \mu$, it follows by the portmanteau theorem that the second summand in (2.11) converges to 0 as $m \to \infty$. By possibly increasing m_0 we obtain that the second summand in (2.11) is at most $\varepsilon/5$ for all $m \geq m_0$. So far we have shown that the second summand in (2.8) is bounded above by $4\varepsilon/5$ for all $m \geq m_0$. As the functions of $\mathbb{M}_{\mathrm{Höl},\alpha;K} := \{h_K : h \in \mathbb{M}_{\mathrm{Höl},\alpha}\}$ are uniformly bounded and equicontinuous, Corollary 11.3.4 in [32] ensures that one can increase m_0 further such that the first summand in (2.8) is bounded above by $\varepsilon/5$ for all $m \geq m_0$. That is, we arrive at (2.7). □

2.1.2 Metric on set of transition functions

Maintain the notation and terminology introduced in Subsection 2.1.1. Fix $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$, and denote by $\overline{\mathcal{P}}_\psi$ the set of all transition functions $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$ satisfying $\int_E \psi(y)\, P_n((x,a), dy) < \infty$ for all $(x,a) \in D_n$ and $n = 0, \ldots, N-1$. That is, $\overline{\mathcal{P}}_\psi$ consists of those transition functions $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$ with $P_n((x,a), \bullet) \in \mathcal{M}_1^\psi(E)$ for all $(x,a) \in D_n$ and $n = 0, \ldots, N-1$. Hence, for the elements $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ of $\overline{\mathcal{P}}_\psi$ all integrals of the shape $\int_E h(y)\, P_n((x,a), dy)$, $h \in \mathbb{M}_\psi(E)$, $(x,a) \in D_n$, $n = 0, \ldots, N-1$, exist and are finite. In particular, for two transition functions $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ and $\boldsymbol{Q} = (Q_n)_{n=0}^{N-1}$ from $\overline{\mathcal{P}}_\psi$ the distance $d_{\mathbb{M}}(P_n((x,a), \bullet), Q_n((x,a), \bullet))$ is well-defined for all $(x,a) \in D_n$ and $n = 0, \ldots, N-1$.

Thus we may define the distance between two transition functions $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ and $\boldsymbol{Q} = (Q_n)_{n=0}^{N-1}$ from $\overline{\mathcal{P}}_\psi$ by

$$d_{\infty,\mathbb{M}}^{\phi}(\boldsymbol{P}, \boldsymbol{Q}) := \max_{n=0,\ldots,N-1} \sup_{(x,a)\in D_n} \frac{1}{\phi(x)} \cdot d_{\mathbb{M}}\Big(P_n\big((x,a), \bullet\big), Q_n\big((x,a), \bullet\big)\Big) \tag{2.12}$$

for another gauge function $\phi : E \to \mathbb{R}_{\geq 1}$. Note that it follows from the discussion below of Display (2.2) that (2.12) defines a semi-metric $d_{\infty,\mathbb{M}}^{\phi} : \overline{\mathcal{P}}_\psi \times \overline{\mathcal{P}}_\psi \to \overline{\mathbb{R}}_{\geq 0}$ on $\overline{\mathcal{P}}_\psi$ which is even a metric if $\mathbb{M}$ separates points in $\mathcal{M}_1^\psi(E)$.

Maybe apart from the factor $1/\phi(x)$, the definition of $d_{\infty,\mathbb{M}}^{\phi}(\boldsymbol{P}, \boldsymbol{Q})$ in (2.12) is quite natural and in line with the definition of a distance introduced by Müller [68, p. 880]. In [68], Müller considers stationary MDMs, so that the transition kernels do not depend on n. He fixed a state x and took the supremum only over all admissible actions a in state x. That is, for any $x \in E$ he defined the distance between $P((x, \cdot), \bullet)$ and $Q((x, \cdot), \bullet)$ by $\sup_{a\in A(x)} d_{\mathbb{M}}(P((x,a), \bullet), Q((x,a), \bullet))$. To obtain a reasonable distance between P_n and Q_n it is however natural to take the supremum of the distance between $P_n((x, \cdot), \bullet)$ and $Q_n((x, \cdot), \bullet)$ w.r.t. $d_{\mathbb{M}}$ uniformly over a *and* over x.

Remark 2.1.8 (i) The factor $1/\phi(x)$ in Display (2.12) causes that the (semi-) metric $d_{\infty,\mathbb{M}}^{\phi}$ is less strict compared to the (semi-) metric $d_{\infty,\mathbb{M}}^{\phi'}$ whenever the gauge function ϕ' satisfies $\phi' \leq \phi$. To put it another way, the 'steeper' the gauge function ϕ the less strict the (semi-) metric $d_{\infty,\mathbb{M}}^{\phi}$. In particular, the (semi-) metric $d_{\infty,\mathbb{M}}^{1}$ which is defined as in (2.12) with $\phi :\equiv 1$ is the most strict one. For a motivation of considering the factor $1/\phi(x)$, see Remark 2.2.2 as well as parts (iii)–(iv) of Remark 2.3.3 and Remark 2.3.4.

(ii) We note that the subset $\mathbb{M}$ of test functions from $\mathbb{M}_\psi(E)$ also influences the shape of the (semi-) metric $d^{\phi}_{\infty,\mathbb{M}}$. In fact, in view of (2.2), the smaller the set $\mathbb{M}$ the less strict the (semi-) metric $d^{\phi}_{\infty,\mathbb{M}}$. ◇

2.1.3 The special case of finite state space and finite action spaces

In this subsection we will explain (using the general framework in Subsection 2.1.2) how the distance between two transition functions can be measured if in the MDM both the state space and the action spaces are finite. Here we will use the notation and the terminology introduced in Section 1.5.

Assume that the state space E as well as the set of all admissible actions $A_n(x)$ for each point of time $n = 0, \ldots, N-1$ and state $x \in E$ are given by (1.23) and (1.24), respectively, where $\mathfrak{e} := \#E \in \mathbb{N}$ and $\mathfrak{t}_{n,i} := \#A_n(x_i) \in \mathbb{N}$. Let $\mathcal{E} := \mathfrak{P}(E)$, and note that the sets A_n as well as D_n from Subsection 1.1.1 are finite for any $n = 0, \ldots, N-1$.

In this case, we measure the distance between two probability measures μ and ν from $\mathcal{M}_1(E)$ by the total variation metric d_{TV} introduced in (2.4), i.e. by

$$d_{\mathrm{TV}}(\mu,\nu) \;=\; \max_{B\in\mathfrak{P}(E)} \big|\mu[B]-\nu[B]\big| \;=\; \frac{1}{2}\sum_{y\in E}\big|\mu[\{y\}]-\nu[\{y\}]\big|.$$

This fits the setting of Subsection 2.1.1 with $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$ and $\psi :\equiv 1$; see Example 2.1.2. Since E was assumed to be finite with $\mathfrak{e} = \#E \in \mathbb{N}$, we may and do identify any probability measure $\mu \in \mathcal{M}_1(E)$ with some element

$$p_\mu \;=\; (p_\mu(1),\ldots,p_\mu(\mathfrak{e})) \tag{2.13}$$

of $\mathbb{R}^{\mathfrak{e}}_{\geq 0,1}$ (with $\mathbb{R}^{\mathfrak{e}}_{\geq 0,1}$ as in Section 1.5). Hence the total variation distance d_{TV} between $\mu,\nu \in \mathcal{M}_1(E)$ can be identified (up to the factor 1/2) with the ℓ_1-distance between p_μ and p_ν:

$$d_{\mathrm{TV}}(\mu,\nu) \;=\; \frac{1}{2}\sum_{i=1}^{\mathfrak{e}}\big|p_\mu(i)-p_\nu(i)\big| \;=\; \frac{1}{2}\,\|p_\mu-p_\nu\|_{\ell_1}. \tag{2.14}$$

For the distance between two transition functions we will employ the metric $d^{1}_{\infty,\mathbb{M}_{\mathrm{TV}}}$, which is defined as in (2.12) with $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$ and $\phi := \psi :\equiv 1$. As already mentioned in Section 1.5, the set $\overline{\mathcal{P}}$ of all transition functions can be represented in the finite setting through (1.25). In particular, under the imposed assumptions, we may identify any transition function $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ from $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}_1$ defined as in Subsection 2.1.2) with an element $\boldsymbol{p}$ as defined in (1.26) from the set $\widetilde{\mathcal{P}}$ introduced in (1.27). Then, imposing (without loss of generality) the metric

$$\begin{aligned} d_{\infty,\ell_1}(\boldsymbol{p},\boldsymbol{q}) \;:=\; \frac{1}{2}\max\Big\{ &\max_{k=1,\ldots,\mathfrak{t}_{0,i_0}} \|p_{0,i_0;a_{0,i_0;k}}-q_{0,i_0;a_{0,i_0;k}}\|_{\ell_1}, \\ &\max_{n=1,\ldots,N-1}\;\max_{i=1,\ldots,\mathfrak{e}}\;\max_{k=1,\ldots\mathfrak{t}_{n,i}} \|p_{n,i;a_{n,i;k}}-q_{n,i;a_{n,i;k}}\|_{\ell_1}\Big\} \end{aligned} \tag{2.15}$$

on $\widetilde{\mathcal{P}}$, it is apparent that the metric d_{∞,ℓ_1} is a special case of the metric $d^{1}_{\infty,\mathbb{M}_{\mathrm{TV}}}$ defined as in (2.12) with $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$ and $\phi := \psi :\equiv 1$. Recall i_0 refers to the index of the initial state $x_0 = x_{i_0} \in E$. That is, in the finite setting of Section 1.5 we will use the metric d_{∞,ℓ_1} given by (2.15) to measure the distance between transition functions.

2.2 'Continuity' in P of the value function

In this section we will prove that the value function of a MDM regarded as a real-valued functional on a suitable set of transition functions is 'Lipschitz continuous' in a certain sense. The notion of 'Lipschitz continuity' will be formally introduced in Subsection 2.2.1. We will also discuss the special case of finite state space and finite action spaces. The motivation of our investigation comes from the work of Müller in [68], where the author proved in [68, Theorem 4.2] that the value function of a stationary MDM depends continuously on the transition probabilities, and he established some bounds for the approximation error. In Subsection 2.2.2 we will formulate our main result (see Theorem 2.2.8 ahead) concerning the 'Lipschitz continuity' of the so-called value functional introduced in Display (2.16) below, which can be seen to some extent as a slight generalization of Theorem 4.2 in [68] for non-stationary MDMs. Throughout this section we suppose that the model components E, $\boldsymbol{A}$, Π, and $\boldsymbol{r}$ of the MDM are fixed.

2.2.1 Definition of 'Lipschitz continuity'

Let ψ be any gauge function, and fix a subset $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$, where $\overline{\mathcal{P}}_\psi$ is defined as in Subsection 2.1.2. In the following we equip the set $\mathcal{P}_\psi$ with the distance $d^{\phi}_{\infty,\mathbb{M}}$ introduced in (2.12) for another gauge function ϕ. In this subsection we present a reasonable notion of 'Lipschitz continuity' for an arbitrary functional $\mathcal{V} : \mathcal{P}_\psi \to \mathbb{R}$. Since this notion is weaker compared to the usual concept of Lipschitz continuity, we will use inverted commas and write 'Lipschitz continuity' instead of Lipschitz continuity.

Definition 2.2.1 ('Lipschitz continuity' in $\boldsymbol{P}$) *Let $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$, ϕ be another gauge function, and fix $\boldsymbol{P} \in \mathcal{P}_\psi$. A map $\mathcal{V} : \mathcal{P}_\psi \to \mathbb{R}$ is said to be 'Lipschitz continuous' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}, \phi)$ if*

$$\big|\mathcal{V}(\boldsymbol{P}_m) - \mathcal{V}(\boldsymbol{P})\big| \;=\; \mathcal{O}\big(d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{P}_m, \boldsymbol{P})\big)$$

holds for every sequence $(\boldsymbol{P}_m) \in \mathcal{P}_\psi^{\mathbb{N}}$ with $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{P}_m, \boldsymbol{P}) \to 0$.

Note that in the setting of Definition 2.2.1 the notation $\mathcal{O}(d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{P}_m, \boldsymbol{P}))$ refers to any real-valued sequence $(c_m)_{m\in\mathbb{N}}$ for which the sequence $(c_m\, d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{P}_m, \boldsymbol{P})^{-1})_{m\in\mathbb{N}}$ is bounded.

Remark 2.2.2 (i) The subset $\mathbb{M}$ ($\subseteq \mathbb{M}_\psi(E)$) and the gauge function ϕ can be considered to a certain extent as factors which influence the 'robustness' of the map $\mathcal{V}$ w.r.t. changes in $\boldsymbol{P}$. Indeed, the smaller the set $\mathbb{M}$ and the 'steeper' the gauge function ϕ, the less strict the (semi-) metric $d^{\phi}_{\infty,\mathbb{M}}$ given by (2.12) (see Remark 2.1.8), and hence the more robust the map $\mathcal{V}$ in $\boldsymbol{P}$. Therefore it is favorable to choose the set $\mathbb{M}$ as small as possible and the gauge function ϕ as 'steep' as possible. However, the smaller $\mathbb{M}$ and the 'steeper' ϕ, the stricter the condition of 'Lipschitz continuity'. To be more precise, if $\mathbb{M}_1 \subseteq \mathbb{M}_2$ and $\phi_1 \geq \phi_2$ then 'Lipschitz continuity' w.r.t. $(\mathbb{M}_1, \phi_1)$ implies 'Lipschitz continuity' w.r.t. $(\mathbb{M}_2, \phi_2)$.

(ii) Note that in our main result of this section (Theorem 2.2.8 below) we can *not* choose the gauge function ϕ 'steeper' than the gauge function ψ which is in this framework the bounding function. In fact, the $(\mathbb{M}, \psi)$-'Lipschitz continuity' of the functionals $\mathcal{V} : \mathcal{P}_\psi \to \mathbb{R}$ in Theorem 2.2.8 can *not* be proven if $d^{\psi}_{\infty,\mathbb{M}}$ is replaced by $d^{\phi}_{\infty,\mathbb{M}}$ in the case that ϕ is 'steeper' than ψ. $\diamond$

We conclude this subsection by discussing how the concept of 'Lipschitz continuity' introduced in Definition 2.2.1 can be simplified for the case that in the MDM both the state space and the action spaces are finite. The latter setting was already discussed in Section 1.5.

To this end, let E be the state space from (1.23) with $\mathfrak{e} := \#E \in \mathbb{N}$, and let $A_n(x_i)$ be for any $i = 1, \dots, \mathfrak{e}$ and $n = 0, \dots, N-1$ the (finite) set of all admissible actions in state x_i at time n given by (1.24). As already discussed in Subsection 2.1.3, we may and do identify any transition function $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ from $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}_1$ defined as in Subsection 2.1.2) with a vector $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ given by (1.26), where $\widetilde{\mathcal{P}}$ is defined as in (1.27). In particular, the metric $d^1_{\infty,\mathbb{M}_{\mathrm{TV}}}$ on $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ introduced in (2.12) with $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$ and $\phi := \psi :\equiv 1$ can be identified with the metric d_{∞,ℓ_1} on $\widetilde{\mathcal{P}}$ given by (2.15). Therefore the concept of 'Lipschitz continuity' (w.r.t. $(\mathbb{M}_{\mathrm{TV}}, \phi)$) stated in Definition 2.2.1 boils down in the finite setting of Section 1.5 to the following notion:

Definition 2.2.3 ('Lipschitz continuity' in $\boldsymbol{p}$) *Let $\boldsymbol{p} \in \widetilde{\mathcal{P}}$. A map $\mathcal{V} : \widetilde{\mathcal{P}} \to \mathbb{R}$ is said to be 'Lipschitz continuous' at $\boldsymbol{p}$ if*

$$\big|\mathcal{V}(\boldsymbol{p}_m) - \mathcal{V}(\boldsymbol{p})\big| \;=\; \mathcal{O}\big(d_{\infty,\ell_1}(\boldsymbol{p}_m, \boldsymbol{p})\big)$$

holds for every sequence $(\boldsymbol{p}_m) \in \widetilde{\mathcal{P}}^{\mathbb{N}}$ with $d_{\infty,\ell_1}(\boldsymbol{p}_m, \boldsymbol{p}) \to 0$.

Analogously to the discussion subsequent to Definition 2.2.1, the notation $\mathcal{O}(d_{\infty,\ell_1}(\boldsymbol{p}_m, \boldsymbol{p}))$ in the setting of Definition 2.2.3 refers to any real-valued sequence $(c_m)_{m\in\mathbb{N}}$ for which the sequence $(c_m\, d_{\infty,\ell_1}(\boldsymbol{p}_m, \boldsymbol{p})^{-1})_{m\in\mathbb{N}}$ is bounded.

2.2.2 'Lipschitz continuity' of the value functional

We now turn back to our general framework of Section 1.1. Recall that E, $\boldsymbol{A}$, Π, and $\boldsymbol{r}$ are fixed, and let $V_n^{\boldsymbol{P};\pi}$ and $V_n^{\boldsymbol{P}}$ be defined as in (1.11) and (1.13), respectively. Moreover let ψ be any gauge function, and fix some $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$.

In view of Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{P}\}$), condition (a) of Assumption 2.2.5 below ensures that Assumption 1.2.1 is satisfied for *any* $\boldsymbol{P} \in \mathcal{P}_\psi$. Then for any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$ we may define under condition (a) of Assumption 2.2.5 functionals $\mathcal{V}_n^{x_n;\pi} : \mathcal{P}_\psi \to \mathbb{R}$ and $\mathcal{V}_n^{x_n} : \mathcal{P}_\psi \to \mathbb{R}$ by

$$\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}) \;:=\; V_n^{\boldsymbol{P};\pi}(x_n) \quad \text{ and } \quad \mathcal{V}_n^{x_n}(\boldsymbol{P}) \;:=\; \sup_{\pi\in\Pi} \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}) \;\big(=\; V_n^{\boldsymbol{P}}(x_n)\big), \tag{2.16}$$

respectively. Note that $\mathcal{V}_n^{x_n}(\boldsymbol{P})$ specifies the maximal value for the expected total reward in the MDM (given state x_n at time n) when the underlying transition function is $\boldsymbol{P}$. By analogy with the name 'value function' we refer to $\mathcal{V}_n^{x_n}$ as *value functional given state x_n at time n*. Part (ii) of Theorem 2.2.8 below shows (under some conditions) that the value functional $\mathcal{V}_n^{x_n}$ is 'Lipschitz continuous' at any fixed $\boldsymbol{P} \in \mathcal{P}_\psi$ in the sense of Definition 2.2.1.

Conditions (b) and (c) of Assumption 2.2.5 involve the so-called *Minkowski (or gauge) functional* $\rho_{\mathbb{M}} : \mathbb{M}_\psi(E) \to \overline{\mathbb{R}}_{\geq 0}$ (see, e.g., [76, p. 25]) defined by

$$\rho_{\mathbb{M}}(h) \;:=\; \inf\big\{\lambda \in \mathbb{R}_{>0} : \, h/\lambda \in \mathbb{M}\big\}, \tag{2.17}$$

where we use the convention $\inf\emptyset := \infty$, $\mathbb{M}$ is any subset of $\mathbb{M}_\psi(E)$, and we set $\mathbb{R}_{>0} := (0, \infty)$. We note that Müller [68] also used the Minkowski functional to formulate his assumptions.

Example 2.2.4 (Minkowski functional) For the sets $\mathbb{M}$ (and the corresponding gauge functions ψ) from Examples 2.1.2–2.1.6 we have $\rho_{\overline{\mathbb{M}}_{\mathrm{TV}}}(h) = \mathrm{sp}(h)$, $\rho_{\overline{\mathbb{M}}_{\mathrm{Kolm}}}(h) = \mathbb{V}_h(\mathbb{R})$, $\rho_{\mathbb{M}_{\mathrm{BL}}}(h) = \|h\|_{\mathrm{BL}}$, $\rho_{\mathbb{M}_{\mathrm{Kant}}}(h) = \|h\|_{\mathrm{Lip}}$, and $\rho_{\mathbb{M}_{\mathrm{Höl},\alpha}}(h) = \|h\|_{\mathrm{Höl},\alpha}$, where as before $\overline{\mathbb{M}}_{\mathrm{TV}}$ and $\overline{\mathbb{M}}_{\mathrm{Kolm}}$ are used to denote the maximal generator of d_{TV} and d_{Kolm}, respectively. The latter three equations are trivial, for the former two equations see [68, p. 880]. ◇

Recall from Subsection 2.1.1 the definition of generators $\mathbb{M}'$ of the (semi-) metric $d_{\mathbb{M}}$ which were introduced subsequent to (2.2). The following Assumption 2.2.5 will be illustrated in Remark 2.2.6 below.

Assumption 2.2.5 *Let $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$ and $\mathbb{M}'$ be any generator of $d_{\mathbb{M}}$. Moreover let $\boldsymbol{P} \in \mathcal{P}_\psi$, and assume that the following three conditions hold.*

(a) *ψ is a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{Q}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for every $\boldsymbol{Q} \in \mathcal{P}_\psi$.*

(b) $\sup_{\pi\in\Pi} \rho_{\mathbb{M}'}(V_n^{\boldsymbol{P};\pi}) < \infty$ *for any $n = 1, \dots, N$.*

(c) $\rho_{\mathbb{M}'}(\psi) < \infty$.

Remark 2.2.6 (i) Condition (a) of Assumption 2.2.5 is in line with the existing literature. In fact, similar conditions as in Definition 1.4.1 (with $\mathcal{P} := \{\boldsymbol{P}\}$) have been imposed many times before; see, for instance, [5, Definition 2.4.1], [68, Definition 2.4], [73, p. 231 ff], and [91].

(ii) In some situations, condition (a) implies condition (b) in Assumption 2.2.5. This is the case, for instance, in the following four settings (the involved sets $\mathbb{M}'$, metrics, and norms were introduced in Examples 2.1.2–2.1.6).

1) $\mathbb{M}' := \overline{\mathbb{M}}_{\mathrm{TV}}$ and $\psi :\equiv 1$.

2) $\mathbb{M}' := \overline{\mathbb{M}}_{\mathrm{Kolm}}$ and $\psi :\equiv 1$, as well as for $n = 1, \dots, N-1$
   - $\int_{\mathbb{R}} V_{n+1}^{\boldsymbol{P};\pi}(y)\, P_n((\,\cdot\,, f_n(\,\cdot\,)), dy)$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, are increasing,
   - $r_n(\,\cdot\,, f_n(\,\cdot\,))$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $r_N(\cdot)$ are increasing.

3) $\mathbb{M}' := \mathbb{M}_{\mathrm{BL}}$ and $\psi :\equiv 1$, as well as for $n = 1, \dots, N-1$
   - $\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \sup_{x\neq y} d_{\mathrm{BL}}(P_n((x, f_n(x)), \bullet), P_n((y, f_n(y)), \bullet))/d_E(x, y) < \infty$,
   - $\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \|r_n(\,\cdot\,, f_n(\,\cdot\,))\|_{\mathrm{Lip}} < \infty$ and $\|r_N\|_{\mathrm{Lip}} < \infty$.

4) $\mathbb{M}' := \mathbb{M}_{\mathrm{Höl},\alpha}$ and $\psi(\cdot) := 1 + d_E(\,\cdot\,, x')^\alpha$, as well as for $n = 1, \dots, N-1$
   - $\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \sup_{x\neq y} d_{\mathrm{Höl},\alpha}(P_n((x, f_n(x)), \bullet), P_n((y, f_n(y)), \bullet))/d_E(x, y)^\alpha < \infty$,
   - $\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \|r_n(\,\cdot\,, f_n(\,\cdot\,))\|_{\mathrm{Höl},\alpha} < \infty$ and $\|r_N\|_{\mathrm{Höl},\alpha} < \infty$,

   for some fixed $x' \in E$ and $\alpha \in (0, 1]$. Recall that $\mathbb{M}_{\mathrm{Höl},\alpha} = \mathbb{M}_{\mathrm{Kant}}$ for $\alpha = 1$.

The proof of (a)⇒(b) relies in setting 1) on Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{P}\}$) and in settings 2)–4) on Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{P}\}$) along with Proposition 1.3.1. The conditions in setting 2) are similar to those in parts (ii)–(iv) of Theorem 2.4.14 in [5], and the conditions in settings 3) and 4) are motivated by the statements in [40, p. 11f].

(iii) In many situations, condition (c) of Assumption 2.2.5 holds trivially. This is the case, for instance, if $\mathbb{M}' \in \{\overline{\mathbb{M}}_{\mathrm{TV}}, \overline{\mathbb{M}}_{\mathrm{Kolm}}, \mathbb{M}_{\mathrm{BL}}\}$ and $\psi :\equiv 1$, or if $\mathbb{M}' := \mathbb{M}_{\mathrm{Höl},\alpha}$ and $\psi(\cdot) := 1 + d_E(\,\cdot\,, x')^\alpha$, for some fixed $x' \in E$ and $\alpha \in (0, 1]$.

(iv) The conditions (b) and (c) of Assumption 2.2.5 can also be verified directly in some cases; see, for instance, Lemma 3.2.10 in Subsection 3.2.4. ◇

In applications it is not necessarily easy to verify condition (b) of Assumption 2.2.5. The following remark may help in some situations; for an application see Subsection 3.2.4.

Remark 2.2.7 In some situations it turns out that for *every* $\boldsymbol{P} \in \mathcal{P}_\psi$ the solution of the optimization problem (1.12) does not change if Π is replaced by a subset $\Pi' \subseteq \Pi$ (being independent of $\boldsymbol{P}$). Then in the definition (1.13) of the value function (at time 0) the set Π can be replaced by the subset Π'. Of course, in this case it suffices to ensure that conditions (a)–(b) of Assumption 2.2.5 are satisfied for the subset Π' instead of Π. ◇

The following theorem shows in particular that (under Assumption 2.2.5) the value functional depends continuously on the transition functions.

Theorem 2.2.8 ('Lipschitz continuity' of $\mathcal{V}_n^{x_n;\pi}$ and $\mathcal{V}_n^{x_n}$ in $\boldsymbol{P}$) *Suppose that Assumption 2.2.5 holds for some $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$ and $\boldsymbol{P} \in \mathcal{P}_\psi$. Then the following two assertions are valid.*

(i) *For any $x_n \in E$, $\pi \in \Pi$, $n = 0, \dots, N$, the map $\mathcal{V}_n^{x_n;\pi} : \mathcal{P}_\psi \to \mathbb{R}$ defined by (2.16) is 'Lipschitz continuous' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}, \psi)$.*

(ii) *For any $x_n \in E$ and $n = 0, \dots, N$, the map $\mathcal{V}_n^{x_n} : \mathcal{P}_\psi \to \mathbb{R}$ defined by (2.16) is 'Lipschitz continuous' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}, \psi)$.*

Remark 2.2.9 (i) It follows from the proof of Theorem 2.2.8 ahead that (under the assumptions of Theorem 2.2.8) the 'Lipschitz continuity' in part (i) of the latter theorem holds even uniformly in $\pi \in \Pi$. That is, for any fixed $\boldsymbol{P} \in \mathcal{P}_\psi$, we have

$$\sup_{\pi \in \Pi} \big| \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}_m) - \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}) \big| \;=\; \mathcal{O}\big(d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m, \boldsymbol{P})\big)$$

for every $x_n \in E$ and $n = 0, \dots, N$ as well as any sequence $(\boldsymbol{P}_m) \in \mathcal{P}_\psi^{\mathbb{N}}$ with $d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m, \boldsymbol{P}) \to 0$.

(ii) In the case where we are interested in minimizing expected total costs in the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ (see Remark 1.2.6(ii)), we obtain under the assumptions (and with similar arguments as in the proof of part (ii)) of Theorem 2.2.8 that the 'Lipschitz continuity' of the corresponding value functional holds. ◇

In the following we provide a proof of Theorem 2.2.8.

Proof of Theorem 2.2.8: We will prove only the assertion in (ii). The claim in part (i) will follow with similar arguments. Let $x_n \in E$ as well as $n = 0, \dots, N$ be arbitrary but fixed. Further let $(\boldsymbol{P}_m)_{m \in \mathbb{N}}$ be any sequence in $\mathcal{P}_\psi$ with $d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m, \boldsymbol{P}) \to 0$. At first, as a simple consequence of the definition of the Minkowski functional $\rho_{\mathbb{M}'}$ (see (2.17)) we have

$$\Big| \int_E h \, d\mu - \int_E h \, d\nu \Big| \;\le\; \rho_{\mathbb{M}'}(h) \cdot d_{\mathbb{M}}(\mu, \nu) \quad \text{for all } h \in \mathbb{M}_\psi(E) \text{ and } \mu, \nu \in \mathcal{M}_1^{\psi}(E), \qquad (2.18)$$

because $\mathbb{M}'$ ($\subseteq \mathbb{M}_\psi(E)$) is a generator of $d_{\mathbb{M}}$ by assumption. Using (2.16), (1.11), and Lemma 1.4.4, we obtain for any $m \in \mathbb{N}$ by rearranging the sums

$$
\begin{aligned}
&\big|\mathcal{V}_n^{x_n}(\boldsymbol{P}_m) - \mathcal{V}_n^{x_n}(\boldsymbol{P})\big| \\
&\quad= \Big|\sup_{\pi\in\Pi}\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}_m) - \sup_{\pi\in\Pi}\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P})\Big| \\
&\quad\le \sup_{\pi\in\Pi}\big|\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}_m) - \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P})\big| \\
&\quad= \sup_{\pi\in\Pi}\big|\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}+(\boldsymbol{P}_m-\boldsymbol{P})) - \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P})\big| \\
&\quad= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\sum_{k=n}^{N-1}\Big(\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}+(\boldsymbol{P}_m-\boldsymbol{P});\pi}\big[r_k(X_k,f_k(X_k))\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[r_k(X_k,f_k(X_k))\big]\Big) \\
&\qquad\qquad + \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}+(\boldsymbol{P}_m-\boldsymbol{P});\pi}\big[r_N(X_N)\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[r_N(X_N)\big]\Big|\Big\} \\
&\quad= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\sum_{k=n+1}^{N-1}\sum_{j=n}^{k-1}\int_E\cdots\int_E r_k(y_k,f_k(y_k))\,P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad\quad \cdots(P_{m;j}-P_j)\big((y_j,f_j(y_j)),dy_{j+1}\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\qquad + \sum_{k=n+2}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1<|J|\le k-n}}\int_E\cdots\int_E\int_E r_k(y_k,f_k(y_k))\,\xi_{k-1,J}^{m;-}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad\quad \xi_{k-2,J}^{m;-}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\qquad + \sum_{j=n}^{N-1}\int_E\cdots\int_E r_N(y_N)\,P_{N-1}\big((y_{N-1},f_{N-1}(y_{N-1})),dy_N\big) \\
&\qquad\quad \cdots(P_{m;j}-P_j)\big((y_j,f_j(y_j)),dy_{j+1}\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\qquad + \sum_{\substack{J\subseteq\{n,\dots,N-1\}\\ 1<|J|\le N-n}}\int_E\cdots\int_E\int_E r_N(y_N)\,\xi_{N-1,J}^{m;-}\big((y_{N-1},f_{N-1}(y_{N-1})),dy_N\big) \\
&\qquad\quad \xi_{N-2,J}^{m;-}\big((y_{N-2},f_{N-2}(y_{N-2})),dy_{N-1}\big)\cdots\xi_{n,J}^{m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\Big\} \\
&\quad= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\sum_{k=n}^{N-1}\int_E\cdots\int_E\int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big) \\
&\qquad\quad P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\qquad + \sum_{k=n+2}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1<|J|\le k-n}}\int_E\cdots\int_E\int_E r_k(y_k,f_k(y_k))\,\xi_{k-1,J}^{m;-}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad\quad \xi_{k-2,J}^{m;-}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\qquad + \sum_{\substack{J\subseteq\{n,\dots,N-1\}\\ 1<|J|\le N-n}}\int_E\cdots\int_E\int_E r_N(y_N)\,\xi_{N-1,J}^{m;-}\big((y_{N-1},f_{N-1}(y_{N-1})),dy_N\big) \\
&\qquad\quad \xi_{N-2,J}^{m;-}\big((y_{N-2},f_{N-2}(y_{N-2})),dy_{N-1}\big)\cdots\xi_{n,J}^{m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \Big| \sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big) \\
& \qquad P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
& \quad + \sum_{k=n+1}^{N-1} \sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1\le|J|\le k-n}} \int_E \cdots \int_E \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big) \\
& \qquad \xi_{k-1,J}^{m;-}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots \xi_{n,J}^{m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\Big\} \\
\le \quad & \sum_{k=n}^{N-1} \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \Big| \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big)\Big| \\
& \qquad P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
& \quad + \sum_{k=n+1}^{N-1} \sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1\le|J|\le k-n}} \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \Big| \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big)\Big| \\
& \qquad \xi_{k-1,J}^{m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots \xi_{n,J}^{m;+}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
=: \quad & S_1(m)+S_2(m),
\end{aligned}
$$

where $\xi_{j,J}^{m;\pm}$ is for any subset $J\subseteq\{0,\dots,N-1\}$ given by

$$
\xi_{j,J}^{m;\pm} := \begin{cases} P_{m;j}\pm P_j & , \quad j\in J \\ P_j & , \quad \text{otherwise} \end{cases} .
$$

It follows from (2.18), part (v) of Lemma 1.4.4 as well as (2.12) that for any $k=n+1,\dots,N-1$ and $m\in\mathbb{N}$

$$
\begin{aligned}
& \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \Big| \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big)\Big| \\
& \qquad\qquad P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
& \quad \le \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\cdot \sup_{x\in E}\frac{1}{\psi(x)}\, d_{\mathbb{M}}\Big(P_{m;k}\big((x,f_k(x)),\bullet\big),P_k\big((x,f_k(x)),\bullet\big)\Big) \\
& \qquad\qquad \cdot \int_E \cdots \int_E \psi(y_k)\,P_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
& \quad \le \sup_{f_k\in\mathbb{F}_k}\sup_{x\in E}\frac{1}{\psi(x)}\, d_{\mathbb{M}}\Big(P_{m;k}\big((x,f_k(x)),\bullet\big),P_k\big((x,f_k(x)),\bullet\big)\Big) \\
& \qquad\qquad \cdot \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\;\cdot\;\sup_{\pi\in\Pi}\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[\psi(X_k)\big] \\
& \quad \le \sup_{(x,a)\in D_k}\frac{1}{\psi(x)}\, d_{\mathbb{M}}\Big(P_{m;k}\big((x,a),\bullet\big),P_k\big((x,a),\bullet\big)\Big) \\
& \qquad\qquad \cdot \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\;\cdot\;\sup_{\pi\in\Pi}\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[\psi(X_k)\big] \\
& \quad \le d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P})\;\cdot\;\sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\;\cdot\;\sup_{\pi\in\Pi}\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[\psi(X_k)\big] \qquad (2.19)
\end{aligned}
$$

because $V_{k+1}^{\boldsymbol{P};\pi}(\cdot)\in\mathbb{M}_\psi(E)$ for any $\pi\in\Pi$ due to Proposition 1.4.3 (applied to $\mathcal{P}:=\{\boldsymbol{P}\}$). Similarly, for any $m\in\mathbb{N}$

$$
\begin{aligned}
&\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\int_E V_{n+1}^{\boldsymbol{P};\pi}(y_{n+1})\,(P_{m;n}-P_n)\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\Big\}\\
&\le\ \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\rho_{\mathbb{M}'}\big(V_{n+1}^{\boldsymbol{P};\pi}\big)\cdot\sup_{x\in E}\frac{1}{\psi(x)}\,d_{\mathbb{M}}\Big(P_{m;n}\big((x,f_n(x)),\bullet\big),P_n\big((x,f_n(x)),\bullet\big)\Big)\cdot\psi(x_n)\Big\}\\
&\le\ \sup_{f_n\in\mathbb{F}_n}\sup_{x\in E}\frac{1}{\psi(x)}\,d_{\mathbb{M}}\Big(P_{m;n}\big((x,f_n(x)),\bullet\big),P_n\big((x,f_n(x)),\bullet\big)\Big)\ \cdot\ \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{n+1}^{\boldsymbol{P};\pi}\big)\ \cdot\ \psi(x_n)\\
&\le\ \sup_{(x,a)\in D_n}\frac{1}{\psi(x)}\,d_{\mathbb{M}}\Big(P_{m;n}\big((x,a),\bullet\big),P_n\big((x,a),\bullet\big)\Big)\ \cdot\ \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{n+1}^{\boldsymbol{P};\pi}\big)\ \cdot\ \sup_{\pi\in\Pi}\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[\psi(X_n)\big]\\
&\le\ d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P})\ \cdot\ \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{n+1}^{\boldsymbol{P};\pi}\big)\ \cdot\ \sup_{\pi\in\Pi}\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[\psi(X_n)\big]
\end{aligned}\tag{2.20}
$$

by part (iii) of Lemma 1.4.4. The second factor in the last line of both (2.19) and (2.20) is (independent of m and) finite due to condition (b) of Assumption 2.2.5. Moreover, the finiteness of the third factor in the last line of both (2.19) and (2.20) (which is also independent of m) follows from Lemma 1.4.4 along with condition (a) of Assumption 2.2.5. Thus $S_1(m)=\mathcal{O}(d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P}))$.

Analogously to (2.19), in view of (2.18), condition (a) of Assumption 2.2.5, and part (c) of Definition 1.4.1 (applied to $\mathcal{P}:=\{\boldsymbol{P}\}$), there exists a finite constant $K_3>0$ such that for any $k=n+1,\dots,N-1$ and $m\in\mathbb{N}$

$$
\begin{aligned}
&\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\ \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\Big|\int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(P_{m;k}-P_k)\big((y_k,f_k(y_k)),dy_{k+1}\big)\Big|\\
&\qquad\qquad \xi_{k-1,J}^{m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots\xi_{n,J}^{m;+}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}\\
&\le\ \sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}} d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P})\ \cdot\ \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\\
&\qquad\cdot\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\psi(y_k)\,\xi_{k-1,J}^{m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\qquad\qquad \xi_{k-2,J}^{m;+}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{m;+}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}\\
&=\ d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P})\ \cdot\ \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\\
&\qquad\cdot\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\ \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\psi(y_k)\,\xi_{k-1,J}^{m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\qquad\qquad \xi_{k-2,J}^{m;+}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{m;+}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}\\
&\le\ d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P})\ \cdot\ \sup_{\pi\in\Pi}\rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big)\\
&\qquad\cdot\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\binom{k-n}{|J|}K_3^{k-n-|J|}\cdot\big(\rho_{\mathbb{M}'}(\psi)\cdot d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P})+2K_3\big)^{|J|}\cdot\psi(x_n)
\end{aligned}
$$

because $\psi\in\mathbb{M}_\psi(E)$. Thus $S_2(m)=\mathcal{O}(d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{P}_m,\boldsymbol{P}))$ by conditions (b) and (c) of Assumption

2.2.5. Hence the assertion follows. This completes the proof of Theorem 2.2.8. $\square$

Next, we reformulate in Corollary 2.2.10 below the statements of Theorem 2.2.8 in the setting of Section 1.5 where in the corresponding MDM both the state space E (given by (1.23)) with $\mathfrak{e} := \#E \in \mathbb{N}$ and the action spaces are finite. In this corollary we will use the notion of 'Lipschitz continuity' introduced in Definition 2.2.3.

Recall from Section 1.5 that in the finite setting any transition function $\boldsymbol{P}$ from the set $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}_1$ defined as in Subsection 2.1.2) can be identified with an element $\boldsymbol{p}$ as defined in (1.26) from the set $\widetilde{\mathcal{P}}$ given by (1.27). Therefore, the functionals $\mathcal{V}_n^{x_n;\pi}$ and $\mathcal{V}_n^{x_n}$ given by (2.16) can be identified in the finite setting with maps $\mathcal{V}_n^{x_i;\pi} : \widetilde{\mathcal{P}} \to \mathbb{R}$ and $\mathcal{V}_n^{x_i} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by

$$\mathcal{V}_n^{x_i;\pi}(\boldsymbol{p}) := V_n^{\boldsymbol{p};\pi}(x_i) \quad \text{and} \quad \mathcal{V}_n^{x_i}(\boldsymbol{p}) := \max_{\pi \in \Pi} \mathcal{V}_n^{x_i;\pi}(\boldsymbol{p}), \tag{2.21}$$

where the policy value function $V_n^{\boldsymbol{p};\pi}(\cdot)$ can be obtained from (1.29). Take into account that the latter functionals are well-defined because $\psi :\equiv 1$ is a bounding function for the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \overline{\mathcal{P}}\}$ (with $\overline{\mathcal{P}}$ given by (1.25)); see the discussion at end of Section 1.5. Similarly, we will refer $\mathcal{V}_n^{x_i}$ to as *value functional given state x_i at time n.*

Corollary 2.2.10 ('Lipschitz continuity' of $\mathcal{V}_n^{x_i;\pi}$ and $\mathcal{V}_n^{x_i}$ in $\boldsymbol{p}$) *Let $\boldsymbol{p} \in \widetilde{\mathcal{P}}$. Then in the setting of Section 1.5 the following two assertions hold.*

(i) *For any $i = 1, \ldots, \mathfrak{e}$, $\pi \in \Pi$, $n = 0, \ldots, N$, the map $\mathcal{V}_n^{x_i;\pi} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by (2.21) is 'Lipschitz continuous' at $\boldsymbol{p}$.*

(ii) *For any $i = 1, \ldots, \mathfrak{e}$ and $n = 0, \ldots, N$, the map $\mathcal{V}_n^{x_i} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by (2.21) is 'Lipschitz continuous' at $\boldsymbol{p}$.*

Proof We intend to apply Theorem 2.2.8. First of all, as discussed above, the gauge function $\psi :\equiv 1$ provides a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for every $\boldsymbol{P} \in \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}$ as in (1.25)). Therefore, condition (a) of Assumption 2.2.5 holds. Thus parts (ii) and (iii) of Remark 2.2.6 entail that conditions (b)–(c) of Assumption 2.2.5 are satisfied for $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$, $\mathbb{M}' := \overline{\mathbb{M}}_{\mathrm{TV}}$, and $\psi :\equiv 1$, where the sets $\mathbb{M}_{\mathrm{TV}}$ as well as $\overline{\mathbb{M}}_{\mathrm{TV}}$ are introduced in Example 2.1.2.

In particular, we have verified the assumptions of Theorem 2.2.8, and an application of parts (i) and (ii) of the latter theorem leads to the assertions in (i) and (ii), respectively. Take into account that it follows from the discussion in Subsection 2.2.1 that in the finite setting of Section 1.5 the notion of 'Lipschitz continuity' (w.r.t. $(\mathbb{M}_{\mathrm{TV}}, \psi)$) in Definition 2.2.1 boils down to the concept of 'Lipschitz continuity' introduced in Definition 2.2.3. $\square$

2.3 'Differentiability' in $\boldsymbol{P}$ of the value function

In this section, we show that the value functional is 'differentiable' in a certain sense. The motivation of our notion of 'differentiability' was discussed subsequent to (2.1). The 'derivative' of the value functional which we propose to regard as a measure for the first-order sensitivity will formally be introduced in Definition 2.3.2 in Subsection 2.3.1. This definition is applicable to quite general finite time horizon MDMs and might look somewhat cumbersome at first glance. However, in the special case of a finite state space and finite action spaces, a situation one faces in many practical applications, the proposed 'differentiability' boils down to a rather intuitive concept. This will be explained in the second part of Subsection 2.3.1. In Subsection 2.3.2 we will specify the 'Hadamard derivative' of the value functional, and we present a backward iteration scheme for the computation of the 'Hadamard derivative'.

2.3.1 Definition of 'differentiability'

Let ψ be any gauge function, and fix some $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$ (with $\overline{\mathcal{P}}_\psi$ defined as in Subsection 2.1.2) being closed under mixtures. The latter means that $(1-\varepsilon)\boldsymbol{P}+\varepsilon\boldsymbol{Q} \in \mathcal{P}_\psi$ for any $\boldsymbol{P},\boldsymbol{Q} \in \mathcal{P}_\psi$ and $\varepsilon \in (0,1)$. We will consider the (semi-) metric space $(\mathcal{P}_\psi, d^\phi_{\infty,\mathbb{M}})$, where (semi-) metric $d^\phi_{\infty,\mathbb{M}}$ is introduced in (2.12) for another gauge function ϕ.

In Definition 2.3.2 below we will introduce a reasonable notion of 'differentiability' for an arbitrary functional $\mathcal{V} : \mathcal{P}_\psi \to L$ taking values in a normed vector space $(L, \|\cdot\|_L)$. It is related to the general functional analytic concept of (tangential) $\mathcal{S}$-differentiability introduced by Sebastião e Silva [82] and Averbukh and Smolyanov [4]; see also [33, 36, 83] for applications. However, $\mathcal{P}_\psi$ is *not* a vector space. This implies that Definition 2.3.2 differs from the classical notion of (tangential) $\mathcal{S}$-differentiability. For that reason we will use inverted commas and write '$\mathcal{S}$-differentiability' instead of $\mathcal{S}$-differentiability. Due to the missing vector space structure, we in particular need to allow the tangent space to depend on the point $\boldsymbol{P} \in \mathcal{P}_\psi$ at which $\mathcal{V}$ is differentiated. The role of the 'tangent space' will be played by the set

$$\mathcal{P}_\psi^{\boldsymbol{P};\pm} := \{\boldsymbol{Q}-\boldsymbol{P} : \boldsymbol{Q} \in \mathcal{P}_\psi\} \tag{2.22}$$

whose elements $\boldsymbol{Q}-\boldsymbol{P} := (Q_0 - P_0, \ldots, Q_{N-1} - P_{N-1})$ can be seen as signed transition functions. Before we introduce our notion of '$\mathcal{S}$-differentiability' in Definition 2.3.2 below, we first need the following terminology.

Definition 2.3.1 *Let $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$, ϕ be another gauge function, and fix $\boldsymbol{P} \in \mathcal{P}_\psi$. A map $\mathcal{W} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to L$ is said to be $(\mathbb{M},\phi)$-continuous if the mapping $\boldsymbol{Q} \mapsto \mathcal{W}(\boldsymbol{Q}-\boldsymbol{P})$ from $\mathcal{P}_\psi$ to L is $(d^\phi_{\infty,\mathbb{M}}, \|\cdot\|_L)$-continuous.*

Note for the following definition that $\boldsymbol{P}+\varepsilon(\boldsymbol{Q}-\boldsymbol{P})$ lies in $\mathcal{P}_\psi$ for any $\boldsymbol{P},\boldsymbol{Q} \in \mathcal{P}_\psi$ and $\varepsilon \in (0,1]$. Recall that $\mathcal{P}_\psi$ was assumed to be closed under mixtures.

Definition 2.3.2 ('$\mathcal{S}$-differentiability' in $\boldsymbol{P}$) *Let $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$, ϕ be another gauge function, and fix $\boldsymbol{P} \in \mathcal{P}_\psi$. Moreover let $\mathcal{S}$ be a system of subsets of $\mathcal{P}_\psi$. A map $\mathcal{V} : \mathcal{P}_\psi \to L$ is said to be '$\mathcal{S}$-differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\phi)$ if there exists an $(\mathbb{M},\phi)$-continuous map $\dot{\mathcal{V}}_{\boldsymbol{P}} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to L$ such that*

$$\lim_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}-\boldsymbol{P})) - \mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \Big\|_L = 0 \quad \textit{uniformly in } \boldsymbol{Q} \in \mathcal{K} \tag{2.23}$$

for every $\mathcal{K} \in \mathcal{S}$ and every sequence $(\varepsilon_m) \in (0,1]^{\mathbb{N}}$ with $\varepsilon_m \to 0$. In this case, $\dot{\mathcal{V}}_{\boldsymbol{P}}$ is called '$\mathcal{S}$-derivative' of $\mathcal{V}$ at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\phi)$.

Note that in Definition 2.3.2 the derivative is *not* required to be linear (in fact the derivative is not even defined on a vector space). This is another point where Definition 2.3.2 differs from the functional analytic definition of (tangential) $\mathcal{S}$-differentiability. However, non-linear derivatives are common in the field of mathematical optimization; see, for instance, [75, 83].

Remark 2.3.3 (i) At least in the case $L = \mathbb{R}$, the '$\mathcal{S}$-derivative' $\dot{\mathcal{V}}_{\boldsymbol{P}}$ evaluated at $\boldsymbol{Q} - \boldsymbol{P}$, i.e. $\dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q} - \boldsymbol{P})$, can be seen as a measure for the first-order sensitivity of the functional $\mathcal{V} : \mathcal{P}_\psi \to \mathbb{R}$ w.r.t. a change of the argument from $\boldsymbol{P}$ to $(1-\varepsilon)\boldsymbol{P}+\varepsilon\boldsymbol{Q}$, with $\varepsilon > 0$ small, for some given transition function $\boldsymbol{Q}$.

(ii) The prefix '$\mathcal{S}$-' in Definition 2.3.2 provides the following information. Since the convergence in (2.23) is required to be uniform in $\boldsymbol{Q} \in \mathcal{K}$, the values of the first-order sensitivities $\dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q} - \boldsymbol{P})$, $\boldsymbol{Q} \in \mathcal{K}$, can be compared with each other with clear conscience for any fixed $\mathcal{K} \in \mathcal{S}$. It is therefore favorable if the sets in $\mathcal{S}$ are large. However, the larger the sets in $\mathcal{S}$, the stricter the condition of '$\mathcal{S}$-differentiability'.

(iii) The subset $\mathbb{M}$ ($\subseteq \mathbb{M}_\psi(E)$) and the gauge function ϕ tell us in a way how 'robust' the '$\mathcal{S}$-derivative' $\dot{\mathcal{V}}_{\boldsymbol{P}}$ is w.r.t. changes in $\boldsymbol{Q}$: The smaller the set $\mathbb{M}$ and the 'steeper' the gauge function ϕ, the less strict the (semi-) metric $d_{\infty,\mathbb{M}}^{\phi}(\boldsymbol{P}, \boldsymbol{Q})$ given by (2.12) (see Remark 2.1.8), and therefore the more robust $\dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q} - \boldsymbol{P})$ in $\boldsymbol{Q}$. It is thus favorable if the set $\mathbb{M}$ is small and the gauge function ϕ is 'steep'. However, the smaller $\mathbb{M}$ and the 'steeper' ϕ, the stricter the condition of $(\mathbb{M}, \phi)$-continuity (and thus of '$\mathcal{S}$-differentiability' w.r.t. $(\mathbb{M}, \phi)$). More precisely, if $\mathbb{M}_1 \subseteq \mathbb{M}_2$ and $\phi_1 \geq \phi_2$ then $(\mathbb{M}_1, \phi_1)$-continuity implies $(\mathbb{M}_2, \phi_2)$-continuity.

(iv) In general the choice of $\mathcal{S}$ and the choice of the pair $(\mathbb{M}, \phi)$ in Definition 2.3.2 do not necessarily depend on each other. However in the specific settings (b) and (c) in Definition 2.3.5, and in particular in the application in Section 3.2, they do.

(v) In the general framework of our main result (Theorem 2.3.11 ahead) we can *not* choose ϕ 'steeper' than the gauge function ψ which plays the role of a bounding function there. Indeed, the proof of $(\mathbb{M}, \psi)$-continuity of the map $\dot{\mathcal{V}}_{\boldsymbol{P}} : \mathcal{P}_{\psi}^{\boldsymbol{P};\pm} \to \mathbb{R}$ in Theorem 2.3.11 does *not* work anymore if $d_{\infty,\mathbb{M}}^{\psi}$ is replaced by $d_{\infty,\mathbb{M}}^{\phi}$ for any gauge function ϕ 'steeper' than ψ. And here it does *not* matter how exactly $\mathcal{S}$ is chosen. ◇

Remark 2.3.4 In the numerical example for the 'Hadamard derivative' of the value functional in Subsection 3.2.5, the set $\{\boldsymbol{Q}_{\Delta,\tau} : \Delta \in [0, \delta]\}$ should be contained in $\mathcal{S}$ (for details see Remark 3.2.14). This set can be shown to be (relatively) compact w.r.t. $d_{\infty,\mathbb{M}}^{\phi}$ for $\mathbb{M} := \mathbb{M}_{\text{Höl},\alpha}$ and $\phi := \psi$ but not for any 'flatter' gauge function ϕ, where $\mathbb{M}_{\text{Höl},\alpha}$ is defined as in Example 2.1.6 and ψ is given by (3.17). So, in this example, and certainly in many other examples, relatively compact subsets of $\mathcal{P}_\psi$ w.r.t. $d_{\infty,\mathbb{M}}^{\phi}$ should be contained in $\mathcal{S}$. It is thus often beneficial to know that the value functional is 'differentiable' in the sense of part (b) of the following Definition 2.3.5. ◇

The terminology in Definition 2.3.5 below is motivated by the functional analytic analogues. Bounded and relatively compact sets in the (semi-) metric space $(\mathcal{P}_\psi, d_{\infty,\mathbb{M}}^{\phi})$ are understood in the conventional way. A set $\mathcal{K} \subseteq \mathcal{P}_\psi$ is said to be bounded (w.r.t. $d_{\infty,\mathbb{M}}^{\phi}$) if there exist $\boldsymbol{P}' \in \mathcal{P}_\psi$

and $\delta > 0$ such that $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q},\boldsymbol{P}') \le \delta$ for every $\boldsymbol{Q} \in \mathcal{K}$. It is said to be relatively compact (w.r.t. $d^{\phi}_{\infty,\mathbb{M}}$) if for every sequence $(\boldsymbol{Q}_m) \in \mathcal{K}^{\mathbb{N}}$ there exists a subsequence $(\boldsymbol{Q}'_m)_{m\in\mathbb{N}}$ of $(\boldsymbol{Q}_m)_{m\in\mathbb{N}}$ such that $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q}'_m,\boldsymbol{Q}) \to 0$ for some $\boldsymbol{Q} \in \mathcal{P}_\psi$. Note that in view of Remark 2.1.8 the system of all bounded sets and the system of all relatively compact sets (w.r.t. $d^{\phi}_{\infty,\mathbb{M}}$) are the larger (the smaller the set $\mathbb{M}$ and/or) the 'steeper' the gauge function ϕ is.

Definition 2.3.5 *In the setting of Definition 2.3.2 we refer to '$\mathcal{S}$-differentiability' as*

(a) *'Gateaux–Lévy differentiability' if $\mathcal{S} = \mathcal{S}_{\mathrm{f}} := \{\mathcal{K} \subseteq \mathcal{P}_\psi : \mathcal{K}$ is finite$\}$.*

(b) *'Hadamard differentiability' if $\mathcal{S} = \mathcal{S}_{\mathrm{rc}} := \{\mathcal{K} \subseteq \mathcal{P}_\psi : \mathcal{K}$ is relatively compact$\}$.*

(c) *'Fréchet differentiability' if $\mathcal{S} = \mathcal{S}_{\mathrm{b}} := \{\mathcal{K} \subseteq \mathcal{P}_\psi : \mathcal{K}$ is bounded$\}$.*

Clearly, in the setting of Definition 2.3.5 we have $\mathcal{S}_{\mathrm{f}} \subseteq \mathcal{S}_{\mathrm{rc}} \subseteq \mathcal{S}_{\mathrm{b}}$. Therefore, 'Fréchet differentiability' (of a map $\mathcal{V} : \mathcal{P}_\psi \to L$ at some fixed $\boldsymbol{P} \in \mathcal{P}_\psi$ w.r.t. $(\mathbb{M},\phi)$) implies 'Hadamard differentiability' which in turn implies 'Gateaux–Lévy differentiability', each with the same 'derivative'.

The last sentence before Definition 2.3.5 and the last sentence in part (iii) of Remark 2.3.3 together imply that 'Hadamard (resp. 'Fréchet) differentiability' w.r.t. $(\mathbb{M},\phi_1)$ implies 'Hadamard (resp. 'Fréchet) differentiability' w.r.t. $(\mathbb{M},\phi_2)$ when $\phi_1 \ge \phi_2$.

The following lemma provides an equivalent characterization of 'Hadamard differentiability' (in the sense of Definitions 2.3.2 and 2.3.5(b)). Its statement will be used to prove (part (ii) of) Theorem 2.3.11 ahead.

Lemma 2.3.6 *Let $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$, ϕ be another gauge function, $\mathcal{V} : \mathcal{P}_\psi \to L$ be any map, and fix $\boldsymbol{P} \in \mathcal{P}_\psi$. Then the following two assertions hold.*

(i) *If $\mathcal{V}$ is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\phi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{\boldsymbol{P}}$, then we have for each triplet $(\boldsymbol{Q},(\boldsymbol{Q}_m),(\varepsilon_m)) \in \mathcal{P}_\psi \times \mathcal{P}_\psi^{\mathbb{N}} \times (0,1]^{\mathbb{N}}$ with $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q}_m,\boldsymbol{Q}) \to 0$ and $\varepsilon_m \to 0$ that*

$$\lim_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P}))-\mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \Big\|_L = 0. \tag{2.24}$$

(ii) *If there exists an $(\mathbb{M},\phi)$-continuous map $\dot{\mathcal{V}}_{\boldsymbol{P}} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to L$ such that (2.24) holds for each triplet $(\boldsymbol{Q},(\boldsymbol{Q}_m),(\varepsilon_m)) \in \mathcal{P}_\psi \times \mathcal{P}_\psi^{\mathbb{N}} \times (0,1]^{\mathbb{N}}$ with $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q}_m,\boldsymbol{Q}) \to 0$ and $\varepsilon_m \to 0$, then $\mathcal{V}$ is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\phi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{\boldsymbol{P}}$.*

Proof For (i), let $\mathcal{V}$ be 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\phi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{\boldsymbol{P}}$. To show that (2.24) holds, pick a triplet $(\boldsymbol{Q},(\boldsymbol{Q}_m),(\varepsilon_m)) \in \mathcal{P}_\psi \times \mathcal{P}_\psi^{\mathbb{N}} \times (0,1]^{\mathbb{N}}$ with $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q}_m,\boldsymbol{Q}) \to 0$ and $\varepsilon_m \to 0$. Then, the set $\mathcal{K} := \{\boldsymbol{Q}_m : m \in \mathbb{N}\}$ $(\subseteq \mathcal{P}_\psi)$ is clearly relatively compact. Using this and the assumption we obtain

$$\begin{aligned}
&\limsup_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P}))-\mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \Big\|_L \\
&\quad\le \limsup_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P}))-\mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}_m-\boldsymbol{P}) \Big\|_L \\
&\qquad + \limsup_{m\to\infty} \big\| \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}_m-\boldsymbol{P}) - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \big\|_L
\end{aligned}$$

$$= \; 0 + 0 \; = \; 0.$$

Hence the claim in (i) follows.

To prove (ii), assume that there exists an $(\mathbb{M}, \phi)$-continuous map $\dot{\mathcal{V}}_{\boldsymbol{P}} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to L$ such that (2.24) holds for each triplet $(\boldsymbol{Q}, (\boldsymbol{Q}_m), (\varepsilon_m)) \in \mathcal{P}_\psi \times \mathcal{P}_\psi^{\mathbb{N}} \times (0,1]^{\mathbb{N}}$ with $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q}_m, \boldsymbol{Q}) \to 0$ and $\varepsilon_m \to 0$. Assume by way of contradiction that $\dot{\mathcal{V}}_{\boldsymbol{P}}$ is *not* the 'Hadamard derivative' of $\mathcal{V}$ at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}, \phi)$, i.e. that there is some relatively compact set $\mathcal{K} \subseteq \mathcal{P}_\psi$ and a sequence $(\varepsilon_m) \in (0,1]^{\mathbb{N}}$ with $\varepsilon_m \to 0$ such that (2.23) does *not* hold uniformly in $\boldsymbol{Q} \in \mathcal{K}$. Then there exist $\delta > 0$ and $(\boldsymbol{Q}_m) \in \mathcal{K}^{\mathbb{N}}$ such that

$$\Big\| \frac{\mathcal{V}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q}_m - \boldsymbol{P})) - \mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}_m - \boldsymbol{P}) \Big\|_L \geq \delta \quad \text{for all } m \in \mathbb{N}. \tag{2.25}$$

Since $\mathcal{K}$ is relatively compact, we can find a subsequence $(\boldsymbol{Q}'_m)_{m\in\mathbb{N}}$ of $(\boldsymbol{Q}_m)_{m\in\mathbb{N}}$ such that $d^{\phi}_{\infty,\mathbb{M}}(\boldsymbol{Q}'_m, \boldsymbol{Q}') \to 0$ for some $\boldsymbol{Q}' \in \mathcal{P}_\psi$. Along with the $(\mathbb{M}, \phi)$-continuity of the map $\dot{\mathcal{V}}_{\boldsymbol{P}} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to L$ and (2.25) (with $\boldsymbol{Q}_m$ replaced by $\boldsymbol{Q}'_m$), we obtain

$$\begin{aligned}
&\liminf_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q}'_m - \boldsymbol{P})) - \mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}' - \boldsymbol{P}) \Big\|_L \\
&= \; \liminf_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q}'_m - \boldsymbol{P})) - \mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}'_m - \boldsymbol{P}) \Big\|_L \\
&\quad + \; \liminf_{m\to\infty} \big\| \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}'_m - \boldsymbol{P}) - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}' - \boldsymbol{P}) \big\|_L \\
&= \; \liminf_{m\to\infty} \Big\| \frac{\mathcal{V}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q}'_m - \boldsymbol{P})) - \mathcal{V}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{\boldsymbol{P}}(\boldsymbol{Q}'_m - \boldsymbol{P}) \Big\|_L + 0 \; \geq \; \delta
\end{aligned}$$

which contradicts the assumption (2.24). In particular, this shows (ii). □

Remark 2.3.7 In contrast to the elaborations in Section A in [57], according to which the notion of quasi-Lipschitz continuity introduced in [57, Definition A.3] can be deduced (under some conditions) by means of Lemma A.5 in [57] from the concept of quasi-Hadamard differentiability introduced in [57, Definition A.1], it is easily seen that in view of Lemma 2.3.6(i) our notion of 'Hadamard differentiability' (in the sense of Definitions 2.3.2 and 2.3.5(b)) does *not* imply the notion of 'Lipschitz continuity' from Definition 2.2.1. ◇

In the rest of this subsection we will consider the special case when in the MDM both the state space as well as the action spaces are finite. By using the discrete setting introduced in Section 1.5, we are able to present the notion of 'differentiability' introduced in Definition 2.3.2 in a more comprehensible way.

Let the state space E be as in (1.23) with $\mathfrak{e} := \#E \in \mathbb{N}$, and let $A_n(x_i)$ given by (1.24) with $\mathfrak{t}_{n,i} := \#A_n(x_i) \in \mathbb{N}$ be for any $i = 1, \ldots, \mathfrak{e}$ and $n = 0, \ldots, N-1$ the set of all admissible actions in state x_i at time n. Recall from Subsection 2.1.3 that we may identify any transition function $\boldsymbol{P} = (P_n)_{n=0}^{N-1}$ from $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}$ as in (1.25)) with an element $\boldsymbol{p}$ from $\widetilde{\mathcal{P}}$ given by (1.26), where $\widetilde{\mathcal{P}} := (\mathbb{R}^{\mathfrak{e}}_{\geq 0,1})^{\times(\mathfrak{d}/\mathfrak{e})}$ (see (1.27)) with $\mathfrak{d} := (\mathfrak{t}_{0,i_0} + \sum_{n=1}^{N-1} \sum_{i=1}^{\mathfrak{e}} \mathfrak{t}_{n,i})\mathfrak{e}$.

In the finite setting it is desirable to consider the classical Fréchet (or total) derivative $\dot{\mathcal{V}}_{\boldsymbol{p}}$ of a map $\mathcal{V} : \widetilde{\mathcal{P}} \to \mathbb{R}$ at $\boldsymbol{p}$ in order to obtain a tool for measuring the first-order sensitivity of $\mathcal{V}$ w.r.t. a change from $\boldsymbol{p}$ to $(1-\varepsilon)\boldsymbol{p}+\varepsilon\boldsymbol{q}$:

$$\dot{\mathcal{V}}_{\boldsymbol{p}}(\boldsymbol{q}-\boldsymbol{p}) \;=\; \lim_{m\to\infty}\frac{\mathcal{V}(\boldsymbol{p}+h_m(\boldsymbol{q}-\boldsymbol{p}))-\mathcal{V}(\boldsymbol{p})}{h_m} \quad \text{uniformly in } \boldsymbol{q}\in\overline{B}_1(\boldsymbol{p}) \tag{2.26}$$

for any $(h_m)\in\mathbb{R}_0^{\mathbb{N}}$ with $h_m\to 0$, where $\overline{B}_1(\boldsymbol{p})$ is the closed ball in $\mathbb{R}^{\mathfrak{d}}$ around $\boldsymbol{p}$ with radius 1 and $\mathbb{R}_0 := \mathbb{R}\setminus\{0\}$. This approach is indeed expedient to some extent. However, one has to note that $\boldsymbol{p}+h_m(\boldsymbol{q}-\boldsymbol{p})$ may lie outside $\mathcal{V}$'s domain $\widetilde{\mathcal{P}}$. To avoid this problem, we replace condition (2.26) by the following variant of (2.26):

$$\dot{\mathcal{V}}_{\boldsymbol{p}}(\boldsymbol{q}-\boldsymbol{p}) = \lim_{m\to\infty}\frac{\mathcal{V}(\boldsymbol{p}+\varepsilon_m(\boldsymbol{q}-\boldsymbol{p}))-\mathcal{V}(\boldsymbol{p})}{\varepsilon_m} \quad \text{uniformly in } \boldsymbol{q}\in\widetilde{\mathcal{P}} \tag{2.27}$$

for any $(\varepsilon_m)\in(0,1]^{\mathbb{N}}$ with $\varepsilon_m\to 0$. Take into account that $\boldsymbol{p}+\varepsilon(\boldsymbol{q}-\boldsymbol{p})$ lies in $\widetilde{\mathcal{P}}$ for any $\boldsymbol{p},\boldsymbol{q}\in\widetilde{\mathcal{P}}$ and $\varepsilon\in(0,1]$. Also note that, if $\mathbb{R}^{\mathfrak{d}}$ is equipped with the max-norm, $\widetilde{\mathcal{P}}$ is contained in $\overline{B}_1(\boldsymbol{p})$ for any $\boldsymbol{p}\in\widetilde{\mathcal{P}}$.

For classical Fréchet (or total) differentiability the derivative $\dot{\mathcal{V}}_{\boldsymbol{p}}$ is required to be linear and continuous. On the one hand, for 'Fréchet differentiability' (see Definition 2.3.9 below) we will also require a sort of continuity, namely that the mapping $\boldsymbol{q}\mapsto\dot{\mathcal{V}}_{\boldsymbol{p}}(\boldsymbol{q}-\boldsymbol{p})$ from $\widetilde{\mathcal{P}}$ to $\mathbb{R}$ is continuous, where $\widetilde{\mathcal{P}}$ is equipped with the relative topology of $\mathbb{R}^{\mathfrak{d}}$. On the other hand, the domain of $\dot{\mathcal{V}}_{\boldsymbol{p}}$ is given by $\widetilde{\mathcal{P}}^{\boldsymbol{p};\pm} := \{\boldsymbol{q}-\boldsymbol{p} : \boldsymbol{q}\in\widetilde{\mathcal{P}}\}$ and thus not a linear space. Therefore linearity of $\dot{\mathcal{V}}_{\boldsymbol{p}}$ is an indefinite property.

Remark 2.3.8 Similarly to (2.1), the quantity $\dot{\mathcal{V}}_{\boldsymbol{p}}(\boldsymbol{q}-\boldsymbol{p})$ can be seen as a measure for the first-order sensitivity of the map $\mathcal{V}:\widetilde{\mathcal{P}}\to\mathbb{R}$ w.r.t. a change from $\boldsymbol{p}$ to $(1-\varepsilon)\boldsymbol{p}+\varepsilon\boldsymbol{q}$, with $\varepsilon>0$ small. For this interpretation it is actually not necessary to require that $\dot{\mathcal{V}}_{\boldsymbol{p}}(\,\cdot\,-\boldsymbol{p})$ is continuous or that the convergence in (2.27) holds uniformly in $\boldsymbol{q}\in\widetilde{\mathcal{P}}$. One can indeed be content with the directional derivative, i.e. with the convergence in (2.27) for fixed $\boldsymbol{q}$. Nevertheless continuity and uniformity are natural wishes in this context, because they ensure stability of the first-order sensitivity w.r.t. small modifications of $\boldsymbol{q}$ as well as comparability of the first-order sensitivity of (infinitely) many different $\boldsymbol{q}$. We refer to the discussion subsequent to (2.1). ◇

It follows from the above discussion that in the case of finite state space and finite action spaces Definition 2.3.9 below gives a suitable notion of 'differentiability'.

Definition 2.3.9 ('Fréchet differentiability' in $\boldsymbol{p}$) *Let $\boldsymbol{p}\in\widetilde{\mathcal{P}}$. A map $\mathcal{V}:\widetilde{\mathcal{P}}\to\mathbb{R}$ is said to be 'Fréchet differentiable' at $\boldsymbol{p}$ if there exists a map $\dot{\mathcal{V}}_{\boldsymbol{p}}:\widetilde{\mathcal{P}}^{\boldsymbol{p};\pm}\to\mathbb{R}$ for which (2.27) holds and for which the mapping $\boldsymbol{q}\mapsto\dot{\mathcal{V}}_{\boldsymbol{p}}(\boldsymbol{q}-\boldsymbol{p})$ from $\widetilde{\mathcal{P}}$ to $\mathbb{R}$ is continuous. In this case $\dot{\mathcal{V}}_{\boldsymbol{p}}$ is called 'Fréchet derivative' of $\mathcal{V}$ at $\boldsymbol{p}$.*

The following remark justifies that in the finite setting of Section 1.5 the concept of 'Fréchet differentiability' introduced in Definition 2.3.9 is only a special case of Definitions 2.3.2 and 2.3.5(b).

Remark 2.3.10 In the setting of Section 1.5, the notion of 'Hadamard differentiability' (w.r.t. $(\mathbb{M}_{\mathrm{TV}},\phi)$) as formulated in Definitions 2.3.2 and 2.3.5(b) boils down to the notion of 'Fréchet differentiability' introduced in Definition 2.3.9 for $L:=\mathbb{R}$ and $\phi:\equiv 1$, where $\mathbb{M}_{\mathrm{TV}}$ is defined as in Example 2.1.2.

Proof Since E is finite by assumption with $\mathfrak{e} = \#E \in \mathbb{N}$, it follows from the discussion in Subsection 2.1.3 that the distance between two probability measures $\mu, \nu \in \mathcal{M}_1(E)$ w.r.t. the total variation metric d_{TV} (see (2.4)) can be identified in view of (2.14) with the ℓ_1-distance between the elements p_μ and p_ν as defined in (2.13). That is, the map $\chi : \mathcal{M}_1(E) \to \mathbb{R}^{\mathfrak{e}}_{\ge 0,1/2}$, $\mu \mapsto p_\mu/2$, provides a surjective isometry (here $\mathbb{R}^{\mathfrak{e}}_{\ge 0,1/2}$ is the set of all vectors from $\mathbb{R}^{\mathfrak{e}}$ whose entries are nonnegative and sum up to $1/2$), and therefore the metric spaces $(\mathcal{M}_1(E), d_{\mathrm{TV}})$ and $(\mathbb{R}^{\mathfrak{e}}_{\ge 0,1/2}, \|\cdot\|_{\ell_1})$ are isometrically isomorphic. This implies in particular that the set $\mathcal{M}_1(E)$ is compact w.r.t. d_{TV}, because $\mathbb{R}^{\mathfrak{e}}_{\ge 0,1/2}$ is clearly compact w.r.t. $\|\cdot\|_{\ell_1}$.

Thus, since the metric $d^1_{\infty,\mathbb{M}_{\mathrm{TV}}}$ given by (2.12) (with $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$ and $\phi := \psi :\equiv 1$) obviously generates in view of (1.25) the product topology on $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}_1$ defined as in Subsection 2.1.2), it follows from Tychonoff's theorem (see, e.g., [32, Theorem 2.2.8]) that $\overline{\mathcal{P}}$ is compact w.r.t. $d^1_{\infty,\mathbb{M}_{\mathrm{TV}}}$ and therefore in particular relatively compact w.r.t. $d^1_{\infty,\mathbb{M}_{\mathrm{TV}}}$. Hence, Definition 2.3.5(b) of 'Hadamard differentiability' (i.e. Definition 2.3.2 with $\mathcal{S} := \mathcal{S}_{\mathrm{rc}}$) simplifies insofar as one can simply require that the convergence in (2.23) holds uniformly in *all* $\boldsymbol{Q} \in \overline{\mathcal{P}}$ for every sequence $(\varepsilon_m) \in (0,1]^{\mathbb{N}}$ with $\varepsilon_m \to 0$. As the metric $d^1_{\infty,\mathbb{M}_{\mathrm{TV}}}$ on $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ can be identified in finite setting with the metric d_{∞,ℓ_1} on $\widetilde{\mathcal{P}}$ given by (2.15), it is apparent that Definition 2.3.9 is a special case of Definition 2.3.2 with $\mathcal{S} := \mathcal{S}_{\mathrm{rc}}$, where $\mathcal{S}_{\mathrm{rc}}$ is defined as in part (b) of Definition 2.3.5. Take into account that in the finite setting 'Fréchet differentiability' and 'Hadamard differentiability' are equivalent. $\diamond$

2.3.2 'Differentiability' of the value functional

We consider again the general framework of Section 1.1, and recall that the components E, $\boldsymbol{A}$, Π, and $\boldsymbol{r}$ of the MDM are fixed. Let ψ be any gauge function, and fix some subset $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$ being closed under mixtures. Moreover let the functionals $\mathcal{V}_n^{x_n;\pi} : \mathcal{P}_\psi \to \mathbb{R}$ and $\mathcal{V}_n^{x_n} : \mathcal{P}_\psi \to \mathbb{R}$ be defined as in (2.16). Take into account that in view of Proposition 1.4.3 (applied to $\mathcal{P} := \{\boldsymbol{P}\}$) the latter functionals are well-defined under condition (a) of Assumption 2.2.5.

Part (ii) of Theorem 2.3.11 below provides (under Assumption 2.2.5) the 'Hadamard derivative' of the value functional $\mathcal{V}_n^{x_n}$ in the sense of Definitions 2.3.2 and 2.3.5(b). Note that in view of Remark 2.2.6 the conditions in Assumption 2.2.5 are not very restrictive. Recall from Definition 1.2.5 that for given $\boldsymbol{P} \in \mathcal{P}_\psi$ and $\delta > 0$ the sets $\Pi(\boldsymbol{P};\delta)$ and $\Pi(\boldsymbol{P})$ consist of all δ-optimal strategies w.r.t. $\boldsymbol{P}$ and of all optimal strategies w.r.t. $\boldsymbol{P}$, respectively.

Theorem 2.3.11 ('Differentiability' of $\mathcal{V}_n^{x_n;\pi}$ and $\mathcal{V}_n^{x_n}$ in $\boldsymbol{P}$) *Suppose that Assumption 2.2.5 holds for some $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$ and $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \mathcal{P}_\psi$. Then the following two assertions are valid.*

(i) *For any $x_n \in E$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$,d an $n = 0,\dots,N$, the map $\mathcal{V}_n^{x_n;\pi} : \mathcal{P}_\psi \to \mathbb{R}$ defined by (2.16) is 'Fréchet differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Fréchet derivative' $\dot{\mathcal{V}}^{x_n;\pi}_{n;\boldsymbol{P}} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to \mathbb{R}$ given by*

$$\begin{aligned}\dot{\mathcal{V}}^{x_n;\pi}_{n;\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \;:=\; & \sum_{k=n+1}^{N-1}\sum_{j=n}^{k-1}\int_E \cdots \int_E r_k(y_k, f_k(y_k))\, P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\ & \cdots (Q_j - P_j)\big((y_j, f_j(y_j)), dy_{j+1}\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big)\end{aligned}$$

$$+\sum_{j=n}^{N-1}\int_E\cdots\int_E r_N(y_N)\,P_{N-1}\big((y_{N-1},f_{N-1}(y_{N-1})),dy_N\big)$$
$$\cdots(Q_j-P_j)\big((y_j,f_j(y_j)),dy_{j+1}\big)\cdots P_n\big((x_n,f_n(x_n)),dy_{n+1}\big). \quad (2.28)$$

(ii) *For any $x_n \in E$ and $n = 0,\dots,N$, the map $\mathcal{V}_n^{x_n} : \mathcal{P}_\psi \to \mathbb{R}$ defined by (2.16) is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to \mathbb{R}$ given by*

$$\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q}-\boldsymbol{P}) := \lim_{\delta\searrow 0}\sup_{\pi\in\Pi(\boldsymbol{P};\delta)}\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q}-\boldsymbol{P}). \quad (2.29)$$

If the set of optimal strategies $\Pi(\boldsymbol{P})$ is non-empty, then the 'Hadamard derivative' admits the representation

$$\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q}-\boldsymbol{P}) = \sup_{\pi\in\Pi(\boldsymbol{P})}\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q}-\boldsymbol{P}) \quad \textit{for all } \boldsymbol{Q}\in\mathcal{P}_\psi. \quad (2.30)$$

Note that in part (ii) of Theorem 2.3.11 the set $\Pi(\boldsymbol{P};\delta)$ shrinks as δ decreases. Therefore the right-hand side of (2.29) is well-defined. The supremum in (2.30) ranges over all optimal strategies w.r.t. $\boldsymbol{P}$. The following Remark 2.3.12 discusses two settings in which at least one optimal strategy can easily be found. If there is even a *unique* optimal strategy $\pi^{\boldsymbol{P}} \in \Pi$ w.r.t. $\boldsymbol{P}$, then $\Pi(\boldsymbol{P})$ is nothing but the singleton $\{\pi^{\boldsymbol{P}}\}$, and in this case the 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ of the (optimal) value functional $\mathcal{V}_0^{x_0}$ at $\boldsymbol{P}$ coincides with $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi^{\boldsymbol{P}}}$. For an evidence of the existence of a unique optimal strategy, see, for instance, part (ii) of Theorem 3.2.5 in Subsection 3.2.3.

Remark 2.3.12 The existence of an optimal strategy is ensured, for instance, in the following two settings:

1) If the sets of all admissible decision rules $\mathbb{F}_0,\dots,\mathbb{F}_{N-1}$ are finite, then it follows from Proposition 4.4.3 in [73] that an optimal strategy can always be found. Note that this situation one often faces in practical applications; see, for instance, the example discussed in Section 3.1.

2) If the MDM $(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r})$ satisfies conditions (a)–(c) of Theorem 1.3.3, then by part (iii) of this theorem an optimal strategy can be found, i.e. $\Pi(\boldsymbol{P})$ is non-empty. For a verification of these conditions, see Theorem 3.2.5 in the example discussed in Section 3.2.

Note that in setting 1) the 'Hadamard derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q}-\boldsymbol{P})$ of $\mathcal{V}_n^{x_n}$ at $\boldsymbol{P}$ can easily be determined by computing the finitely many values $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q}-\boldsymbol{P})$, $\pi\in\Pi(\boldsymbol{P})$, and taking their maximum; see Corollary 2.3.21 ahead for details. ◇

Remark 2.3.13 (i) The 'Fréchet differentiability' in part (i) of Theorem 2.3.11 holds even uniformly in $\pi\in\Pi$; see Theorem 2.3.17 for the precise meaning.

(ii) We do not know if it is possible to replace 'Hadamard differentiability' by 'Fréchet differentiability' in part (ii) of Theorem 2.3.11. The following arguments rather cast doubt on this possibility. The proof of part (ii) is based on the decomposition of the value functional $\mathcal{V}_n^{x_n}$ in Display (2.35) ahead and a suitable chain rule, where the decomposition (2.35) involves the sup-functional Ψ introduced in (2.36) below. However, Corollary 1 in [27] (see also Proposition 4.6.5 in [81]) shows that in normed vector spaces sup-functionals are in general *not* Fréchet differentiable. This could be an indication that 'Fréchet differentiable' of the value functional indeed fails. We can not make a reliable statement in this regard.

(iii) Recall that 'Hadamard (resp. Fréchet) differentiability' w.r.t. $(\mathbb{M}, \psi)$ implies 'Hadamard (resp. Fréchet) differentiability' w.r.t. $(\mathbb{M}, \phi)$ for any gauge function $\phi \leq \psi$. However, for any such ϕ 'Hadamard (resp. Fréchet) differentiability' w.r.t. $(\mathbb{M}, \phi)$ is less meaningful than w.r.t. $(\mathbb{M}, \psi)$. Indeed, when using $d_{\infty,\mathbb{M}}^{\phi}$ with $\phi \leq \psi$ instead of $d_{\infty,\mathbb{M}}^{\psi}$, the sets $\mathcal{K}$ for whose elements the first-order sensitivities can be compared with each other with clear conscience are smaller and the 'derivative' is less robust.

(iv) In the case where we are interested in minimizing expected total costs in the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi$ $\boldsymbol{X}, \boldsymbol{r})$ (see Remark 1.2.6(ii)), we obtain under the assumptions (and with the same arguments as in the proof of part (ii)) of Theorem 2.3.11 that the 'Hadamard derivative' of the corresponding value functional is given by (2.29) (resp. (2.30)) with "sup" replaced by "inf". $\diamond$

In applications it is not necessarily easy to specify the set $\Pi(\boldsymbol{P})$ of all optimal strategies w.r.t. $\boldsymbol{P}$. While in most cases an optimal strategy can be found with little effort (one can use the Bellman equation; see part (i) of Theorem 1.3.3 in Section 1.3), it is typically more involved to specify *all* optimal strategies or to show that the optimal strategy is unique. The following remark may help in some situations; for an application see Subsection 3.2.4.

Remark 2.3.14 In some situations it turns out that for *every* $\boldsymbol{P} \in \mathcal{P}_\psi$ the solution of the optimization problem (1.12) does not change if Π is replaced by a subset $\Pi' \subseteq \Pi$ (being independent of $\boldsymbol{P}$). Then in the definition (1.13) of the value function (at time 0) the set Π can be replaced by the subset Π', and it follows (under the assumptions of Theorem 2.3.11)) that in the representation (2.30) of the 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ of $\mathcal{V}_0^{x_0}$ at $\boldsymbol{P}$ the set $\Pi(\boldsymbol{P})$ can be replaced by the set $\Pi'(\boldsymbol{P})$ of all optimal strategies w.r.t. $\boldsymbol{P}$ from the subset Π'. Of course, in this case it suffices to guarantee that conditions (a)–(b) of Assumption 2.2.5 hold for the subset Π' instead of Π. $\diamond$

The following two Remarks 2.3.15 and 2.3.16 give two alternative representations (see (2.31) and (2.32)) of the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}$ of $\mathcal{V}_n^{x_n;\pi}$ at $\boldsymbol{P}$ in (2.28). Display (2.31) provides a more compact representation compared to (2.28) and will be beneficial for the proof of Theorem 2.3.11 (see Lemma 2.3.18 below). Moreover the representation (2.32) offers a possibility to determine the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}$ recursively (which is of interest for practical purposes) and will be used to derive the 'Hadamard derivative' of the value functional of the terminal wealth problem in Display (3.20) (see the proof of Theorem 3.2.11 below).

Remark 2.3.15 (Representation I) By rearranging the sums in (2.28) and using the iteration scheme in Display (2.34) below, we obtain under the assumptions of Theorem 2.3.11 that for every fixed $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \mathcal{P}_\psi$ the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}$ of $\mathcal{V}_n^{x_n;\pi}$ at $\boldsymbol{P}$ can be represented as

$$\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P}) \quad = \quad \sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\, (Q_k - P_k)\big((y_k, f_k(y_k)), dy_{k+1}\big)$$

$$P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big) \qquad (2.31)$$

for every $x_n \in E$, $\boldsymbol{Q} = (Q_n)_{n=0}^{N-1} \in \mathcal{P}_\psi$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 0, \dots, N$. $\diamond$

Note that it follows from the discussion below of Proposition 1.3.1 in Section 1.3 that under condition (a) of Assumption 2.2.5 we may apply the iteration scheme (2.34) ahead to get the representation (2.31).

Remark 2.3.16 (Representation II) For every fixed $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \mathcal{P}_\psi$, and under the assumptions of Theorem 2.3.11, the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}$ of $\mathcal{V}_n^{x_n;\pi}$ at $\boldsymbol{P}$ admits the representation

$$\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P}) \;=\; \dot{V}_n^{\boldsymbol{P},\boldsymbol{Q};\pi}(x_n) \qquad (2.32)$$

for every $x_n \in E$, $\boldsymbol{Q} = (Q_n)_{n=0}^{N-1} \in \mathcal{P}_\psi$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 0, \dots, N$, where $(\dot{V}_k^{\boldsymbol{P},\boldsymbol{Q};\pi})_{k=0}^{N}$ is the solution of the following *backward iteration scheme*

$$\begin{aligned} \dot{V}_N^{\boldsymbol{P},\boldsymbol{Q};\pi}(\cdot) \;&:=\; 0, \\ \dot{V}_k^{\boldsymbol{P},\boldsymbol{Q};\pi}(\cdot) \;&:=\; \int_E \dot{V}_{k+1}^{\boldsymbol{P},\boldsymbol{Q};\pi}(y)\, P_k\big((\,\cdot\,, f_k(\cdot)), dy\big) \\ &\quad\; + \int_E V_{k+1}^{\boldsymbol{P};\pi}(y)\,(Q_k - P_k)\big((\,\cdot\,, f_k(\cdot)), dy\big), \qquad k = 0, \dots, N-1. \end{aligned} \qquad (2.33)$$

Indeed, it is easily seen that $\dot{V}_n^{\boldsymbol{P},\boldsymbol{Q};\pi}(x_n)$ coincides with the right-hand side of (2.31). Note that it can be verified iteratively by means of condition (a) of Assumption 2.2.5 and Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{Q}\}$) that $\dot{V}_n^{\boldsymbol{P},\boldsymbol{Q};\pi}(\cdot) \in \mathbb{M}_\psi(E)$ for every $\boldsymbol{Q} \in \mathcal{P}_\psi$, $\pi \in \Pi$, and $n = 0, \dots, N$. In particular, this implies that the integrals on the right-hand side of (2.33) exist and are finite. Also note that the iteration scheme (2.33) involves the family $(V_k^{\boldsymbol{P};\pi})_{k=1}^{N}$ which itself can be seen as the solution of a backward iteration scheme:

$$\begin{aligned} V_N^{\boldsymbol{P};\pi}(\cdot) \;&:=\; r_N(\cdot), \\ V_k^{\boldsymbol{P};\pi}(\cdot) \;&:=\; r_k(\,\cdot\,, f_k(\cdot)) + \int_E V_{k+1}^{\boldsymbol{P};\pi}(y)\, P_k\big((\,\cdot\,, f_k(\cdot)), dy\big), \qquad k = 1, \dots, N-1; \end{aligned} \qquad (2.34)$$

see Proposition 1.3.1 in Section 1.3. $\diamond$

Now, let us turn to the proof of Theorem 2.3.11. In virtue of condition (a) of Assumption 2.2.5, the value functional $\mathcal{V}_n^{x_n}$ introduced in (2.16) admits for any $x_n \in E$ and $n = 0, \dots, N$ the representation

$$\mathcal{V}_n^{x_n} \;=\; \Psi \circ \Upsilon_n^{x_n} \qquad (2.35)$$

with maps $\Upsilon_n^{x_n} : \mathcal{P}_\psi \to \ell^\infty(\Pi)$ and $\Psi : \ell^\infty(\Pi) \to \mathbb{R}$ defined by

$$\Upsilon_n^{x_n}(\boldsymbol{P}) \;:=\; \big(\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P})\big)_{\pi\in\Pi} \quad \text{and} \quad \Psi\big((w(\pi))_{\pi\in\Pi}\big) \;:=\; \sup_{\pi\in\Pi} w(\pi), \qquad (2.36)$$

where $\ell^\infty(\Pi)$ stands for the space of all bounded real-valued functions on Π equipped with the sup-norm $\|\cdot\|_\infty$. It is easily seen that condition (a) of Assumption 2.2.5 along with Proposition 1.4.3

ensure that the map $\Upsilon_n^{x_n}$ is well-defined for any $x_n \in E$ and $n = 0, \dots, N$, i.e. that $(\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}))_{\pi\in\Pi} \in \ell^\infty(\Pi)$ for any $x_n \in E$, $\boldsymbol{P} \in \mathcal{P}_\psi$, and $n = 0, \dots, N$.

Theorem 2.3.17 below shows that under the assumptions of Theorem 2.3.11 and for any $x_n \in E$ and $n = 0, \dots, N$ the map $\Upsilon_n^{x_n}$ is 'Fréchet differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}, \psi)$ (in the sense of Definitions 2.3.2 and 2.3.5(c)) with 'Fréchet derivative' $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to \ell^\infty(\Pi)$ given by

$$\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q} - \boldsymbol{P}) := \big(\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P})\big)_{\pi\in\Pi}. \tag{2.37}$$

Note that the well-definiteness of $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}$ is ensured by condition (a) of Assumption 2.2.5 along with Definition 1.4.1. Together with the Hadamard differentiability of the map Ψ (which is known from [75]), we will see later that this implies assertion (ii) of Theorem 2.3.11. We emphasize that the claim in part (i) of Theorem 2.3.11 is an immediate consequence of the following Theorem 2.3.17.

Theorem 2.3.17 *Suppose that Assumption 2.2.5 holds for some $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$ and $\boldsymbol{P} \in \mathcal{P}_\psi$. Then for any $x_n \in E$ and $n = 0, \dots, N$, the map $\Upsilon_n^{x_n} : \mathcal{P}_\psi \to \ell^\infty(\Pi)$ defined by (2.36) is 'Fréchet differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}, \psi)$ with 'Fréchet derivative' $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to \ell^\infty(\Pi)$ given by (2.37).*

The statement of Theorem 2.3.17 is a direct consequence of Lemmas 2.3.18 and 2.3.20 ahead.

Lemma 2.3.18 *Under the assumptions of Theorem 2.3.17 (except condition (c) of Assumption 2.2.5) and for any fixed $x_n \in E$ and $n = 0, \dots, N$, the map $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to \ell^\infty(\Pi)$ given by (2.37) is $(\mathbb{M}, \psi)$-continuous.*

Proof Let $(\boldsymbol{Q}_m)_{m\in\mathbb{N}}$ be any sequence in $\mathcal{P}_\psi$ with $d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{Q}_m, \boldsymbol{Q}) \to 0$ for some $\boldsymbol{Q} \in \mathcal{P}_\psi$. Using the representation (2.31), we obtain for any $m \in \mathbb{N}$

$$\begin{aligned}
&\|\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q}_m - \boldsymbol{P}) - \dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q} - \boldsymbol{P})\|_\infty \\
&= \sup_{\pi\in\Pi} \big|\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q}_m - \boldsymbol{P}) - \dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P})\big| \\
&= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{\Big|\sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(Q_{m;k} - P_k)\big((y_k, f_k(y_k)), dy_{k+1}\big) \\
&\qquad P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
&\qquad - \sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(Q_k - P_k)\big((y_k, f_k(y_k)), dy_{k+1}\big) \\
&\qquad P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big|\Big\} \\
&= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{\Big|\sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(Q_{m;k} - Q_k)\big((y_k, f_k(y_k)), dy_{k+1}\big) \\
&\qquad P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big|\Big\} \\
&\le \sum_{k=n}^{N-1} \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{\int_E \cdots \int_E \Big|\int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(Q_{m;k} - Q_k)\big((y_k, f_k(y_k)), dy_{k+1}\big)\Big| \\
&\qquad P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\}.
\end{aligned}$$

Analogously to (2.19)–(2.20), we observe for any $k = n, \dots, N-1$ and $m \in \mathbb{N}$

$$
\begin{aligned}
&\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \Big| \int_E V_{k+1}^{\boldsymbol{P};\pi}(y_{k+1})\,(Q_{m;k}-Q_k)\big((y_k, f_k(y_k)), dy_{k+1}\big)\Big| \\
&\qquad P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\} \\
&\leq\; d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{Q}_m, \boldsymbol{Q}) \cdot \sup_{\pi\in\Pi} \rho_{\mathbb{M}'}\big(V_{k+1}^{\boldsymbol{P};\pi}\big) \cdot \sup_{\pi\in\Pi} \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[\psi(X_k)\big].
\end{aligned}
\tag{2.38}
$$

The second factor in the last line of formula Display (2.38) is (independent of m and) finite due to condition (b) of Assumption 2.2.5. Moreover, the finiteness of the third factor in the last line of (2.38) (which is also independent of m) follows from Lemma 1.4.4 along with condition (a) of Assumption 2.2.5. Therefore, we arrive at $\|\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q}_m - \boldsymbol{P}) - \dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q} - \boldsymbol{P})\|_\infty \to 0$ as $m \to \infty$. □

Recall Definition 1.4.1 for the following lemma. Bounded sets in the (semi-) metric space $(\mathcal{P}_\psi, d_{\infty,\mathbb{M}}^{\psi})$ were introduced above of Definition 2.3.5.

Lemma 2.3.19 *Under the assumptions of Theorem 2.3.17 (except condition (b) of Assumption 2.2.5) let $\mathcal{K} \subseteq \mathcal{P}_\psi$ be a bounded set (w.r.t. $d_{\infty,\mathbb{M}}^{\psi}$). Then ψ is a bounding function for the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{Q}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{Q} \in \mathcal{K}\}$.*

Proof Note at first that conditions (a) and (b) of Definition 1.4.1 (which are independent of any transition function) are satisfied due to condition (a) of Assumption 2.2.5. Thus it suffices for the claim to show that condition (c) of Definition 1.4.1 is satisfied for any bounded set $\mathcal{K}$ which plays the role of $\mathcal{P}$ there. For any bounded set $\mathcal{K}$ we can find by definition some $\boldsymbol{P'} = (P'_n)_{n=0}^{N-1} \in \mathcal{P}_\psi$ and $\delta > 0$ such that

$$
d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{Q}, \boldsymbol{P'}) \leq \delta \qquad \text{for every } \boldsymbol{Q} \in \mathcal{K}. \tag{2.39}
$$

Letting $K_3 > 0$ denote the finite constant in condition (c) of Definition 1.4.1 for the singleton $\mathcal{P} := \{\boldsymbol{P'}\}$, and using (2.18), (2.39) as well as condition (c) of Assumption 2.2.5, we obtain for any $(x, a) \in D_n$, $\boldsymbol{Q} = (Q_n)_{n=0}^{N-1} \in \mathcal{K}$, and $n = 0, \dots, N-1$

$$
\begin{aligned}
\int_E \psi(y)\, Q_n\big((x,a), dy\big) \;&\leq\; \Big| \int_E \psi(y)\,(Q_n - P'_n)\big((x,a), dy\big)\Big| + \int_E \psi(y)\, P'_n\big((x,a), dy\big) \\
&\leq\; \rho_{\mathbb{M}'}(\psi) \cdot \frac{1}{\psi(x)}\, d_{\mathbb{M}}\Big(Q_n\big((x,a), \bullet\big), P'_n\big((x,a), \bullet\big)\Big) \cdot \psi(x) + K_3\psi(x) \\
&\leq\; \rho_{\mathbb{M}'}(\psi) \cdot d_{\infty,\mathbb{M}}^{\psi}(\boldsymbol{Q}, \boldsymbol{P'}) \cdot \psi(x) + K_3\psi(x) \;\leq\; \widetilde{K}_3\psi(x)
\end{aligned}
$$

for $\widetilde{K}_3 := \rho_{\mathbb{M}'}(\psi) \cdot \delta + K_3$, because $\psi \in \mathbb{M}_\psi(E)$. Thus condition (c) of Definition 1.4.1 holds for $\mathcal{P} := \mathcal{K}$. □

Lemma 2.3.20 *Under the assumptions of Theorem 2.3.17 and for any fixed $x_n \in E$ and $n = 0, \dots, N$,*

$$
\lim_{m\to\infty} \Big\| \frac{\Upsilon_n^{x_n}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q} - \boldsymbol{P})) - \Upsilon_n^{x_n}(\boldsymbol{P})}{\varepsilon_m} - \dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q} - \boldsymbol{P}) \Big\|_\infty = 0 \quad \textit{uniformly in } \boldsymbol{Q} \in \mathcal{K}
$$

for every bounded set $\mathcal{K} \subseteq \mathcal{P}_\psi$ and every sequence $(\varepsilon_m) \in (0,1]^{\mathbb{N}}$ with $\varepsilon_m \to 0$.

Proof Let $\mathcal{K} \subseteq \mathcal{P}_\psi$ be a fixed bounded set and $(\varepsilon_m) \in (0,1]^{\mathbb{N}}$ such that $\varepsilon_m \to 0$. First of all, note that it can be verified easily by means of condition (a) of Assumption 2.2.5 and Lemma 2.3.19 that $\Upsilon_n^{x_n}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q} - \boldsymbol{P}))\, (= (\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q} - \boldsymbol{P})))_{\pi\in\Pi}) \in \ell^\infty(\Pi)$ as well as $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q} - \boldsymbol{P})\, (= (\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P}))_{\pi\in\Pi}) \in \ell^\infty(\Pi)$ for any $m \in \mathbb{N}$ and $\boldsymbol{Q} \in \mathcal{K}$. In view of Lemma 1.4.4, we get for any $m \in \mathbb{N}$, $\boldsymbol{Q} = (Q_n)_{n=0}^{N-1} \in \mathcal{K}$, and $\pi = (f_n)_{n=0}^{N-1} \in \Pi$

$$
\begin{aligned}
&\Big|\frac{\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P} + \varepsilon_m(\boldsymbol{Q} - \boldsymbol{P})) - \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P})}{\varepsilon_m} - \dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P})\Big| \\
&= \Big|\frac{1}{\varepsilon_m}\sum_{k=n}^{N-1}\Big(\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}-\boldsymbol{P});\pi}\big[r_k(X_k, f_k(X_k))\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[r_k(X_k, f_k(X_k))\big]\Big) \\
&\qquad + \frac{1}{\varepsilon_m}\Big(\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}-\boldsymbol{P});\pi}\big[r_N(X_N)\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}\big[r_N(X_N)\big]\Big) - \dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P})\Big| \\
&= \Big|\sum_{k=n+1}^{N-1}\sum_{j=n}^{k-1}\int_E\cdots\int_E r_k(y_k, f_k(y_k))\, P_{k-1}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad \cdots (Q_j - P_j)\big((y_j, f_j(y_j)), dy_{j+1}\big)\cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
&\quad + \frac{1}{\varepsilon_m}\sum_{k=n+2}^{N-1}\sum_{\substack{J\subseteq\{n,\ldots,k-1\}\\ 1<|J|\le k-n}}\varepsilon_m^{|J|}\int_E\int_E\cdots\int_E r_k(y_k, f_k(y_k))\,\xi_{k-1,J}^{\boldsymbol{Q}}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad \cdots \xi_{n+1,J}^{\boldsymbol{Q}}\big((y_{n+1}, f_{n+1}(y_{n+1})), dy_{n+2}\big)\,\xi_{n,J}^{\boldsymbol{Q}}\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
&\quad + \sum_{j=n}^{N-1}\int_E\int_E\cdots\int_E r_N(y_N)\, P_{N-1}\big((y_{N-1}, f_{N-1}(y_{N-1})), dy_N\big) \\
&\qquad \cdots (Q_j - P_j)\big((y_j, f_j(y_j)), dy_{j+1}\big)\cdots P_n\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
&\quad + \frac{1}{\varepsilon_m}\sum_{\substack{J\subseteq\{n,\ldots,N-1\}\\ 1<|J|\le N-n}}\varepsilon_m^{|J|}\int_E\int_E\cdots\int_E r_N(y_N)\,\xi_{N-1,J}^{\boldsymbol{Q}}\big((y_{N-1}, f_{N-1}(y_{N-1})), dy_N\big) \\
&\qquad \cdots \xi_{n+1,J}^{\boldsymbol{Q}}\big((y_{n+1}, f_{n+1}(y_{n+1})), dy_{n+2}\big)\,\xi_{n,J}^{\boldsymbol{Q}}\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
&\quad - \dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P})\Big| \\
&\le \big|\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P}) - \dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q} - \boldsymbol{P})\big| \\
&\quad + \sum_{k=n+2}^{N-1}\sum_{\substack{J\subseteq\{n,\ldots,k-1\}\\ 1<|J|\le k-n}}\varepsilon_m^{|J|-1}\Big|\int_E\int_E\cdots\int_E r_k(y_k, f_k(y_k))\,\xi_{k-1,J}^{\boldsymbol{Q}}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad \cdots \xi_{n+1,J}^{\boldsymbol{Q}}\big((y_{n+1}, f_{n+1}(y_{n+1})), dy_{n+2}\big)\,\xi_{n,J}^{\boldsymbol{Q}}\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big| \\
&\quad + \sum_{\substack{J\subseteq\{n,\ldots,N-1\}\\ 1<|J|\le N-n}}\varepsilon_m^{|J|-1}\Big|\int_E\int_E\cdots\int_E r_N(y_N)\,\xi_{N-1,J}^{\boldsymbol{Q}}\big((y_{N-1}, f_{N-1}(y_{N-1})), dy_N\big) \\
&\qquad \cdots \xi_{n+1,J}^{\boldsymbol{Q}}\big((y_{n+1}, f_{n+1}(y_{n+1})), dy_{n+2}\big)\,\xi_{n,J}^{\boldsymbol{Q}}\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big| \\
&=: S_1(\boldsymbol{Q}, \pi) + S_2(m, \boldsymbol{Q}, \pi) + S_3(m, \boldsymbol{Q}, \pi),
\end{aligned}
$$

where $S_1(\boldsymbol{Q},\pi)=0$ and $\xi_{j,J}^{\boldsymbol{Q}}$ is for any subset $J\subseteq\{0,\dots,N-1\}$ given by

$$\xi_{j,J}^{\boldsymbol{Q}} := \begin{cases} Q_j - P_j & , \quad j\in J \\ P_j & , \quad \text{otherwise} \end{cases}.$$

In view of condition (a) of Assumption 2.2.5 and Lemma 2.3.19 we find finite constants $K_1, K_3, \widetilde{K}_3 > 0$ such that for every $m\in\mathbb{N}$, $\boldsymbol{Q}\in\mathcal{K}$, and $\pi\in\Pi$

$$S_2(m,\boldsymbol{Q},\pi) \;\le\; \varepsilon_m\cdot\left\{K_1\sum_{k=n+2}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1<|J|\le k-n}}\varepsilon_m^{|J|-2}\binom{k-n}{|J|}K_3^{k-n-|J|}\cdot\sum_{l=0}^{|J|}\binom{|J|}{l}K_3^l\widetilde{K}_3^{|J|-l}\psi(x_n)\right\}.$$

Hence $\lim_{m\to\infty}S_2(m,\boldsymbol{Q},\pi)=0$ uniformly in $\boldsymbol{Q}\in\mathcal{K}$ and $\pi\in\Pi$. Analogously we find some finite constant $K_2>0$ such that

$$S_3(m,\boldsymbol{Q},\pi) \;\le\; \varepsilon_m\cdot\left\{K_2\sum_{\substack{J\subseteq\{n,\dots,N-1\}\\ 1<|J|\le N-n}}\varepsilon_m^{|J|-2}\binom{N-n}{|J|}K_3^{N-n-|J|}\cdot\sum_{l=0}^{|J|}\binom{|J|}{l}K_3^l\widetilde{K}_3^{|J|-l}\psi(x_n)\right\}$$

for every $m\in\mathbb{N}$, $\boldsymbol{Q}\in\mathcal{K}$, and $\pi\in\Pi$, and thus $\lim_{m\to\infty}S_3(m,\boldsymbol{Q},\pi)=0$ uniformly in $\boldsymbol{Q}\in\mathcal{K}$ and $\pi\in\Pi$. Hence, the assertion follows. □

So far we have shown that the claim in part (i) of Theorem 2.3.11 holds. In the following we will prove the assertion in part (ii) of Theorem 2.3.11, that is, we intend to show that the value functional $\mathcal{V}_n^{x_n}$ is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n}$ given by (2.29).

Proof of part (ii) of Theorem 2.3.11: The key of the proof will be (2.35) which says that $\mathcal{V}_n^{x_n}$ can be represented as a composition of the functionals Ψ and $\Upsilon_n^{x_n}$ defined in (2.36). Proposition 1 in [75] ensures that Ψ is Hadamard differentiable (in the sense of [75]) at every $(w(\pi))_{\pi\in\Pi}\in\ell^\infty(\Pi)$ with (possibly nonlinear) Hadamard derivative $\dot{\Psi}_{(w(\pi))_{\pi\in\Pi}}:\ell^\infty(\Pi)\to\mathbb{R}$ given by

$$\dot{\Psi}_{(w(\pi))_{\pi\in\Pi}}\big((z(\pi))_{\pi\in\Pi}\big) \;:=\; \lim_{\delta\searrow 0}\sup_{\pi\in\Pi((w(\pi))_{\pi\in\Pi};\delta)} z(\pi), \tag{2.40}$$

where $\Pi((w(\pi))_{\pi\in\Pi};\delta)$ denotes the set of all $\pi\in\Pi$ for which $\sup_{\sigma\in\Pi}w(\sigma)-\delta\le w(\pi)$. Moreover Theorem 2.3.17 implies that $\Upsilon_n^{x_n}$ is in particular 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Hadamard derivative' $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}$ given by (2.37).

In view of (2.35) and the shape of $\dot{\Psi}_{(w(\pi))_{\pi\in\Pi}}$ and $\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}$, 'Hadamard differentiability' of $\mathcal{V}_n^{x_n}$ at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n}$ given by (2.29) (resp. (2.30)) can be identified with 'Hadamard differentiability' of the map $\Psi\circ\Upsilon_n^{x_n}:\mathcal{P}_\psi\to\mathbb{R}$ at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Hadamard derivative' $(\Psi\circ\dot{\Upsilon}_n^{x_n})_{\boldsymbol{P}}:\mathcal{P}_\psi^{\boldsymbol{P};\pm}\to\mathbb{R}$ given by

$$\big(\Psi\circ\dot{\Upsilon}_n^{x_n}\big)_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \;:=\; \dot{\Psi}_{\Upsilon_n^{x_n}(\boldsymbol{P})}\circ\dot{\Upsilon}_{n;\boldsymbol{P}}^{x_n}(\boldsymbol{Q}-\boldsymbol{P}). \tag{2.41}$$

Take into account that by (2.37) and (2.40)

$$\big(\Psi\circ\dot{\Upsilon}_n^{x_n}\big)_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \;=\; \dot{\Psi}_{(\mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}))_{\pi\in\Pi}}\big((\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}(\boldsymbol{Q}-\boldsymbol{P}))_{\pi\in\Pi}\big)$$

$$= \lim_{\delta\searrow 0} \sup_{\pi\in\Pi(\boldsymbol{P};\delta)} \dot{\mathcal{V}}^{x_n;\pi}_{n;\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P})$$

for all $\boldsymbol{Q} \in \mathcal{P}_\psi$, and that, if in addition the set $\Pi(\boldsymbol{P})$ is non-empty,

$$\big(\dot{\Psi \circ \Upsilon^{x_n}_n}\big)_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \,=\, \sup_{\pi\in\Pi(\boldsymbol{P})} \dot{\mathcal{V}}^{x_n;\pi}_{n;\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P})$$

for every $\boldsymbol{Q} \in \mathcal{P}_\psi$.

In the remainder of the proof we will show that the composite map $\Psi \circ \Upsilon^{x_n}_n$ is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M},\psi)$ with 'Hadamard derivative' $(\dot{\Psi \circ \Upsilon^{x_n}_n})_{\boldsymbol{P}}$ given by (2.41). We first note that the map $(\dot{\Psi \circ \Upsilon^{x_n}_n})_{\boldsymbol{P}}$ is $(\mathbb{M},\psi)$-continuous by Lemma 2.3.18 and the $(\|\cdot\|_\infty, |\cdot|)$-continuity of the mapping $(z(\pi))_{\pi\in\Pi} \mapsto \dot{\Psi}_{\Upsilon^{x_n}_n(\boldsymbol{P})}((z(\pi))_{\pi\in\Pi})$. In view of part (ii) of Lemma 2.3.6, for the desired 'Hadamard differentiability' of $\Psi \circ \Upsilon^{x_n}_n$ at $\boldsymbol{P}$ it therefore suffices to show that

$$\lim_{m\to\infty} \Big| \frac{\Psi \circ \Upsilon^{x_n}_n(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P})) - \Psi \circ \Upsilon^{x_n}_n(\boldsymbol{P})}{\varepsilon_m} - \big(\dot{\Psi \circ \Upsilon^{x_n}_n}\big)_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \Big| \,=\, 0$$

for any fixed triplet $(\boldsymbol{Q},(\boldsymbol{Q}_m),(\varepsilon_m)) \in \mathcal{P}_\psi \times \mathcal{P}_\psi^{\mathbb{N}} \times (0,1]^{\mathbb{N}}$ with $d^{\psi}_{\infty,\mathbb{M}}(\boldsymbol{Q}_m,\boldsymbol{Q}) \to 0$ and $\varepsilon_m \to 0$. For any such fixed triplet and any $m \in \mathbb{N}$ we have

$$\frac{\Psi \circ \Upsilon^{x_n}_n(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P})) - \Psi \circ \Upsilon^{x_n}_n(\boldsymbol{P})}{\varepsilon_m} \,=\, \frac{\Psi(\Upsilon^{x_n}_n(\boldsymbol{P})+\varepsilon_m v_m) - \Psi(\Upsilon^{x_n}_n(\boldsymbol{P}))}{\varepsilon_m},$$

where $v_m := \varepsilon_m^{-1}(\Upsilon^{x_n}_n(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P})) - \Upsilon^{x_n}_n(\boldsymbol{P}))\,(\in \ell^\infty(\Pi))$. If we set $v := \dot{\Upsilon}^{x_n}_{n;\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P})\,(\in \ell^\infty(\Pi))$, then by Theorem 2.3.17 and part (i) of Lemma 2.3.6

$$\lim_{m\to\infty} \|v_m - v\|_\infty \,=\, \lim_{m\to\infty} \Big\| \frac{\Upsilon^{x_n}_n(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P})) - \Upsilon^{x_n}_n(\boldsymbol{P})}{\varepsilon_m} - \dot{\Upsilon}^{x_n}_{n;\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \Big\|_\infty \,=\, 0.$$

Thus, since Ψ is Hadamard differentiable at (in particular) $\Upsilon^{x_n}_n(\boldsymbol{P})\,(\in \ell^\infty(\Pi))$ (see the discussion above), we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \Big| \frac{\Psi \circ \Upsilon^{x_n}_n(\boldsymbol{P}+\varepsilon_m(\boldsymbol{Q}_m-\boldsymbol{P})) - \Psi \circ \Upsilon^{x_n}_n(\boldsymbol{P})}{\varepsilon_m} - \big(\dot{\Psi \circ \Upsilon^{x_n}_n}\big)_{\boldsymbol{P}}(\boldsymbol{Q}-\boldsymbol{P}) \Big| \\
&\quad = \lim_{m\to\infty} \Big| \frac{\Psi(\Upsilon^{x_n}_n(\boldsymbol{P})+\varepsilon_m v_m) - \Psi(\Upsilon^{x_n}_n(\boldsymbol{P}))}{\varepsilon_m} - \dot{\Psi}_{\Upsilon^{x_n}_n(\boldsymbol{P})}(v) \Big| \,=\, 0.
\end{aligned}$$

This finishes the proof of part (ii) of Theorem 2.3.11. □

The following Corollary 2.3.21 presents the statements in Theorem 2.3.11 in the special case of finite state space and finite action spaces. In this corollary we will use the notion of 'Fréchet differentiability' introduced in Definition 2.3.9. Let $\mathcal{V}^{x_i;\pi}_n$ and $\mathcal{V}^{x_i}_n$ be the functionals defined as in (2.21). Recall from Section 1.5 that in the finite setting for given $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ (with $\boldsymbol{p}$ as in (1.26)) the set $\Pi(\boldsymbol{p})$ of all optimal strategies w.r.t. $\boldsymbol{p}$ is non-empty (and finite).

Corollary 2.3.21 ('Fréchet differentiability' of $\mathcal{V}^{x_i;\pi}_n$ and $\mathcal{V}^{x_i}_n$ in $\boldsymbol{p}$) *Let $\boldsymbol{p} \in \widetilde{\mathcal{P}}$. Then in the setting of Section 1.5 the following two assertions hold.*

(i) *For any* $i = 1, \ldots, \mathfrak{e}$, $\pi \in \Pi$, *and* $n = 0, \ldots, N$, *the map* $\mathcal{V}_n^{x_i;\pi} : \widetilde{\mathcal{P}} \to \mathbb{R}$ *defined by (2.21) is 'Fréchet differentiable' at* $\boldsymbol{p}$ *with 'Fréchet derivative'* $\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi} : \widetilde{\mathcal{P}}^{\boldsymbol{p};\pm} \to \mathbb{R}$ *given by*

$$\begin{aligned}\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi}(\boldsymbol{q}-\boldsymbol{p}) \;:=\; & \sum_{k=n+1}^{N-1}\sum_{j=n}^{k-1}\Big(\sum_{i_{n+1}=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}} r_k(x_{i_k}, f_k(x_{i_k}))\, p_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\\ & \cdots\big(q_{j,i_j;f_j(x_{i_j})}(i_{j+1}) - p_{j,i_j;f_j(x_{i_j})}(i_{j+1})\big)\cdots p_{n,i;f_n(x_i)}(i_{n+1})\Big)\\ & + \sum_{j=n}^{N-1}\Big(\sum_{i_{n+1}=1}^{\mathfrak{e}}\cdots\sum_{i_N=1}^{\mathfrak{e}} r_N(x_{i_N})\, p_{N-1,i_{N-1};f_{N-1}(x_{i_{N-1}})}(i_N)\\ & \cdots\big(q_{j,i_j;f_j(x_{i_j})}(i_{j+1}) - p_{j,i_j;f_j(x_{i_j})}(i_{j+1})\big)\cdots p_{n,i;f_n(x_i)}(i_{n+1})\Big).\end{aligned} \tag{2.42}$$

(ii) *For any* $i = 1, \ldots, \mathfrak{e}$ *and* $n = 0, \ldots, N$, *the map* $\mathcal{V}_n^{x_i} : \widetilde{\mathcal{P}} \to \mathbb{R}$ *defined by (2.21) is 'Fréchet differentiable' at* $\boldsymbol{p}$ *with 'Fréchet derivative'* $\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i} : \widetilde{\mathcal{P}}^{\boldsymbol{p};\pm} \to \mathbb{R}$ *given by*

$$\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i}(\boldsymbol{q}-\boldsymbol{p}) \;:=\; \max_{\pi\in\Pi(\boldsymbol{p})} \dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi}(\boldsymbol{q}-\boldsymbol{p}). \tag{2.43}$$

Proof At first, it follows from the discussion in Section 1.5 that in the finite setting any transition function $\boldsymbol{P}$ from the set $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}$ as in (1.25)) can be identified with a vector $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ (with $\widetilde{\mathcal{P}}$ as in (1.27)) given by (1.26). In the same section we discussed that the gauge function ψ given by (1.30) is in particular a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{Q}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for any $\boldsymbol{Q} \in \overline{\mathcal{P}}$. Therefore condition (a) of Assumption 2.2.5 is satisfied. According to Remark 2.2.6(ii), conditions (b) and (c) of Assumption 2.2.5 hold for $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$, $\mathbb{M}' := \overline{\mathbb{M}}_{\mathrm{TV}}$, and $\psi :\equiv 1$. Hence, in the finite setting, the conditions of Assumption 2.2.5 (with $\mathbb{M} := \mathbb{M}_{\mathrm{TV}}$, $\mathbb{M}' := \overline{\mathbb{M}}_{\mathrm{TV}}$, and $\psi :\equiv 1$) are always fulfilled.

Thus the assumptions of Theorem 2.3.11 hold, and an application of parts (i) and (ii) of the latter theorem entails that the assertions in parts (i) and (ii) hold, respectively. Take into account that in the finite setting in view of Remark 2.3.10 the notion of 'Hadamard differentiability' (w.r.t. $(\mathbb{M}_{\mathrm{TV}}, \psi)$) introduced in Definitions 2.3.2 and 2.3.5(b) boils down to the concept of 'Fréchet differentiability' from Definition 2.3.9. Also note that in the finite setting 'Fréchet differentiability' and 'Hadamard differentiability' are equivalent. □

We conclude this subsection with the following two Remarks 2.3.22 and 2.3.23 which are immediate consequences of Remarks 2.3.15 and 2.3.16 as well as (the proof of) Corollary 2.3.21, respectively.

Remark 2.3.22 For any fixed $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi}$ of $\mathcal{V}_n^{x_i}$ at $\boldsymbol{p}$ admits in the finite setting of Section 1.5 the representation

$$\begin{aligned}\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi}(\boldsymbol{q}-\boldsymbol{p}) \;=\; & \sum_{k=n}^{N-1}\Big(\sum_{i_{n+1}=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}}\sum_{i_{k+1}=1}^{\mathfrak{e}} V_{k+1}^{\boldsymbol{p};\pi}(x_{i_{k+1}})\,\big(q_{k,i_k;f_k(x_{i_k})}(i_{k+1}) - p_{k,i_k;f_k(x_{i_k})}(i_{k+1})\big)\\ & p_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\cdots p_{n,i;f_n(x_i)}(i_{n+1})\Big)\end{aligned}$$

for every $i = 1, \ldots, \mathfrak{e}$, $\boldsymbol{q} \in \widetilde{\mathcal{P}}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 0, \ldots, N$. Note that $V_n^{\boldsymbol{p};\pi}(\cdot)$ can be computed by the backward iteration scheme (1.29). ◇

Remark 2.3.23 For any fixed $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi}$ of $\mathcal{V}_n^{x_i}$ at $\boldsymbol{p}$ can be computed in the finite setting of Section 1.5 via a iteration scheme. Indeed, according to Remark 2.3.16, for every $i = 1, \ldots, \mathfrak{e}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 0, \ldots, N$, we have

$$\dot{\mathcal{V}}_{n;\boldsymbol{p}}^{x_i;\pi}(\boldsymbol{q} - \boldsymbol{p}) = \dot{V}_n^{\boldsymbol{p},\boldsymbol{q};\pi}(x_i)$$

for

$$\begin{aligned} \dot{V}_N^{\boldsymbol{p},\boldsymbol{q};\pi}(x_i) &:= 0, \\ \dot{V}_n^{\boldsymbol{p},\boldsymbol{q};\pi}(x_i) &:= \sum_{j=1}^{\mathfrak{e}} \dot{V}_{n+1}^{\boldsymbol{p},\boldsymbol{q};\pi}(x_j)\, p_{n,i;f_n(x_i)}(j) \\ &\qquad + \sum_{j=1}^{\mathfrak{e}} V_{n+1}^{\boldsymbol{p};\pi}(x_j) \left(q_{n,i;f_n(x_i)}(j) - p_{n,i;f_n(x_i)}(j) \right), \quad n = 0, \ldots, N-1, \end{aligned} \tag{2.44}$$

$i = 1, \ldots, \mathfrak{e}$, where the $V_n^{\boldsymbol{p};\pi}(\cdot)$ is given by the iteration scheme (1.29). ◇

Chapter 3

Examples of finite horizon discrete time Markov decision optimization problems

In this chapter we will apply the theory and results of Chapters 1–2 to the so-called stochastic inventory control problem and the terminal wealth problem. These two stochastic control problems can be seen as two examples of finite horizon discrete time Markov decision optimization problems. The terminal wealth problem in Section 3.2 motivates the general set-up chosen in Chapters 1–2, while the stochastic inventory control problem presented in Section 3.1 justifies the consideration of the special case of finite state space and finite action spaces in the framework of Chapters 1–2.

3.1 Stochastic inventory control problem

In this section we will consider an inventory control problem, which is a classical example in discrete dynamic optimization; see, e.g., [12, 38, 73] and references cited therein. At first, we introduce in Subsection 3.1.1 a (simple) inventory control model and formulate the corresponding inventory control problem. Thereafter, in Subsection 3.1.2, we will explain how the inventory control model can be embedded into the finite setting of Section 1.5. In Subsection 3.1.3 we show that the value functional of the inventory control problem is 'Lipschitz continuous' and 'Fréchet differentiable'. Finally, Subsection 3.1.4 is devoted to some numerical examples for the 'Fréchet derivative' of the value functional. As already motivated in the main introduction, we will illustrate in this subsection a situation where in the MDM the 'true' transition function is replaced by a less complex variant.

3.1.1 Basic inventory control model, and the target

Consider an N-period inventory control system (with $N \in \mathbb{N}$ fixed) where a supplier of a single product seeks optimal inventory management to meet random commodity demand in such a way that a measure of profit over a time horizon of N periods is maximized. For the formulation of the model, let $I_1, \ldots, I_N$ be $\mathbb{N}_0$-valued independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with known probability distributions $\mathfrak{m}_1, \ldots, \mathfrak{m}_N$. The random variable I_{n+1} can be seen as the random demand of the single product in the period between time n and time $n+1$. Denote by $\mathfrak{p}_{n+1} = (\mathfrak{p}_{n+1;k})_{k \in \mathbb{N}_0}$ the counting density of I_{n+1}, i.e.

$$\mathfrak{p}_{n+1;k} := \mathfrak{m}_{n+1}[\{k\}], \qquad k \in \mathbb{N}_0, \tag{3.1}$$

and note that $\mathfrak{p}_{n+1} \in \mathbb{R}^{\mathbb{N}_0}_{\geq 0,1}$. Here $\mathbb{R}^{\mathbb{N}_0}_{\geq 0,1}$ stands for the space of all real-valued sequences whose entries are nonnegative and sum up to 1. Let $\mathcal{F}_0$ be the trivial σ-algebra, and set $\mathcal{F}_n := \sigma(I_1, \dots, I_n)$, $n = 1, \dots, N$.

We suppose that within each period of time the available inventory level of the single product is restricted to K units (for some fixed $K \in \mathbb{N}$) and that there is no backlogging of unsatisfied demand at the end of each period. The latter means that if at the end of a period the demand exceeds the current inventory, then the whole inventory is sold and the surplus demand gets lost.

Given an initial inventory level $y_0 \in \{0, \dots, K\}$, the supplier intends to find optimal order quantities according to an order strategy to maximize some measure of profit. By *order strategy* we mean an $(\mathcal{F}_n)$-adapted $\{0, \dots, K\}$-valued stochastic process $\varphi = (\varphi_n)_{n=0}^{N-1}$, where φ_n specifies the amount of ordered units of the single product at the beginning of period n. Here we suppose that the delivery of any order occurs instantaneously. Since excess demand is lost by assumption, the corresponding *inventory (level) process* $Y^\varphi = (Y_0^\varphi, \dots, Y_N^\varphi)$ associated with $\varphi = (\varphi_n)_{n=0}^{N-1}$ is defined by

$$Y_0^\varphi := y_0 \quad \text{and} \quad Y_{n+1}^\varphi := Y_n^\varphi + \varphi_n - \min\{I_{n+1}, Y_n^\varphi + \varphi_n\}, \qquad n = 0, \dots, N-1. \tag{3.2}$$

Note that $\min\{I_{n+1}, Y_n^\varphi + \varphi_n\}$ corresponds to the amount of units of the single product sold in the period between time n and time $n+1$. Hence we refer to the process $Z^\varphi := (Z_0^\varphi, \dots, Z_N^\varphi)$ given by

$$Z_0^\varphi := 0 \quad \text{and} \quad Z_{n+1}^\varphi := \min\{I_{n+1}, Y_n^\varphi + \varphi_n\}, \qquad n = 0, \dots, N-1 \tag{3.3}$$

as *sales process* associated with $\varphi = (\varphi_n)_{n=0}^{N-1}$.

In view of (3.2) and since the inventory capacity is restricted to K units, we may and do identify any order strategy with an $(\mathcal{F}_n)$-adapted $\{0, \dots, K\}$-valued stochastic process $\varphi = (\varphi_n)_{n=0}^{N-1}$ satisfying $\varphi_0 \in \{0, \dots, K - y_0\}$ and $\varphi_n \in \{0, \dots, K - Y_n^\varphi\}$ for all $n = 1, \dots, N-1$. We restrict ourselves to *Markovian* order strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ which means that φ_n only depends on n and $(Y_n^\varphi, Z_n^\varphi)$. To put it another way, we suppose that for any $n = 0, \dots, N-1$ there is some map $f_n : \{0, \dots, K\}^2 \to \{0, \dots, K\}$ such that $\varphi_n = f_n(Y_n^\varphi, Z_n^\varphi)$. Hence, for given strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) the process $X^\varphi := (Y^\varphi, Z^\varphi)$ is a $\{0, \dots, K\}^2$-valued $(\mathcal{F}_n)$-Markov process whose one-step transition probability for the transition from state $x = (y, z) \in \{0, \dots, K\}^2$ at time $n \in \{0, \dots, N-1\}$ to state $x' = (y', z') \in \{0, \dots, K\}^2$ at time $n+1$ is given by

$$\mathfrak{m}_{n+1} \circ \eta_{(y, f_n(x))}^{-1}[\{z'\}] \cdot \mathbb{1}_{\{y' = y + f_n(x) - z'\}},$$

where

$$\eta_{(y, f_n(x))}(t) := \min\{t, y + f_n(x)\}, \qquad t \in \mathbb{N}_0. \tag{3.4}$$

The supplier's aim is to find an order strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) for which the expected total profit is maximized. Here the profit can be seen as the difference between the sales revenue and the costs for ordering and holding the single product. For the sake of simplicity, we suppose that the sales revenue as well as the ordering and holding costs are known and linear in each period. Hence, we are interested in those order strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) for which the expectation of

$$\sum_{k=0}^{N-1} \big\{u^{\mathsf{rev}}(Z_k^\varphi) - u^{\mathsf{ord}}(f_k(Y_k^\varphi, Z_k^\varphi)) - u^{\mathsf{hol}}(Y_k^\varphi, f_k(Y_k^\varphi, Z_k^\varphi))\big\} + \big\{u^{\mathsf{rev}}(Z_N^\varphi) - u^{\mathsf{hol}}(Y_N^\varphi, 0)\big\} \tag{3.5}$$

under $\mathbb{P}$ is maximized, where $u^{\mathsf{rev}}, u^{\mathsf{ord}} : \{0,\dots,K\} \to \mathbb{N}_0$ and $u^{\mathsf{hol}} : \{0,\dots,K\}^2 \to \mathbb{N}_0$ are for some fixed $s_{\mathsf{rev}}, c_{\mathsf{ord}}, c_{\mathsf{fix}}, c_{\mathsf{hol}} \in \mathbb{N}$ defined by

$$u^{\mathsf{rev}}(z) := s_{\mathsf{rev}} \cdot z, \quad u^{\mathsf{ord}}(a) := (c_{\mathsf{fix}} + c_{\mathsf{ord}} \cdot a)\mathbb{1}_{\{a>0\}}, \quad u^{\mathsf{hol}}(y,a) := c_{\mathsf{hol}} \cdot (y+a).$$

Note here that $s_{\mathsf{rev}}, c_{\mathsf{ord}}, c_{\mathsf{fix}}$, and c_{hol} denote the sales revenue, the ordering costs, the fixed ordering costs, and the holding costs per unit of the single product, respectively.

3.1.2 Markov decision model, and optimal order strategies

The setting introduced in Subsection 3.1.1 can be embedded into the setting of Section 1.5 as follows. Let

$$(E,\mathcal{E}) := \big(\{x_1,\dots,x_{\mathfrak{e}}\}, \mathfrak{P}(\{x_1,\dots,x_{\mathfrak{e}}\})\big)$$

with $\mathfrak{e} = (K+1)^2$ as well as

$$A_n(x_i) := \{a_{n,i;1},\dots,a_{n,i;\mathfrak{t}_{n,i}}\}$$

with $a_{n,i;k} := k-1$ and $\mathfrak{t}_{n,i} = \mathfrak{t}_i := K - y_i + 1$ for any $i = 1,\dots,\mathfrak{e}$ and $n = 0,\dots,N-1$. Here $\{x_1,\dots,x_{\mathfrak{e}}\}$ corresponds to the enumeration $x_1,\dots,x_{\mathfrak{e}}$ of $\{0,\dots,K\}^2$ given by $x_i = (y_i,z_i)$ with $y_i := \lceil i/(K+1)\rceil - 1$ and $z_i := i - (K+1)\lceil i/(K+1)\rceil + K$ (here $\lceil\cdot\rceil$ is the ceiling function), $i = 1,\dots,\mathfrak{e}$. Then $A_n = \{0,\dots,K\}$ for every $n = 0,\dots,N-1$. The set $\overline{\mathbb{F}}_n$ of all decision rules consists of maps $f_n : \{x_1,\dots,x_{\mathfrak{e}}\} \to \{0,\dots,K\}$ satisfying

$$f_n(x_i) \;=\; f_n(y_i,z_i) \in \{0,\dots,K-y_i\} \qquad \text{for all } i = 1,\dots,\mathfrak{e} \tag{3.6}$$

(in particular $\overline{\mathbb{F}}_n$ is independent of n). For any $n = 0,\dots,N-1$, let the set $\mathbb{F}_n$ of all admissible decision rules at time n be equal to $\overline{\mathbb{F}}_n$, and set $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$.

For any $i = 1,\dots,\mathfrak{e}$, $k = 1,\dots,\mathfrak{t}_i$, and $n = 0,\dots,N-1$, let the component $p_{n,i;a_{n,i;k}} = (p_{n,i;a_{n,i;k}}(1), \dots, p_{n,i;a_{n,i;k}}(\mathfrak{e}))$ of the vector $\boldsymbol{p}$ from (1.26) be given by

$$p_{n,i;a_{n,i;k}}(j) \;:=\; \mathfrak{m}_{n+1} \circ \eta^{-1}_{(y_i,a_{n,i;k})}[\{z_j\}] \cdot \mathbb{1}_{\{y_j = y_i + a_{n,i;k} - z_j\}}, \qquad j = 1,\dots,\mathfrak{e} \tag{3.7}$$

for some $\mathfrak{m}_{n+1} \in \mathcal{M}_1(\mathbb{R},\mathbb{N}_0)$, where the map $\eta_{(y_i,a_{n,i;k})} : \mathbb{N}_0 \to \{0,\dots,K\}$ is defined as in (3.4) and $\mathcal{M}_1(\mathbb{R},\mathbb{N}_0)$ refers to the set of all $\mu \in \mathcal{M}_1(\mathbb{R})$ satisfying $\mu[\mathbb{N}_0] = 1$. In virtue of (3.1) and (3.4) it is easily seen that the one-step transition probability in (3.7) can be represented as

$$p_{n,i;a_{n,i;k}}(j) \;=\; \eta^{\mathfrak{p}_{n+1}}_{(y,a_{n,i;k})}(z_j) \cdot \mathbb{1}_{\{y_j = y_i + a_{n,i;k} - z_j\}} \tag{3.8}$$

for any $i,j = 1,\dots,\mathfrak{e}$, $k = 1,\dots,\mathfrak{t}_i$, and $n = 0,\dots,N-1$, where $\mathfrak{p}_{n+1} = (\mathfrak{m}_{n+1}[\{k\}])_{k\in\mathbb{N}_0}$ and

$$\eta^{\mathfrak{p}_{n+1}}_{(y,a)}(z') \;:=\; \begin{cases} 0 & , \quad z' > y+a \\ \mathfrak{p}_{n+1;z'} & , \quad z' < y+a \\ \sum_{\ell=z'}^{\infty} \mathfrak{p}_{n+1;\ell} & , \quad z' = y+a \end{cases} . \tag{3.9}$$

Thus $\boldsymbol{p} \in \widetilde{\mathcal{P}}$, where the set $\widetilde{\mathcal{P}}$ is introduced in (1.27). Note that any element $\boldsymbol{p}$ of $\widetilde{\mathcal{P}}$ is generated via (3.8)–(3.9) by some N-tuple $\mathfrak{p} = (\mathfrak{p}_1,\dots,\mathfrak{p}_N)$ of counting densities $\mathfrak{p}_1,\dots,\mathfrak{p}_N$ on $\mathbb{N}_0$; here $\mathfrak{p}_1,\dots,\mathfrak{p}_N$ should be seen as the counting densities of $I_1,\dots,I_N$. The value in (3.7) should be seen as the probability of a transition from state (y_i, z_i) to state (y_j, z_j) in time between n and $n+1$ (this transition probability is even independent of z_i).

For any $i = 1, \ldots, \mathfrak{e}$ and $k = 1, \ldots, \mathfrak{t}_i$, set

$$\begin{aligned} r_0(x_i, a_{0,i;k}) &:= -u^{\mathsf{ord}}(a_{0,i;k}) - u^{\mathsf{hol}}(y_i, a_{0,i;k}), \\ r_n(x_i, a_{n,i;k}) &:= u^{\mathsf{rev}}(z_i) - u^{\mathsf{ord}}(a_{n,i;k}) - u^{\mathsf{hol}}(y_i, a_{n,i;k}),\ n = 1, \ldots, N-1, \\ r_N(x_i) &:= u^{\mathsf{rev}}(z_i) - u^{\mathsf{hol}}(y_i, 0). \end{aligned} \tag{3.10}$$

Then for every fixed $i_0 \in \{1, \ldots, \mathfrak{e}\}$ and $\boldsymbol{p} \in \widetilde{\mathcal{P}}$, the stochastic inventory control problem introduced at the very end of Subsection 3.1.1 reads as

$$V_0^{\boldsymbol{p};\pi}(x_{i_0}) \longrightarrow \max \text{ (in } \pi \in \Pi) ! \tag{3.11}$$

where $V_0^{\boldsymbol{p};\pi}(x_{i_0})$ is given by (1.29) along with (3.10) ($x_{i_0} \in E$ is the initial state). A strategy $\pi^{\boldsymbol{p}} \in \Pi$ is called an *optimal order strategy w.r.t.* $\boldsymbol{p}$ if it solves the maximization problem (3.11). Note that it follows from the discussion in Section 1.5 that there exists for every $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ an optimal order strategy $\pi^{\boldsymbol{p}} \in \Pi$ w.r.t. $\boldsymbol{p}$.

Recall that in the finite setting of Section 1.5 the gauge function $\psi :\equiv 1$ (see also (1.30)) provides in view of Remark 1.4.2(i) a bounding function for the family of MDMs $\{(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r}) : \boldsymbol{P} \in \overline{\mathcal{P}}\}$ (with $\overline{\mathcal{P}}$ given by (1.25)). Note here that $\boldsymbol{X}$ plays the role of the process $X^{\varphi} = (Y^{\varphi}, Z^{\varphi})$ introduced in (3.2)–(3.3), and that any order strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ may be identified with some $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ through $\varphi_n = f_n(Y_n^{\varphi}, Z_n^{\varphi})$. Also note that in the setting above any transition function $\boldsymbol{P}$ from $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}_1$ defined as in Subsection 2.1.2) can be identified with a vector $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ given by (1.26) whose components are of the form (3.7).

Remark 3.1.1 In the inventory control model introduced in Subsection 3.1.1 we only consider Markovian order strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ which may be identified with some $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ via $\varphi_n = f_n(Y_n^{\varphi}, Z_n^{\varphi})$. Of course, one could suppose that the decision rules of an order strategy π also depend on past actions and past values of the inventory process Y^{φ} and the sales process Z^{φ}. However, in view of Remark 1.2.6(i), the corresponding history-dependent order strategies would not improve the optimal value of the inventory control problem (3.11). $\diamond$

3.1.3 'Lipschitz continuity' and 'Fréchet differentiability' of the value functional

Maintain the notation and terminology introduced in Subsections 3.1.1–3.1.2. In this subsection we will show that the value function of the inventory control problem (3.11) regarded as a real-valued functional is 'Lipschitz continuous' as well as 'Fréchet differentiable' at (fixed) $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ (with $\boldsymbol{p}$ as in Subsection 3.1.2); see part (ii) of Theorems 3.1.2 and 3.1.3 below.

Since the setting of Subsections 3.1.1–3.1.2 matches the finite setting of Subsection 2.1.3, we may use the metric d_{∞,ℓ_1} defined in (2.15) to measure the distance between transition functions.

For the formulation of Theorems 3.1.2 and 3.1.3 recall from (2.21) the definition of the functionals $\mathcal{V}_0^{x_{i_0};\pi}$ and $\mathcal{V}_0^{x_{i_0}}$. In the finite setting of Subsection 3.1.2 we know that

$$\mathcal{V}_0^{x_{i_0};\pi}(\boldsymbol{p}) = V_0^{\boldsymbol{p};\pi}(x_{i_0}) \quad \text{and} \quad \mathcal{V}_0^{x_{i_0}}(\boldsymbol{p}) = \max_{\pi \in \Pi} \mathcal{V}_0^{x_{i_0};\pi}(\boldsymbol{p}) \tag{3.12}$$

for every $i_0 = 1, \ldots, \mathfrak{e}$, $\boldsymbol{p} \in \widetilde{\mathcal{P}}$, and $\pi \in \Pi$. Here $V_0^{\boldsymbol{p};\pi}(\cdot)$ can be computed via (1.29) with (3.10). Note that $i_0 \in \{1, \ldots, \mathfrak{e}\}$ refers to the index of the initial state $x_{i_0} \in E$.

Part (ii) of the following Theorem 3.1.2 shows that the value functional $\mathcal{V}_0^{x_{i_0}}$ of the inventory control problem (3.11) is 'Lipschitz continuous' at fixed $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ in the sense of Definition 2.2.3. Its statement is an immediate consequence of Corollary 2.2.10.

Theorem 3.1.2 ('Lipschitz continuity' of $\mathcal{V}_0^{x_{i_0};\pi}$ and $\mathcal{V}_0^{x_{i_0}}$ in $\boldsymbol{p}$) *In the setting above let $i_0 \in \{1, \ldots, \mathfrak{e}\}$, $\pi \in \Pi$, and $\boldsymbol{p} \in \widetilde{\mathcal{P}}$. Then the following two assertions hold.*

(i) *The map $\mathcal{V}_0^{x_{i_0};\pi} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by (3.12) is 'Lipschitz continuous' at $\boldsymbol{p}$.*

(ii) *The map $\mathcal{V}_0^{x_{i_0}} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by (3.12) is 'Lipschitz continuous' at $\boldsymbol{p}$.*

Next, part (ii) of Theorem 3.1.3 specifies the 'Fréchet derivative' of the value functional $\mathcal{V}_0^{x_{i_0}}$ of the inventory control problem (3.11). In this theorem we will use the notion of 'Fréchet differentiability' introduced in Definition 2.3.9. The assertions in Theorem 3.1.3 can be deduced from Corollary 2.3.21.

Theorem 3.1.3 ('Fréchet differentiability' of $\mathcal{V}_0^{x_{i_0};\pi}$ and $\mathcal{V}_0^{x_{i_0}}$ in $\boldsymbol{p}$) *In the setting above let $i_0 \in \{1, \ldots, \mathfrak{e}\}$, $\pi \in \Pi$, and $\boldsymbol{p} \in \widetilde{\mathcal{P}}$. Then the following two assertions hold.*

(i) *The map $\mathcal{V}_0^{x_{i_0};\pi} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by (3.12) is 'Fréchet differentiable' at $\boldsymbol{p}$ with 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi} : \widetilde{\mathcal{P}}^{\boldsymbol{p};\pm} \to \mathbb{R}$ given by (2.42) along with (3.10).*

(ii) *The map $\mathcal{V}_0^{x_{i_0}} : \widetilde{\mathcal{P}} \to \mathbb{R}$ defined by (3.12) is 'Fréchet differentiable' at $\boldsymbol{p}$ with 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0}} : \widetilde{\mathcal{P}}^{\boldsymbol{p};\pm} \to \mathbb{R}$ given by (2.43) along with (3.10).*

If there exists for some given $\boldsymbol{p} \in \widetilde{\mathcal{P}}$ a *unique* optimal trading strategy $\pi^{\boldsymbol{p}} \in \Pi$ w.r.t. $\boldsymbol{p}$, then $\Pi(\boldsymbol{p}) = \{\pi^{\boldsymbol{p}}\}$ and part (ii) of Theorem 3.1.3 implies that in this case the 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0}}$ of the optimal value (functional) $\mathcal{V}_0^{x_{i_0}}$ at $\boldsymbol{p}$ coincides with $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}$.

3.1.4 Numerical examples for the 'Fréchet derivative'

In this subsection we quantify by means of the 'Fréchet derivative' of the value functional $\mathcal{V}_0^{x_{i_0}}$ (given by Theorem 3.1.3(ii)) the effect of incorporating an unlikely but significant change in the demand of the single product on the optimal value of the corresponding stochastic inventory control problem (3.11).

To this end, let us take up the numerical example at p. 41 in [73] where $N := 3$, $K := 4$, $s_{\mathsf{rev}} := 8$, $c_{\mathsf{ord}} := 2$, $c_{\mathsf{fix}} := 4$, and $c_{\mathsf{hol}} := 1$. We fix $i_0 \in \{1, \ldots, \mathfrak{e}\}$ (with $\mathfrak{e} := (K+1)^2$) as well as $\mathfrak{p} := (\mathfrak{p}_\bullet, \mathfrak{p}_\bullet, \mathfrak{p}_\bullet)$ with $\mathfrak{p}_\bullet := (0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0 \ldots)$, and denote by $\boldsymbol{p}$ the unique element of $\widetilde{\mathcal{P}}$ generated by $\mathfrak{p}$ through (3.8)–(3.9). This choice of $\mathfrak{p}$ means that in each period the demand is 1, 2, or 3 with probability $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. Table 3.1 provides the (unique) optimal order strategy $\pi^{\boldsymbol{p}} = (f_0^{\boldsymbol{p}}, f_1^{\boldsymbol{p}}, f_2^{\boldsymbol{p}})$, and the second column of Table 3.2 displays the maximal expected total reward $\mathcal{V}_0^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{p})$ of the inventory control problem (3.11) for all possible initial inventory levels $y_0 := y_{i_0} \in \{0, \ldots, 4\}$. Moreover, the last two columns in Table 3.2 display the 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}(\cdot)$ of $\mathcal{V}_0^{x_{i_0};\pi^{\boldsymbol{p}}}$ at $\boldsymbol{p}$ evaluated at direction $\boldsymbol{q}_{(0)} - \boldsymbol{p}$ and at direction $\boldsymbol{q}_{(4)} - \boldsymbol{p}$ (calculated with the iteration scheme (2.44)), again for all possible initial inventory levels y_0. Here $\boldsymbol{q}_{(0)}$ and $\boldsymbol{q}_{(4)}$ are generated through (3.8) and (3.9) by $\mathfrak{q}_{(0)} := (\mathfrak{q}_{(0)\bullet}, \mathfrak{q}_{(0)\bullet}, \mathfrak{q}_{(0)\bullet})$ and $\mathfrak{q}_{(4)} := (\mathfrak{q}_{(4)\bullet}, \mathfrak{q}_{(4)\bullet}, \mathfrak{q}_{(4)\bullet})$ respectively, where $\mathfrak{q}_{(0)\bullet} := (1, 0, 0, \ldots)$ and $\mathfrak{q}_{(4)\bullet} := (0, 0, 0, 0, 1, 0, 0, \ldots)$. As the optimal strategy $\pi^{\boldsymbol{p}}$ is unique in our example, this implies $\Pi(\boldsymbol{p}) = \{\pi^{\boldsymbol{p}}\}$ and thus in view of part (ii) of Theorem 3.1.3 we even have $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0}}(\cdot) = \dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}(\cdot)$.

(y,z)	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(2,0)$	$\cdots$	$(4,4)$
$f_0^{\boldsymbol{p}}$	4	4	4	4	4	3	3	3	3	3	0	$\cdots$	0
$f_1^{\boldsymbol{p}}$	4	4	4	4	4	3	3	3	3	3	0	$\cdots$	0
$f_2^{\boldsymbol{p}}$	2	2	2	2	2	0	0	0	0	0	0	$\cdots$	0

Table 3.1: Optimal order strategy $\pi^{\boldsymbol{p}} = (f_0^{\boldsymbol{p}}, f_1^{\boldsymbol{p}}, f_2^{\boldsymbol{p}})$ for $\boldsymbol{p}$ as above.

Note that for $j \in \{0, 4\}$ the 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0}}$ evaluated at $\boldsymbol{q}_{(j)} - \boldsymbol{p}$, i.e. $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0}}(\boldsymbol{q}_{(j)} - \boldsymbol{p})$ (in our case it equals $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{q}_{(j)} - \boldsymbol{p})$), quantifies the first-order sensitivity of $\mathcal{V}_0^{x_{i_0}}(\boldsymbol{p})$ (respectively of $\mathcal{V}_0^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{p})$) w.r.t. a change of the underlying probability transition function from $\boldsymbol{p}$ to $\boldsymbol{p}_{(j)} := (1-\varepsilon)\boldsymbol{p} + \varepsilon \boldsymbol{q}_{(j)}$ with $\varepsilon \in (0,1)$ small. It can be easily seen that $\boldsymbol{p}_{(j)}$ is generated via (3.8)–(3.9) through $\mathfrak{p} := (\mathfrak{p}_\bullet, \mathfrak{p}_\bullet, \mathfrak{p}_\bullet)$ replaced by $\mathfrak{p}_{(j)} := (\mathfrak{p}_{(j)\bullet}, \mathfrak{p}_{(j)\bullet}, \mathfrak{p}_{(j)\bullet})$, where $\mathfrak{p}_{(j)\bullet} := (1-\varepsilon)\mathfrak{p}_\bullet + \varepsilon \mathfrak{q}_{(j)\bullet}$. That is, the change from $\boldsymbol{p}$ to $\boldsymbol{p}_{(j)}$ means that the formerly impossible demand j now gets assigned small but strictly positive probability ε in each period.

$x_{i_0} = (y_{i_0}, z_{i_0})$	$\mathcal{V}_0^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{p})$	$\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{q}_{(0)} - \boldsymbol{p})$	$\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{q}_{(4)} - \boldsymbol{p})$
$(0, \cdot)$	16.5313	−34.0938	16.0313
$(1, \cdot)$	18.5313	−34.0938	16.0313
$(2, \cdot)$	23.1250	−39.8125	14.0000
$(3, \cdot)$	26.1094	−37.3906	15.6094
$(4, \cdot)$	28.5313	−34.0938	16.0313

Table 3.2: Optimal value $\mathcal{V}_0^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{p})$ and the value $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}(\boldsymbol{q}_{(j)} - \boldsymbol{p})$ (in our example it equals $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0}}(\boldsymbol{q}_{(j)} - \boldsymbol{p})$) of the 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{p}}^{x_{i_0};\pi^{\boldsymbol{p}}}$ evaluated at $\boldsymbol{q}_{(j)} - \boldsymbol{p}$ with $\boldsymbol{q}_{(j)}$ as above, $j \in \{0, 4\}$, in dependence of the initial inventory level y_{i_0}.

As appears from Table 3.2, the negative effect of incorporating demand 0 into the counting density $\mathfrak{p}_\bullet$ with small probability ε is roughly twice as large as the positive effect of incorporating demand 4 into $\mathfrak{p}_\bullet$ with the same small probability ε, no matter what the initial inventory level looks like. So, when worrying about robustness of the optimal value w.r.t. changes in the demand's counting density $\mathfrak{p}_\bullet$, it seems to be somewhat more important to analyse in detail the adequacy of the assumption that an absent demand is impossible than the adequacy of the assumption that a demand of 4 is impossible.

3.2 Terminal wealth optimization problem

In this section we illustrate the theory and results from Chapters 1–2 to assess the impact of one or more than one unlikely but substantial shock in the dynamics of an asset on the optimal value of a terminal wealth problem in a (simple) financial market model free of shocks. Shocks in (discrete time) financial market models in the context of a terminal wealth problem have already been discussed several times in the existing literature, see for instance [23] and [53], where in the latter reference the authors even considered the continuous time case. At first, we introduce in Subsection 3.2.1 the basic financial market model and formulate subsequently the terminal wealth problem as a classical optimization problem in mathematical finance. The market model is in line with standard literature as [5, Chapter 4] or [35, Chapter 5]. To keep the presentation as clear as possible we restrict ourselves to a simple variant of the market model (only one risky asset). In Subsection 3.2.2 we will see that the market model can be embedded into the MDM of Section 1.1. It turns out that the existence (and computation) of an optimal (trading) strategy can be obtained by solving iteratively N one-stage investment problems; see Subsection 3.2.3. In Subsection 3.2.4 we will show that the (optimal) value functional of the terminal wealth problem is 'Lipschitz continuous' as well as 'Hadamard differentiable', and Subsection 3.2.5 provides some numerical examples for the 'Hadamard derivative' of the (optimal) value functional.

3.2.1 Basic financial market model, and the target

Consider an N-period financial market (with $N \in \mathbb{N}$ fixed) consisting of one riskless bond $S^0 = (S_0^0, \ldots, S_N^0)$ and one risky asset $S = (S_0, \ldots, S_N)$. Further we assume that the value of the bond evolves deterministically according to

$$S_0^0 = 1, \qquad S_{n+1}^0 = \mathfrak{r}_{n+1} S_n^0, \qquad n = 0, \ldots, N-1$$

for some fixed constants $\mathfrak{r}_1, \ldots, \mathfrak{r}_N \in \mathbb{R}_{\geq 1}$, and that the value of the asset evolves stochastically according to

$$S_0 = s_0, \qquad S_{n+1} = \mathfrak{R}_{n+1} S_n, \qquad n = 0, \ldots, N-1$$

for some fixed constant $s_0 \in \mathbb{R}_{>0}$ and independent $\mathbb{R}_{\geq 0}$-valued random variables $\mathfrak{R}_1, \ldots, \mathfrak{R}_N$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with known probability distributions $\mathfrak{m}_1, \ldots, \mathfrak{m}_N$, respectively. Note that the constants $\mathfrak{r}_1, \ldots, \mathfrak{r}_N$ and the random variables $\mathfrak{R}_1, \ldots, \mathfrak{R}_N$ can be interpreted as deterministic and stochastic interest rates, respectively.

Throughout this section we will assume that the financial market satisfies the following Assumption 3.2.1, where u_α (with $\alpha \in (0,1)$ fixed) is introduced in (3.16) below. In Examples 3.2.7 and 3.2.8 we will discuss specific financial market models which satisfy Assumption 3.2.1.

Assumption 3.2.1 *The following three assertions hold for any $n = 0, \ldots, N-1$.*

(a) $\int_{\mathbb{R}_{\geq 0}} u_\alpha \, d\mathfrak{m}_{n+1} < \infty$.

(b) $\mathfrak{R}_{n+1} > 0$ $\mathbb{P}$-*a.s.*

(c) $\mathbb{P}[\mathfrak{R}_{n+1} \neq \mathfrak{r}_{n+1}] = 1$.

Note that condition (a) of Assumption 3.2.1 requires that the expectation of $u_\alpha(\mathfrak{R}_{n+1})$ under $\mathbb{P}$ is finite. Also note that condition (b) of Assumption 3.2.1 is in line with the existing literature; see, for instance, [5, p. 61]. Condition (c) of Assumption 3.2.1 is used to ensure the uniqueness of a solution to the reduced optimization problem in (3.21) ahead; see Lemma 3.2.4 below.

We emphasize that for any $n = 0, \ldots, N-1$ the value $\mathfrak{r}_{n+1}$ (resp. $\mathfrak{R}_{n+1}$) corresponds to the relative price change S^0_{n+1}/S^0_n (resp. S_{n+1}/S_n) of the bond (resp. asset) in the period between time n and $n+1$. In the sequel, we let $\mathcal{F}_0$ be the trivial σ-algebra, and set $\mathcal{F}_n := \sigma(S_0, \ldots, S_n)$, $n = 1, \ldots, N$. Note that $\mathcal{F}_n = \sigma(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ for any $n = 1, \ldots, N$.

Now, an agent invests a given amount of capital $x_0 \in \mathbb{R}_{\geq 0}$ in the bond and the asset according to some self-financing trading strategy. By *trading strategy* we mean an $(\mathcal{F}_n)$-adapted $\mathbb{R}^2_{\geq 0}$-valued stochastic process $\varphi = (\varphi^0_n, \varphi_n)_{n=0}^{N-1}$, where φ^0_n (resp. φ_n) specifies the amount of capital that is invested in the bond (resp. asset) during the time interval $[n, n+1)$. Here we require that both φ^0_n and φ_n are nonnegative for any n, which means that taking loans and short sellings of the asset are excluded. The corresponding *portfolio (or wealth) process* $X^\varphi = (X^\varphi_0, \ldots, X^\varphi_N)$ associated with $\varphi = (\varphi^0_n, \varphi_n)_{n=0}^{N-1}$ is given by

$$X^\varphi_0 := \varphi^0_0 + \varphi_0 \quad \text{and} \quad X^\varphi_{n+1} := \varphi^0_n \mathfrak{r}_{n+1} + \varphi_n \mathfrak{R}_{n+1}, \qquad n = 0, \ldots, N-1. \tag{3.13}$$

A trading strategy $\varphi = (\varphi^0_n, \varphi_n)_{n=0}^{N-1}$ is said to be *self-financing w.r.t. the initial capital* x_0 if $x_0 = \varphi^0_0 + \varphi_0$ and $X^\varphi_n = \varphi^0_n + \varphi_n$ for all $n = 1, \ldots, N$. It is easily seen that for any self-financing trading strategy $\varphi = (\varphi^0_n, \varphi_n)_{n=0}^{N-1}$ w.r.t. x_0 the corresponding portfolio process admits the representation

$$X^\varphi_0 = x_0 \quad \text{and} \quad X^\varphi_{n+1} = \mathfrak{r}_{n+1} X^\varphi_n + \varphi_n(\mathfrak{R}_{n+1} - \mathfrak{r}_{n+1}) \quad \text{for } n = 0, \ldots, N-1. \tag{3.14}$$

Note that $X^\varphi_n - \varphi_n$ corresponds to the amount of capital which is invested in the bond during the time interval $[n, n+1)$. Also note that it can be verified easily by means of Remark 3.1.6 in [5] that under condition (c) of Assumption 3.2.1 the financial market introduced above is free of arbitrage opportunities.

In view of (3.14), we may and do identify a self-financing trading strategy w.r.t. x_0 with an $(\mathcal{F}_n)$-adapted $\mathbb{R}_{\geq 0}$-valued stochastic process $\varphi = (\varphi_n)_{n=0}^{N-1}$ satisfying $\varphi_0 \in [0, x_0]$ and $\varphi_n \in [0, X^\varphi_n]$ for all $n = 1, \ldots, N-1$. We restrict ourselves to *Markovian* self-financing trading strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ w.r.t. x_0 which means that φ_n only depends on n and X^φ_n. To put it another way, we assume that for any $n = 0, \ldots, N-1$ there exists some Borel measurable map $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\varphi_n = f_n(X^\varphi_n)$. Then, in particular, X^φ is an $\mathbb{R}_{\geq 0}$-valued $(\mathcal{F}_n)$-Markov process whose one-step transition probability at time $n \in \{0, \ldots, N-1\}$ given state $x \in \mathbb{R}_{\geq 0}$ and strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) is given by

$$\mathfrak{m}_{n+1} \circ \eta^{-1}_{n,(x, f_n(x))},$$

where

$$\eta_{n,(x,f_n(x))}(y) := \mathfrak{r}_{n+1} x + f_n(x)(y - \mathfrak{r}_{n+1}), \qquad y \in \mathbb{R}_{\geq 0}. \tag{3.15}$$

The agent's aim is to find a self-financing trading strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) w.r.t. x_0 for which her expected utility of the discounted terminal wealth is maximized. We assume that the agent is risk averse and that her attitude towards risk is set via the *power utility* function $u_\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$u_\alpha(y) := y^\alpha \tag{3.16}$$

for some fixed $\alpha \in (0,1)$ (as in condition (a) of Assumption 3.2.1). The coefficient α determines the degree of risk aversion of the agent: the smaller the coefficient α, the greater her risk aversion. Hence the agent is interested in those self-financing trading strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) w.r.t. x_0 for which the expectation of $u_\alpha(X_N^\varphi / S_N^0)$ under $\mathbb{P}$ is maximized.

Remark 3.2.2 In the following we will assume for notational simplicity that $\mathfrak{r}_1, \ldots, \mathfrak{r}_N$ are fixed and that $\mathfrak{m}_1, \ldots, \mathfrak{m}_N$ are a sort of model parameters. In this case the factor $1/S_N^0$ in $u_\alpha(X_N^\varphi / S_N^0)$ in Display (3.19) ahead is superfluous; it indeed does not influence the maximization problem or any 'derivative' of the optimal value. On the other hand, if also the (Dirac-) distributions of $\mathfrak{r}_1, \ldots, \mathfrak{r}_N$ would be allowed to be variable, then this factor could matter for the derivative of the optimal value w.r.t. changes in the (deterministic) dynamics of the bond S^0. $\diamond$

3.2.2 Markov decision model, and optimal trading strategies

The setting introduced in Subsection 3.2.1 can be embedded into the setting of Chapters 1–2 as follows. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_N \in \mathbb{R}_{\geq 1}$ be a priori fixed constants. Further let

$$(E, \mathcal{E}) := (\mathbb{R}_{\geq 0}, \mathcal{B}(\mathbb{R}_{\geq 0})) \quad \text{and} \quad A_n(x) := [0, x], \qquad x \in \mathbb{R}_{\geq 0},\, n = 0, \ldots, N-1,$$

and note that $A_n(x)$ does not depend on n. Then $A_n = \mathbb{R}_{\geq 0}$ and $D_n = D := \{(x,a) \in \mathbb{R}_{\geq 0}^2 : a \in [0,x]\}$ for any $n = 0, \ldots, N-1$. Let $\mathcal{A}_n := \mathcal{B}(\mathbb{R}_{\geq 0})$. In particular, $\mathcal{D}_n = \mathcal{B}(\mathbb{R}_{\geq 0}^2) \cap D$ and the set $\overline{\mathbb{F}}_n$ of all decision rules at time n consists of all those Borel measurable maps $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which satisfy $f_n(x) \in [0,x]$ for all $x \in \mathbb{R}_{\geq 0}$ (in particular $\overline{\mathbb{F}}_n$ is independent of n). For any $n = 0, \ldots, N-1$, let the set $\mathbb{F}_n$ of all admissible decision rules at time n be equal to $\overline{\mathbb{F}}_n$. Let as before $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$.

Consider the gauge function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ defined by

$$\psi(x) := 1 + u_\alpha(x). \tag{3.17}$$

Let $\mathcal{P}_\psi$ be the set of all transition functions $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$ consisting of transition kernels of the shape

$$P_n\big((x,a), \bullet\big) := \mathfrak{m}_{n+1} \circ \eta_{n,(x,a)}^{-1}[\,\bullet\,], \qquad (x,a) \in D_n,\, n = 0, \ldots, N-1 \tag{3.18}$$

for some $\mathfrak{m}_{n+1} \in \mathcal{M}_1^\alpha(\mathbb{R}, \mathbb{R}_{\geq 0})$, where the map $\eta_{n,(x,a)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as in (3.15) and $\mathcal{M}_1^\alpha(\mathbb{R}, \mathbb{R}_{\geq 0})$ is the set of all $\mu \in \mathcal{M}_1(\mathbb{R})$ satisfying $\mu[\mathbb{R}_{\geq 0}] = 1$ as well as $\int_{\mathbb{R}_{\geq 0}} u_\alpha \, d\mu < \infty$. It is easily seen that $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$ (with $\overline{\mathcal{P}}_\psi$ defined as in Subsection 2.1.2) and that $(1-\varepsilon)\boldsymbol{P} + \varepsilon \boldsymbol{Q} \in \mathcal{P}_\psi$ for any $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{P}_\psi$ and $\varepsilon \in (0,1)$. In particular, $\mathcal{P}_\psi$ is closed under mixtures.

Moreover, set

$$\begin{aligned} r_n(x,a) &:\equiv 0, \qquad (x,a) \in D_n,\, n = 0, \ldots, N-1, \\ r_N(x) &:= u_\alpha(x/S_N^0), \qquad x \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{3.19}$$

Then, for every fixed $x_0 \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{P} \in \mathcal{P}_\psi$ the terminal wealth problem introduced in the paragraph above Remark 3.2.2 reads as

$$\mathbb{E}^{x_0, \boldsymbol{P}; \pi}[r_N(X_N)] \longrightarrow \max \text{ (in } \pi \in \Pi)\,! \tag{3.20}$$

A strategy $\pi^{\boldsymbol{P}} \in \Pi$ is called an *optimal (self-financing) trading strategy w.r.t. $\boldsymbol{P}$ (and x_0)* if it solves the maximization problem (3.20). Note that it follows from Lemma 3.2.10(i) below that the gauge function ψ given by (3.17) provides a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for every $\boldsymbol{P} \in \mathcal{P}_\psi$. Note here that $\boldsymbol{X}$ plays the role of the portfolio process X^φ introduced in (3.13), and that for some fixed $x_0 \in \mathbb{R}_{\geq 0}$ any self-financing trading strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ w.r.t. x_0 may be identified with some $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ through $\varphi_n = f_n(X_n^\varphi)$.

Remark 3.2.3 In the financial market model introduced in Subsection 3.2.1 we restrict ourselves to Markovian self-financing trading strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ w.r.t. x_0 which may be identified with some $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ via $\varphi_n = f_n(X_n^\varphi)$. Of course, one could also assume that the decision rules of a trading strategy π also depend on past actions and past values of the portfolio process X^φ. However, as already discussed in part (i) of Remark 1.2.6, the corresponding history-dependent trading strategies do not lead to an improved optimal value of the terminal wealth problem (3.20). ◇

3.2.3 Existence and computation of optimal trading strategies

In the following we discuss the existence and computation of solutions to the terminal wealth problem (3.20), maintaining the notation introduced in Subsections 3.2.1–3.2.2. We will adapt the arguments of Section 4.2 in [5]. As before $\mathfrak{r}_1, \ldots, \mathfrak{r}_N \in \mathbb{R}_{\geq 1}$ are fixed constants.

Basically the existence of an optimal trading strategy of the terminal wealth problem (3.20) can be ensured with the help of a suitable analogue of Theorem 4.2.2 in [5]. In order to specify the optimal trading strategy explicitly one has to determine the local maximizers in the Bellman equation; see Theorem 1.3.3(i) in Section 1.3. However this is not necessarily easy. On the other hand, part (ii) of Theorem 3.2.5 ahead (a variant of Theorem 4.2.6 in [5]) shows that, for our particular choice of the utility function (recall (3.16)), the optimal investment in the asset at time $n \in \{0, \ldots, N-1\}$ has a rather simple form insofar as it depends linearly on the wealth. The respective coefficient can be obtained by solving the one-stage optimization problem in (3.21) ahead. That is, instead of finding the optimal amount of capital (possibly depending on the wealth) to be invested in the asset, it suffices to find the optimal fraction of the wealth (being independent of the wealth itself) to be invested in the asset.

For the formulation of the one-stage optimization problem, we note that every transition function $\boldsymbol{P} \in \mathcal{P}_\psi$ is generated through (3.18) by some $(\mathfrak{m}_1, \ldots, \mathfrak{m}_N) \in \mathcal{M}_1^\alpha(\mathbb{R}, \mathbb{R}_{\geq 0})^N$. For every $\boldsymbol{P} \in \mathcal{P}_\psi$, we use $(\mathfrak{m}_1^{\boldsymbol{P}}, \ldots, \mathfrak{m}_N^{\boldsymbol{P}})$ to denote any such set of 'parameters'. Now, consider for any $\boldsymbol{P} \in \mathcal{P}_\psi$ and $n = 0, \ldots, N-1$ the optimization problem

$$v_n^{\boldsymbol{P};\gamma} := \int_{\mathbb{R}_{\geq 0}} (u_\alpha \circ \eta_n^\gamma)(y)\, \mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy) \;\longrightarrow\; \max\ (\text{in } \gamma \in [0,1])\,! \tag{3.21}$$

where the map $\eta_n^\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is for any $\gamma \in [0,1]$ and $n = 0, \ldots, N-1$ defined by

$$\eta_n^\gamma(y) := 1 + \gamma(y/\mathfrak{r}_{n+1} - 1). \tag{3.22}$$

Note that the integral on the left-hand side in (3.21) (exists and) is finite (this follows from Displays (3.23)–(3.24) ahead) and should be seen as the expectation of $u_\alpha \circ \eta_n^\gamma(\mathfrak{R}_{n+1})$ under $\mathbb{P}$.

The following lemma shows in particular that

$$v_n^{\boldsymbol{P}} \;:=\; \sup_{\gamma\in[0,1]} v_n^{\boldsymbol{P};\gamma}$$

is the maximal value of the optimization problem (3.21).

Lemma 3.2.4 *For any $\boldsymbol{P}\in\mathcal{P}_\psi$ and $n=0,\ldots,N-1$, there exists a unique solution $\gamma_n^{\boldsymbol{P}}\in[0,1]$ to the optimization problem (3.21).*

Proof Let $\boldsymbol{P}\in\mathcal{P}_\psi$ and $n=0,\ldots,N-1$. Define a map $\mathfrak{f}_n^{\boldsymbol{P}}:\mathbb{R}_{\ge 0}\times[0,1]\to\mathbb{R}_{\ge 0}$ through

$$\mathfrak{f}_n^{\boldsymbol{P}}(y,\gamma) \;:=\; (u_\alpha\circ\eta_n^\gamma)(y). \tag{3.23}$$

Note that $\mathfrak{f}_n^{\boldsymbol{P}}(\,\cdot\,,\gamma)$ $(=u_\alpha(1+\gamma(\,\cdot\,/\mathfrak{r}_{n+1}-1)))$ is clearly Borel measurable for any $\gamma\in[0,1]$, and it is easily seen that in view of (3.22)

$$|\mathfrak{f}_n^{\boldsymbol{P}}(y,\gamma)| \;=\; u_\alpha\big((1-\gamma)+\gamma(y/\mathfrak{r}_{n+1})\big) \;\le\; u_\alpha(1+y)$$

for every $y\in\mathbb{R}_{\ge 0}$ and $\gamma\in[0,1]$. Therefore, the function $\mathfrak{f}_n^{\boldsymbol{P}}$ is absolutely dominated by the Borel measurable function $\mathfrak{h}:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ defined by $\mathfrak{h}(y):=u_\alpha(1+y)$. Moreover set $\overline{\mathfrak{m}}_{\boldsymbol{P}} := \max_{k=0,\ldots,N-1}\int_{\mathbb{R}_{\ge 0}} u_\alpha\, d\mathfrak{m}_{k+1}^{\boldsymbol{P}}$, and note that $\overline{\mathfrak{m}}_{\boldsymbol{P}}\in\mathbb{R}_{>0}$ (by conditions (a)–(b) of Assumption 3.2.1). Since $\mathfrak{h}$ satisfies

$$\int_{\mathbb{R}_{\ge 0}} \mathfrak{h}(y)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy) \;\le\; 1+\int_{\mathbb{R}_{\ge 0}} u_\alpha(y)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy) \;\le\; 1+\overline{\mathfrak{m}}_{\boldsymbol{P}} \;<\; \infty \tag{3.24}$$

(i.e. $\mathfrak{h}$ is $\mathfrak{m}_{n+1}^{\boldsymbol{P}}$-integrable) and $\mathfrak{f}_n^{\boldsymbol{P}}(y,\,\cdot\,)$ is continuous on $[0,1]$ for any $y\in\mathbb{R}_{\ge 0}$, we may apply the continuity lemma (see, e.g., [6, Lemma 16.1]) to obtain that the mapping $\mathfrak{F}_n^{\boldsymbol{P}}:[0,1]\to\mathbb{R}_{>0}$ given by $\mathfrak{F}_n^{\boldsymbol{P}}(\gamma):=\int_{\mathbb{R}_{\ge 0}}\mathfrak{f}_n^{\boldsymbol{P}}(y,\gamma)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)$ is continuous. Along with the compactness of the set $[0,1]$, this ensures the existence of a solution $\gamma_n^{\boldsymbol{P}}\in[0,1]$ to the optimization problem (3.21).

Moreover it can be verified easily by means of condition (c) of Assumption 3.2.1 that $\mathfrak{F}_n^{\boldsymbol{P}}$ is strictly concave; take into account that $\int_{\mathbb{R}_{\ge 0}}\mathfrak{f}_n^{\boldsymbol{P}}(y,\gamma)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)$ can be seen for any $\gamma\in[0,1]$ as the expectation of $u_\alpha\circ\eta_n^\gamma(\mathfrak{R}_{n+1})$ under $\mathbb{P}$. This implies that the solution $\gamma_n^{\boldsymbol{P}}$ is even unique. □

Part (i) of the following Theorem 3.2.5 involves the value function introduced in (1.13). In the present setting this function has in view of (3.19) a comparatively simple form:

$$V_n^{\boldsymbol{P}}(x_n) \;=\; \sup_{\pi\in\Pi}\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}[r_N(X_N)] \tag{3.25}$$

for any $x_n\in\mathbb{R}_{\ge 0}$, $\boldsymbol{P}\in\mathcal{P}_\psi$, and $n=0,\ldots,N$. Part (ii) of this theorem involves the subset Π_{lin} of Π which consists of all *linear trading strategies*, i.e. of all $\pi\in\Pi$ of the form $\pi=(f_n^{\boldsymbol{\gamma}})_{n=0}^{N-1}$ for some $\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$, where

$$f_n^{\boldsymbol{\gamma}}(x) \;:=\; \gamma_n\cdot x, \qquad x\in\mathbb{R}_{\ge 0},\, n=0,\ldots,N-1. \tag{3.26}$$

In part (i) and elsewhere we will use the convention that the product over the empty set is 1.

Theorem 3.2.5 (Optimal trading strategy) *Let $\boldsymbol{P} \in \mathcal{P}_\psi$. Then the following two assertions hold.*

(i) *The value function $V_n^{\boldsymbol{P}}$ given by (3.25) admits the representation*

$$V_n^{\boldsymbol{P}}(x_n) \;=\; \mathfrak{v}_n^{\boldsymbol{P}}\, u_\alpha(x_n/S_n^0)$$

for any $x_n \in \mathbb{R}_{\geq 0}$ and $n = 0, \ldots, N-1$, where $\mathfrak{v}_n^{\boldsymbol{P}} := \prod_{k=n}^{N-1} v_k^{\boldsymbol{P}}$.

(ii) *For any $n = 0, \ldots, N-1$, let $\gamma_n^{\boldsymbol{P}} \in [0,1]$ be the unique solution to the optimization problem (3.21) and define a decision rule $f_n^{\boldsymbol{P}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ at time n through*

$$f_n^{\boldsymbol{P}}(x) \;:=\; \gamma_n^{\boldsymbol{P}} x, \qquad x \in \mathbb{R}_{\geq 0}. \tag{3.27}$$

Then $\pi^{\boldsymbol{P}} := (f_n^{\boldsymbol{P}})_{n=0}^{N-1} \in \Pi$ with $\pi^{\boldsymbol{P}} \in \Pi_{\rm lin}$ forms an optimal trading strategy w.r.t. $\boldsymbol{P}$. Moreover, there is no further optimal trading strategy w.r.t. $\boldsymbol{P}$ which belongs to $\Pi_{\rm lin}$.

The second assertion of part (ii) of Theorem 3.2.5 will be beneficial for the proof of Theorem 3.2.11 below; for details see Remark 3.2.12.

For the proof of Theorem 3.2.5 we need the following Lemma 3.2.6. Note that the policy value function introduced in (1.11) admits in view of (3.19) the representation

$$V_n^{\boldsymbol{P};\pi}(x_n) \;=\; \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P};\pi}[r_N(X_N)] \tag{3.28}$$

for any $x_n \in \mathbb{R}_{\geq 0}$, $\boldsymbol{P} \in \mathcal{P}_\psi$, $\pi \in \Pi$, and $n = 0, \ldots, N$. Also note that any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol{\gamma}} := (f_n^{\gamma})_{n=0}^{N-1} \in \Pi_{\rm lin}$ through (3.26). Finally, let $v_n^{\boldsymbol{P};\gamma_n}$ be defined as on the left-hand side of (3.21), and set $v_n^{\boldsymbol{P};\boldsymbol{\gamma}} := v_n^{\boldsymbol{P};\gamma_n}$ for any $n = 0, \ldots, N-1$.

Lemma 3.2.6 *Let $\boldsymbol{P} \in \mathcal{P}_\psi$ and $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$. Then the policy value function $V_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}$ given by (3.28) admits the representation*

$$V_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(x_n) \;=\; \mathfrak{v}_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(x_n/S_n^0) \tag{3.29}$$

for any $x_n \in \mathbb{R}_{\geq 0}$ and $n = 0, \ldots, N$, where $\mathfrak{v}_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}} := \prod_{k=n}^{N-1} v_k^{\boldsymbol{P};\boldsymbol{\gamma}} = \prod_{k=n}^{N-1} v_k^{\boldsymbol{P};\gamma_k}$.

Proof We prove the assertion in (3.29) by (backward) induction on n. For $n = N$ we obtain by means of (3.28), part (iii) of Lemma 1.4.4, and (3.19)

$$V_N^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(x_N) \;=\; r_N(x_N) \;=\; \mathfrak{v}_N^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(x_N/S_N^0)$$

for any $x_N \in \mathbb{R}_{\geq 0}$, where $\mathfrak{v}_N^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}} := 1$. Now, suppose that the assertion in (3.29) holds for $k \in \{n+1, \ldots, N\}$. Note that $V_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(\cdot) \in \mathbb{M}'$ (with $\mathbb{M}'$ defined as in (3.31) ahead) by choosing $\vartheta := \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}$ ($\in \mathbb{R}_{>0}$) as well as $\kappa := S_{n+1}^0$ ($\in \mathbb{R}_{\geq 1}$), and that $\mathbb{M}' \subseteq \mathbb{M}_n^{\boldsymbol{P}}(\mathbb{R}_{\geq 0})$ (with $\mathbb{M}_n^{\boldsymbol{P}}(\mathbb{R}_{\geq 0})$ as in Section 1.3); see the proof of Theorem 3.2.5 below. Then, in view of part (i) of Proposition 1.3.1, (3.15), and (3.22), for any $x_n \in \mathbb{R}_{\geq 0}$ we get

$$\begin{aligned} V_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(x_n) \;&=\; \mathcal{T}_{n,f_n^{\gamma}}^{\boldsymbol{P}} V_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(x_n) \;=\; \int_{\mathbb{R}_{\geq 0}} V_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(y)\, P_n\big((x_n, f_n^{\gamma}(x_n)), dy\big) \\ &=\; \int_{\mathbb{R}_{\geq 0}} \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(y/S_{n+1}^0)\, \mathfrak{m}_{n+1}^{\boldsymbol{P}} \circ \eta_{n,(x_n, f_n^{\gamma}(x_n))}^{-1}(dy) \end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_\gamma} \cdot \int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x_n + f_n^\gamma(x_n)(y-\mathfrak{r}_{n+1})}{S_{n+1}^0}\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&= \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_\gamma} \cdot \int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x_n + \gamma_n\,x_n(y-\mathfrak{r}_{n+1})}{\mathfrak{r}_{n+1}S_n^0}\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&= \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_\gamma}\,u_\alpha(x_n/S_n^0)\cdot \int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(1+\gamma_n\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&= \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_\gamma}\,u_\alpha(x_n/S_n^0)\,v_n^{\boldsymbol{P};\gamma} \;=\; \mathfrak{v}_n^{\boldsymbol{P};\pi_\gamma}\,u_\alpha(x_n/S_n^0),
\end{aligned}
\tag{3.30}
$$

where we used for the fifth "=" the definition of the map f_n^γ in (3.26). For the last step we employed $\mathfrak{v}_n^{\boldsymbol{P};\pi_\gamma} = \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_\gamma} v_n^{\boldsymbol{P};\gamma}$. Thus we have verified the representation of the map $V_n^{\boldsymbol{P};\pi_\gamma}$ in (3.29). □

Now, we are in the position to prove Theorem 3.2.5.

Proof of Theorem 3.2.5: (i): We intend to apply Theorem 1.3.3 (see Section 1.3). Let $\mathbb{M}_n^{\boldsymbol{P}} := \mathbb{M}'$ and $\mathbb{F}_n' := \mathbb{F}'$ for any $n = 0,\ldots,N-1$, where

$$
\begin{aligned}
\mathbb{M}' &:= \big\{h \in \mathbb{R}^{\mathbb{R}_{\geq 0}} : h(x) = \vartheta\,u_\alpha(x/\kappa),\ x\in\mathbb{R}_{\geq 0},\ \text{for some } \vartheta\in\mathbb{R}_{>0}, \kappa\in\mathbb{R}_{\geq 1}\big\},\\
\mathbb{F}' &:= \big\{f\in\mathbb{F} : f(x) = \gamma\,x,\ x\in\mathbb{R}_{\geq 0},\ \text{for some } \gamma\in[0,1]\big\}
\end{aligned}
\tag{3.31}
$$

with $\mathbb{F} := \mathbb{F}_n$ (recall that $\mathbb{F}_n = \overline{\mathbb{F}}_n$ and that $\overline{\mathbb{F}}_n$ is independent of n). It can be verified easily by means of condition (a) of Assumption 3.2.1 that $\mathbb{M}_n^{\boldsymbol{P}} = \mathbb{M}'$ is a subset of $\mathbb{M}_n^{\boldsymbol{P}}(\mathbb{R}_{\geq 0})$ for any $n = 0,\ldots,N-1$, where $\mathbb{M}_n^{\boldsymbol{P}}(\mathbb{R}_{\geq 0})$ is defined as in (1.16) in Section 1.3. Moreover we obviously have $\mathbb{F}_n' = \mathbb{F}' \subseteq \mathbb{F} = \mathbb{F}_n$ for any $n = 0,\ldots,N-1$.

Below we will show that conditions (a)–(c) of Theorem 1.3.3 are met. Thus we may apply part (i) of Theorem 1.3.3 (Bellman equation) to obtain part (i) of Theorem 3.2.5. In fact, for $n = N$ we have

$$
V_N^{\boldsymbol{P}}(x_N) \;=\; r_N(x_N) \;=\; \mathfrak{v}_N^{\boldsymbol{P}}\,u_\alpha(x_N/S_N^0)
$$

for any $x_N \in \mathbb{R}_{\geq 0}$ (by (3.19)), where $\mathfrak{v}_N^{\boldsymbol{P}} := 1$. Now, suppose that the assertion holds for $k \in \{n+1,\ldots,N\}$. Then, using again part (i) of Theorem 1.3.3, we have in view of (3.15) for any $x_n \in \mathbb{R}_{\geq 0}$

$$
\begin{aligned}
V_n^{\boldsymbol{P}}(x_n) &= \mathcal{T}_n^{\boldsymbol{P}} V_{n+1}^{\boldsymbol{P}}(x_n) \;=\; \sup_{f_n\in\mathbb{F}_n} \mathcal{T}_{n,f_n}^{\boldsymbol{P}} V_{n+1}^{\boldsymbol{P}}(x_n)\\
&= \sup_{f_n\in\mathbb{F}_n} \int_{\mathbb{R}_{\geq 0}} V_{n+1}^{\boldsymbol{P}}(y)\,P_n\big((x_n, f_n(x_n)), dy\big)\\
&= \sup_{f_n\in\mathbb{F}_n} \int_{\mathbb{R}_{\geq 0}} \mathfrak{v}_{n+1}^{\boldsymbol{P}}\,u_\alpha(y/S_{n+1}^0)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}\circ\eta_{n,(x_n,f_n(x_n))}^{-1}(dy)\\
&= \mathfrak{v}_{n+1}^{\boldsymbol{P}}\cdot \sup_{f_n\in\mathbb{F}_n} \int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x_n + f_n(x_n)(y-\mathfrak{r}_{n+1})}{\mathfrak{r}_{n+1}S_n^0}\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy).
\end{aligned}
\tag{3.32}
$$

For $x_n = 0$ we have $f_n(x_n) = 0$ for any $f_n \in \mathbb{F}_n$ and therefore (in view of (3.32)) $V_n^{\boldsymbol{P}}(x_n) = 0$. For $x_n \in \mathbb{R}_{>0}$ we obtain from (3.32)

$$
V_n^{\boldsymbol{P}}(x_n) \;=\; \mathfrak{v}_{n+1}^{\boldsymbol{P}}\,u_\alpha(x_n/S_n^0)\cdot \sup_{f_n\in\mathbb{F}_n} \int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(1+\frac{f_n(x_n)}{x_n}\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)
$$

$$
\begin{aligned}
&= \mathfrak{v}_{n+1}^{\boldsymbol{P}}\, u_\alpha(x_n/S_n^0) \cdot \sup_{\gamma\in[0,1]} \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(1+\gamma\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&= \mathfrak{v}_{n+1}^{\boldsymbol{P}}\, u_\alpha(x_n/S_n^0)\, v_n^{\boldsymbol{P}} \;=\; \mathfrak{v}_n^{\boldsymbol{P}}\, u_\alpha(x_n/S_n^0), \qquad (3.33)
\end{aligned}
$$

where we used for the second "=" that the value of $f_n(x_n)$ ranges over the interval $[0, x_n]$ when f_n ranges over $\mathbb{F}_n$; we can then indeed replace $f_n(x_n)$ by γx_n when "$\sup_{f_n\in\mathbb{F}_n}$" is replaced by "$\sup_{\gamma\in[0,1]}$". For the last step we employed $\mathfrak{v}_n^{\boldsymbol{P}} = \mathfrak{v}_{n+1}^{\boldsymbol{P}} v_n^{\boldsymbol{P}}$. Hence we have verified the representation of the value function asserted in part (i). It remains to show that conditions (a)–(c) of Theorem 1.3.3 (in Section 1.3) are indeed satisfied.

(a): In view of (3.19) we obtain $r_N \in \mathbb{M}'$ by choosing $\vartheta := 1$ ($\in \mathbb{R}_{>0}$) and $\kappa := S_N^0$ ($\in \mathbb{R}_{\ge 1}$). In particular, $r_N \in \mathbb{M}_{N-1}^{\boldsymbol{P}}$.

(b): Let $n \in \{1, \ldots, N-1\}$ and $h \in \mathbb{M}_n^{\boldsymbol{P}} = \mathbb{M}'$, i.e. $h(x) = \vartheta\, u_\alpha(x/\kappa)$, $x \in \mathbb{R}_{\ge 0}$, for some $\vartheta \in \mathbb{R}_{>0}$ and $\kappa \in \mathbb{R}_{\ge 1}$. Then as in (3.32) we obtain for any $x \in \mathbb{R}_{\ge 0}$

$$
\begin{aligned}
\mathcal{T}_n^{\boldsymbol{P}} h(x) &= \sup_{f_n\in\mathbb{F}_n} \mathcal{T}_{n,f_n}^{\boldsymbol{P}} h(x)\\
&= \vartheta \cdot \sup_{f_n\in\mathbb{F}_n} \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x + f_n(x)(y-\mathfrak{r}_{n+1})}{\kappa}\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy). \qquad (3.34)
\end{aligned}
$$

For $x = 0$ we have $f_n(x) = 0$ for any $f_n \in \mathbb{F}_n$ and therefore (in view of (3.34)) $\mathcal{T}_n^{\boldsymbol{P}} h(x) = 0$. For $x \in \mathbb{R}_{>0}$ we obtain from (3.34) (analogously to (3.33))

$$
\begin{aligned}
\mathcal{T}_n^{\boldsymbol{P}} h(x) &= \vartheta\, \mathfrak{r}_{n+1}^\alpha\, u_\alpha(x/\kappa) \cdot \sup_{f_n\in\mathbb{F}_n} \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(1+\frac{f_n(x)}{x}\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&= \vartheta\, \mathfrak{r}_{n+1}^\alpha\, u_\alpha(x/\kappa) \cdot \sup_{\gamma\in[0,1]} \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(1+\gamma\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&= \vartheta\, \mathfrak{r}_{n+1}^\alpha\, u_\alpha(x/\kappa)\, v_n^{\boldsymbol{P}} \;=\; \widetilde{\vartheta}\, u_\alpha(x/\kappa), \qquad (3.35)
\end{aligned}
$$

where $\widetilde{\vartheta} := \vartheta \mathfrak{r}_{n+1}^\alpha v_n^{\boldsymbol{P}} \in \mathbb{R}_{>0}$ is finite due to (3.23)–(3.24). Altogether we have shown that $\mathcal{T}_n^{\boldsymbol{P}} h \in \mathbb{M}'$. In particular, $\mathcal{T}_n^{\boldsymbol{P}} h \in \mathbb{M}_{n-1}^{\boldsymbol{P}}$.

(c): Let $n \in \{0, \ldots, N-1\}$ and $h \in \mathbb{M}_n^{\boldsymbol{P}} = \mathbb{M}'$ (with corresponding ϑ and κ as in (b)). Moreover, let $f_n^{\boldsymbol{P}}$ be the map as defined in (3.27), and note that $f_n^{\boldsymbol{P}} \in \mathbb{F}_n$. Then, similarly to (3.34), we have for any $x \in \mathbb{R}_{\ge 0}$ and $f_n \in \mathbb{F}_n$

$$
\mathcal{T}_{n,f_n}^{\boldsymbol{P}} h(x) \;=\; \vartheta \cdot \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x + f_n(x)(y-\mathfrak{r}_{n+1})}{\kappa}\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy).
$$

For $x = 0$ we obviously have $\mathcal{T}_{n,f_n}^{\boldsymbol{P}} h(x) = 0$ and thus $\mathcal{T}_{n,f_n^{\boldsymbol{P}}}^{\boldsymbol{P}} h(x) = \mathcal{T}_n^{\boldsymbol{P}} h(x)$. For $x \in \mathbb{R}_{>0}$ we have similarly to (3.35) that for any $f_n \in \mathbb{F}_n$

$$
\mathcal{T}_{n,f_n}^{\boldsymbol{P}} h(x) \;=\; \vartheta\, \mathfrak{r}_{n+1}^\alpha\, u_\alpha(x/\kappa) \cdot \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(1+\frac{f_n(x)}{x}\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy).
$$

By Lemma 3.2.4, the map $\gamma \mapsto \int_{\mathbb{R}_{\ge 0}} (u_\alpha \circ \eta_n^\gamma)(y)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)$ has exactly one maximal point, $\gamma_n^{\boldsymbol{P}}$, in $[0,1]$. Thus, since the second line in (3.35) coincides with $\mathcal{T}_n^{\boldsymbol{P}} h(x)$, we obtain $\mathcal{T}_{n,f_n^{\boldsymbol{P}}}^{\boldsymbol{P}} h(x) = \mathcal{T}_n^{\boldsymbol{P}} h(x)$

also for any $x \in \mathbb{R}_{>0}$. Therefore the map $f_n^{\boldsymbol{P}}$ provides a maximizer $f_n^{\boldsymbol{P}} \in \mathbb{F}_n$ of h w.r.t. $\boldsymbol{P}$ with $f_n^{\boldsymbol{P}} \in \mathbb{F}'$ $(= \mathbb{F}_n')$.

(ii): In the proof of (i) we have seen that the assumptions of Theorem 1.3.3 are fulfilled. Thus, part (i) of this theorem gives $V_{n+1}^{\boldsymbol{P}} \in \mathbb{M}_n^{\boldsymbol{P}}$ for any $n = 0, \ldots, N-1$. In particular, the above elaborations under (c) show that for any $n = 0, \ldots, N-1$ the map $f_n^{\boldsymbol{P}}$ defined by (3.27) provides a maximizer $f_n^{\boldsymbol{P}} \in \mathbb{F}_n$ of $V_{n+1}^{\boldsymbol{P}}$ w.r.t. $\boldsymbol{P}$ with $f_n^{\boldsymbol{P}} \in \mathbb{F}'$ $(= \mathbb{F}_n')$. Hence, part (iii) of Theorem 1.3.3 ensures that the strategy $\pi^{\boldsymbol{P}} := (f_n^{\boldsymbol{P}})_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ forms an optimal trading strategy w.r.t. $\boldsymbol{P}$.

For the second part of the assertion we assume that there exists another optimal trading strategy $\widetilde{\pi}^{\boldsymbol{P}}$ w.r.t. $\boldsymbol{P}$ with $\widetilde{\pi}^{\boldsymbol{P}} \in \Pi_{\mathrm{lin}}$. Then, by definition of Π_{lin}, there exists $\widetilde{\boldsymbol{\gamma}}^{\boldsymbol{P}} = (\widetilde{\gamma}_n^{\boldsymbol{P}})_{n=0}^{N-1} \in [0,1]^N$ such that $\widetilde{\pi}^{\boldsymbol{P}} = \pi_{\widetilde{\boldsymbol{\gamma}}^{\boldsymbol{P}}} := (f_n^{\widetilde{\boldsymbol{\gamma}}^{\boldsymbol{P}}})_{n=0}^{N-1}$. In particular, we have $V_0^{\boldsymbol{P}}(x_0) = V_0^{\boldsymbol{P};\pi_{\widetilde{\boldsymbol{\gamma}}^{\boldsymbol{P}}}}(x_0)$ for any $x_0 \in \mathbb{R}_{\geq 0}$. Along with part (i) of this theorem and Lemma 3.2.6 (recall that $S_0^0 = 1$), this implies $\mathfrak{v}_0^{\boldsymbol{P}} u_\alpha(x_0) = \mathfrak{v}_0^{\boldsymbol{P};\pi_{\widetilde{\boldsymbol{\gamma}}^{\boldsymbol{P}}}} u_\alpha(x_0)$ for every $x_0 \in \mathbb{R}_{>0}$ and thus $\mathfrak{v}_0^{\boldsymbol{P}} = \mathfrak{v}_0^{\boldsymbol{P};\pi_{\widetilde{\boldsymbol{\gamma}}^{\boldsymbol{P}}}}$, i.e.

$$\prod_{k=0}^{N-1} v_k^{\boldsymbol{P}} = \prod_{k=0}^{N-1} v_k^{\boldsymbol{P};\widetilde{\gamma}_k^{\boldsymbol{P}}}. \tag{3.36}$$

Below we will show that (3.36) implies

$$v_n^{\boldsymbol{P}} = v_n^{\boldsymbol{P};\widetilde{\gamma}_n^{\boldsymbol{P}}} \quad \text{for all } n = 0, \ldots, N-1. \tag{3.37}$$

Then it follows from (3.37) that for any $n = 0, \ldots, N-1$ the fraction $\widetilde{\gamma}_n^{\boldsymbol{P}} \in [0,1]$ is a solution to the optimization problem (3.21). However, according to Lemma 3.2.4, this optimization problem has exactly one solution, $\gamma_n^{\boldsymbol{P}}$, in $[0,1]$. Hence $\widetilde{\gamma}_n^{\boldsymbol{P}} = \gamma_n^{\boldsymbol{P}}$ for any $n = 0, \ldots, N-1$ and we arrive at $\widetilde{\pi}^{\boldsymbol{P}} = \pi^{\boldsymbol{P}}$ which implies that $\pi^{\boldsymbol{P}}$ is unique among all $\pi \in \Pi_{\mathrm{lin}}(\boldsymbol{P})$.

It remains to show that (3.36) implies (3.37). Assume by way of contradiction that (3.37) does *not* hold, i.e. there exists $n \in \{0, \ldots, N-1\}$ such that $v_n^{\boldsymbol{P}} \neq v_n^{\boldsymbol{P};\widetilde{\gamma}_n^{\boldsymbol{P}}}$. Then

$$v_n^{\boldsymbol{P}} = \sup_{\gamma \in [0,1]} v_n^{\boldsymbol{P};\gamma} > v_n^{\boldsymbol{P};\widetilde{\gamma}_n^{\boldsymbol{P}}}$$

because the reverse inequality would lead to a contradiction of the maximality of $v_n^{\boldsymbol{P}}$. By assumption (3.36), this implies that there exists $k \in \{0, \ldots, N-1\}$ with $k \neq n$ such that

$$v_k^{\boldsymbol{P}} = \sup_{\gamma \in [0,1]} v_k^{\boldsymbol{P};\gamma} < v_k^{\boldsymbol{P};\widetilde{\gamma}_k^{\boldsymbol{P}}}.$$

This, however, contradicts the maximality of $v_k^{\boldsymbol{P}}$. Hence (3.36) indeed implies (3.37). This completes the proof of Theorem 3.2.5. □

We conclude this subsection with the following two Examples 3.2.7 and 3.2.8 which illustrate part (ii) of Theorem 3.2.5.

Example 3.2.7 (Cox–Ross–Rubinstein model) In the setting of Subsection 3.2.2 let $\mathfrak{r}_1 = \cdots = \mathfrak{r}_N = \mathfrak{r}$ for some $\mathfrak{r} \in \mathbb{R}_{\geq 1}$. Moreover let $\boldsymbol{P} \in \overline{\mathcal{P}}$ be any transition function defined as in (3.18) with $\mathfrak{m}_1 = \cdots = \mathfrak{m}_N = \mathfrak{m}_{\boldsymbol{P}}$ for some $\mathfrak{m}_{\boldsymbol{P}} := p_{\boldsymbol{P}} \delta_{\mathrm{u}_{\boldsymbol{P}}} + (1 - p_{\boldsymbol{P}}) \delta_{\mathrm{d}_{\boldsymbol{P}}}$, where $p_{\boldsymbol{P}} \in [0,1]$ and

$\mathsf{d}_{\boldsymbol{P}}, \mathsf{u}_{\boldsymbol{P}} \in \mathbb{R}_{>0}$ are some given constants (depending on $\boldsymbol{P}$) satisfying $\mathsf{d}_{\boldsymbol{P}} < \mathfrak{r} < \mathsf{u}_{\boldsymbol{P}}$. Then the corresponding MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ is stationary, and we have $\boldsymbol{P} \in \mathcal{P}_{\psi}$. Moreover conditions (a)–(c) of Assumption 3.2.1 are clearly satisfied. In particular, the corresponding financial market is arbitrage-free and the optimization problem (3.21) simplifies to (up to the factor $\mathfrak{r}^{-\alpha}$)

$$\big\{p_{\boldsymbol{P}}\, u_\alpha(\mathfrak{r} + \gamma(\mathsf{u}_{\boldsymbol{P}} - \mathfrak{r})) + (1 - p_{\boldsymbol{P}})\, u_\alpha(\mathfrak{r} + \gamma(\mathsf{d}_{\boldsymbol{P}} - \mathfrak{r}))\big\} \longrightarrow \max \text{ (in } \gamma \in [0,1])\,! \tag{3.38}$$

Lemma 3.2.4 ensures that (3.38) has a unique solution, $\gamma_{\mathsf{CRR}}^{\boldsymbol{P}}$, and it can be checked easily (see, e.g., [5, p. 86]) that this solution admits the representation

$$\gamma_{\mathsf{CRR}}^{\boldsymbol{P}} = \begin{cases} 0 & , \;\; p_{\boldsymbol{P}} \in [0, p_{\boldsymbol{P},0}] \\ \frac{\mathfrak{r}}{(\mathfrak{r}-\mathsf{d}_{\boldsymbol{P}})(\mathsf{u}_{\boldsymbol{P}}-\mathfrak{r})} \cdot \frac{p_{\boldsymbol{P}}^{\kappa_\alpha}(\mathsf{u}_{\boldsymbol{P}}-\mathfrak{r})^{\kappa_\alpha} - (1-p_{\boldsymbol{P}})^{\kappa_\alpha}(\mathfrak{r}-\mathsf{d}_{\boldsymbol{P}})^{\kappa_\alpha}}{p_{\boldsymbol{P}}^{\kappa_\alpha}(\mathsf{u}_{\boldsymbol{P}}-\mathfrak{r})^{\kappa_\alpha \alpha} + (1-p_{\boldsymbol{P}})^{\kappa_\alpha}(\mathfrak{r}-\mathsf{d}_{\boldsymbol{P}})^{\kappa_\alpha \alpha}} & , \;\; p_{\boldsymbol{P}} \in (p_{\boldsymbol{P},0}, p_{\boldsymbol{P},1}) \\ 1 & , \;\; p_{\boldsymbol{P}} \in [p_{\boldsymbol{P},1}, 1] \end{cases} , \tag{3.39}$$

where $\kappa_\alpha := (1-\alpha)^{-1}$ and

$$p_{\boldsymbol{P},0} := \frac{\mathfrak{r} - \mathsf{d}_{\boldsymbol{P}}}{\mathsf{u}_{\boldsymbol{P}} - \mathsf{d}_{\boldsymbol{P}}} \,(> 0) \quad \text{ and } \quad p_{\boldsymbol{P},1} := \frac{\mathsf{u}_{\boldsymbol{P}}^{1-\alpha}(\mathfrak{r} - \mathsf{d}_{\boldsymbol{P}})}{\mathsf{u}_{\boldsymbol{P}}^{1-\alpha}(\mathfrak{r} - \mathsf{d}_{\boldsymbol{P}}) + \mathsf{d}_{\boldsymbol{P}}^{1-\alpha}(\mathsf{u}_{\boldsymbol{P}} - \mathfrak{r})} \,(< 1).$$

Note that only fractions from the interval $[0,1]$ are admissible, and that the expression in the middle line in (3.39) lies in $(0,1)$ when $p_{\boldsymbol{P}} \in (p_{\boldsymbol{P},0}, p_{\boldsymbol{P},1})$. Thus, part (ii) of Theorem 3.2.5 shows that the strategy $\pi_{\mathsf{CRR}}^{\boldsymbol{P}}$ defined by (3.27) (with $\gamma_n^{\boldsymbol{P}}$ replaced by $\gamma_{\mathsf{CRR}}^{\boldsymbol{P}}$) is optimal w.r.t. $\boldsymbol{P}$ and unique among all $\pi \in \Pi_{\mathrm{lin}}(\boldsymbol{P})$. ◇

In the following example the bond and the asset evolve according to the ordinary differential equation and the Itô stochastic differential equation

$$d\mathfrak{e}_t^0 = \nu \mathfrak{e}_t^0\, dt \quad \text{ and } \quad d\mathfrak{e}_t = \mu \mathfrak{e}_t\, dt + \sigma \mathfrak{e}_t\, d\mathfrak{B}_t,$$

respectively, where $\nu, \mu \in \mathbb{R}_{\geq 0}$ and $\sigma \in \mathbb{R}_{>0}$ are constants and $\mathfrak{B}$ is a one-dimensional standard Brownian motion. We assume that the trading period is (without loss of generality) the unit interval $[0,1]$ and that the bond and the asset can be traded only at N equidistant time points in $[0,1]$, namely at $t_{N,n} := n/N$, $n = 0, \ldots, N-1$. Then, in particular, the relative price changes $\mathfrak{r}_{n+1} := S_{n+1}^0/S_n^0 = \mathfrak{e}_{t_{N,n+1}}^0/\mathfrak{e}_{t_{N,n}}^0$ and $\mathfrak{R}_{n+1} := S_{n+1}/S_n = \mathfrak{e}_{t_{N,n+1}}/\mathfrak{e}_{t_{N,n}}$ are given by

$$\exp\big\{\nu(t_{N,n+1} - t_{N,n})\big\}$$

and

$$\exp\big\{(\mu - \sigma^2/2)(t_{N,n+1} - t_{N,n}) + \sigma(\mathfrak{B}_{t_{N,n+1}} - \mathfrak{B}_{t_{N,n}})\big\},$$

respectively. In particular, $\mathfrak{r}_{n+1} = \exp(\nu/N)$ and $\mathfrak{R}_{n+1}$ is distributed according to the log-normal distribution $\mathrm{LN}_{(\mu-\sigma^2/2)/\mathrm{N}, \sigma^2/\mathrm{N}}$ for any $n = 0, \ldots, N-1$.

Example 3.2.8 (Black–Scholes–Merton model) In the setting of Subsection 3.2.2 let $\mathfrak{r}_1 = \cdots = \mathfrak{r}_N = \mathfrak{r}$ for $\mathfrak{r} := \exp(\nu/N)$, where $\nu \in \mathbb{R}_{\geq 0}$. Moreover let $\boldsymbol{P} \in \overline{\mathcal{P}}$ be any transition function defined as in (3.18) with $\mathfrak{m}_1 = \cdots = \mathfrak{m}_N = \mathfrak{m}_{\boldsymbol{P}}$ for $\mathfrak{m}_{\boldsymbol{P}} := \mathrm{LN}_{(\mu_{\boldsymbol{P}} - \sigma_{\boldsymbol{P}}^2/2)/\mathrm{N}, \sigma_{\boldsymbol{P}}^2/\mathrm{N}}$, where $\mu_{\boldsymbol{P}} \in \mathbb{R}_{\geq 0}$ and $\sigma_{\boldsymbol{P}} \in \mathbb{R}_{>0}$ are some given constants (depending on $\boldsymbol{P}$) satisfying $\mu_{\boldsymbol{P}} > (1-\alpha)\sigma_{\boldsymbol{P}}^2$. Then the corresponding MDM $(E, \boldsymbol{A}, \boldsymbol{P}, \Pi, \boldsymbol{X}, \boldsymbol{r})$ is stationary, and we have $\boldsymbol{P} \in \mathcal{P}_\psi$. Moreover it is easily seen that conditions (a)–(c) of Assumption 3.2.1 hold. In particular, the corresponding financial market is arbitrage-free and the optimization problem (3.21) now reads as

$$\int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(1 + \gamma\Big(\frac{y}{\mathfrak{r}} - 1\Big)\Big)\varphi^{\mathrm{LN}}_{(\mu_{\boldsymbol{P}} - \sigma^2_{\boldsymbol{P}}/2)/N, \sigma^2_{\boldsymbol{P}}/N}(y)\,\ell(dy) \longrightarrow \max\ (\text{in } \gamma \in [0,1])\,! \tag{3.40}$$

where $\varphi^{\mathrm{LN}}_{(\mu_{\boldsymbol{P}} - \sigma^2_{\boldsymbol{P}}/2)/N, \sigma^2_{\boldsymbol{P}}/N}$ stands for the standard Lebesgue density of the log-normal distribution $\mathrm{LN}_{(\mu_{\boldsymbol{P}} - \sigma^2_{\boldsymbol{P}}/2)/N, \sigma^2_{\boldsymbol{P}}/N}$ defined as in Display (5.71) in Subsection 5.2.4. Lemma 3.2.4 ensures that (3.40) has a unique solution, $\gamma^{\boldsymbol{P}}_{\mathsf{BSM}}$, and it is known (see, e.g., [67, 72]) that this solution is given by

$$\gamma^{\boldsymbol{P}}_{\mathsf{BSM}} = \begin{cases} 0 & , \quad \nu \in [\mu_{\boldsymbol{P}}, \infty) \\ \frac{1}{1-\alpha}\,\frac{\mu_{\boldsymbol{P}} - \nu}{\sigma^2_{\boldsymbol{P}}} & , \quad \nu \in (\nu_{\boldsymbol{P},\alpha}, \mu_{\boldsymbol{P}}) \\ 1 & , \quad \nu \in [0, \nu_{\boldsymbol{P},\alpha}] \end{cases}, \tag{3.41}$$

where $\nu_{\boldsymbol{P},\alpha} := \mu_{\boldsymbol{P}} - (1-\alpha)\sigma^2_{\boldsymbol{P}}\,(\in (0, \mu_{\boldsymbol{P}}))$. Note that only fractions from the interval $[0,1]$ are admissible, and that the expression in the middle line in (3.41) is called *Merton ratio* and lies in $(0,1)$ when $\nu \in (\nu_{\boldsymbol{P},\alpha}, \mu_{\boldsymbol{P}})$. Thus, part (ii) of Theorem 3.2.5 shows that the strategy $\pi^{\boldsymbol{P}}_{\mathsf{BSM}}$ defined by (3.27) (with $\gamma^{\boldsymbol{P}}_n$ replaced by $\gamma^{\boldsymbol{P}}_{\mathsf{BSM}}$) is optimal w.r.t. $\boldsymbol{P}$ and unique among all $\pi \in \Pi_{\mathrm{lin}}(\boldsymbol{P})$. $\diamond$

3.2.4 'Lipschitz continuity' and 'Hadamard differentiability' of the value functional

Maintain the notation and terminology introduced in Subsections 3.2.1–3.2.3. In the following we will show that the value function of the terminal wealth problem (3.20) regarded as a real-valued functional is 'Lipschitz continuous' as well as 'Hadamard differentiable' at (fixed) $\boldsymbol{P} \in \mathcal{P}_\psi$; see part (ii) of Theorems 3.2.9 and 3.2.11 below. Recall that $\alpha \in (0,1)$ introduced in (3.16) is fixed and determines the degree of risk aversion of the agent.

By the choice of the gauge function ψ (see (3.17)) we may choose $\mathbb{M} := \mathbb{M}' := \mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}$ (with $\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}$ introduced in Example 2.1.6) in the setting of Subsections 2.2.2 and 2.3.2. Note that ψ given by (3.17) coincides with the corresponding gauge function in Example 2.1.6 for $x' := 0$. That is, in the end the metric $d^{\psi}_{\infty, \mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}$ (as defined in (2.12)) on $\mathcal{P}_\psi$ is used to measure the distance between transition functions.

For the formulation of Theorems 3.2.9 and 3.2.11 recall from (2.16) the definition of the functionals $\mathcal{V}^{x_0;\pi}_0$ and $\mathcal{V}^{x_0}_0$, where the maps $V^{\boldsymbol{P};\pi}_0$ and $V^{\boldsymbol{P}}_0$ are given by (1.11) and (1.13), respectively. In the specific setting of Subsection 3.2.2 we know from (3.25) and (3.28) that

$$\mathcal{V}^{x_0;\pi}_0(\boldsymbol{P}) \,=\, V^{\boldsymbol{P};\pi}_0(x_0) \,=\, \mathbb{E}^{x_0,\boldsymbol{P};\pi}[r_N(X_N)] \quad \text{and} \quad \mathcal{V}^{x_0}_0(\boldsymbol{P}) \,=\, \sup_{\pi \in \Pi} \mathcal{V}^{x_0;\pi}_0(\boldsymbol{P}) \tag{3.42}$$

for any $x_0 \in \mathbb{R}_{\geq 0}$, $\boldsymbol{P} \in \mathcal{P}_\psi$, and $\pi \in \Pi$.

Part (ii) of Theorem 2.2.8 shows that the value functional of the terminal wealth problem (3.20) is 'Lipschitz continuous' in the sense of Definition 2.2.1. Note that any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol{\gamma}} := (f^{\gamma}_n)_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ through (3.26).

Theorem 3.2.9 ('Lipschitz continuity' of $\mathcal{V}^{x_0;\pi_{\boldsymbol{\gamma}}}_0$ and $\mathcal{V}^{x_0}_0$ in $\boldsymbol{P}$) *In the setting above let $x_0 \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, and $\boldsymbol{P} \in \mathcal{P}_\psi$. Then the following two assertions hold.*

(i) *The map* $\mathcal{V}_0^{x_0;\pi_{\boldsymbol{\gamma}}}:\mathcal{P}_\psi\to\mathbb{R}$ *defined by (3.42) is 'Lipschitz continuous' at* $\boldsymbol{P}$ *w.r.t.* $(\mathbb{M}_{\text{Höl},\alpha},\psi)$.

(ii) *The map* $\mathcal{V}_0^{x_0}:\mathcal{P}_\psi\to\mathbb{R}$ *defined by (3.42) is 'Lipschitz continuous' at* $\boldsymbol{P}$ *w.r.t.* $(\mathbb{M}_{\text{Höl},\alpha},\psi)$.

The proof of Theorem 3.2.9 relies on the following lemma. Recall Definition 1.4.1, and note that $\rho_{\mathbb{M}_{\text{Höl},\alpha}}$ refers to the Minkowski functional introduced in (2.17) with $\mathbb{M}:=\mathbb{M}_{\text{Höl},\alpha}$. Also note that we used $(\mathfrak{m}_1^{\boldsymbol{P}},\ldots,\mathfrak{m}_N^{\boldsymbol{P}})$ to denote any set of 'parameters' which generates through (3.18) the transition function $\boldsymbol{P}\in\mathcal{P}_\psi$.

Lemma 3.2.10 *In the setting above the following two assertions hold.*

(i) ψ *is a bounding function for the MDM* $(E,\boldsymbol{A},\boldsymbol{P},\Pi,\boldsymbol{X},\boldsymbol{r})$ *for every* $\boldsymbol{P}\in\mathcal{P}_\psi$.

(ii) *For any fixed* $\boldsymbol{P}\in\mathcal{P}_\psi$ *we have* $\sup_{\pi\in\Pi_{\text{lin}}}\rho_{\mathbb{M}_{\text{Höl},\alpha}}(V_n^{\boldsymbol{P};\pi})<\infty$ *for every* $n=1,\ldots,N$.

(iii) $\rho_{\mathbb{M}_{\text{Höl},\alpha}}(\psi)<\infty$.

Proof For (i) let $\boldsymbol{P}=(P_n)_{n=0}^{N-1}\in\mathcal{P}_\psi$ be arbitrary but fixed. Since $r_n\equiv 0$ for any $n=0,\ldots,N-1$ (by (3.19)), there exists a finite constant $K_1>0$ such that

$$|r_n(x,a)|\;\le\;K_1\;\le\;K_1\big(1+u_\alpha(x)\big)\;=\;K_1\psi(x)$$

for every $(x,a)\in D_n$ and $n=0,\ldots,N-1$.

Moreover, in view of (3.19), we can find some finite constant $K_2>0$ such that

$$|r_N(x)|\;=\;\big(1/u_\alpha(S_N^0)\big)\cdot u_\alpha(x)\;\le\;u_\alpha(x)\;\le\;K_2\big(1+u_\alpha(x)\big)\;=\;K_2\psi(x)$$

for every $x\in\mathbb{R}_{\ge 0}$ and $n=0,\ldots,N-1$.

Next, set $\bar{\mathfrak{r}}:=\max_{k=0,\ldots,N-1}\mathfrak{r}_{k+1}$, and note that $\bar{\mathfrak{r}}\in\mathbb{R}_{\ge 1}$. Using equations (3.15)–(3.19), we find some finite constant $K_3>0$ (depending on $\boldsymbol{P}$) such that

$$\begin{aligned}
&\int_{\mathbb{R}_{\ge 0}}\psi(y)\,P_n\big((x,a),dy\big)\;=\;1+\int_{\mathbb{R}_{\ge 0}}u_\alpha(y)\,\mathfrak{m}_{n+1}\circ\eta_{n,(x,a)}^{-1}\,[dy]\\
&=\;1+\int_{\mathbb{R}_{\ge 0}}u_\alpha\big(\mathfrak{r}_{n+1}x+a(y-\mathfrak{r}_{n+1})\big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\;=\;1+\mathfrak{r}_{n+1}^\alpha\cdot\int_{\mathbb{R}_{\ge 0}}u_\alpha\Big(x+a\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\\
&\le\;1+\bar{\mathfrak{r}}^\alpha\,u_\alpha(x)\cdot\int_{\mathbb{R}_{\ge 0}}u_\alpha(1+y)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\;\le\;1+\bar{\mathfrak{r}}^\alpha\,u_\alpha(x)\cdot\Big(1+\int_{\mathbb{R}_{\ge 0}}u_\alpha(y)\,\mathfrak{m}_{n+1}^{\boldsymbol{P}}(dy)\Big)\\
&\le\;1+\bar{\mathfrak{r}}^\alpha\,u_\alpha(x)\cdot(1+\overline{\mathfrak{m}}_{\boldsymbol{P}})\;\le\;K_3\psi(x)
\end{aligned}$$

for every $(x,a)\in D_n$ and $n=0,\ldots,N-1$, where $\overline{\mathfrak{m}}_{\boldsymbol{P}}$ is defined as in the proof of Lemma 3.2.4. Take into account that $\alpha\in(0,1)$ introduced in (3.16) is fixed. Consequently, conditions (a)–(c) of Definition 1.4.1 are satisfied for $\mathcal{P}:=\{\boldsymbol{P}\}$.

To prove (ii) let $n\in\{1,\ldots,N\}$ be arbitrary but fixed. Since any $\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$ induces a linear trading strategy $\pi=\pi_{\boldsymbol{\gamma}}:=(f_n^{\boldsymbol{\gamma}})_{n=0}^{N-1}\in\Pi_{\text{lin}}$ through (3.26), it suffices in view of Example 2.2.4 to show that

$$\sup_{\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N}\|V_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\|_{\text{Höl},\alpha}<\infty.\tag{3.43}$$

First of all, it is easily seen that the terminal reward function r_N given by (3.19) is contained in $\mathbb{M}_{\text{Höl},\alpha}$. Thus $\|r_N\|_{\text{Höl},\alpha}\le 1$. Moreover, in view of Lemma 3.2.6 and (3.19), we have $V_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(\cdot)=$

$\mathfrak{v}_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}} r_N(\cdot)$ for any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$, where $\mathfrak{v}_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}} := \prod_{k=n}^{N-1} v_k^{\boldsymbol{P};\boldsymbol{\gamma}}$. Then in view of (3.23)–(3.24)

$$\begin{aligned}\|V_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\|_{\mathrm{H\ddot{o}l},\alpha} \,&=\, \|\mathfrak{v}_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}} r_N\|_{\mathrm{H\ddot{o}l},\alpha} \,=\, |\mathfrak{v}_n^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}|\, \|r_N\|_{\mathrm{H\ddot{o}l},\alpha} \,=\, \prod_{k=n}^{N-1} |v_k^{\boldsymbol{P};\boldsymbol{\gamma}}|\, \|r_N\|_{\mathrm{H\ddot{o}l},\alpha} \\ &\le\, \prod_{k=n}^{N-1} \int_{\mathbb{R}_{\geq 0}} u_\alpha\Big(1+\gamma_k\Big(\frac{y}{\mathfrak{r}_{k+1}^{\boldsymbol{P}}}-1\Big)\Big)\, \mathfrak{m}_{k+1}^{\boldsymbol{P}}(dy) \,\le\, (1+\overline{\mathfrak{m}}_{\boldsymbol{P}})^{N-n}\end{aligned}$$

for any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$, where we used in the second "=" the absolute homogeneity of the semi-norm $\|\cdot\|_{\mathrm{H\ddot{o}l},\alpha}$ (as defined in Example 2.1.6). Hence, we arrive at (3.43).

For (iii) note that it can be shown easily that the gauge function ψ belongs to $\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}$, i.e. $\|\psi\|_{\mathrm{H\ddot{o}l},\alpha} \le 1$. Thus, in view of Example 2.2.4, we have $\rho_{\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}(\psi) = \|\psi\|_{\mathrm{H\ddot{o}l},\alpha} \le 1 < \infty$. □

Let us turn to the proof of Theorem 3.2.9.

Proof of Theorem 3.2.9: We intend to apply Theorem 2.2.8. At first, note that the first assertion of part (ii) of Theorem 3.2.5 along with Remark 2.2.7 entail that the value functional $\mathcal{V}_0^{x_0}$ given by (3.42) admits the representation

$$\mathcal{V}_0^{x_0}(\boldsymbol{P}) \,=\, \sup_{\pi\in\Pi_{\mathrm{lin}}} \mathcal{V}_0^{x_0;\pi}(\boldsymbol{P}). \tag{3.44}$$

Therefore it suffices to verify conditions (a)–(b) of Assumption 2.2.5 for Π_{lin} in place of Π.

By Lemma 3.2.10(i) we know that ψ given by (3.17) provides a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{Q}, \Pi_{\mathrm{lin}}, \boldsymbol{X}, \boldsymbol{r})$ for any $\boldsymbol{Q} \in \mathcal{P}_\psi$. Take into account that a bounding function (see Definition 1.4.1) is independent of the set of all (admissible) strategies. Moreover parts (ii) and (iii) of Lemma 3.2.10 ensure that conditions (b)–(c) of Assumption 2.2.5 are satisfied for $\mathbb{M} := \mathbb{M}' := \mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}$, ψ given by (3.17), and Π_{lin} instead of Π.

Therefore, the assumptions of Theorem 2.2.8 hold, and an application of parts (i) and (ii) of the latter theorem implies the claims in parts (i) and (ii), respectively. Thus we have proved Theorem 3.2.9. □

The following Theorem 3.2.11 specifies the 'Hadamard derivative' of the optimal value functional of the terminal wealth problem (3.20) at (fixed) $\boldsymbol{P}$. For the formulation of this theorem, let $v_n^{\boldsymbol{P};\gamma_n}$ be defined as on the left-hand side of (3.21), and set $v_n^{\boldsymbol{P};\boldsymbol{\gamma}} := v_n^{\boldsymbol{P};\gamma_n}$ for any $n = 0,\dots,N-1$. Moreover, for any $n = 0,\dots,N-1$ denote by $\gamma_n^{\boldsymbol{P}}$ the unique solution to the optimization problem (3.21) (Lemma 3.2.4 ensures the existence of a unique solution), and set $\boldsymbol{\gamma}^{\boldsymbol{P}} := (\gamma_n^{\boldsymbol{P}})_{n=0}^{N-1}$. Finally recall Definitions 2.3.2 and 2.3.5(b)–(c).

Theorem 3.2.11 ('Differentiability' of $\mathcal{V}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ and $\mathcal{V}_0^{x_0}$ in $\boldsymbol{P}$) *In the setting above let $x_0 \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, and $\boldsymbol{P} \in \mathcal{P}_\psi$. Then the following two assertions hold.*

(i) *The map $\mathcal{V}_0^{x_0;\pi_{\boldsymbol{\gamma}}} : \mathcal{P}_\psi \to \mathbb{R}$ defined by (3.42) is 'Fréchet differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}, \psi)$ with 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi_{\boldsymbol{\gamma}}} : \mathcal{P}_\psi^{\boldsymbol{P};\pm} \to \mathbb{R}$ given by*

$$\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi_{\boldsymbol{\gamma}}}(\boldsymbol{Q}-\boldsymbol{P}) \,=\, \dot{\mathfrak{v}}_0^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} \cdot u_\alpha(x_0),$$

where

$$\dot{\mathfrak{v}}_0^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} := \sum_{k=0}^{N-1} v_{N-1}^{\boldsymbol{P};\boldsymbol{\gamma}} \cdots (v_k^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_k^{\boldsymbol{P};\boldsymbol{\gamma}}) \cdots v_0^{\boldsymbol{P};\boldsymbol{\gamma}}.$$

(ii) *The map $\mathcal{V}_0^{x_0} : \mathcal{P}_\psi \to \mathbb{R}$ defined by (3.42) is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}_{\mathrm{Höl},\alpha}, \psi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0} : \mathcal{P}_\psi{}^{\boldsymbol{P};\pm} \to \mathbb{R}$ given by*

$$\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}-\boldsymbol{P}) = \sup_{\pi\in\Pi_{\mathrm{lin}}(\boldsymbol{P})} \dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi}(\boldsymbol{Q}-\boldsymbol{P}) = \dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi_{\boldsymbol{\gamma}^{\boldsymbol{P}}}}(\boldsymbol{Q}-\boldsymbol{P}). \tag{3.45}$$

Remark 3.2.12 Basically, part (ii) of Theorem 2.3.11 yields the first "=" in (3.45) with $\Pi_{\mathrm{lin}}(\boldsymbol{P})$ replaced by $\Pi(\boldsymbol{P})$. Since part (ii) of Theorem 3.2.5 ensures that for any $\boldsymbol{P} \in \mathcal{P}_\psi$ there exists an optimal trading strategy which belongs to Π_{lin}, we may replace for *any* $\boldsymbol{P} \in \mathcal{P}_\psi$ in the representation (3.25) of the value function $V_0^{\boldsymbol{P}}(x_0)$ the set Π by Π_{lin} $(\subseteq \Pi)$. Therefore one can use Theorem 2.3.11 to derive the first "=" in (3.45). In the proof of Theorem 3.2.11 we will see that the second "=" in (3.45) is ensured by the second assertion in part (ii) of Theorem 3.2.5. ◇

For the evidence of Theorem 3.2.11 we will need the following lemma.

Lemma 3.2.13 *In the setting above let $\boldsymbol{P} \in \mathcal{P}_\psi$ and $\boldsymbol{\gamma} \in [0,1]^N$. Then the solution $(\dot{V}_k^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}})_{k=0}^N$ of the backward iteration scheme (2.33) admits the representation*

$$\dot{V}_n^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}}(x_n) = \dot{\mathfrak{v}}_n^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} \cdot u_\alpha(x_n/S_n^0) \tag{3.46}$$

for any $x_n \in \mathbb{R}_{\geq 0}$, $\boldsymbol{Q} \in \mathcal{P}_\psi$, and $n = 0, \ldots, N$, where

$$\dot{\mathfrak{v}}_n^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} := \sum_{k=n}^{N-1} v_{N-1}^{\boldsymbol{P};\boldsymbol{\gamma}} \cdots (v_k^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_k^{\boldsymbol{P};\boldsymbol{\gamma}}) \cdots v_n^{\boldsymbol{P};\boldsymbol{\gamma}}.$$

Proof We prove the assertion in (3.46) by (backward) induction on n. Note that in view of Lemma 3.2.10(i) and Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{Q}\}$) all occurring integrals in the following (exist and) are finite; see the discussion in Remark 2.3.16. For $n = N$, the assertion in (3.46) is valid because of (2.33) and by the choice $\dot{\mathfrak{v}}_N^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} := 0$. Now, assume that the assertion in (3.46) holds for $k \in \{n+1, \ldots, N\}$. Then, analogously to equation (3.30), we obtain by means of (2.33) and Lemma 3.2.6

$$\begin{aligned}
&\dot{V}_n^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}}(x_n) \\
&= \int_{\mathbb{R}_{\geq 0}} \dot{V}_{n+1}^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}}(y)\, P_n\big((x_n, f_n^{\gamma}(x_n)), dy\big) + \int_{\mathbb{R}_{\geq 0}} V_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}(y)\, (Q_n - P_n)\big((x_n, f_n^{\gamma}(x_n)), dy\big) \\
&= \int_{\mathbb{R}_{\geq 0}} \dot{\mathfrak{v}}_{n+1}^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(y/S_{n+1}^0)\, \mathfrak{m}_{n+1}^{\boldsymbol{P}} \circ \eta_{n,(x_n,f_n^{\gamma}(x_n))}^{-1}(dy) \\
&\quad + \int_{\mathbb{R}_{\geq 0}} \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(y/S_{n+1}^0) \mathfrak{m}_{n+1}^{\boldsymbol{Q}} \circ \eta_{n,(x_n,f_n^{\gamma}(x_n))}^{-1}(dy) - \int_{\mathbb{R}_{\geq 0}} \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(y/S_{n+1}^0) \mathfrak{m}_{n+1}^{\boldsymbol{P}} \circ \eta_{n,(x_n,f_n^{\gamma}(x_n))}^{-1}(dy) \\
&= \dot{\mathfrak{v}}_{n+1}^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} \cdot u_\alpha(x_n/S_n^0)\, v_n^{\boldsymbol{P};\boldsymbol{\gamma}} + \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\, u_\alpha(x_n/S_n^0)\, (v_n^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_n^{\boldsymbol{P};\boldsymbol{\gamma}}) = \dot{\mathfrak{v}}_n^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} \cdot u_\alpha(x_n/S_n^0)
\end{aligned}$$

for every $x_n \in \mathbb{R}_{\geq 0}$, where

$$\begin{aligned}
\dot{\mathfrak{v}}_n^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} &:= \dot{\mathfrak{v}}_{n+1}^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}}\, v_n^{\boldsymbol{P};\boldsymbol{\gamma}} + \mathfrak{v}_{n+1}^{\boldsymbol{P};\pi_{\boldsymbol{\gamma}}}\,(v_n^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_n^{\boldsymbol{P};\boldsymbol{\gamma}}) \\
&= \sum_{k=n+1}^{N-1} v_{N-1}^{\boldsymbol{P};\boldsymbol{\gamma}} \cdots (v_k^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_k^{\boldsymbol{P};\boldsymbol{\gamma}}) \cdots v_{n+1}^{\boldsymbol{P};\boldsymbol{\gamma}}\, v_n^{\boldsymbol{P};\boldsymbol{\gamma}} + \prod_{k=n+1}^{N-1} v_k^{\boldsymbol{P};\boldsymbol{\gamma}}\,(v_n^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_n^{\boldsymbol{P};\boldsymbol{\gamma}}) \\
&= \sum_{k=n}^{N-1} v_{N-1}^{\boldsymbol{P};\boldsymbol{\gamma}} \cdots (v_k^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_k^{\boldsymbol{P};\boldsymbol{\gamma}}) \cdots v_n^{\boldsymbol{P};\boldsymbol{\gamma}}.
\end{aligned}$$

This shows the assertion. □

With the help of Lemma 3.2.13 we are now able to prove Theorem 3.2.11.

Proof of Theorem 3.2.11: We intend to apply Theorem 2.3.11. First of all, note that Lemma 3.2.10 ensures that conditions (a)–(c) of Assumption 2.2.5 are satisfied for $\mathbb{M} := \mathbb{M}' := \mathbb{M}_{\mathrm{Höl},\alpha}$, ψ given by (3.17), and Π_{lin} instead of Π. Take into account that a bounding function (see Definition 1.4.1) is independent of the set of all (admissible) strategies. In particular, we have verified the assumptions of Theorem 2.3.11.

(i): It is an immediate consequence of part (i) of Theorem 2.3.11 that the functional $\mathcal{V}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ defined as in (3.42) is 'Fréchet differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}_{\mathrm{Höl},\alpha}, \psi)$. The corresponding 'Fréchet derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi_{\boldsymbol{\gamma}}}$ of $\mathcal{V}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ at $\boldsymbol{P}$ admits in view of Remark 2.3.16 and Lemma 3.2.13 (recall that $S_0^0 = 1$) the representation

$$\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi_{\boldsymbol{\gamma}}}(\boldsymbol{Q}-\boldsymbol{P}) \;=\; \dot{V}_0^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}}(x_0) \;=\; \dot{\mathfrak{v}}_0^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} \cdot u_\alpha(x_0)$$

for every $\boldsymbol{Q} \in \mathcal{P}_\psi$, where $\dot{\mathfrak{v}}_0^{\boldsymbol{P},\boldsymbol{Q};\pi_{\boldsymbol{\gamma}}} := \sum_{k=0}^{N-1} v_{N-1}^{\boldsymbol{P};\boldsymbol{\gamma}} \cdots (v_k^{\boldsymbol{Q};\boldsymbol{\gamma}} - v_k^{\boldsymbol{P};\boldsymbol{\gamma}}) \cdots v_0^{\boldsymbol{P};\boldsymbol{\gamma}}$.

(ii): For any $n = 0, \ldots, N-1$ let $\gamma_n^{\boldsymbol{P}} \in [0,1]$ be the unique solution to the optimization problem (3.21), and set $\boldsymbol{\gamma}^{\boldsymbol{P}} := (\gamma_n^{\boldsymbol{P}})_{n=0}^{N-1} \in [0,1]^N$. Then it follows from the first assertion in part (ii) of Theorem 3.2.5 that the linear trading strategy $\pi^{\boldsymbol{P}} = \pi_{\boldsymbol{\gamma}^{\boldsymbol{P}}} := (f_n^{\boldsymbol{\gamma}^{\boldsymbol{P}}})_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ defined by (3.26) is optimal w.r.t. $\boldsymbol{P}$. Therefore, the value functional $\mathcal{V}_0^{x_0}$ defined by (3.42) admits in view of Remark 2.3.14 the representation (3.44). As a consequence, part (ii) of Theorem 2.3.11 implies that the value functional $\mathcal{V}_0^{x_0}$ is 'Hadamard differentiable' at $\boldsymbol{P}$ w.r.t. $(\mathbb{M}_{\mathrm{Höl},\alpha}, \psi)$ with 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ given by

$$\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}-\boldsymbol{P}) \;=\; \sup_{\pi \in \Pi_{\mathrm{lin}}(\boldsymbol{P})} \dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi}(\boldsymbol{Q}-\boldsymbol{P}) \tag{3.47}$$

for any $\boldsymbol{Q} \in \mathcal{P}_\psi$. By the second assertion in part (ii) of Theorem 3.2.5 we have $\Pi_{\mathrm{lin}}(\boldsymbol{P}) = \{\pi_{\boldsymbol{\gamma}^{\boldsymbol{P}}}\}$ and therefore the representation of the 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ in (3.47) simplifies to

$$\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}-\boldsymbol{P}) \;=\; \sup_{\pi \in \Pi_{\mathrm{lin}}(\boldsymbol{P})} \dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi}(\boldsymbol{Q}-\boldsymbol{P}) \;=\; \dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0;\pi_{\boldsymbol{\gamma}^{\boldsymbol{P}}}}(\boldsymbol{Q}-\boldsymbol{P})$$

for every $\boldsymbol{Q} \in \mathcal{P}_\psi$. This completes the proof of Theorem 3.2.11. □

3.2.5 Numerical examples for the 'Hadamard derivative'

In this subsection we quantify by means of the 'Hadamard derivative' of the optimal value functional $\mathcal{V}_0^{x_0}$ (given by Theorem 3.2.11(ii)) the effect of incorporating an unlikely but significant jump in the dynamics $S = (S_0, \dots, S_N)$ of an asset price on the optimal value of the corresponding terminal wealth problem (3.20). At the end of this subsection we will also study the effect of incorporating more than one price jump.

We specifically focus on the setting of the time discretized Black–Scholes–Merton model from Example 3.2.8 with (mainly) $N = 12$. That is, we let $\mathfrak{r}_1 = \cdots = \mathfrak{r}_N = \mathfrak{r}$ for $\mathfrak{r} := \exp(\nu/N)$, where $\nu \in \mathbb{R}_{\geq 0}$. Moreover let $\boldsymbol{P}$ correspond to $\mathfrak{m}_1 = \cdots = \mathfrak{m}_N = \mathfrak{m}_{\boldsymbol{P}}$ for $\mathfrak{m}_{\boldsymbol{P}} := \mathrm{LN}_{(\mu_{\boldsymbol{P}} - \sigma_{\boldsymbol{P}}^2/2)/\mathrm{N}, \sigma_{\boldsymbol{P}}^2/\mathrm{N}}$, where $\mu_{\boldsymbol{P}} \in \mathbb{R}_{\geq 0}$ and $\sigma_{\boldsymbol{P}} \in \mathbb{R}_{>0}$ are chosen such that $\mu_{\boldsymbol{P}} > (1-\alpha)\sigma_{\boldsymbol{P}}^2$. In fact we let specifically $\mu_{\boldsymbol{P}} = 0.05$ and $\sigma_{\boldsymbol{P}} = 0.2$. This set of parameters is often used in numerical examples in the field of mathematical finance; see, e.g., [64, p. 898]. For the initial state we choose $x_0 = 1$. For the drift ν of the bond we will consider different values, all of them lying in $\{0.01, 0.02, 0.03, 0.035, 0.04\}$. Moreover, we let (mainly) $\alpha \in \{0.25, 0.5, 0.75\}$. Recall that α determines the degree of risk aversion of the agent; a small α corresponds to high risk aversion.

By a price jump at a fixed time $n \in \{0, \dots, N-1\}$ we mean that the asset's return $\mathfrak{R}_{n+1}$ is not anymore drawn from $\mathfrak{m}_{\boldsymbol{P}}$ but is given by a deterministic value $\Delta \in \mathbb{R}_{\geq 0}$ esstentially 'away' from 1. As appears from Table 3.3, in the case $N = 12$ it seems to be reasonable to speak of a 'jump' at least if $\Delta \leq 0.8$ or $\Delta \geq 1.25$. The probability under $\mathfrak{m}_{\boldsymbol{P}}$ for a realized return smaller than 0.8 (resp. larger than 1.25) is smaller than 0.0001. A realized return of ≤ 0.5 (resp. ≥ 1.5) is practically impossible; its probability under $\mathfrak{m}_{\boldsymbol{P}}$ is smaller than 10^{-30} (resp. 10^{-10}). That is, the choice $\Delta = 0.5$ or $\Delta = 1.5$ doubtlessly corresponds to a significant price jump.

t	10^{-30}	10^{-10}	0.0001	0.0005	0.005	0.01	0.025	0.05
$F_{\mathfrak{m}_{\boldsymbol{P}}}^{-1}(t)$	0.5172	0.6944	0.8088	0.8290	0.8639	0.8765	0.8952	0.9116
$F_{\mathfrak{m}_{\boldsymbol{P}}}^{-1}(1-t)$	1.9433	1.4474	1.2426	1.2126	1.1632	1.1466	1.1226	1.1024

Table 3.3: Some quantiles of the distribution $\mathfrak{m}_{\boldsymbol{P}}$ of the asset's return in the time discretized ($N = 12$) Black–Scholes–Merton model ($\mu_{\boldsymbol{P}} = 0.05$, $\sigma_{\boldsymbol{P}} = 0.2$).

If at a fixed time $\tau \in \{0, \dots, N-1\}$ a formerly nearly impossible 'jump' Δ can now occur with probability ε, then instead of $\mathfrak{m}_{\tau+1} = \mathfrak{m}_{\boldsymbol{P}}$ one has $\mathfrak{m}_{\tau+1} = (1-\varepsilon)\mathfrak{m}_{\boldsymbol{P}} + \varepsilon\delta_\Delta$. That is, instead of $\boldsymbol{P}$ the transition function is now given by $(1-\varepsilon)\boldsymbol{P} + \varepsilon\boldsymbol{Q}_{\Delta,\tau}$ with $\boldsymbol{Q}_{\Delta,\tau}$ generated through (3.18) by $\mathfrak{m}_{n+1} = \mathfrak{m}_{\boldsymbol{Q}_{\Delta,\tau;n}}$, $n = 0, \dots, N-1$, where

$$\mathfrak{m}_{\boldsymbol{Q}_{\Delta,\tau;n}} := \begin{cases} \delta_\Delta & , \quad n = \tau \\ \mathfrak{m}_{\boldsymbol{P}} & , \quad \text{otherwise} \end{cases} . \tag{3.48}$$

By part (ii) of Theorem 3.2.11 the 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ of the optimal value functional $\mathcal{V}_0^{x_0}$ evaluated at $\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P}$ can be written as

$$\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P}) \quad = \quad \sum_{k=0}^{N-1} v_{N-1}^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} \cdots (v_k^{\boldsymbol{Q}_{\Delta,\tau};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} - v_k^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}}) \cdots v_0^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}}$$

$$= \quad v_{N-1}^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} \cdots (v_\tau^{\boldsymbol{Q}_{\Delta,\tau};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} - v_\tau^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}}) \cdots v_0^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} \tag{3.49}$$

with $\boldsymbol{\gamma}_{\mathsf{BSM}}^{\boldsymbol{P}} := (\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}, \ldots, \gamma_{\mathsf{BSM}}^{\boldsymbol{P}})$, where $\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}$ is given by (3.41). The involved factors are

$$v_n^{\boldsymbol{P};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} = \begin{cases} 1 & , \quad \nu \in [\mu_{\boldsymbol{P}}, \infty) \\ \int_{\mathbb{R}_{\geq 0}} u_\alpha\big(1 + \frac{1}{1-\alpha}\frac{\mu_{\boldsymbol{P}}-\nu}{\sigma_{\boldsymbol{P}}^2}(\frac{y}{\mathfrak{r}} - 1)\big)\varphi^{\mathrm{LN}}_{(\mu_{\boldsymbol{P}}-\sigma_{\boldsymbol{P}}^2/2)/N, \sigma_{\boldsymbol{P}}^2/N}(y)\,\ell(dy) & , \quad \nu \in (\nu_{\boldsymbol{P},\alpha}, \mu_{\boldsymbol{P}}) \\ \mathfrak{r}^{-\alpha}\exp\big\{\frac{\alpha}{N}(\mu_{\boldsymbol{P}} - \frac{\sigma_{\boldsymbol{P}}^2}{2}) + \frac{(\alpha\sigma_{\boldsymbol{P}})^2}{2N}\big\} & , \quad \nu \in [0, \nu_{\boldsymbol{P},\alpha}], \end{cases}$$

$$v_n^{\boldsymbol{Q}_{\Delta,\tau};\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}} = \begin{cases} 1 & , \quad \nu \in [\mu_{\boldsymbol{P}}, \infty) \\ \int_{\mathbb{R}_{\geq 0}} u_\alpha\big(1 + \frac{1}{1-\alpha}\frac{\mu_{\boldsymbol{P}}-\nu}{\sigma_{\boldsymbol{P}}^2}(\frac{y}{\mathfrak{r}} - 1)\big)\,\mathfrak{m}_{\boldsymbol{Q}_{\Delta,\tau;n}}(dy) & , \quad \nu \in (\nu_{\boldsymbol{P},\alpha}, \mu_{\boldsymbol{P}}) \\ \mathfrak{r}^{-\alpha}\int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\,\mathfrak{m}_{\boldsymbol{Q}_{\Delta,\tau;n}}(dy) & , \quad \nu \in [0, \nu_{\boldsymbol{P},\alpha}] \end{cases} \tag{3.50}$$

for $n = 0, \ldots, N-1$, where $\nu_{\boldsymbol{P},\alpha} := \mu_{\boldsymbol{P}} - (1-\alpha)\sigma_{\boldsymbol{P}}^2$ $(\in (0, \mu_{\boldsymbol{P}}))$.

Note that $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P})$ is independent of τ, which can be seen from (3.48)–(3.50). That is, the effect of a jump is independent of the time at which the jump takes place. Also note that $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P}) \equiv 0$ when $\nu \in [\mu_{\boldsymbol{P}}, \infty)$. This is not surprising, because in this case the optimal fraction $\gamma_{\mathsf{BSM}}^{\boldsymbol{P}}$ to be invested into the asset is equal to 0 (see (3.41)) and the agent performs a complete investment in the bond at each trading time n.

Remark 3.2.14 As mentioned before, the 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ evaluated at $\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P}$ can be seen as the first-order sensitivity of the optimal value $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ w.r.t. a change of $\boldsymbol{P}$ to $(1-\varepsilon)\boldsymbol{P} + \varepsilon\boldsymbol{Q}_{\Delta,\tau}$, with $\varepsilon > 0$ small. It is a natural wish to compare these values for different $\Delta \in \mathbb{R}_{\geq 0}$. Lemma 3.2.15 below shows that the family $\{\boldsymbol{Q}_{\Delta,\tau} : \Delta \in [0, \delta]\}$ is relatively compact w.r.t. $d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\psi}$ (the proof does *not* work if $d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\psi}$ is replaced by $d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\phi}$ for any gauge function ϕ 'flatter' than ψ) for any fixed $\delta \in \mathbb{R}_{>0}$ (and $\tau \in \{0, \ldots, N-1\}$, $\alpha \in (0,1)$). As a consequence the approximation (2.1) with $\boldsymbol{Q} = \boldsymbol{Q}_{\Delta,\tau}$ holds uniformly in $\Delta \in [0, \delta]$, and therefore the values $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P})$, $\Delta \in [0, \delta]$, can be compared in view of Remark 2.3.3(ii) with each other with clear conscience. $\diamond$

Recall for the following lemma that relatively compact sets in the metric space $(\mathcal{P}_\psi, d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\psi})$ were introduced above of Definition 2.3.5.

Lemma 3.2.15 *In the setting above, the family* $\{\boldsymbol{Q}_{\Delta,\tau} : \Delta \in [0, \delta]\}$ *is relatively compact (w.r.t.* $d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\psi}$*) for any fixed* $\delta \in \mathbb{R}_{>0}$, $\tau \in \{0, \ldots, N-1\}$, *and* $\alpha \in (0,1)$.

Proof We will show that the set $\mathcal{K}_{\tau,\delta} := \{\boldsymbol{Q}_{\Delta,\tau} = (Q_{\Delta,\tau;n})_{n=0}^{N-1} : \Delta \in [0, \delta]\}$ $(\subseteq \mathcal{P}_\psi)$ is compact (w.r.t. $d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\psi}$). In particular, this implies that $\mathcal{K}_{\tau,\delta}$ is relatively compact (w.r.t. $d_{\infty,\mathbb{M}_{\mathrm{H\ddot{o}l},\alpha}}^{\psi}$).

Consider any sequence in $\mathcal{K}_{\tau,\delta}$. That is, in other words, pick any sequence $(\Delta_m) \in [0, \delta]^{\mathbb{N}}$ and consider the sequence $(\boldsymbol{Q}_{\Delta_m,\tau}) \in \mathcal{K}_{\tau,\delta}^{\mathbb{N}}$. Since $[0, \delta]$ is compact and thus sequentially compact (w.r.t. the Euclidean distance), we can find a subsequence $(\Delta_m')_{m\in\mathbb{N}}$ of $(\Delta_m)_{m\in\mathbb{N}}$ and some $\Delta_0 \in [0, \delta]$ such that $\Delta_m' \to \Delta_0$. Then $(\boldsymbol{Q}_{\Delta_m',\tau})_{m\in\mathbb{N}}$ is a subsequence of $(\boldsymbol{Q}_{\Delta_m,\tau})_{m\in\mathbb{N}}$, and $\boldsymbol{Q}_{\Delta_0,\tau} \in \mathcal{K}_{\tau,\delta}$. Thus in view of equations (3.15)–(3.18) and (3.48)

$$\Big|\int_{\mathbb{R}_{\geq 0}} h(y)\,Q_{\Delta_m',\tau;n}\big((x,a),dy\big) - \int_{\mathbb{R}_{\geq 0}} h(y)\,Q_{\Delta_0,\tau;n}\big((x,a),dy\big)\Big|$$

$$
\begin{aligned}
&= \Big|\int_{\mathbb{R}_{\geq 0}} h(y)\,\delta_{\Delta'_m}\circ\eta^{-1}_{\tau,(x,a)}(dy) - \int_{\mathbb{R}_{\geq 0}} h(y)\,\delta_{\Delta_0}\circ\eta^{-1}_{\tau,(x,a)}(dy)\Big| \\
&= \Big|\int_{\mathbb{R}_{\geq 0}} h\big(\eta_{\tau,(x,a)}(y)\big)\,\delta_{\Delta'_m}(dy) - \int_{\mathbb{R}_{\geq 0}} h\big(\eta_{\tau,(x,a)}(y)\big)\,\delta_{\Delta_0}(dy)\Big| = \big|h\big(\eta_{\tau,(x,a)}(\Delta'_m)\big) - h\big(\eta_{\tau,(x,a)}(\Delta_0)\big)\big| \\
&\leq \big|\eta_{\tau,(x,a)}(\Delta'_m) - \eta_{\tau,(x,a)}(\Delta_0)\big|^\alpha = a^\alpha\,|\Delta'_m - \Delta_0|^\alpha \leq x^\alpha\,|\Delta'_m - \Delta_0|^\alpha \leq \psi(x)\,|\Delta'_m - \Delta_0|^\alpha
\end{aligned}
$$

for any $h \in \mathbb{M}_{\text{Höl},\alpha}$, $(x,a) \in D_n$, $n = 0,\ldots,N-1$, and $m \in \mathbb{N}$. In view of (2.12), this implies $d^{\psi}_{\infty,\mathbb{M}_{\text{Höl},\alpha}}(\boldsymbol{Q}_{\Delta'_m,\tau}, \boldsymbol{Q}_{\Delta_0,\tau}) \to 0$. Hence, the assertion follows. □

By Remark 3.2.14 and (3.49) we are able to compare the effect of incorporating different 'jumps' Δ in the dynamics $S = (S_0,\ldots,S_N)$ of an asset price on the optimal value $\mathcal{V}_0^{x_0}(\boldsymbol{P})$.

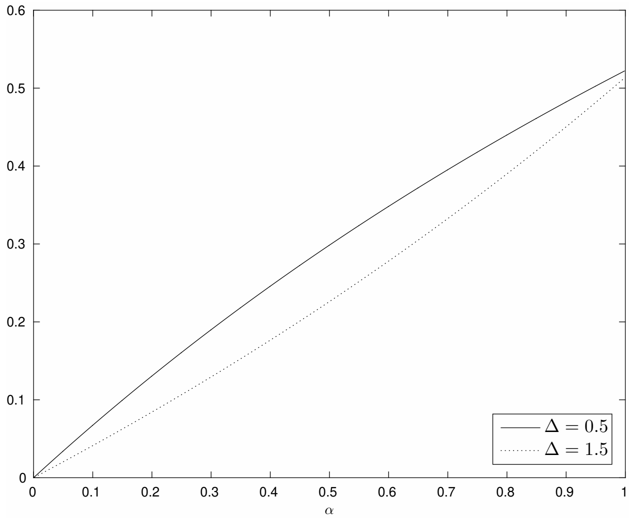

Figure 3.1: 'Hadamard derivative' $\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau}-\boldsymbol{P})$ (for $\Delta = 1.5$) and negative 'Hadamard derivative' $-\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau}-\boldsymbol{P})$ (for $\Delta = 0.5$) for $N = 12$, $\nu = 0.01$, $\mu_{\boldsymbol{P}} = 0.05$, and $\sigma_{\boldsymbol{P}} = 0.2$ in dependence of the risk aversion parameter α.

As appears from Figure 3.1 the negative effect of incorporating a 'jump' $\Delta = 0.5$ in the dynamics $S = (S_0,\ldots,S_N)$ of an asset price on $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ is larger than the positive effect of incorporating a 'jump' $\Delta = 1.5$ for every choice of the agent's degree of risk aversion. Figure 3.1 also shows the unsurprising effect that a high risk aversion (small value of α) leads to a negligible sensitivity.

Next we compare the values of $\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau}-\boldsymbol{P})$ for trading horizons $N \in \{4, 12, 52\}$ in dependence of the drift ν of the bond and the 'jump' Δ. This choices of N correspond respectively to a quarterly, monthly, and weekly time discretization. We will restrict ourselves to 'jumps' $\Delta \leq 0.8$. On the one hand, this ensures that the 'jumps' are significant; see the discussion above. On the other hand, as just discerned from Figure 3.1, the effect of jumps 'down' are more significant than jumps 'up'.

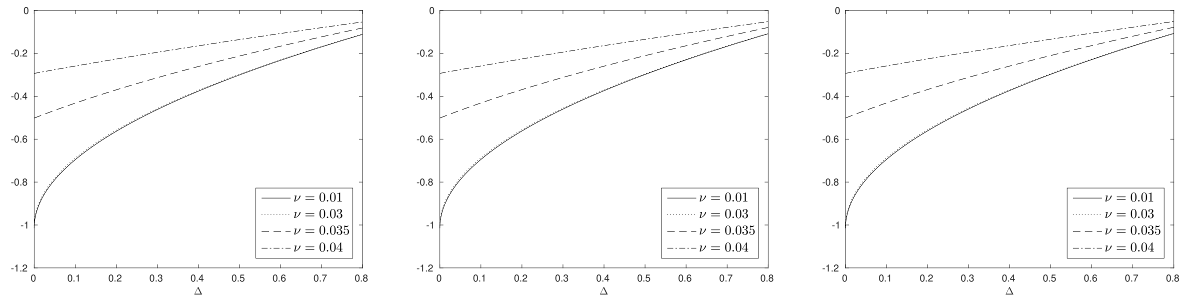

Figure 3.2: 'Hadamard derivative' $\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P})$ for $\alpha = 0.5$, $\mu_{\boldsymbol{P}} = 0.05$, and $\sigma_{\boldsymbol{P}} = 0.2$ in dependence of the 'jump' Δ and the drift ν of the bond, showing $N = 4$ in the first, $N = 12$ in the second, and $N = 52$ in the third column.

From Figure 3.2 one can see that for each trading time N and any $\Delta \in [0, 0.8]$ the (negative) effect of incorporating a 'jump' Δ in the dynamics $S = (S_0, \dots, S_N)$ of an asset price on $\mathcal{V}^{x_0}_0(\boldsymbol{P})$ is the smaller the smaller the spread between the drift $\mu_{\boldsymbol{P}}$ of the asset and the drift ν of the bond. There is only a tiny (nearly invisible) difference between the 'Hadamard derivative' $\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P})$ for the trading times $N \in \{4, 12, 52\}$. So the fineness of the discretization seems to play a minor part. Next we compare the values of $\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P})$ for the drift $\nu \in \{0.02, 0.03, 0.04\}$ of the bond in dependence of the risk aversion parameter α and the 'jump' Δ.

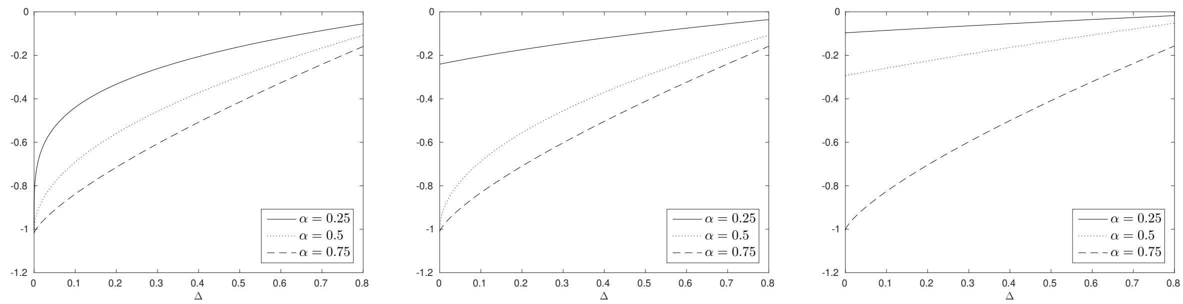

Figure 3.3: 'Hadamard derivative' $\dot{\mathcal{V}}^{x_0}_{0;\boldsymbol{P}}(\boldsymbol{Q}_{\Delta,\tau} - \boldsymbol{P})$ for $N = 12$, $\mu_{\boldsymbol{P}} = 0.05$, and $\sigma_{\boldsymbol{P}} = 0.2$ in dependence of the 'jump' Δ and risk aversion parameter α, showing $\nu = 0.02$ in the first, $\nu = 0.03$ in the second, and $\nu = 0.04$ in the third column.

As appears from Figure 3.3, for any $\Delta \in [0, 0.8]$ the (negative) effect of incorporating a 'jump' Δ in the dynamics $S = (S_0, \dots, S_N)$ of an asset price on $\mathcal{V}^{x_0}_0(\boldsymbol{P})$ is the smaller the higher the agent's risk aversion (i.e. the smaller the value of α), no matter what the drift $\nu \in \{0.02, 0.03, 0.04\}$ of the bond looks like. Take into account that the extent of this effect is influenced via (3.49)–(3.50) by the optimal fraction $\gamma^{\boldsymbol{P}}_{\mathsf{BSM}}$ to be invested into the asset which in turn depends on the risk aversion parameter α (see (3.41)).

To conclude this subsection, let us briefly touch on the case where more than one jump may appear. More precisely, instead of $\boldsymbol{Q}_{\Delta,\tau}$ (with $\tau \in \{0, \dots, N-1\}$) consider the transition function $\boldsymbol{Q}_{\Delta,\boldsymbol{\tau}(\ell)}$

(with $1 \le \ell \le N$, $\boldsymbol{\tau}(\ell) = (\tau_1, \ldots, \tau_\ell)$, $\tau_1, \ldots, \tau_\ell \in \{0, \ldots, N-1\}$ pairwise distinct) which is still generated by means of (3.48) but with the difference that at the ℓ different times $\tau_1, \ldots, \tau_\ell$ the distribution $\mathfrak{m}_{\boldsymbol{P}}$ is replaced by δ_Δ. Just as in the case $\ell = 1$, it turns out that it does not matter at which times $\tau_1, \ldots, \tau_\ell$ exactly these ℓ jumps occur. Figure 3.4 shows the value of $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}_{\Delta,\boldsymbol{\tau}(\ell)} - \boldsymbol{P})$ in dependence on ℓ and Δ.

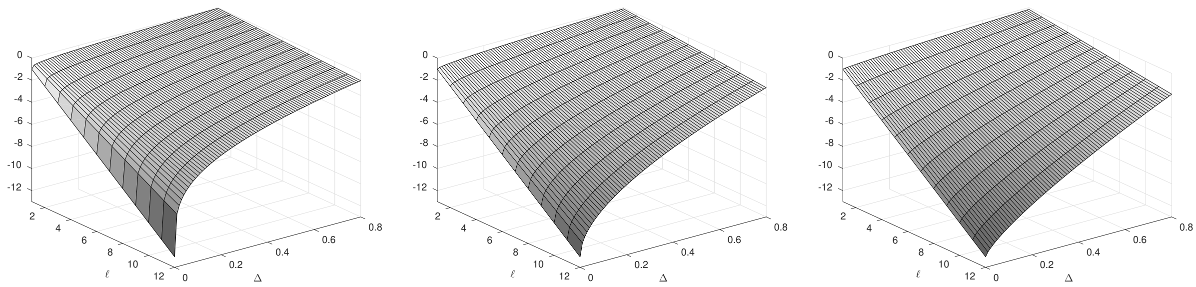

Figure 3.4: ‘Hadamard derivative’ $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}(\boldsymbol{Q}_{\Delta,\boldsymbol{\tau}(\ell)} - \boldsymbol{P})$ for $N = 12$ in dependence on $\ell \in \{1, \ldots, N\}$ and $\Delta \in [0, 0.8]$ showing $\alpha = 0.25$ and $\nu = 0.02$ (left), $\alpha = 0.5$ and $\nu = 0.03$ (middle), and $\alpha = 0.75$ and $\nu = 0.04$ (right).

As appears figure 3.4 it seems that for any $\Delta \in [0, 0.8]$ the first-order sensitivity of $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ w.r.t. a change of $\boldsymbol{P}$ to $(1-\varepsilon)\boldsymbol{P} + \varepsilon\boldsymbol{Q}_{\Delta,\boldsymbol{\tau}(\ell)}$ (with $\varepsilon > 0$ small) increases approximately linearly in ℓ,

Chapter 4

Statistical estimation of the optimal value in a specific Markov decision model

In the last chapters we have considered the situation where in the MDM the 'true' transition function is replaced by a less complex version. In this framework our elaborations in Chapter 2 showed that the (optimal) value function of a MDM regarded as a real-valued functional on some set of transition functions is sensitive w.r.t. changes in the underlying transition function, and we used these results (more precisely the 'derivative' of the value functional) to evaluate model reductions in the transition function with respect to their influences on the optimal value of the MDM. These elaborations are motivated by the fact that in many real applications the transition probabilities (and thus the corresponding transition function) are unknown and must be estimated using statistical methods which can lead to incorrect estimates for the optimal value of the corresponding MDM, for example due to the lack of missing historical data. Detached from these investigations, users, such as operations engineers, become also increasingly interested in a concrete and easy to handle statistical estimation of the optimal value of a MDM with unknown transition function. In this chapter we would like to go into this in more detail.

The objective of our following elaborations is the statistical estimation of the optimal value $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ (for some given initial state $x_0 \in E$) of a MDM in which the corresponding transition function $\boldsymbol{P}$ is *not* known. Therefore our main task is to find a suitable estimator for the unknown transition function $\boldsymbol{P}$. If $\widehat{\boldsymbol{P}}_m$ corresponds to an appropriate estimator for the transition function $\boldsymbol{P}$, then a natural choice for an estimator for the optimal value $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ will be the plug-in estimator $\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m)$. In the sequel, we want to establish asymptotic properties of the (plug-in) estimator $\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m)$.

One possible approach to determine the asymptotics of the estimator $\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m)$ is to make use of the regularity results from Chapter 2. In fact, by means of the 'Lipschitz continuity' property of the value functional (known from Theorem 2.2.8 in Subsection 2.2.2) we obtain strong consistency of the sequence of estimators $(\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m))_{m\in\mathbb{N}}$ for the optimal value $\mathcal{V}_0^{x_0}(\boldsymbol{P})$ if (under Assumption 2.2.5 for some gauge function ψ and $\mathbb{M} \subseteq \mathbb{M}_\psi(E)$) the sequence $(\widehat{\boldsymbol{P}}_m)_{m\in\mathbb{N}}$ satisfies a strong law w.r.t. the (semi-) metric $d_{\infty,\mathbb{M}}^{\psi}$ given by (2.12) with $\phi := \psi$. However, a verification of such a strong law for the sequence $(\widehat{\boldsymbol{P}}_m)_{m\in\mathbb{N}}$ is in general difficult, as this property depends on the choice of the set $\mathbb{M}$ (and gauge function ψ). Besides this, for the asymptotic error distribution of the sequence of estimators $(\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m))_{m\in\mathbb{N}}$, we can *not* apply part (ii) of Theorem 2.3.11 and an adapted functional delta method because in view of

$$\sqrt{m}\big(\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m) - \mathcal{V}_0^{x_0}(\boldsymbol{P})\big) \;=\; \frac{\mathcal{V}_0^{x_0}(\boldsymbol{P} + \frac{1}{\sqrt{m}}(\sqrt{m}(\widehat{\boldsymbol{P}}_m - \boldsymbol{P}))) - \mathcal{V}_0^{x_0}(\boldsymbol{P})}{1/\sqrt{m}}$$

the expression $\sqrt{m}(\widehat{\boldsymbol{P}}_m - \boldsymbol{P})$ is in general *not* contained in the 'tangent space' $\mathcal{P}_{\psi}^{\boldsymbol{P};\pm}$ (see (2.22)) which corresponds to the domain of the 'Hadamard derivative' $\dot{\mathcal{V}}_{0;\boldsymbol{P}}^{x_0}$ of the (optimal) value functional $\mathcal{V}_0^{x_0}$. Therefore we can *not* determine in general the asymptotic distribution of the sequence of plug-in estimator $(\mathcal{V}_0^{x_0}(\widehat{\boldsymbol{P}}_m))_{m\in\mathbb{N}}$.

However, in order to be able to carry out a meaningful statistical inference for the optimal value in the unknown transition function, we will restrict ourselves in the following to a specific MDM in which the corresponding transition function is generated only by some (unknown) single distribution function F. In this particular case we are in the position to perform a detailed study of the asymptotics of the corresponding estimator for the optimal value of the simple MDM. That is, if $\boldsymbol{P}_F$ denotes the (unknown) transition function whose corresponding transition probabilities are governed by F, it suffices for the statistical investigation of the optimal value $\mathcal{V}_0^{x_0}(\boldsymbol{P}_F)$ to estimate the (unknown) distribution function F. If $\widehat{F}_m$ is a reasonable estimator for F, then $\boldsymbol{P}_{\widehat{F}_m}$ can be a reasonable estimator for $\boldsymbol{P}_F$. In this case, a canonical estimator for the optimal value $\mathcal{W}_0^{x_0}(F) := \mathcal{V}_0^{x_0}(\boldsymbol{P}_F)$ is given by the plug-in estimator $\mathcal{W}_0^{x_0}(\widehat{F}_m) := \mathcal{V}_0^{x_0}(\boldsymbol{P}_{\widehat{F}_m})$. In Sections 4.4–4.5, we will describe in detail two methods by which the (unknown) distribution function and thus the optimal value $\mathcal{W}_0^{x_0}(F)$ of a simple MDM can be statistically estimated, and we present some asymptotic properties of the corresponding estimator $\mathcal{W}_0^{x_0}(\widehat{F}_m)$.

The rest of this chapter is organized as follows. At first, in Sections 4.1–4.2 we will introduce based on the elaborations in Section 1.1 the underlying Markov decision model in which the corresponding transition probability function is governed by some single distribution function, and define the value function which specifies the optimal value of a simplified version of the optimization problem (1.12), where $\boldsymbol{P}$ is replaced by $\boldsymbol{P}_F$. Afterwards, in Section 4.3, we will show that the (optimal) value functional $\mathcal{W}_0^{x_0}$ is continuous and functionally differentiable in a certain sense. These regularity results will be used in Sections 4.4–4.5 to derive asymptotic properties of the plug-in estimator $\mathcal{W}_0^{x_0}(\widehat{F}_m)$ for the optimal value $\mathcal{W}_0^{x_0}(F)$ in a nonparametric and a parametric framework.

4.1 Basic Markov decision model

Based on the elaborations in Section 1.1 we formally introduce in this section our discrete time Markov decision model (MDM) with finite time horizon in which the transition function (and thus the transition probabilities) are governed by an (unknown) single distribution function. The model components of the underlying MDM will be analogously defined as in Subsections 1.1.1–1.1.3.

Let $N \in \mathbb{N}$ be again a fixed time horizon in discrete time, and let E be a non-empty set referred to as state space which is equipped with a σ-algebra $\mathcal{E}$. Let $\boldsymbol{A}$ be the family of all non-empty sets $A_n(x)$, $x \in E$, $n = 0, \ldots, N-1$, where $A_n(x)$ consists of all admissible actions in state x at time n. Moreover let A_n and D_n be for every $n = 0, \ldots, N-1$ the sets of all (allowable) actions and possible state-action combinations at time n as defined in (1.1).

Further let $\overline{\boldsymbol{F}}$ be the set of all distribution functions on $\mathbb{R}$, and fix a subset $\boldsymbol{F} \subseteq \overline{\boldsymbol{F}}$. Let $\boldsymbol{P}_F = (P_n^F)_{n=0}^{N-1} \in \overline{\mathcal{P}}$ be for any $F \in \boldsymbol{F}$ a fixed transition function (as introduced in Subsection 1.1.1) which consists of (one-step) transition probabilities $P_n^F((x,a), \bullet)$, $(x,a) \in D_n$, $n = 0, \dots, N-1$, that are parametrized by the distribution function F. Recall from Subsection 1.1.1 that $\overline{\mathcal{P}}$ stands for the set of all transition functions. Let $\overline{\mathbb{F}}_n$ be for every $n = 0, \dots, N-1$ a non-empty set consisting of all (deterministic and Markovian) decision rules at time n as defined in Subsection 1.1.1, and fix $\mathbb{F}_n \subseteq \overline{\mathbb{F}}_n$. Note that the elements of $\mathbb{F}_n$ can be seen as admissible decision rules at time n. Further, we set $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$, and recall that any element $\pi = (f_n)_{n=0}^{N-1}$ of Π is an (admissible) strategy.

Let $(\Omega, \mathcal{F}) := (E^{N+1}, \mathcal{E}^{\otimes(N+1)})$ and $\boldsymbol{X} = (X_n)_{n=0}^N$ be the map defined on Ω through (1.5). It follows from Lemma 1.1.1 that the random variable $\boldsymbol{X}$ corresponds for any initial state $x_0 \in E$, distribution function $F \in \boldsymbol{F}$, and strategy $\pi \in \Pi$ to the (finite horizon discrete time) Markov decision process (MDP) under law $\mathbb{P}^{x_0, \boldsymbol{P}_F, \pi}$, where the probability measure $\mathbb{P}^{x_0, \boldsymbol{P}_F, \pi}$ on $(\Omega, \mathcal{F})$ is defined as in (1.4). Moreover, the vector $\boldsymbol{r} := (r_n)_{n=0}^N$ contains of $(\mathcal{D}_n, \mathcal{B}(\mathbb{R}))$- and $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable maps $r_n : D_n \to \mathbb{R}$, $n = 0, \dots, N-1$, and $r_N : E \to \mathbb{R}$, respectively. As before, r_n and r_N correspond to the one-stage- and the terminal reward function, respectively.

Then, similarly to Definition 1.1.3, the sextuple

$$(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r}) \tag{4.1}$$

is called (finite horizon discrete time) Markov decision model (MDM) associated with state space E, the family of all action spaces $\boldsymbol{A}$, transition function $\boldsymbol{P}_F \in \overline{\mathcal{P}}$, set of admissible strategies Π, and reward functions $\boldsymbol{r}$.

4.2 Value function and optimal strategies

Using the notation of Section 4.1 we introduce in this section the value function of the MDM $(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r})$ which can be derived from an analogous sequential optimization control problem as in Section 1.2 based on the expected total reward criterion.

Let $\boldsymbol{F} \subseteq \overline{\boldsymbol{F}}$ be a fixed subset, and let ν be any measure on $\mathcal{B}(\mathbb{R})$. Denote by $\boldsymbol{F}(\nu)$ the subset of all $F \in \boldsymbol{F}$ satisfying

$$\int_{\mathbb{R}_{<0}} F \, d\nu < \infty \quad \text{and} \quad \int_{\mathbb{R}_{\geq 0}} (1 - F) \, d\nu < \infty. \tag{4.2}$$

For our subsequent analysis we let $\boldsymbol{P}_F = (P_n^F)_{n=0}^{N-1} \in \overline{\mathcal{P}}$ be for any $F \in \boldsymbol{F}(\nu)$ a fixed transition function. In the following $\boldsymbol{P}_F$ will be referred to as *transition function associated with* $F \in \boldsymbol{F}(\nu)$. Moreover let $\psi : E \to \mathbb{R}_{\geq 1}$ be any gauge function (as introduced in Section 1.4), and fix some subset $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$, where the set $\overline{\mathcal{P}}_\psi$ is defined as in Subsection 2.1.2.

In the sequel, we will always assume that the following Assumption 4.2.1 holds. Recall Definition 1.4.1, and note that the conditions in Assumption 4.2.1 will be illustrated later in the examples of Chapter 5. It follows from the discussion in Remark 2.2.6(i) that condition (a) of Assumption 4.2.1 is in line with the classical literature on MDMs.

Assumption 4.2.1 *The following two assertions hold for any* $F \in \boldsymbol{F}(\nu)$.

(a) *ψ is a bounding function for the MDM* $(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r})$.

(b) $\boldsymbol{P}_F \in \mathcal{P}_\psi$.

Under Assumption 4.2.1, it follows from Proposition 1.4.3 (applied to $\mathcal{P} := \{\boldsymbol{P}_F\}$) that we may define in a MDM $(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for any $F \in \boldsymbol{F}(\nu)$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 0, \ldots, N$ a map $V_n^{F;\pi} : E \to \mathbb{R}$ via

$$V_n^{F;\pi}(x_n) := \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\Big[\sum_{k=n}^{N-1} r_k(X_k, f_k(X_k)) + r_N(X_N)\Big]. \tag{4.3}$$

Recall that $\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}$ corresponds to the expectation w.r.t. the factorized conditional distribution $\mathbb{P}^{x_0,\boldsymbol{P}_F;\pi}[\,\bullet\,\|X_n = x_n]$; see Lemma 1.1.1. Similarly to Section 1.2, the value $V_n^{F;\pi}(x_n)$ specifies the expected total reward from time n to N of $\boldsymbol{X}$ under $\mathbb{P}^{x_0,\boldsymbol{P}_F;\pi}$ when strategy π is used and $\boldsymbol{X}$ is in state x_n at time n. Hence the map $V_n^{F;\pi}$ given by (4.3) will be referred to as *policy value function (at time n)*.

Then similarly to (1.12) and for any fixed $F \in \boldsymbol{F}(\nu)$, we are looking for those strategies $\pi \in \Pi$ for which the policy value function $V_0^{F;\pi}(x_0)$ at time 0 is maximal for all initial states $x_0 \in E$:

$$V_0^{F;\pi}(x_0) \longrightarrow \max \;\; (\text{in } \pi \in \Pi)\,! \tag{4.4}$$

In particular, the maximal expected total reward from time n to N of $\boldsymbol{X}$ under $\mathbb{P}^{x_0,\boldsymbol{P}_F;\pi}$ when strategy π is used and $\boldsymbol{X}$ is in state x_n at time n will be described by the map $V_n^F : E \to \mathbb{R}$ defined by

$$V_n^F(x_n) := \sup_{\pi \in \Pi} V_n^{F;\pi}(x_n) \tag{4.5}$$

which will be referred to as *value function (at time n)*. Note that it follows from Proposition 1.4.3 (applied to $\mathcal{P} := \{\boldsymbol{P}_F\}$) that under Assumption 4.2.1 the value function V_n^F is well-defined. Also note that the value function V_n^F is in general not $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable. It follows from the discussion in Remark 1.2.4 that in some situations the measurability holds true.

Moreover we will say that for any fixed $F \in \boldsymbol{F}(\nu)$ a strategy $\pi^F \in \Pi$ is *optimal w.r.t. F* if $V_0^{F;\pi^F}(x_0) = V_0^F(x_0)$ for all $x_0 \in E$. In this case $V_0^{F;\pi^F}(x_0)$ is called optimal value (function), and the set of all optimal strategies w.r.t. F will be denoted by $\Pi(F)$. Moreover, for given $\delta > 0$, a strategy $\pi^{F;\delta} \in \Pi$ is said to be δ-optimal w.r.t. F if $V_0^F(x_0) - \delta \leq V_0^{F;\pi^{F;\delta}}(x_0)$ for all $x_0 \in E$, and we denote by $\Pi(F;\delta)$ the set of all δ-optimal strategies w.r.t. F. To conclude this section we note that the discussion subsequent to Definition 1.2.5 and the elaborations in Remark 1.2.6 can be transferred in an analogous way to the setting introduced in Section 4.1.

4.3 Regularity of the value function

Maintain the notation and terminology introduced in Sections 4.1–4.2. Let ν be any measure on $\mathcal{B}(\mathbb{R})$, and fix $\boldsymbol{F} \subseteq \overline{\boldsymbol{F}}$. Moreover let $\boldsymbol{F}(\nu)$ be the subset of all $F \in \boldsymbol{F}$ satisfying (4.2), and let ψ be any gauge function. Note that we fixed for any $F \in \boldsymbol{F}(\nu)$ a transition function $\boldsymbol{P}_F \in \overline{\mathcal{P}}$, and recall Assumption 4.2.1.

In this section we will investigate the (policy) value function of the MDM $(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r})$ (with $F \in \boldsymbol{F}(\nu)$) regarded as a real-valued functional on a set of distribution functions for continuity and differentiability. By means of these regularity results, we are able to derive asymptotic properties of certain plug-in estimators for the value function in nonparametric and parametric statistical models, respectively; see Sections 4.4 and 4.5.

To this end, we consider in the following for any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$ maps $\mathcal{W}_n^{x_n;\pi} : \boldsymbol{F}(\nu) \to \mathbb{R}$ and $\mathcal{W}_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by

$$\mathcal{W}_n^{x_n;\pi}(F) := V_n^{F;\pi}(x_n) \quad \text{and} \quad \mathcal{W}_n^{x_n}(F) := \sup_{\pi \in \Pi} \mathcal{W}_n^{x_n;\pi}(F) \; \big(= V_n^F(x_n) \big), \tag{4.6}$$

where $V_n^{F;\pi}$ as well as V_n^F are introduced in (4.3) and (4.5), respectively. Note that it follows from the discussion in Section 4.2 that (under Assumption 4.2.1) the maps $\mathcal{W}_n^{x_n;\pi}$ and $\mathcal{W}_n^{x_n}$ given by (4.6) are well-defined for any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$, respectively. Also note that under Assumption 4.2.1) we have $\mathcal{W}_n^{x_n;\pi}(F) = \mathcal{V}_n^{x_n;\pi}(\boldsymbol{P}_F)$ as well as $\mathcal{W}_n^{x_n}(F) = \mathcal{V}_n^{x_n}(\boldsymbol{P}_F)$ for every $F \in \boldsymbol{F}(\nu)$, $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$, where the functionals $\mathcal{V}_n^{x_n;\pi}$ and $\mathcal{V}_n^{x_n}$ are introduced in (2.16). Analogously to the notion 'value function' we will refer in the sequel to $\mathcal{W}_n^{x_n}$ as *value functional given state x_n at time n*. If $\pi^F \in \Pi$ is for some given $F \in \boldsymbol{F}(\nu)$ an optimal strategy w.r.t. F, then $\mathcal{W}_0^{x_0}(F)$ $(= \mathcal{W}_0^{x_0;\pi^F}(F))$ corresponds (for any $x_0 \in E$) to the optimal value.

Let $\boldsymbol{L}_0(\nu)$ be the space of all Borel measurable maps $h \in \mathbb{R}^{\mathbb{R}}$ modulo the equivalence relation of ν-almost sure identity. Furthermore, let $\boldsymbol{L}_1(\nu)$ be the subspace of all $h \in \boldsymbol{L}_0(\nu)$ for which

$$\|h\|_{1,\nu} := \int |h(y)| \, \nu(dy) \tag{4.7}$$

is finite. Here and in the sequel we will suppress the range of integration if it is the whole real line. It follows from Corollary 4.1.2 and Theorem 4.1.3 in [20] that the map $\| \cdot \|_{1,\nu} : \boldsymbol{L}_1(\nu) \to \mathbb{R}_{\geq 0}$ defined by (4.7) provides a complete norm on $\boldsymbol{L}_1(\nu)$. That is, the vector space $\boldsymbol{L}_1(\nu)$ equipped with the norm $\| \cdot \|_{1,\nu}$ is a Banach space.

4.3.1 'Continuity' in F of the value function

In this subsection we will use the notion of 'Lipschitz continuity' introduced in Definition 4.3.1 below. Since this notion is weaker compared to the usual concept of Lipschitz continuity, we will use inverted commas and write 'Lipschitz continuity' in place of Lipschitz continuity.

To explain our concept of 'Lipschitz continuity' more explicitly, we note that we use in the sequel the norm $\| \cdot \|_{1,\nu}$ to measure the distance between distribution functions from $\boldsymbol{F}(\nu)$. Take into account that $F - G \in \boldsymbol{L}_1(\nu)$ holds for every $F, G \in \boldsymbol{F}(\nu)$. Let $(L, \| \cdot \|_L)$ be a normed vector space.

Definition 4.3.1 ('Lipschitz continuity' in F) *Let $F \in \boldsymbol{F}(\nu)$. A map $\mathcal{W} : \boldsymbol{F}(\nu) \to L$ is said to be 'Lipschitz continuous' at F w.r.t. $(\| \cdot \|_{1,\nu}, \| \cdot \|_L)$ if*

$$\big\| \mathcal{W}(F_m) - \mathcal{W}(F) \big\|_L = \mathcal{O}\big(\|F_m - F\|_{1,\nu} \big)$$

holds for every sequence $(F_m) \in \boldsymbol{F}(\nu)^{\mathbb{N}}$ with $\|F_m - F\|_{1,\nu} \to 0$.

Similarly to the elaborations in Subsection 2.2.1, the notation $\mathcal{O}(\|F_m - F\|_{1,\nu})$ refers in the setting of Definition 4.3.1 to any real-valued sequence $(c_m)_{m\in\mathbb{N}}$ for which the sequence $(c_m\,\|F_m - F\|_{1,\nu}^{-1})_{m\in\mathbb{N}}$ is bounded.

Remark 4.3.2 We note that the concept of quasi-Lipschitz continuity (along $\boldsymbol{L}_1(\nu)$) in the sense of Definition A.3(iii) in Section A is more general compared to the notion of 'Lipschitz continuity' introduced in Definition 4.3.1. ◇

For any fixed $F \in \boldsymbol{F}(\nu)$ as well as any $x \in E$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, and $n = 0, \dots, N-1$, we will consider in the following the maps $\Lambda_n^{F;(\pi,x)} : \boldsymbol{F}(\nu) \to \mathbb{R}$ and $\Phi_n^{(\pi,x)} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by

$$\Lambda_n^{F;(\pi,x)}(G) := \int_E V_{n+1}^{F;\pi}(y)\, P_n^G\big((x, f_n(x)), dy\big) \quad \text{and} \quad \Phi_n^{(\pi,x)}(G) := \int_E \psi(y)\, P_n^G\big((x, f_n(x)), dy\big). \tag{4.8}$$

It follows from conditions (a)–(b) of Assumption 4.2.1 along with Definition 1.4.1 as well as Proposition 1.4.3 (applied to $\mathcal{P} := \{\boldsymbol{P}_F\}$) that the maps $\Lambda_n^{F;(\pi,x)}$ and $\Phi_n^{(\pi,x)}$ in (4.8) are well-defined. Using the same arguments one can even show that

$$\sup_{\pi\in\Pi} \|\Lambda_n^{F;(\pi,\,\cdot\,)}(G)\|_\psi < \infty \quad \text{and} \quad \sup_{\pi\in\Pi} \|\Phi_n^{(\pi,\,\cdot\,)}(G)\|_\psi < \infty \tag{4.9}$$

for any fixed $F \in \boldsymbol{F}(\nu)$, and every $n = 0, \dots, N-1$ and $G \in \boldsymbol{F}(\nu)$. Recall from Section 1.4 that $\|\cdot\|_\psi$ refers to the weighted sup-norm introduced in (1.18).

Part (ii) of Theorem 4.3.3 shows that the value functional $\mathcal{W}_n^{x_n}$ is 'Lipschitz continuous' at fixed $F \in \boldsymbol{F}(\nu)$ w.r.t. $(\|\cdot\|_{1,\nu}, |\cdot|)$. Conditions (a)–(b) of this theorem involve for any fixed $F \in \boldsymbol{F}(\nu)$ as well as any $n = 0, \dots, N-1$ the maps $\boldsymbol{\Lambda}_n^F : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ and $\boldsymbol{\Phi}_n : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ defined by

$$\boldsymbol{\Lambda}_n^F(G) := \big(\Lambda_n^{F;(\pi,x)}(G)\big)_{(\pi,x)\in\Pi\times E} \quad \text{and} \quad \boldsymbol{\Phi}_n(G) := \big(\Phi_n^{(\pi,x)}(G)\big)_{(\pi,x)\in\Pi\times E}, \tag{4.10}$$

where $\Lambda_n^{F;(\pi,x)}$ and $\Phi_n^{(\pi,x)}$ are given by (4.8) and $\ell_\psi^\infty(\Pi \times E)$ stands for the space of all bounded real-valued functions on $\Pi \times E$ equipped with the norm $\|\cdot\|_{\infty,\psi}$ defined by (4.11). It is easily seen that the assignment

$$\|h\|_{\infty,\psi} := \sup_{\pi\in\Pi} \|h(\pi,\,\cdot\,)\|_\psi, \qquad h = (h(\pi,x))_{(\pi,x)\in\Pi\times E} \in \ell_\psi^\infty(\Pi \times E) \tag{4.11}$$

defines a map $\|\cdot\|_{\infty,\psi} : \ell_\psi^\infty(\Pi \times E) \to \mathbb{R}_{\ge 0}$ which indeed provides a norm on $\ell_\psi^\infty(\Pi \times E)$. It follows from (4.9) and (4.11) that the maps $\boldsymbol{\Lambda}_n^F$ and $\boldsymbol{\Phi}_n$ given by (4.10) are well-defined for any fixed $F \in \boldsymbol{F}(\nu)$ and any $n = 0, \dots, N-1$.

Theorem 4.3.3 ('Lipschitz continuity' of $\mathcal{W}_n^{x_n;\pi}$ and $\mathcal{W}_n^{x_n}$ in F) *Under Assumption 4.2.1 let $F \in \boldsymbol{F}(\nu)$, and assume that the following two conditions hold for any $n = 0, \dots, N-1$.*

(a) *The map $\boldsymbol{\Lambda}_n^F : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ defined by (4.10) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\nu}, \|\cdot\|_{\infty,\psi})$.*

(b) *The map $\boldsymbol{\Phi}_n : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ defined by (4.10) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\nu}, \|\cdot\|_{\infty,\psi})$.*

Then the following two assertions hold.

(i) *For any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$, the map $\mathcal{W}_n^{x_n;\pi} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by (4.6) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\nu}, |\cdot|)$.*

(ii) *For any $x_n \in E$ and $n = 0, \dots, N$, the map $\mathcal{W}_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by (4.6) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\nu}, |\cdot|)$.*

If $\widehat{F}_m$ is a reasonable estimator for the unknown distribution function F, then by the 'Lipschitz continuity' of the value functional $\mathcal{W}_n^{x_n}$ (known from part (ii) Theorem 4.3.3) we are in the position to easily derive in several situations a strong law of the (plug-in) estimator $\mathcal{W}_n^{x_n}(\widehat{F}_m)$ for the aspect $\mathcal{W}_n^{x_n}(F)$; see Subsections 4.4.1 and 4.5.1 for details.

Remark 4.3.4 (i) It follows from the proof of Theorem 4.3.3 below that (under the assumptions of Theorem 4.3.3) the 'Lipschitz continuity' of the map $\mathcal{W}_n^{x_n;\pi}$ in part (i) of Theorem 4.3.3 holds even uniformly in $\pi \in \Pi$. That is, for any fixed $F \in \boldsymbol{F}(\nu)$, we have

$$\sup_{\pi\in\Pi} \big|\mathcal{W}_n^{x_n;\pi}(F_m) - \mathcal{W}_n^{x_n;\pi}(F)\big| \;=\; \mathcal{O}\big(\|F_m - F\|_{1,\nu}\big)$$

for every $x_n \in E$ and $n = 0, \dots, N$ as well as any sequence $(F_m) \in \boldsymbol{F}(\nu)^{\mathbb{N}}$ with $\|F_m - F\|_{1,\nu} \to 0$.

(ii) The proof of Theorem 4.3.3 below reveals that (under the assumptions of Theorem 4.3.3) the claims in (i) and (ii) of the latter theorem are also valid if in conditions (a)–(b) of Theorem 4.3.3 the norm $\|\cdot\|_{\infty,\psi}$ is replaced by the norm $\|\cdot\|_{\infty,1}$ which is defined as in (4.11) with $\psi :\equiv 1$. However, the shape of the norm $\|\cdot\|_{\infty,\psi}$ is motivated by the elaborations in Section 5.2. More precisely, we can *not* verify the statements in Theorem 5.2.2 in Subsection 5.2.2 if in conditions (a)–(b) of Theorem 4.3.3 this norm is replaced by the (stricter) norm $\|\cdot\|_{\infty,1}$. ◇

In applications, conditions (a) and (b) of Theorem 4.3.3 may be difficult to verify. Against this background, the following remark might be helpful in some situations; for an illustration see Subsection 5.2.2.

Remark 4.3.5 In some situations it turns out that for *every* $F \in \boldsymbol{F}(\nu)$ the solution of the optimization problem (4.4) does not change if Π is replaced by a subset $\Pi' \subseteq \Pi$ (being independent of F). Then in the definition (4.5) of the value function (at time 0) the set Π can be replaced by the subset Π'. Of course, in this case it suffices to ensure that conditions (a)–(b) of Theorem 4.3.3 are satisfied for the subset Π' instead of Π. ◇

Now, let us turn to the proof of Theorem 4.3.3.

Proof of Theorem 4.3.3: We will prove only the assertion in (ii). The claim in part (i) will follow with similar arguments. Let $x_n \in E$ as well as $n = 0, \dots, N$ be arbitrary but fixed. Moreover let $(F_m)_{m\in\mathbb{N}}$ be any sequence in $\boldsymbol{F}(\nu)$ with $\|F_m - F\|_{1,\nu} \to 0$. Using analogous arguments as in the proof of Theorem 2.2.8, we obtain by means of (4.6), (4.3), and (4.8) for any $m \in \mathbb{N}$

$$\big|\mathcal{W}_n^{x_n}(F_m) - \mathcal{W}_n^{x_n}(F)\big|$$

$$
\begin{aligned}
&= \Big|\sup_{\pi\in\Pi}\mathcal{W}_n^{x_n;\pi}(F_m) - \sup_{\pi\in\Pi}\mathcal{W}_n^{x_n;\pi}(F)\Big| \\
&\le \sup_{\pi\in\Pi}\big|\mathcal{W}_n^{x_n;\pi}(F_m) - \mathcal{W}_n^{x_n;\pi}(F)\big| \\
&= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\sum_{k=n}^{N-1}\Big(\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F+(\boldsymbol{P}_{F_m}-\boldsymbol{P}_F);\pi}\big[r_k(X_k,f_k(X_k))\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\big[r_k(X_k,f_k(X_k))\big]\Big) \\
&\qquad\qquad + \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F+(\boldsymbol{P}_{F_m}-\boldsymbol{P}_F);\pi}\big[r_N(X_N)\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\big[r_N(X_N)\big]\Big|\Big\} \\
&= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\sum_{k=n}^{N-1}\int_E\cdots\int_E\int_E V_{k+1}^{F;\pi}(y_{k+1})\,(P_k^{F_m}-P_k^F)\big((y_k,f_k(y_k)),dy_{k+1}\big) \\
&\qquad P_{k-1}^F\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n^F\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\quad + \sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1\le|J|\le k-n}}\int_E\cdots\int_E\int_E V_{k+1}^{F;\pi}(y_{k+1})\,(P_k^{F_m}-P_k^F)\big((y_k,f_k(y_k)),dy_{k+1}\big) \\
&\qquad \xi_{k-1,J}^{F,m;-}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots\xi_{n,J}^{F,m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\Big\} \\
&= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\Big|\sum_{k=n}^{N-1}\int_E\cdots\int_E\int_E\big(\Lambda_k^{F;(\pi,y_k)}(F_m)-\Lambda_k^{F;(\pi,y_k)}(F)\big)\,P_{k-1}^F\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad P_{k-2}^F\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P_n^F\big((x_n,f_n(x_n)),dy_{n+1}\big) \\
&\quad + \sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1\le|J|\le k-n}}\int_E\cdots\int_E\int_E\big(\Lambda_k^{F;(\pi,y_k)}(F_m)-\Lambda_k^{F;(\pi,y_k)}(F)\big)\,\xi_{k-1,J}^{F,m;-}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad \xi_{k-2,J}^{F,m;-}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{F,m;-}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\Big\} \\
&\le \sum_{k=n}^{N-1}\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\big|\Lambda_k^{F;(\pi,y_k)}(F_m)-\Lambda_k^{F;(\pi,y_k)}(F)\big|\,P_{k-1}^F\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad P_{k-2}^F\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P_n^F\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
&\quad + \sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1\le|J|\le k-n}}\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\big|\Lambda_k^{F;(\pi,y_k)}(F_m)-\Lambda_k^{F;(\pi,y_k)}(F)\big|\xi_{k-1,J}^{F,m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad \xi_{k-2,J}^{F,m;+}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{F,m;+}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
&= \sup_{\pi\in\Pi}\big|\Lambda_n^{F;(\pi,x_n)}(F_m)-\Lambda_n^{F;(\pi,x_n)}(F)\big| \\
&\quad + \sum_{k=n+1}^{N-1}\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\big|\Lambda_k^{F;(\pi,y_k)}(F_m)-\Lambda_k^{F;(\pi,y_k)}(F)\big|\,P_{k-1}^F\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big) \\
&\qquad P_{k-2}^F\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P_n^F\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\} \\
&\quad + \sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\ 1\le|J|\le k-n}}\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\big|\Lambda_k^{F;(\pi,y_k)}(F_m)-\Lambda_k^{F;(\pi,y_k)}(F)\big|\xi_{k-1,J}^{F,m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)
\end{aligned}
$$

$$\xi^{F,m;+}_{k-2,J}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi^{F,m;+}_{n,J}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}$$
$$=: S_1(m)+S_2(m)+S_3(m),$$

where $\xi^{F,m;\pm}_{j,J}$ is for any subset $J\subseteq\{0,\dots,N-1\}$ given by

$$\xi^{F,m;\pm}_{j,J}:=\begin{cases} P^{F_m}_j\pm P^F_j & ,\ j\in J\\ P^F_j & ,\ \text{otherwise}\end{cases}.$$

By condition (a) there exists a finite constant $C_{\mathbf{\Lambda}}>0$ such that in view of (4.10)–(4.11)

$$\begin{aligned}\sup_{\pi\in\Pi}\big|\Lambda^{F;(\pi,x_n)}_n(F_m)-\Lambda^{F;(\pi,x_n)}_n(F)\big| &\le \sup_{\pi\in\Pi}\sup_{x\in E}\frac{1}{\psi(x)}\cdot\big|\Lambda^{F;(\pi,x)}_n(F_m)-\Lambda^{F;(\pi,x)}_n(F)\big|\cdot\psi(x_n)\\ &= \|\mathbf{\Lambda}^F_n(F_m)-\mathbf{\Lambda}^F_n(F)\|_{\infty,\psi}\ \le\ C_{\mathbf{\Lambda}}\|F_m-F\|_{1,\nu}\cdot\psi(x_n)\end{aligned}$$

for every $m\in\mathbb{N}$. Thus $S_1(m)=\mathcal{O}(\|F_m-F\|_{1,\nu})$.

Further, in view of condition (a), Lemma 1.4.4(v), and (4.10)–(4.11), we obtain for any $k=n+1,\dots,N-1$ and $m\in\mathbb{N}$

$$\begin{aligned}&\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\big|\Lambda^{F;(\pi,y_k)}_k(F_m)-\Lambda^{F;(\pi,y_k)}_k(F)\big|\,P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\ &\qquad P^F_{k-2}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}\\ &\le\ \sup_{\pi\in\Pi}\sup_{x\in E}\frac{1}{\psi(x)}\cdot\big|\Lambda^{F;(\pi,x)}_k(F_m)-\Lambda^{F;(\pi,x)}_k(F)\big|\\ &\qquad\cdot\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\psi(y_k)\,P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}\\ &=\ \|\mathbf{\Lambda}^F_n(F_m)-\mathbf{\Lambda}^F_n(F)\|_{\infty,\psi}\cdot\sup_{\pi\in\Pi}\mathbb{E}^{x_0,\boldsymbol{P}_F;\pi}_{n,x_n}\big[\psi(X_k)\big]\\ &\le\ C_{\mathbf{\Lambda}}\|F_m-F\|_{1,\nu}\cdot\sup_{\pi\in\Pi}\mathbb{E}^{x_0,\boldsymbol{P}_F;\pi}_{n,x_n}\big[\psi(X_k)\big].\end{aligned}\tag{4.12}$$

Note that the second factor in the last line of (4.12) is (independent of m and) bounded by condition (a) of Assumption 4.2.1 along with part (c) of Definition 1.4.1 (applied to $\mathcal{P}:=\{\boldsymbol{P}_F\}$). Therefore we have $S_2(m)=\mathcal{O}(\|F_m-F\|_{1,\nu})$.

In view of conditions (a)–(b), condition (a) of Assumption 4.2.1, part (c) of Definition 1.4.1 (applied to $\mathcal{P}:=\{\boldsymbol{P}_F\}$), and (4.10)–(4.11), for every $k=n+1,\dots,N-1$ we find some finite constant $C_{\mathbf{\Phi}}>0$ such that

$$\begin{aligned}&\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\ \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\big|\Lambda^{F;(\pi,y_k)}_k(F_m)-\Lambda^{F;(\pi,y_k)}_k(F)\big|\,\xi^{F,m;+}_{k-1,J}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\ &\qquad\xi^{F,m;+}_{k-2,J}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi^{F,m;+}_{n,J}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}\\ &\le\ \sup_{\pi\in\Pi}\sup_{x\in E}\frac{1}{\psi(x)}\cdot\big|\Lambda^{F;(\pi,x)}_k(F_m)-\Lambda^{F;(\pi,x)}_k(F)\big|\\ &\qquad\cdot\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\ \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi}\Big\{\int_E\cdots\int_E\int_E\psi(y_k)\,\xi^{F,m;+}_{k-1,J}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\end{aligned}$$

$$\xi_{k-2,J}^{F,m;+}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi_{n,J}^{F,m;+}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big\}$$

$$\begin{aligned}
&\le \|\boldsymbol{\Lambda}_n^F(F_m)-\boldsymbol{\Lambda}_n^F(F)\|_{\infty,\psi}\\
&\quad\cdot\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\binom{k-n}{|J|}K_3^{k-n-|J|}\big(\|\boldsymbol{\Phi}_n(F_m)-\boldsymbol{\Phi}_n(F)\|_{\infty,\psi}+2K_3\big)^{|J|}\cdot\psi(x_n)\\
&\le C_{\boldsymbol{\Lambda}}\|F_m-F\|_{1,\nu}\\
&\quad\cdot\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\binom{k-n}{|J|}K_3^{k-n-|J|}\big(C_{\boldsymbol{\Phi}}\|F_m-F\|_{1,\nu}+2K_3\big)^{|J|}\cdot\psi(x_n)
\end{aligned}$$

for every $m\in\mathbb{N}$. Hence $S_3(m)=\mathcal{O}(\|F_m-F\|_{1,\nu})$. Thus the assertion follows. This completes the proof of Theorem 4.3.3. □

The following Corollary 4.3.6 reduces the statements in Theorem 4.3.3 to the case when in the setting of Sections 4.1–4.2 both the state space as well as the action spaces are finite. That is, let E be given by (1.23) with $\mathfrak{e}:=\#E\in\mathbb{N}$, and let $A_n(x_i)$ given by (1.24) be for any $i=1,\dots,\mathfrak{e}$ and $n=0,\dots,N-1$ the finite set of all admissible actions in state x_i at time n. Note that it follows from the discussion in Section 1.5 that in the finite setting for any $F\in\boldsymbol{F}(\nu)$ the transition function $\boldsymbol{P}_F$ from $\overline{\mathcal{P}}_1=\overline{\mathcal{P}}$ (with $\overline{\mathcal{P}}$ given by (1.25)) can be identified with some vector $\boldsymbol{p}_F\in\widetilde{\mathcal{P}}$ defined as in (1.26) whose components are parametrized by F. Recall that the set $\widetilde{\mathcal{P}}$ is defined as in (1.27).

Therefore, the functionals $\mathcal{W}_n^{x_n;\pi}$ and $\mathcal{W}_n^{x_n}$ given by (4.6) can be identified in the finite setting with maps $\mathcal{W}_n^{x_i;\pi}:\boldsymbol{F}(\nu)\to\mathbb{R}$ and $\mathcal{W}_n^{x_i}:\boldsymbol{F}(\nu)\to\mathbb{R}$ defined by

$$\mathcal{W}_n^{x_i;\pi}(F):=V_n^{F;\pi}(x_i)\quad\text{and}\quad\mathcal{W}_n^{x_i}(F):=\max_{\pi\in\Pi}\mathcal{W}_n^{x_i;\pi}(F)\tag{4.13}$$

for every $i=1,\dots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\dots,N$, where the policy value function $V_n^{F;\pi}(\cdot):=V_n^{\boldsymbol{p}_F;\pi}(\cdot)$ can be computed by (1.29) (with $\boldsymbol{p}_F$ in place of $\boldsymbol{p}$). Take into account that (under condition (a) of Corollary 4.3.6 below) the latter functionals are well-defined because it follows from the discussion at end of Section 1.5 that $\psi:\equiv 1$ is a bounding function for the MDM $(E,\boldsymbol{A},\boldsymbol{P}_F,\Pi,\boldsymbol{X},\boldsymbol{r})$ for any $F\in\boldsymbol{F}(\nu)$.

Moreover the maps $\Lambda_n^{F;(\pi,x_i)}:\boldsymbol{F}(\nu)\to\mathbb{R}$ and $\Phi_n^{(\pi,x_i)}:\boldsymbol{F}(\nu)\to\mathbb{R}$ defined as in (4.8) admit (under condition (a) of Corollary 4.3.6) in the finite setting above the representations

$$\Lambda_n^{F;(\pi,x_i)}(G)=\sum_{j=1}^{\mathfrak{e}}V_{n+1}^{F;\pi}(x_j)\,p_{n,i;f_n(x_i)}^G(j)\quad\text{and}\quad\Phi_n^{(\pi,x_i)}(G)=\sum_{j=1}^{\mathfrak{e}}p_{n,i;f_n(x_i)}^G(j)\equiv 1\tag{4.14}$$

for fixed $F\in\boldsymbol{F}(\nu)$, for any $i=1,\dots,\mathfrak{e}$, $\pi=(f_n)_{n=0}^{N-1}\in\Pi$, $n=0,\dots,N-1$, and $G\in\boldsymbol{F}(\nu)$.

Corollary 4.3.6 ('Lipschitz continuity' of $\mathcal{W}_n^{x_i;\pi}$ and $\mathcal{W}_n^{x_i}$ in F) *Let $F\in\boldsymbol{F}(\nu)$, and assume that in the finite setting above the following two conditions hold.*

(a) *$\boldsymbol{p}_G\in\widetilde{\mathcal{P}}$ for every $G\in\boldsymbol{F}(\nu)$.*

(b) *For any $i=1,\dots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\dots,N-1$, the map $\Lambda_n^{F;(\pi,x_i)}:\boldsymbol{F}(\nu)\to\mathbb{R}$ defined by (4.14) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\nu},|\cdot|)$.*

Then the following two assertions hold.

(i) *For any* $i = 1, \dots, \mathfrak{e}$, $\pi \in \Pi$, *and* $n = 0, \dots, N$, *the map* $\mathcal{W}_n^{x_i;\pi} : \boldsymbol{F}(\nu) \to \mathbb{R}$ *defined by (4.13) is 'Lipschitz continuous' at* F *w.r.t.* $(\|\cdot\|_{1,\nu}, |\cdot|)$.

(ii) *For any* $i = 1, \dots, \mathfrak{e}$ *and* $n = 0, \dots, N$, *the map* $\mathcal{W}_n^{x_i} : \boldsymbol{F}(\nu) \to \mathbb{R}$ *defined by (4.13) is 'Lipschitz continuous' at* F *w.r.t.* $(\|\cdot\|_{1,\nu}, |\cdot|)$.

Proof First of all, it follows from the above discussion that in the finite setting $\psi :\equiv 1$ provides a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{P}_G, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for every $G \in \boldsymbol{F}(\nu)$. In particular, condition (a) of Assumption 4.2.1 holds. Further condition (a) matches condition (b) of Assumption 4.2.1 in the finite setting. Moreover, in view of (4.10), (4.11), (1.18), (4.14), and the choice of the bounding function $\psi :\equiv 1$, condition (b) corresponds to condition (a) of Theorem 4.3.3 in the finite setting. By (4.10) and (4.14) we observe that condition (b) of Theorem 4.3.3 is satisfied. Thus the assertions in (i) and (ii) follow from parts (i) and (ii) of the latter theorem, respectively. □

4.3.2 Differentiability in F of the value function

We will use in the following the notion of quasi-Hadamard differentiability in Definition 4.3.7 below. The latter concept of differentiability, which was introduced by [13, 15], is stronger compared to the classical notion of tangential Hadamard differentiability; see, for instance, [83, 87].

The following definition can be deduced from Definition A.1(iii) (and Remark A.2) in Section A. Take into account that $\|\cdot\|_{1,\nu}$ does *not* provide a norm on all of $\boldsymbol{L}_0(\nu)$ but only on $\boldsymbol{L}_1(\nu)$. Let $(L, \|\cdot\|_L)$ be a normed vector space.

Definition 4.3.7 (Quasi-Hadamard differentiability in F) *Let* $F \in \boldsymbol{F}(\nu)$. *A map* $\mathcal{W} : \boldsymbol{F}(\nu) \to L$ *is said to be quasi-Hadamard differentiable at* F *tangentially to* $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ *if there exists a continuous map* $\dot{\mathcal{W}}_F : \boldsymbol{L}_1(\nu) \to L$ *such that*

$$\lim_{m\to\infty} \Big\| \frac{\mathcal{W}(F + \varepsilon_m h_m) - \mathcal{W}(F)}{\varepsilon_m} - \dot{\mathcal{W}}_F(h) \Big\|_L = 0$$

holds for every triplet $(h, (h_m), (\varepsilon_m)) \in \boldsymbol{L}_1(\nu) \times \boldsymbol{L}_1(\nu)^{\mathbb{N}} \times \mathbb{R}_{>0}^{\mathbb{N}}$ *satisfying* $\|h_m - h\|_{1,\nu} \to 0$, $\varepsilon_m \to 0$ *as well as* $(F + \varepsilon_m h_m) \subseteq \boldsymbol{F}(\nu)$. *In this case, the map* $\dot{\mathcal{W}}_F$ *is called quasi-Hadamard derivative of* $\mathcal{W}$ *at* F *tangentially to* $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$.

Part (ii) of Theorem 4.3.8 provides (under some assumptions) the quasi-Hadamard derivative of the value functional $\mathcal{W}_n^{x_n}$. Recall from (4.10) the definitions of the maps $\boldsymbol{\Lambda}_n^F : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ and $\boldsymbol{\Phi}_n : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$. Also recall that for given $F \in \boldsymbol{F}(\nu)$ and $\delta > 0$ the sets $\Pi(F; \delta)$ and $\Pi(F)$ consists of all δ-optimal strategies w.r.t. F and of all optimal strategies w.r.t. F, respectively. Let $0_{\boldsymbol{L}_0(\nu)}$ be the null in $\boldsymbol{L}_0(\nu)$, and recall that $\mathbb{M}(E)$ consists of all $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable maps $h \in \mathbb{R}^E$. Finally, let $\|\cdot\|_{\infty,\psi}$ be the norm introduced in (4.11).

Theorem 4.3.8 (Quasi-Hadamard differentiability of $\mathcal{W}_n^{x_n;\pi}$ and $\mathcal{W}_n^{x_n}$ in F) *Under Assumption 4.2.1 let* $F \in \boldsymbol{F}(\nu)$, *and assume that the following three conditions hold.*

(a) *For any* $x \in E$, $\pi \in \Pi$, *and* $n = 0, \ldots, N-1$ *there exists a map* $\dot{\Lambda}_{n;F}^{F;(\pi,x)} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ *satisfying* $\dot{\Lambda}_{n;F}^{F;(\pi,x)}(0_{\boldsymbol{L}_0(\nu)}) = 0$, $\dot{\Lambda}_{n;F}^{F;(\pi,\,\cdot\,)}(h) \in \mathbb{M}(E)$ *as well as* $\sup_{\pi\in\Pi} \|\dot{\Lambda}_{n;F}^{F;(\pi,\,\cdot\,)}(h)\|_\psi \leq C_{\dot{\Lambda}}$ *for all* $h \in \boldsymbol{L}_1(\nu)$, *where* $C_{\dot{\Lambda}} > 0$ *is a finite constant (depending on* n *and* h*).*

(b) *For any* $n = 0, \ldots, N-1$, *the map* $\boldsymbol{\Lambda}_n^F : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ *defined by (4.10) is quasi-Hadamard differentiable at* F *tangentially to* $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ *with quasi-Hadamard derivative* $\dot{\boldsymbol{\Lambda}}_{n;F}^F : \boldsymbol{L}_1(\nu) \to \ell_\psi^\infty(\Pi \times E)$ *given by*

$$\dot{\boldsymbol{\Lambda}}_{n;F}^F(h) := \big(\dot{\Lambda}_{n;F}^{F;(\pi,x)}(h)\big)_{(\pi,x)\in\Pi\times E}, \tag{4.15}$$

where $\dot{\Lambda}_{n;F}^{F;(\pi,x)}$ *is as in condition (a).*

(c) *The map* $\boldsymbol{\Phi}_n : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ *defined by (4.10) is 'Lipschitz continuous' at* F *w.r.t.* $(\|\cdot\|_{1,\nu}, \|\cdot\|_{\infty,\psi})$.

Then the following two assertions hold.

(i) *For any* $x_n \in E$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, *and* $n = 0, \ldots, N$, *the map* $\mathcal{W}_n^{x_n;\pi} : \boldsymbol{F}(\nu) \to \mathbb{R}$ *defined by (4.6) is quasi-Hadamard differentiable at* F *tangentially to* $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ *with quasi-Hadamard derivative* $\dot{\mathcal{W}}_{n;F}^{x_n;\pi} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ *given by*

$$\begin{aligned}\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h) \;:=\; & \sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E \dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h)\, P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\ & P_{k-2}^F\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big).\end{aligned} \tag{4.16}$$

(ii) *For any* $x_n \in E$ *and* $n = 0, \ldots, N$, *the map* $\mathcal{W}_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ *defined by (4.6) is quasi-Hadamard differentiable at* F *tangentially to* $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ *with quasi-Hadamard derivative* $\dot{\mathcal{W}}_{n;F}^{x_n} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ *given by*

$$\dot{\mathcal{W}}_{n;F}^{x_n}(h) := \lim_{\delta\searrow 0} \sup_{\pi\in\Pi(F;\delta)} \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h). \tag{4.17}$$

If the set of optimal strategies $\Pi(F)$ *is non-empty, then the quasi-Hadamard derivative admits the representation*

$$\dot{\mathcal{W}}_{n;F}^{x_n}(h) = \sup_{\pi\in\Pi(F)} \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h) \quad \textit{for all } h \in \boldsymbol{L}_1(\nu). \tag{4.18}$$

Note that the representation of the quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}$ in (4.16) has a certain analogy to the representation of the 'Fréchet derivative' $\dot{\mathcal{V}}_{n;\boldsymbol{P}}^{x_n;\pi}$ given by (2.31). Also note that in part (ii) of Theorem 4.3.8 the set $\Pi(F;\delta)$ becomes the smaller the smaller δ is. In particular, this implies that the right-hand side of (4.17) is well-defined. We point out that the supremum in (4.18) ranges over all optimal strategies w.r.t. F. Remark 2.3.12 discusses (for $\boldsymbol{P}_F$ in place of $\boldsymbol{P}$) two settings in which one can find at least one optimal strategy. If there exists even a unique optimal strategy $\pi^F \in \Pi$ w.r.t. F, then the set $\Pi(F)$ reduces to the singleton $\{\pi^F\}$, and in this case the quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0}$ of the (optimal) value functional $\mathcal{W}_0^{x_0}$ at F is equal to $\dot{\mathcal{W}}_{0;F}^{x_0;\pi^F}$. It is discussed in the example of Section 5.2 that there exists a unique optimal strategy.

Note that the quasi-Hadamard differentiability of the value functional shown in Theorem 4.3.8(ii) provides a key tool to easily derive results on the asymptotic of the (plug-in) estimator $\mathcal{W}_n^{x_n}(\widehat{F}_m)$ for the aspect $\mathcal{W}_n^{x_n}(F)$, where $\widehat{F}_m$ corresponds to a reasonable estimator for the (unknown) distribution function F; see Subsections 4.4.2, 4.4.3, and 4.5.2 for details.

Remark 4.3.9 (i) Note that the quasi-Hadamard differentiability in part (i) of Theorem 4.3.8 holds even uniformly in $\pi \in \Pi$. We refer to Theorem 4.3.12 for the precise meaning.

(ii) In the case that in the setting (and under the assumptions) of Theorem 4.3.8 the map $\dot{\Lambda}_{n;F}^{F;(\pi,x)}$: $\boldsymbol{L}_1(\nu) \to \mathbb{R}$ in condition (a) is linear for any $x \in E$, $\pi \in \Pi$, and $n = 0, \ldots, N-1$, then it follows from the representation (4.16) that the quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(\cdot)$ of $\mathcal{W}_n^{x_n;\pi}$ at F is also linear. The linearity of the quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n}(\cdot)$ of $\mathcal{W}_n^{x_n}$ at F in part (ii) of Theorem 4.3.8 is ensured if in addition the set of all optimal strategies $\Pi(F)$ is unique. The latter property becomes important for deriving the so-called bootstrap consistency (in probability) of a certain point estimator based on the map $\mathcal{W}_n^{x_n}$; for an example, see Subsection 4.4.3.

(iii) It follows from the proof of Theorem 4.3.12 below that (under the assumptions of Theorem 4.3.8) the assertions in part (i) and (ii) of Theorem 4.3.8 also hold if condition (a) of the latter theorem holds for the usual sup-norm $\|\cdot\|_\infty$ in place of the weighted sup-norm $\|\cdot\|_\psi$ (given by (1.18)) and if in conditions (b)–(c) of Theorem 4.3.8 the norm $\|\cdot\|_{\infty,\psi}$ (on $\ell_\psi^\infty(\Pi \times E)$) is replaced by the norm $\|\cdot\|_{\infty,1}$ as defined in (4.11) with $\psi :\equiv 1$. The choice of the norms $\|\cdot\|_\psi$ as well as $\|\cdot\|_{\infty,\psi}$ is motivated by the fact that in the example of Section 5.2 conditions (a)–(c) of Theorem 4.3.8 can *not* be verified for the (stricter) norms $\|\cdot\|_\infty$ as well as $\|\cdot\|_{\infty,1}$; see (the proof of) Theorem 5.2.6 in Subsection 5.2.2 for further details.

(iv) Note that condition (c) of Theorem 4.3.8 is nothing but condition (b) of Theorem 4.3.3. Moreover it can be deduced from Lemma A.5 along with Definition A.3 (and Remark 4.3.2) that conditions (a)–(b) of Theorem 4.3.8 imply condition (a) of Theorem 4.3.3. ◇

In practice it can be cumbersome to determine the set $\Pi(F)$ of all optimal strategies w.r.t. F. While in many cases an optimal strategy can easily be found by means of the Bellman equation (see part (i) of Theorem 1.3.3 in Section 1.3), it is more difficult to specify all optimal strategies or to prove that an optimal strategy is unique. The following remark may help in some situations; see Subsection 5.2.2 for an application.

Remark 4.3.10 In some situations it turns out that for *every* $F \in \boldsymbol{F}(\nu)$ the solution of the optimization problem in Display (4.4) does not change if Π is replaced by a subset $\Pi' \subseteq \Pi$ (being independent of F). Then in the definition (4.5) of the value function (at time 0) the set Π can be replaced by the subset Π', and it follows (under the assumptions of Theorem 4.3.8) that in the representation (4.18) of the quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0}$ of $\mathcal{W}_0^{x_0}$ at F the set $\Pi(F)$ can be replaced by the set $\Pi'(F)$ of all optimal strategies w.r.t. F from the subset Π'. Of course, in this case it suffices to ensure that conditions (a)–(c) of Theorem 4.3.8 are satisfied for the subset Π' instead of Π. ◇

It is an immediate consequence of Theorem 4.3.8 that for every $x_n \in E$, $\pi \in \Pi$, and $n = 0, \ldots, N$ the functional $\mathcal{W}_n^{x_n;\pi}$ (resp. $\mathcal{W}_n^{x_n}$) is even quasi-Hadamard differentiable at some $F \in \boldsymbol{F}(\nu)$ tangentially to any subspace of $\boldsymbol{L}_1(\nu)$ which is equipped with a norm dominating the norm $\|\cdot\|_{1,\nu}$ given by (4.7). The following example (an analogue of Example 2.5 in [59]) illustrates this statement. Note that a map $\phi : \mathbb{R} \to \mathbb{R}_{\geq 1}$ is called a *weight function* if ϕ is continuous as well as non-increasing on $\mathbb{R}_{\leq 0}$ and non-decreasing on $\mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\leq 0} := (-\infty, 0]$.

Example 4.3.11 Let ϕ be any weight function. Moreover let $\boldsymbol{D}$ be the space of all bounded càdlàg functions on $\mathbb{R}$, and denote by $\boldsymbol{D}_\phi$ the subspace of all $h \in \boldsymbol{D}$ satisfying $\|h\|_{1/\phi} < \infty$ and $\lim_{|x|\to\infty} |h(x)| = 0$. Note that $\|h\|_{1/\phi} = \|h\phi\|_\infty$ (by (1.18)), where $\|\cdot\|_\infty$ stands for the usual sup-norm. Also note that $\lim_{|x|\to\infty} |h(x)| = 0$ clearly holds for all $h \in \boldsymbol{D}$ with $\|h\|_{1/\phi} < \infty$ when $\lim_{|x|\to\infty} \phi(x) = \infty$. If $C_\phi := \int 1/\phi \, d\nu < \infty$, then in view of

$$\|h\|_{1,\nu} \;=\; \int |h(y)|\, \nu(dy) \;\leq\; C_\phi \|h\|_{1/\phi}$$

for all $h \in \boldsymbol{D}_\phi$, we have $\boldsymbol{D}_\phi \subseteq \boldsymbol{L}_1(\nu)$. That is, on the space $\boldsymbol{D}_\phi$, the norm $\|\cdot\|_{1/\phi}$ is stricter than the norm $\|\cdot\|_{1,\nu}$. Hence, for every $x_n \in E$, $\pi \in \Pi$, and $n = 0, \ldots, N$ the map $\mathcal{W}_n^{x_n;\pi}$ (resp. $\mathcal{W}_n^{x_n}$) given by (4.6) is quasi-Hadamard differentiable at some $F \in \boldsymbol{F}(\nu)$ tangentially to $\boldsymbol{D}_\phi\langle\boldsymbol{D}_\phi\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n;\pi} : \boldsymbol{D}_\phi \to \mathbb{R}$ (resp. $\dot{\mathcal{W}}_{n;F}^{x_n} : \boldsymbol{D}_\phi \to \mathbb{R}$) given by (4.16) (resp. (4.17)) restricted to $h \in \boldsymbol{D}_\phi$. ◇

In the following we will prove Theorem 4.3.8. Analogously to the proof of Theorem 2.3.11, we note that under Assumption 4.2.1 the functional $\mathcal{W}_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ given by (4.6) can be represented for any $x_n \in E$ and $n = 0, \ldots, N$ as a composition

$$\mathcal{W}_n^{x_n} = \Psi \circ \Upsilon_n^{x_n} \tag{4.19}$$

with maps $\Upsilon_n^{x_n} : \boldsymbol{F}(\nu) \to \ell^\infty(\Pi)$ and $\Psi : \ell^\infty(\Pi) \to \mathbb{R}$ defined by

$$\Upsilon_n^{x_n}(F) \;:=\; \big(\mathcal{W}_n^{x_n;\pi}(F)\big)_{\pi\in\Pi} \quad \text{and} \quad \Psi\big((w(\pi))_{\pi\in\Pi}\big) \;:=\; \sup_{\pi\in\Pi} w(\pi). \tag{4.20}$$

Recall that $\ell^\infty(\Pi)$ refer to the space of all bounded real-valued functions on Π equipped with the sup-norm $\|\cdot\|_\infty$. It can be verified easily by means of (4.6), condition (a) of Assumption 4.2.1, and Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{P}_F\}$) that the map $\Upsilon_n^{x_n}$ is well-defined for any $x_n \in E$ and $n = 0, \ldots, N$, i.e. that $(\mathcal{W}_n^{x_n;\pi}(F))_{\pi\in\Pi} \in \ell^\infty(\Pi)$ for every $x_n \in E$, $n = 0, \ldots, N$, and $F \in \boldsymbol{F}(\nu)$. Take into account that $\boldsymbol{P}_F \in \mathcal{P}_\psi$ $(\subseteq \overline{\mathcal{P}})$ for every $F \in \boldsymbol{F}(\nu)$ by condition (b) of Assumption 4.2.1.

In Theorem 4.3.12 we will show that under the assumptions of Theorem 4.3.8 and for any $x_n \in E$ and $n = 0, \ldots, N$ the map $\Upsilon_n^{x_n}$ is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ (in the sense of Definition 4.3.7) with quasi-Hadamard derivative $\dot{\Upsilon}_{n,F}^{x_n} : \boldsymbol{L}_1(\nu) \to \ell^\infty(\Pi)$ given by

$$\dot{\Upsilon}_{n,F}^{x_n}(h) \;:=\; \big(\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h)\big)_{\pi\in\Pi} \tag{4.21}$$

(the well-definiteness of $\dot{\Upsilon}_{n,F}^{x_n}$ will be proven in Lemma 4.3.13 below). In view of the Hadamard differentiability of the map Ψ (which is known from [75, Proposition 1]) we claim that this is sufficient for the proof of part (ii) of Theorem 4.3.8; see below. Assertion (i) of Theorem 4.3.8 can be deduced from the following Theorem 4.3.12. Its statement is an immediate consequence of Lemmas 4.3.13–4.3.14 below.

Theorem 4.3.12 *Let $F \in \boldsymbol{F}(\nu)$, and suppose that the assumptions of Theorem 4.3.8 hold. Then for any $x_n \in E$ and $n = 0, \dots, N$, the map $\Upsilon_n^{x_n} : \boldsymbol{F}(\nu) \to \ell^\infty(\Pi)$ defined by (4.20) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\Upsilon}_{n,F}^{x_n} : \boldsymbol{L}_1(\nu) \to \ell^\infty(\Pi)$ given by (4.21).*

Lemma 4.3.13 *Under the assumptions of Theorem 4.3.12 (except condition (c) of Theorem 4.3.8) and for any fixed $x_n \in E$ and $n = 0, \dots, N$, the map $\dot{\Upsilon}_{n,F}^{x_n} : \boldsymbol{L}_1(\nu) \to \ell^\infty(\Pi)$ given by (4.21) is $(\|\cdot\|_{1,\nu}, \|\cdot\|_\infty)$-continuous.*

Proof First of all, note that $(\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h))_{\pi\in\Pi} \in \ell^\infty(\Pi)$ holds for every $h \in \boldsymbol{L}_1(\nu)$ by conditions (a)–(b) of Assumption 4.2.1, the second part of condition (a) of Theorem 4.3.8, and the representation of the quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}$ in (4.16).

Now, let $(h_m)_{m\in\mathbb{N}}$ be any sequence in $\boldsymbol{L}_1(\nu)$ with $\|h_m - h\|_{1,\nu} \to 0$ for some $h \in \boldsymbol{L}_1(\nu)$. Using the representation (4.16), we first get for every $m \in \mathbb{N}$

$$
\begin{aligned}
\big\|\dot{\Upsilon}_{n,F}^{x_n}(h_m) - \dot{\Upsilon}_{n,F}^{x_n}(h)\big\|_\infty &= \sup_{\pi\in\Pi} \big|\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h_m) - \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h)\big| \\
&= \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{\Big| \sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E \big(\dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h_m) - \dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h)\big)\, P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad P_{k-2}^F\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big|\Big\} \\
&\le \sum_{k=n}^{N-1} \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \int_E \big|\dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h_m) - \dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h)\big|\, P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad P_{k-2}^F\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\} \\
&= \sup_{\pi\in\Pi} \big|\dot{\Lambda}_{n;F}^{F;(\pi,x_n)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi,x_n)}(h)\big| \\
&\quad + \sum_{k=n+1}^{N-1} \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{\int_E \cdots \int_E \int_E \big|\dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h_m) - \dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h)\big| P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad P_{k-2}^F\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\} \\
&=: S_1(m) + S_2(m).
\end{aligned}
$$

By conditions (a)–(b) of Theorem 4.3.8, we know that the map $\boldsymbol{\Lambda}_n^F : \boldsymbol{F}(\nu) \to \ell_\psi^\infty(\Pi \times E)$ defined by (4.10) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\boldsymbol{\Lambda}}_{n;F}^F : \boldsymbol{L}_1(\nu) \to \ell_\psi^\infty(\Pi \times E)$ given by (4.15). In particular, this implies that the map $\dot{\boldsymbol{\Lambda}}_{n;F}^F$ is continuous w.r.t. $(\|\cdot\|_{1,\nu}, \|\cdot\|_{\infty,\psi})$, that is, we get

$$
\|\dot{\boldsymbol{\Lambda}}_{n;F}^F(h_m) - \dot{\boldsymbol{\Lambda}}_{n;F}^F(h)\|_{\infty,\psi} \;\to\; 0 \quad \text{as } m \to \infty. \tag{4.22}
$$

Using (4.22), (4.15), and (4.11) along with the estimate

$$
\begin{aligned}
\sup_{\pi\in\Pi} \big|\dot{\Lambda}_{n;F}^{F;(\pi,x_n)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi,x_n)}(h)\big| &\le \sup_{\pi\in\Pi} \sup_{x\in E} \frac{1}{\psi(x)} \big|\dot{\Lambda}_{n;F}^{F;(\pi,x)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi,x)}(h)\big| \cdot \psi(x_n) \\
&= \|\dot{\boldsymbol{\Lambda}}_{n;F}^F(h_m) - \dot{\boldsymbol{\Lambda}}_{n;F}^F(h)\|_{\infty,\psi} \cdot \psi(x_n),
\end{aligned}
$$

we obtain $\lim_{m\to\infty} S_1(m) = 0$. Moreover, for any $k = n+1, \ldots, N-1$ and $m \in \mathbb{N}$ we obtain by means of part (v) of Lemma 1.4.4 (applied to $\boldsymbol{P}_F$)

$$
\begin{aligned}
&\sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \int_E \big|\dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h_m) - \dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h)\big| \, P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
&\qquad\qquad P_{k-2}^F\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\} \\
\leq\ & \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \sup_{x\in E} \frac{1}{\psi(x)} \big|\dot{\Lambda}_{k;F}^{F;(\pi,x)}(h_m) - \dot{\Lambda}_{k;F}^{F;(\pi,x)}(h)\big| \qquad\qquad (4.23)\\
&\quad \cdot \int_E \cdots \int_E \psi(y_k) P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\} \\
\leq\ & \sup_{\pi\in\Pi} \sup_{x\in E} \frac{1}{\psi(x)} \big|\dot{\Lambda}_{k;F}^{F;(\pi,x)}(h_m) - \dot{\Lambda}_{k;F}^{F;(\pi,x)}(h)\big| \\
&\quad \cdot \sup_{\pi=(f_n)_{n=0}^{N-1}\in\Pi} \Big\{ \int_E \cdots \int_E \psi(y_k) P_{k-1}^F\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \cdots P_n^F\big((x_n, f_n(x_n)), dy_{n+1}\big)\Big\}. \\
=\ & \|\dot{\boldsymbol{\Lambda}}_{n;F}^F(h_m) - \dot{\boldsymbol{\Lambda}}_{n;F}^F(h)\|_{\infty,\psi} \cdot \sup_{\pi\in\Pi} \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\big[\psi(X_k)\big]. \qquad\qquad (4.24)
\end{aligned}
$$

The second factor in the last line of formula display (4.23) is (independent of m and) finite due to Assumption 4.2.1 along with Lemma 1.4.4(v) (applied to $\boldsymbol{P}_F$) and part (c) of Definition 1.4.1 (with $\mathcal{P} := \{\boldsymbol{P}_F\}$). Thus $\lim_{m\to\infty} S_2(m) = 0$ by (4.22). Hence the assertion follows. □

Lemma 4.3.14 *Under the assumptions of Theorem 4.3.12 and for any fixed $x_n \in E$ and $n = 0, \ldots, N$,*

$$
\lim_{m\to\infty} \Big\| \frac{\Upsilon_n^{x_n}(F + \varepsilon_m h_m) - \Upsilon_n^{x_n}(F)}{\varepsilon_m} - \dot{\Upsilon}_{n;F}^{x_n}(h) \Big\|_\infty = 0
$$

for each triplet $(h, (h_m), (\varepsilon_m)) \in \boldsymbol{L}_1(\nu) \times \boldsymbol{L}_1(\nu)^{\mathbb{N}} \times \mathbb{R}_{>0}^{\mathbb{N}}$ satisfying $\|h_m - h\|_{1,\nu} \to 0$, $\varepsilon_m \to 0$ as well as $(F + \varepsilon_m h_m) \subseteq \boldsymbol{F}(\nu)$.

Proof Let $(h, (h_m), (\varepsilon_m)) \in \boldsymbol{L}_1(\nu) \times \boldsymbol{L}_1(\nu)^{\mathbb{N}} \times \mathbb{R}_{>0}^{\mathbb{N}}$ be any triplet with $\|h_m - h\|_{1,\nu} \to 0$, $\varepsilon_m \to 0$ as well as $(F + \varepsilon_m h_m) \subseteq \boldsymbol{F}(\nu)$. At first, note that $\Upsilon_n^{x_n}(F + \varepsilon_m h_m)$ $(= (\mathcal{W}_n^{x_n;\pi}(F + \varepsilon_m h_m))_{\pi\in\Pi}) \in \ell^\infty(\Pi)$ holds for every $m \in \mathbb{N}$ by Proposition 1.4.3 (with $\mathcal{P} := \{\boldsymbol{P}_{F+\varepsilon_m h_m}\}$). Take into account that the latter result is applicable by conditions (a)–(b) of Assumption 4.2.1 because $\boldsymbol{P}_{F+\varepsilon_m h_m} \in \mathcal{P}_\psi$ for every $m \in \mathbb{N}$. Proceeding as in the proof Theorem 2.2.8, we obtain in view of (4.6), (4.3), (4.8) as well as (4.16) for any $m \in \mathbb{N}$ and $\pi \in \Pi$

$$
\begin{aligned}
&\Big|\frac{\mathcal{W}_n^{x_n;\pi}(F + \varepsilon_m h_m) - \mathcal{W}_n^{x_n;\pi}(F)}{\varepsilon_m} - \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h)\Big| \\
=\ & \Big|\frac{1}{\varepsilon_m}\Big(\sum_{k=n}^{N-1} \Big(\mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F+(\boldsymbol{P}_{F+\varepsilon_m h_m}-\boldsymbol{P}_F);\pi}\big[r_k(X_k, f_k(X_k))\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\big[r_k(X_k, f_k(X_k))\big]\Big) \\
&\quad + \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F+(\boldsymbol{P}_{F+\varepsilon_m h_m}-\boldsymbol{P}_F);\pi}\big[r_N(X_N)\big] - \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\big[r_N(X_N)\big]\Big) - \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h)\Big| \\
=\ & \Big|\frac{1}{\varepsilon_m}\Big(\sum_{k=n}^{N-1} \int_E \cdots \int_E \int_E V_{k+1}^{F;\pi}(y_{k+1})\, (P_k^{F+\varepsilon_m h_m} - P_k^F)\big((y_k, f_k(y_k)), dy_{k+1}\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&\quad +\sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\int_E\cdots\int_E\int_E V^{F;\pi}_{k+1}(y_{k+1})\,(P^{F+\varepsilon_m h_m}_k-P^F_k)\big((y_k,f_k(y_k)),dy_{k+1}\big)\\
&\quad \xi^{F,m;-}_{k-1,J}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots\xi^{F,m;-}_{n,J}\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big)\\
&\quad -\dot{\mathcal{W}}^{x_n;\pi}_{n;F}(h)\Big|\\
&=\Big|\sum_{k=n}^{N-1}\int_E\cdots\int_E\int_E\Big(\frac{\Lambda^{F;(\pi,y_k)}_k(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,y_k)}_k(F)}{\varepsilon_m}\Big)\,P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\quad P^F_{k-2}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&\quad +\sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\int_E\cdots\int_E\int_E\Big(\frac{\Lambda^{F;(\pi,y_k)}_k(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,y_k)}_k(F)}{\varepsilon_m}\Big)\xi^{F,m;-}_{k-1,J}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\quad \xi^{F,m;-}_{k-2,J}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi^{F,m;-}_{n,J}\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&\quad -\sum_{k=n}^{N-1}\int_E\cdots\int_E\int_E\dot{\Lambda}^{F;(\pi,y_k)}_{k;F}(h)\,P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\Big|\\
&\le\sum_{k=n}^{N-1}\int_E\cdots\int_E\int_E\Big|\frac{\Lambda^{F;(\pi,y_k)}_k(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,y_k)}_k(F)}{\varepsilon_m}-\dot{\Lambda}^{F;(\pi,y_k)}_{k;F}(h)\Big|\,P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\quad P^F_{k-2}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&\quad +\sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\int_E\cdots\int_E\int_E\Big|\frac{\Lambda^{F;(\pi,y_k)}_k(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,y_k)}_k(F)}{\varepsilon_m}\Big|\,\xi^{F,m;+}_{k-1,J}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\quad \xi^{F,m;+}_{k-2,J}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi^{F,m;+}_{n,J}\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&=\Big|\frac{\Lambda^{F;(\pi,x_n)}_n(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,x_n)}_n(F)}{\varepsilon_m}-\dot{\Lambda}^{F;(\pi,x_n)}_{n;F}(h)\Big|\\
&\quad +\sum_{k=n+1}^{N-1}\int_E\cdots\int_E\int_E\Big|\frac{\Lambda^{F;(\pi,y_k)}_k(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,y_k)}_k(F)}{\varepsilon_m}-\dot{\Lambda}^{F;(\pi,y_k)}_{k;F}(h)\Big|P^F_{k-1}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\quad P^F_{k-2}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P^F_n\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&\quad +\sum_{k=n+1}^{N-1}\sum_{\substack{J\subseteq\{n,\dots,k-1\}\\1\le|J|\le k-n}}\int_E\cdots\int_E\int_E\Big|\frac{\Lambda^{F;(\pi,y_k)}_k(F+\varepsilon_m h_m)-\Lambda^{F;(\pi,y_k)}_k(F)}{\varepsilon_m}\Big|\xi^{F,m;+}_{k-1,J}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\quad \xi^{F,m;+}_{k-2,J}\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots\xi^{F,m;+}_{n,J}\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&=:S_1(m,\pi)+S_2(m,\pi)+S_3(m,\pi),
\end{aligned}
$$

where $\xi^{F,m;\pm}_{j,J}$ is for any subset $J\subseteq\{0,\dots,N-1\}$ given by

$$
\xi^{F,m;\pm}_{j,J}:=\begin{cases} P^{F+\varepsilon_m h_m}_j\pm P^F_j & ,\quad j\in J\\ P^F_j & ,\quad \text{otherwise}\end{cases}.
$$

In view of condition (a)–(b) of Theorem 4.3.8, the map $\boldsymbol{\Lambda}^F_n:\boldsymbol{F}(\nu)\to\ell^\infty_\psi(\Pi\times E)$ defined by (4.10) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\boldsymbol{\Lambda}}^F_{n;F}:\boldsymbol{L}_1(\nu)\to\ell^\infty_\psi(\Pi\times E)$ given by (4.15). As a consequence

$$\sup_{\pi\in\Pi}\Big\|\frac{\Lambda_n^{F;(\pi,\,\cdot\,)}(F+\varepsilon_m h_m)-\Lambda_n^{F;(\pi,\,\cdot\,)}(F)}{\varepsilon_m}-\dot{\Lambda}_{n;F}^{F;(\pi,\,\cdot\,)}(h)\Big\|_\psi\ \to\ 0\quad\text{ as } m\to\infty \tag{4.25}$$

by (4.11), (4.10), and (4.15). Using (4.25) and the estimate

$$\begin{aligned}
&\Big|\frac{\Lambda_n^{F;(\pi,x_n)}(F+\varepsilon_m h_m)-\Lambda_n^{F;(\pi,x_n)}(F)}{\varepsilon_m}-\dot{\Lambda}_{n;F}^{F;(\pi,x_n)}(h)\Big|\\
&\le\ \sup_{x\in E}\frac{1}{\psi(x)}\cdot\Big|\frac{\Lambda_n^{F;(\pi,x)}(F+\varepsilon_m h_m)-\Lambda_n^{F;(\pi,x)}(F)}{\varepsilon_m}-\dot{\Lambda}_{n;F}^{F;(\pi,x)}(h)\Big|\cdot\psi(x_n)\\
&=\ \Big\|\frac{\Lambda_n^{F;(\pi,\,\cdot\,)}(F+\varepsilon_m h_m)-\Lambda_n^{F;(\pi,\,\cdot\,)}(F)}{\varepsilon_m}-\dot{\Lambda}_{n;F}^{F;(\pi,\,\cdot\,)}(h)\Big\|_\psi\cdot\ \psi(x_n)
\end{aligned}$$

for every $\pi\in\Pi$ and $m\in\mathbb{N}$, we obtain $\lim_{m\to\infty}S_1(m,\pi)=0$ uniformly in $\pi\in\Pi$.

Moreover we observe for any $\pi\in\Pi$, $k=n+1,\ldots,N-1$, and $m\in\mathbb{N}$

$$\begin{aligned}
&\int_E\cdots\int_E\int_E\Big|\frac{\Lambda_k^{F;(\pi,y_k)}(F+\varepsilon_m h_m)-\Lambda_k^{F;(\pi,y_k)}(F)}{\varepsilon_m}-\dot{\Lambda}_{k;F}^{F;(\pi,y_k)}(h)\Big|\,P_{k-1}^F\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\\
&\qquad P_{k-2}^F\big((y_{k-2},f_{k-2}(y_{k-2})),dy_{k-1}\big)\cdots P_n^F\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&\le\ \sup_{x\in E}\frac{1}{\psi(x)}\cdot\Big|\frac{\Lambda_k^{F;(\pi,x)}(F+\varepsilon_m h_m)-\Lambda_k^{F;(\pi,x)}(F)}{\varepsilon_m}-\dot{\Lambda}_{k;F}^{F;(\pi,x)}(h)\Big|\\
&\quad\cdot\int_E\cdots\int_E\psi(y_k)\,P_{k-1}^F\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)\cdots P_n^F\big((x_n,f_n(x_n)),dy_{n+1}\big)\\
&=\ \Big\|\frac{\Lambda_k^{F;(\pi,\,\cdot\,)}(F+\varepsilon_m h_m)-\Lambda_k^{F;(\pi,\,\cdot\,)}(F)}{\varepsilon_m}-\dot{\Lambda}_{k;F}^{F;(\pi,\,\cdot\,)}(h)\Big\|_\psi\cdot\ \mathbb{E}_{n,x_n}^{x_0,\boldsymbol{P}_F;\pi}\big[\psi(X_k)\big]
\end{aligned}\tag{4.26}$$

by part (v) of Lemma 1.4.4 (applied to $\boldsymbol{P}_F$). Since the second factor in the last line of (4.26) is uniformly bounded in $\pi\in\Pi$ due to Assumption 4.2.1 along with Lemma 1.4.4(v) (applied to $\boldsymbol{P}_F$) and Definition 1.4.1(c) (with $\mathcal{P}:=\{\boldsymbol{P}_F\}$), Display (4.25) implies that $\lim_{m\to\infty}S_2(m,\pi)=0$ uniformly in $\pi\in\Pi$.

As a consequence of condition (c) of Theorem 4.3.8 and (4.10), there exists a finite constant $C_{\boldsymbol{\Phi}}>0$ such that

$$\sup_{\pi\in\Pi}\big\|\Phi_n^{(\pi,\,\cdot\,)}(F+\varepsilon_m h_m)-\Phi_n^{(\pi,\,\cdot\,)}(F)\big\|_\psi\ \le\ C_{\boldsymbol{\Phi}}\,\|(F+\varepsilon_m h_m)-F\|_{1,\nu}\ \le\ C_{\boldsymbol{\Phi}}\varepsilon_m\cdot\sup_{\ell\in\mathbb{N}}\|h_\ell\|_{1,\nu} \tag{4.27}$$

for every $m\in\mathbb{N}$. Note that the latter bound is finite because $\|h_m-h\|_{1,\nu}\to 0$ (by assumption). In view of the second part of condition (a) of Theorem 4.3.8, (4.8), (4.27), conditions (a)–(b) of Assumption 4.2.1 along with Definition 1.4.1 (applied to $\mathcal{P}:=\{\boldsymbol{P}_F\}$), for every $\pi\in\Pi$, $k=n+1,\ldots,N-1$, and $m\in\mathbb{N}$ we get

$$\sum_{\substack{J\subseteq\{n,\ldots,k-1\}\\1\le|J|\le k-n}}\int_E\cdots\int_E\int_E\Big|\frac{\Lambda_k^{F;(\pi,y_k)}(F+\varepsilon_m h_m)-\Lambda_k^{F;(\pi,y_k)}(F)}{\varepsilon_m}\Big|\,\xi_{k-1,J}^{F,m;+}\big((y_{k-1},f_{k-1}(y_{k-1})),dy_k\big)$$

$$
\begin{aligned}
& \qquad \xi_{k-2,J}^{F,m;+}\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots \xi_{n,J}^{F,m;+}\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
\leq\ & \sup_{x\in E} \frac{1}{\psi(x)} \cdot \Big| \frac{\Lambda_k^{F;(\pi,x)}(F+\varepsilon_m h_m) - \Lambda_k^{F;(\pi,x)}(F)}{\varepsilon_m} \Big| \\
& \cdot \sum_{\substack{J\subseteq\{n,\ldots,k-1\}\\ 1\leq|J|\leq k-n}} \int_E \cdots \int_E \int_E \psi(y_k)\, \xi_{k-1,J}^{F,m;+}\big((y_{k-1}, f_{k-1}(y_{k-1})), dy_k\big) \\
& \qquad \xi_{k-2,J}^{F,m;+}\big((y_{k-2}, f_{k-2}(y_{k-2})), dy_{k-1}\big) \cdots \xi_{n,J}^{F,m;+}\big((x_n, f_n(x_n)), dy_{n+1}\big) \\
\leq\ & \sup_{x\in E} \frac{1}{\psi(x)} \cdot \Big| \frac{\Lambda_k^{F;(\pi,x)}(F+\varepsilon_m h_m) - \Lambda_k^{F;(\pi,x)}(F)}{\varepsilon_m} - \dot\Lambda_{k;F}^{F;(\pi,x)}(0_{\boldsymbol{L}_0(\nu)}) \Big| \\
& \cdot \sum_{\substack{J\subseteq\{n,\ldots,k-1\}\\ 1\leq|J|\leq k-n}} \binom{k-n}{|J|} K_3^{k-n-|J|} \big(\big\|\Phi_n^{(\pi,\,\cdot\,)}(F+\varepsilon_m h_m) - \Phi_n^{(\pi,\,\cdot\,)}(F)\big\|_\psi + 2K_3\big)^{|J|} \cdot \psi(x_n) \\
\leq\ & \Big\| \frac{\Lambda_k^{F;(\pi,\,\cdot\,)}(F+\varepsilon_m h_m) - \Lambda_k^{F;(\pi,\,\cdot\,)}(F)}{\varepsilon_m} - \dot\Lambda_{k;F}^{F;(\pi,\,\cdot\,)}(0_{\boldsymbol{L}_0(\nu)}) \Big\|_\psi \\
& \cdot \sum_{\substack{J\subseteq\{n,\ldots,k-1\}\\ 1\leq|J|\leq k-n}} \binom{k-n}{|J|} K_3^{k-n-|J|} \big(C_{\boldsymbol{\Phi}}\varepsilon_m \cdot \sup_{\ell\in\mathbb{N}} \|h_\ell\|_{1,\nu} + 2K_3\big)^{|J|} \cdot \psi(x_n). \qquad (4.28)
\end{aligned}
$$

By conditions (a)–(b) of Theorem 4.3.8 along with (4.11), (4.10), and (4.15), the first factor in the last line of (4.28) converges to 0 as $m \to \infty$ uniformly in $\pi \in \Pi$. Thus, since $\sup_{\ell\in\mathbb{N}} \|h_\ell\|_{1,\nu} < \infty$ (recall $\|h_m - h\|_{1,\nu} \to 0$), we may conclude $\lim_{m\to\infty} S_3(m,\pi) = 0$ uniformly in $\pi \in \Pi$. Thus the assertion follows. □

It remains to show that the claim in part (ii) of Theorem 4.3.8 holds.

Proof of part (ii) of Theorem 4.3.8: Let $x_n \in E$ and $n = 0, \ldots, N$ be arbitrary but fixed. In the sequel, we will verify that the map $\mathcal{W}_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by (4.6) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{n,F}^{x_n} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ given by (4.17).

For the proof we will use (4.19) which says that $\mathcal{W}_n^{x_n}$ can be represented as a composition of the functionals Ψ and $\Upsilon_n^{x_n}$ given by (4.20). Note that Proposition 1 in [75] guarantees that Ψ is Hadamard differentiable (in the sense of [75]) at every $(w(\pi))_{\pi\in\Pi} \in \ell^\infty(\Pi)$ with (possibly nonlinear) Hadamard derivative $\dot\Psi_{(w(\pi))_{\pi\in\Pi}} : \ell^\infty(\Pi) \to \mathbb{R}$ given by (2.40). Further it follows from Theorem 4.3.12 that $\Upsilon_n^{x_n}$ is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot\Upsilon_n^{x_n}$ given by (4.21).

In view of (4.19) and the shape of $\dot\Psi_{(w(\pi))_{\pi\in\Pi}}$ and $\dot\Upsilon_{n,F}^{x_n}$, quasi-Hadamard differentiability of $\mathcal{W}_n^{x_n}$ at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ can be identified with quasi-Hadamard differentiability of the map $\Psi \circ \Upsilon_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $(\Psi \circ \dot{\Upsilon}_n^{x_n})_F : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ given by

$$
(\Psi \circ \dot{\Upsilon}_n^{x_n})_F(h) \,:=\, \dot\Psi_{\Upsilon_n^{x_n}(F)} \circ \dot\Upsilon_{n,F}^{x_n}(h). \qquad (4.29)
$$

Indeed, using (4.21) and (2.40) we observe

$$
(\Psi \circ \dot{\Upsilon}_n^{x_n})_F(h) \,=\, \dot\Psi_{(\mathcal{W}_n^{x_n;\pi}(F))_{\pi\in\Pi}}\big((\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h))_{\pi\in\Pi}\big) \,=\, \lim_{\delta\searrow 0} \sup_{\pi\in\Pi(F;\delta)} \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h)
$$

for every $h \in \boldsymbol{L}_1(\nu)$, and that if in addition the set $\Pi(F)$ is non-empty

$$(\Psi \circ \dot{\Upsilon}_n^{x_n})_F(h) \;=\; \sup_{\pi\in\Pi(F)} \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(h)$$

for all $h \in \boldsymbol{L}_1(\nu)$.

Now, an application of the chain rule in the form of Lemma A.2 in [57] yields that the map $\Psi \circ \Upsilon_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$ is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $(\Psi \circ \dot{\Upsilon}_n^{x_n})_F : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ given by (4.29). This completes the proof of Theorem 4.3.8. □

Corollary 4.3.15 below is a special case of Theorem 4.3.8 when in the MDM in (4.1) the state space E as well as the action spaces are finite. That is, let E be given by (1.23) with $\mathfrak{e} := \#E \in \mathbb{N}$ and let $A_n(x_i)$ be the (finite) set of all admissible actions in state $x_i \in E$ at time $n = 0,\dots,N-1$ given by (1.24). The latter framework is discussed in detail in Section 1.5 for general MDMs. Let the set $\widetilde{\mathcal{P}}$ be introduced as in (1.27), and recall the definitions of the functionals $\mathcal{W}_n^{x_i;\pi}$ and $\mathcal{W}_n^{x_i}$ from (4.13). Finally, let $\Pi(F)$ be the set of all optimal strategies w.r.t. F which solves the optimization problem (1.28) (with $\boldsymbol{p}_F$ in place of $\boldsymbol{p}$), and note that this set is non-empty and finite (see the discussion in Section 1.5).

Corollary 4.3.15 (Quasi-Hadamard differentiability of $\mathcal{W}_n^{x_i;\pi}$ and $\mathcal{W}_n^{x_i}$ in F) *Let $F \in \boldsymbol{F}(\nu)$, and assume that in the setting above the following two conditions hold.*

(a) $\boldsymbol{p}_G \in \widetilde{\mathcal{P}}$ *for every* $G \in \boldsymbol{F}(\nu)$.

(b) *For any $i = 1,\dots,\mathfrak{e}$, $\pi \in \Pi$, and $n = 0,\dots,N-1$, the map $\Lambda_n^{F;(\pi,x_i)} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by (4.14) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\Lambda}_{n;F}^{F;(\pi,x_i)} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ satisfying $\dot{\Lambda}_{n;F}^{F;(\pi,x_i)}(0_{\boldsymbol{L}_0(\nu)}) = 0$ as well as $|\dot{\Lambda}_{n;F}^{F;(\pi,x_i)}(h)| \le C_{\dot{\Lambda}}$ for all $h \in \boldsymbol{L}_1(\nu)$, where $C_{\dot{\Lambda}} > 0$ is a finite constant (depending on n and h).*

Then the following two assertions hold.

(i) *For any $i = 1,\dots,\mathfrak{e}$, $\pi \in \Pi$, and $n = 0,\dots,N$, the map $\mathcal{W}_n^{x_i;\pi} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by (4.13) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_i;\pi} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ given by*

$$\begin{aligned}\dot{\mathcal{W}}_{n;F}^{x_i;\pi}(h) \;:=\; & \sum_{k=n}^{N-1} \sum_{i_{n+1}=1}^{\mathfrak{e}} \cdots \sum_{i_{k-1}=1}^{\mathfrak{e}} \sum_{i_k=1}^{\mathfrak{e}} \dot{\Lambda}_{k;F}^{F;(\pi,x_{i_k})}(h)\, p_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}^{F}(i_k) \\ & p_{k-2,i_{k-2};f_{k-2}(x_{i_{k-2}})}^{F}(i_{k-1}) \cdots p_{n,i;f_n(x_i)}^{F}(i_{n+1}).\end{aligned} \tag{4.30}$$

(ii) *For any $i = 1,\dots,\mathfrak{e}$ and $n = 0,\dots,N$, the map $\mathcal{W}_n^{x_i} : \boldsymbol{F}(\nu) \to \mathbb{R}$ defined by (4.13) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ given by*

$$\dot{\mathcal{W}}_{n;F}^{x_i}(h) \;:=\; \max_{\pi\in\Pi(F)} \dot{\mathcal{W}}_{n;F}^{x_i;\pi}(h).$$

Proof At first, it follows from the proof of Corollary 4.3.6 that conditions (a)–(b) of Assumption 4.2.1 are satisfied in the finite setting. Take into account that condition (a) of Corollary 4.3.15 coincides with condition (a) of Corollary 4.3.6. In the same proof we have shown that condition (b) of Theorem 4.3.3 holds in the finite setting. Thus, in view of Remark 4.3.9(iv), condition (c) of Theorem 4.3.8 is also satisfied. Finally, in virtue of (4.10), (4.11), (1.18), (4.14), and the choice of the bounding function $\psi :\equiv 1$, condition (b) clearly matches conditions (a)–(b) of Theorem 4.3.8 in the finite setting. Therefore, an application of the latter theorem entails that the assertions in part (i) and (ii) hold. □

In the following two Sections 4.4 and 4.5 we will use the regularity results from Theorems 4.3.3 and 4.3.8 to perform a statistical estimation of the optimal value (function) of a simple MDM in which the transition function is generated only by an (unknown) single distribution function F. Therefore the objective of these sections is the estimation of the unknown distribution function F which in turn provides an estimate for the (optimal) value $\mathcal{W}_0^{x_0}(F)$ of the (simple) MDM from (4.1) for some given initial state $x_0 \in E$.

Recall from (4.6) the definitions of the functionals $\mathcal{W}_n^{x_n;\pi} : \boldsymbol{F}(\nu) \to \mathbb{R}$ and $\mathcal{W}_n^{x_n} : \boldsymbol{F}(\nu) \to \mathbb{R}$. If $\widehat{F}_m$ corresponds to a reasonable estimator for the unknown distribution function F ($\in \boldsymbol{F}(\nu)$) satisfying $\widehat{F}_m \in \boldsymbol{F}(\nu)$, then for any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \ldots, N$ the plug-in estimator $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ (resp. $\mathcal{W}_n^{x_n}(\widehat{F}_m)$) can be regarded as a reasonable (point) estimator for the aspect $\mathcal{W}_n^{x_n;\pi}(F)$ (resp. $\mathcal{W}_n^{x_n}(F)$). In Sections 4.4–4.5 we show for the estimators $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ and $\mathcal{W}_n^{x_n}(\widehat{F}_m)$ several asymptotic properties, such as strong consistency, asymptotic error distribution, and bootstrap consistency (in probability).

4.4 Nonparametric estimation of $\mathcal{W}_0^{x_0}(F)$

In this section, we will deal with a nonparametric statistical model and consider the empirical distribution function $\widehat{F}_m$ as the canonical estimator for the unknown distribution function F. As already mentioned in the main introduction, a similar estimation approach in a MDM whose transition probabilities are governed by a family of (unknown) distribution functions has already been performed in [26].

Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. real-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by F the common distribution function of $Y_1, Y_2, \ldots$. In particular F is an element of the set $\overline{\boldsymbol{F}}$ of all distribution functions on $\mathbb{R}$. The random variables Y_i can be seen as historical observations (or simulated data) drawn from the unknown distribution function F which in turn governs the random transition mechanism of the MDP at all decision epochs. In practice this means that the Y_i's can be extracted from the observations of the transition probabilities of the MDP.

The estimator for the marginal distribution function F of the sequence $(Y_i)_{i\in\mathbb{N}}$ based on sample size $m \in \mathbb{N}$ will be in the following the empirical distribution function $\widehat{F}_m$ of $Y_1, \ldots, Y_m$ defined by

$$\widehat{F}_m(\omega) := \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{[Y_i(\omega),\infty)}, \qquad \omega \in \Omega. \tag{4.31}$$

Note that (4.31) clearly defines a map $\widehat{F}_m : \Omega \to \overline{\boldsymbol{F}}$. In this case, for any $x_n \in E$, $\pi \in \Pi$, and

$n = 0, \dots, N$, the plug-in estimator $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ (resp. $\mathcal{W}_n^{x_n}(\widehat{F}_m)$) can be regarded as a reasonable nonparametric estimator for $\mathcal{W}_n^{x_n;\pi}(F)$ (resp. $\mathcal{W}_n^{x_n}(F)$) if we can ensure that $F \in \boldsymbol{F}(\nu)$ and $\widehat{F}_m(\omega, \cdot\,) \in \boldsymbol{F}(\nu)$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$.

4.4.1 Strong consistency

The following theorem gives a strong law for the sequences of plug-in estimators $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_{m\in\mathbb{N}}$ for the aspects $\mathcal{W}_n^{x_n;\pi}(F)$ and $\mathcal{W}_n^{x_n}(F)$, respectively. Its assertions are (under some assumptions) an immediate consequence of Theorem 4.3.3 along with an appropriate strong law for the sequence of empirical distribution functions $(\widehat{F}_m)_{m\in\mathbb{N}}$ (see condition (c) of Theorem 4.4.1). Recall again the definition of the norm $\|\cdot\|_{1,\nu}$ introduced in (4.7), and note that $F-G \in \boldsymbol{L}_1(\nu)$ if $F, G \in \boldsymbol{F}(\nu)$.

Theorem 4.4.1 (Strong consistency of $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))$) *Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of real-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by F the common distribution function of the Y_i. Moreover let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1, \dots, Y_m$ as defined in (4.31), and assume that the following three conditions hold.*

(a) *$F \in \boldsymbol{F}$, and $\int_{\mathbb{R}_{<0}} F\, d\nu < \infty$ as well as $\int_{\mathbb{R}_{\geq 0}} (1-F)\, d\nu < \infty$ (that is $F \in \boldsymbol{F}(\nu)$).*

(b) *$\widehat{F}_m(\omega, \cdot\,) \in \boldsymbol{F}(\nu)$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$.*

(c) *$\|\widehat{F}_m - F\|_{1,\nu} \to 0 \quad \mathbb{P}$-a.s.*

Then under the assumptions of Theorem 4.3.3 the following two assertions hold.

(i) *For any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$, the sequence of estimators $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_n^{x_n;\pi}(F)$ under $\mathbb{P}$ in the sense that*

$$\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m) \to \mathcal{W}_n^{x_n;\pi}(F) \quad \mathbb{P}\text{-a.s.}$$

(ii) *For any $x_n \in E$ and $n = 0, \dots, N$, the sequence of estimators $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_n^{x_n}(F)$ under $\mathbb{P}$ in the sense that*

$$\mathcal{W}_n^{x_n}(\widehat{F}_m) \to \mathcal{W}_n^{x_n}(F) \quad \mathbb{P}\text{-a.s.}$$

Note that for the statements in parts (i) and (ii) of Theorem 4.4.1 it is *not* necessary that the estimators $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ and $\mathcal{W}_n^{x_n}(\widehat{F}_m)$ are $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

Part (ii) of Theorem 4.4.1 provides the following information. If there exists an optimal strategy $\pi^F \in \Pi$ w.r.t. F, then under conditions (a)–(b) of Theorem 4.4.1 and the assumptions of Theorem 4.3.3, the sequence of estimators $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent (under $\mathbb{P}$) for the optimal value $\mathcal{W}_0^{x_0;\pi^F}(F)$ of the optimization problem (4.4) whenever the sequence of empirical distribution functions $(\widehat{F}_m)_{m\in\mathbb{N}}$ is strongly consistent (under $\mathbb{P}$) for F w.r.t. the norm $\|\cdot\|_{1,\nu}$. However, the existence of an optimal strategy w.r.t. F and the latter convergence are not trivially satisfied in general. In the examples of Chapter 5 it is shown that these conditions hold; see Subsections 5.1.3 and 5.2.3 for details.

The following remark discusses an approach that could help in practice to estimate the (exact) optimal strategy $\pi^F \in \Pi$ w.r.t. F.

Remark 4.4.2 In the nonparametric setting above an optimal strategy could be calculated approximately using the Bellman equation in part (i) of Theorem 1.3.3 in Section 1.3 (applied to $\boldsymbol{P}_F$), where the corresponding transition probabilities are computed by means of the empirical distribution function $\widehat{F}_m$ based on the observed data $(Y_i)_{i\in\mathbb{N}}$. As a consequence, the resulting strategy $\pi^{\widehat{F}_m} \in \Pi$ can be seen as an approximate solution to the optimization problem (4.4) and thus an estimator for an exact (but unknown) optimal strategy $\pi^F \in \Pi$ w.r.t. F. At this point it could be of interest how well the estimated optimal strategy $\pi^{\widehat{F}_m}$ approximates the true optimal strategy π^F. To answer this question, one needs to know how sensitive the optimal strategy is w.r.t. changes in the transition probabilities. However, to the best of my knowledge there is no result which shows this sensitivity. ◇

Remark 4.4.3 (i) It follows from the discussion in Remark 4.3.4(i) that (under the assumptions of Theorem 4.4.1) the assertion in part (i) of Theorem 4.4.1 holds even uniformly in $\pi \in \Pi$. That is, for every $x_n \in E$ and $n = 0, \dots, N$ we have $\sup_{\pi\in\Pi} |\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m) - \mathcal{W}_n^{x_n;\pi}(F)| \to 0 \quad \mathbb{P}$-a.s.

(ii) Note that we get even stronger results in parts (i) and (ii) of Theorem 4.4.1 if condition (c) of the latter theorem is replaced by the following slightly stronger condition:

(c') $m^r\|\widehat{F}_m - F\|_{1,\nu} \to 0 \quad \mathbb{P}$-a.s. for every $r < \frac{1}{2}$.

In fact, we obtain by means of Theorem 4.3.3 that under the assumptions of Theorem 4.4.1 with (c') in place of (c) the statements

(i') $m^r\big(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m) - \mathcal{W}_n^{x_n;\pi}(F)\big) \to 0 \quad \mathbb{P}$-a.s.

(ii') $m^r\big(\mathcal{W}_n^{x_n}(\widehat{F}_m) - \mathcal{W}_n^{x_n}(F)\big) \to 0 \quad \mathbb{P}$-a.s.

hold for any $x_n \in E$, $\pi \in \Pi$, $n = 0, \dots, N$, and $r < \frac{1}{2}$. ◇

4.4.2 Asymptotic error distribution

In this subsection we determine in Theorem 4.4.4 below the asymptotic error distribution of the sequences of the (plug-in) estimators $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_{m\in\mathbb{N}}$. The key will be a special functional delta-method in the form of [59]. In fact, we will derive the asymptotic error distribution of the latter sequences of estimators by applying this functional delta-method along with Theorem 4.3.8 and a central limit theorem for the empirical process (see Theorem 4.4.6 ahead).

Let $\boldsymbol{L}_1(\nu)$ again be the space of all maps $h \in \boldsymbol{L}_0(\nu)$ with $\|h\|_{1,\nu} < \infty$, where the norm $\|\cdot\|_{1,\nu}$ is defined by (4.7). In this subsection we will assume that the Borel measure ν is locally finite. Recall that any locally finite measure on $\mathcal{B}(\mathbb{R})$ is finite on bounded intervals and thus clearly σ-finite. Note that it follows from Corollary 4.2.2 in [20] that the Banach space $(\boldsymbol{L}_1(\nu), \|\cdot\|_{1,\nu})$ is separable. In the following, let $\mathcal{B}(\boldsymbol{L}_1(\nu))$ be the Borel σ-algebra on $\boldsymbol{L}_1(\nu)$ w.r.t. the norm $\|\cdot\|_{1,\nu}$.

For the formulation of Theorem 4.4.4, recall from [2, 62] that an $(\boldsymbol{L}_1(\nu), \mathcal{B}(\boldsymbol{L}_1(\nu)))$-valued random variable B on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ is called an $\boldsymbol{L}_1(\nu)$-valued Gaussian random variable if $\Xi(B)$ is a real-valued Gaussian random variable for any $(\|\cdot\|_{1,\nu},|\cdot|)$-continuous linear functional $\Xi:\boldsymbol{L}_1(\nu)\to\mathbb{R}$, that is, if $\int h(t)B(\,\cdot\,,t)\,\nu(dt)$ is a real-valued Gaussian random variable for every $h\in\boldsymbol{L}_\infty(\nu)$. Here $\boldsymbol{L}_\infty(\nu)$ denotes the space of all bounded maps in $\boldsymbol{L}_0(\nu)$. We note that the covariance operator of an $\boldsymbol{L}_1(\nu)$-valued Gaussian random variable B is the map $\Gamma_{B,\nu}(h_1,h_2):\boldsymbol{L}_\infty(\nu)\times\boldsymbol{L}_\infty(\nu)\to\mathbb{R}$ defined by

$$\begin{aligned}&\Gamma_{B,\nu}(h_1,h_2)\\&\quad:=\ \check{\mathbb{E}}\Big[\Big(\int h_1(t_1)(B(\,\cdot\,,t_1)-\check{\mathbb{E}}[B(\,\cdot\,,t_1)])\,\nu(dt_1)\Big)\Big(\int h_2(t_2)(B(\,\cdot\,,t_2)-\check{\mathbb{E}}[B(\,\cdot\,,t_2)])\,\nu(dt_2)\Big)\Big];\end{aligned}$$

see, for example, [2]. Also note that an $\boldsymbol{L}_1(\nu)$-valued random variable B on some probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$ is said to be *centred* if $\check{\mathbb{E}}[B(\,\cdot\,,t)]\equiv 0$ for all $t\in\mathbb{R}$.

Part (ii) of Theorem 4.4.4 determines (under some assumptions) the asymptotic error distribution of the sequence of estimators $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_{m\in\mathbb{N}}$ for the aspect $\mathcal{W}_0^{x_0}(F)$. Here and in the sequel convergence in distribution will be denoted by $\rightsquigarrow$, where we refer to Section 2 in [19] for the notion of weak convergence in metric spaces that are equipped with a Borel σ-algebra.

Theorem 4.4.4 (Asymptotic error distribution of $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))$) *Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of real-valued random variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, and denote by F the common distribution function of the Y_i. Moreover let $\widehat{F}_m$ be for every $m\in\mathbb{N}$ the empirical distribution function of $Y_1,\ldots,Y_m$ as defined in (4.31), and assume that the following three conditions hold.*

(a) *$F\in\boldsymbol{F}$, and $\int\sqrt{F(1-F)}\,d\nu<\infty$ (in particular $F\in\boldsymbol{F}(\nu)$).*

(b) *$\widehat{F}_m(\omega,\,\cdot\,)\in\boldsymbol{F}(\nu)$ for every $\omega\in\Omega$ and $m\in\mathbb{N}$.*

(c) *For every $x_n\in E$, $\pi\in\Pi$, $n=0,\ldots,N$, and $m\in\mathbb{N}$, the estimators $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ and $\mathcal{W}_n^{x_n}(\widehat{F}_m)$ are $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable.*

Then under the assumptions of Theorem 4.3.8 the following two assertions hold.

(i) *For any $x_n\in E$, $\pi\in\Pi$, and $n=0,\ldots,N$, we have that $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and*

$$\sqrt{m}\big(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)-\mathcal{W}_n^{x_n;\pi}(F)\big)\ \rightsquigarrow\ \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F)\qquad\text{in }(\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|),$$

where $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}$ is given by (4.16) and B_F is an $\boldsymbol{L}_1(\nu)$-valued centred Gaussian random variable on some probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$ with covariance operator given by

$$\Gamma_{B_F,\nu}(h_1,h_2)\ =\ \int_{\mathbb{R}^2}h_1(t_1)\,C_F(t_1,t_2)\,h_2(t_2)\,(\nu\otimes\nu)(d(t_1,t_2))\qquad\text{for all }h_1,h_2\in\boldsymbol{L}_\infty(\nu),\tag{4.32}$$

where

$$C_F(t_1,t_2)\ :=\ F(t_1\wedge t_2)(1-F(t_1\vee t_2)),\qquad t_1,t_2\in\mathbb{R}.\tag{4.33}$$

(ii) *For any $x_n\in E$ and $n=0,\ldots,N$, we have that $\dot{\mathcal{W}}_{n;F}^{x_n}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and*

$$\sqrt{m}\big(\mathcal{W}_n^{x_n}(\widehat{F}_m)-\mathcal{W}_n^{x_n}(F)\big)\ \rightsquigarrow\ \dot{\mathcal{W}}_{n;F}^{x_n}(B_F)\qquad\text{in }(\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|),$$

where $\dot{\mathcal{W}}_{n;F}^{x_n}$ is given by (4.17) and B_F is as in (i).

The measurability assumption in condition (c) of the latter theorem is not very restrictive. We refer to Subsections 5.1.3 and 5.2.3 for a verification of this condition.

In the examples of Chapter 5 we will see that part (ii) of Theorem 4.4.4 can be used to construct an asymptotic confidence interval for the (optimal) value $\mathcal{W}_0^{x_0}(F)$. However, since (in these examples) the asymptotic error distribution of the corresponding plug-in estimator $\mathcal{W}_0^{x_0}(\widehat{F}_m)$ depends on the unknown distribution function F in a rather complex manner, it is therefore expected that the bootstrap results presented in the next subsection could lead to a more efficient method than the method based on a nonparametric estimation of the distribution of $\dot{\mathcal{W}}_{0;F}^{x_0}(B_F)$ in the unknown distribution function F.

Now, we intend to prove Theorem 4.4.4. Its proof relies on Theorem 4.4.6 below. For the latter theorem, however, we need an additional lemma.

Recall that a real-valued stochastic process ξ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with index set $\mathbb{R}$ is a map $\xi : \Omega \times \mathbb{R} \to \mathbb{R}$ such that the coordinate $\omega \mapsto \xi(\omega, t)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for all $t \in \mathbb{R}$. The process ξ will be called *measurable* if the map $\xi : \Omega \times \mathbb{R} \to \mathbb{R}$ is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable. Note that any real-valued stochastic process with right-continuous paths is measurable. In particular, the empirical distribution function $\widehat{F}_m$ defined by (4.31) may be regarded as a real-valued measurable stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$.

The statement of the following lemma can be proven in the same way as Lemma C.1 in [59].

Lemma 4.4.5 *Suppose that the stochastic process ξ is measurable and that $\xi(\omega, \cdot) \in \boldsymbol{L}_1(\nu)$ for every $\omega \in \Omega$. Then the mapping $\omega \mapsto \xi(\omega, \cdot)$ from Ω to $\boldsymbol{L}_1(\nu)$ is $(\mathcal{F}, \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable. In particular, the process ξ can be seen as an $(\boldsymbol{L}_1(\nu), \mathcal{B}(\boldsymbol{L}_1(\nu)))$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.*

The following Theorem 4.4.6, which can be deduced from Corollary 2.4 in [29], provides a central limit theorem for the empirical process $\sqrt{m}(\widehat{F}_m - F)$.

Theorem 4.4.6 *With the notation and under the assumptions of Theorem 4.4.4 (except condition (c) of Theorem 4.4.4) we have*

$$\sqrt{m}(\widehat{F}_m - F) \rightsquigarrow B_F \quad \text{in } (\boldsymbol{L}_1(\nu), \mathcal{B}(\boldsymbol{L}_1(\nu)), \|\cdot\|_{1,\nu}) \tag{4.34}$$

for an $\boldsymbol{L}_1(\nu)$-valued centred Gaussian random variable B_F on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\nu}$ given by (4.32).

Note that in the setting of Theorem 4.4.6 the $\boldsymbol{L}_1(\nu)$-valued centred Gaussian random variable B_F jumps where F jumps. Also note that the integrability condition $\int \sqrt{F(1-F)}\, d\nu < \infty$ in condition (a) of Theorem 4.4.4 clearly implies that $\int_{\mathbb{R}_{<0}} F\, d\nu < \infty$ as well as $\int_{\mathbb{R}_{\geq 0}} (1-F)\, d\nu < \infty$. Thus $F \in \boldsymbol{F}(\nu)$ by the first part of condition (a) of Theorem 4.4.4. Therefore it follows from Lemma 4.4.5 that under conditions (a)–(b) of Theorem 4.4.4 the empirical process $\sqrt{m}(\widehat{F}_m - F)$ can be seen as an $(\boldsymbol{L}_1(\nu), \mathcal{B}(\boldsymbol{L}_1(\nu)))$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ for every $m \in \mathbb{N}$. Recall that $F - G \in \boldsymbol{L}_1(\nu)$ if $F, G \in \boldsymbol{F}(\nu)$.

Proof of Theorem 4.4.6: For the assertions in (4.32) and (4.34) we intend to apply Corollary 2.4 in [29]. First, we may define an i.i.d. sequence $(Z_i)_{i \in \mathbb{N}_0}$ of real-valued random variables by

$Z_i := Y_{i+1}$, and it follows from Lemma 10.2 in [46] that the latter sequence can be extended to an i.i.d. sequence $(Z_i)_{i\in\mathbb{Z}}$. Take into account that every sequence of identically distributed random variables is clearly stationary in the sense of [46, p. 179]. Now, let $X_i : \Omega \times \mathbb{R} \to \mathbb{R}$ be for any $i \in \mathbb{Z}$ a real-valued stochastic process defined by

$$X_i(\omega, t) := \mathbb{1}_{[Z_i(\omega),\infty)}(t) - F(t).$$

Note that F is the common distribution function of the random variables Z_i, $i \in \mathbb{Z}$, and that the mapping $t \mapsto X_i(\omega, t)$ is right-continuous and thus $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable for every $i \in \mathbb{Z}$ and $\omega \in \Omega$. In particular, X_i is measurable for any $i \in \mathbb{Z}$. Moreover, we get

$$\begin{aligned}
&\|X_i(\omega, \cdot\,)\|_{1,\nu}\\
&= \int_{\mathbb{R}_{<0}} \big|\mathbb{1}_{[Z_i(\omega),\infty)}(t) - F(t)\big|\,\nu(dt) + \int_{\mathbb{R}_{\geq 0}} \big|\mathbb{1}_{[Z_i(\omega),\infty)}(t) - F(t)\big|\,\nu(dt)\\
&\leq \int_{\mathbb{R}_{<0}} \mathbb{1}_{[Z_i(\omega),\infty)}(t)\,\nu(dt) + \int_{\mathbb{R}_{<0}} F(t)\,\nu(dt) + \int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\nu(dt)\\
&\quad + \int_{\mathbb{R}_{\geq 0}} \big(1 - \mathbb{1}_{[Z_i(\omega),\infty)}(t)\big)\,\nu(dt)\\
&= \nu\big[[Z_i(\omega), \infty) \cap \mathbb{R}_{<0}\big] + \int_{\mathbb{R}_{<0}} F(t)\,\nu(dt) + \int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\nu(dt)\\
&\quad + \int_{\mathbb{R}_{\geq 0}} \big(1 - \mathbb{1}_{[Z_i(\omega),\infty)}(t)\big)\,\nu(dt)\,\mathbb{1}_{\{Z_i(\omega)<0\}} + \int_{\mathbb{R}_{\geq 0}} \big(1 - \mathbb{1}_{[Z_i(\omega),\infty)}(t)\big)\,\nu(dt)\,\mathbb{1}_{\{Z_i(\omega)>0\}}\\
&\quad + \int_{\mathbb{R}_{\geq 0}} \big(1 - \mathbb{1}_{[Z_i(\omega),\infty)}(t)\big)\,\nu(dt)\,\mathbb{1}_{\{Z_i(\omega)=0\}}\\
&= \nu\big[[Z_i(\omega), 0)\big]\,\mathbb{1}_{\{Z_i(\omega)<0\}} + \int_{\mathbb{R}_{<0}} F(t)\,\nu(dt) + \int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\nu(dt)\\
&\quad + \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{[Z_i(\omega),0)}(t)\,\nu(dt)\,\mathbb{1}_{\{Z_i(\omega)<0\}} + \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{[0,Z_i(\omega))}(t)\,\nu(dt)\,\mathbb{1}_{\{Z_i(\omega)>0\}} + 0\\
&= \nu\big[[Z_i(\omega), 0)\big]\,\mathbb{1}_{\{Z_i(\omega)<0\}} + \int_{\mathbb{R}_{<0}} F(t)\,\nu(dt) + \int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\nu(dt)\\
&\quad + \nu\big[[Z_i(\omega), 0) \cap \mathbb{R}_{\geq 0}\big]\,\mathbb{1}_{\{Z_i(\omega)<0\}} + \nu\big[[0, Z_i(\omega))\big]\,\mathbb{1}_{\{Z_i(\omega)>0\}}\\
&= \int_{\mathbb{R}_{<0}} F(t)\,\nu(dt) + \int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\nu(dt)\\
&\quad + \nu\big[[Z_i(\omega), 0)\big]\,\mathbb{1}_{\{Z_i(\omega)<0\}} + \nu\big[[0, Z_i(\omega))\big]\,\mathbb{1}_{\{Z_i(\omega)>0\}}\\
&< \infty
\end{aligned}$$

for any $\omega \in \Omega$ and $i \in \mathbb{Z}$ because $F \in \boldsymbol{F}(\nu)$ (by condition (a)) and ν is locally finite. Hence $X_i(\omega, \cdot\,) \in \boldsymbol{L}_1(\nu)$ for every $\omega \in \Omega$ and $i \in \mathbb{Z}$. As a consequence and in view of

$$\mathbb{E}[X_i(\cdot\,, t)] = \mathbb{E}[\mathbb{1}_{[Z_i(\cdot\,),\infty)}(t)] - F(t) = \mathbb{P}\big[\{Z_i(\cdot\,) \in (-\infty, t]\}\big] - F(t) = F(t) - F(t) = 0 \quad (4.35)$$

for every $i \in \mathbb{Z}$ and $t \in \mathbb{R}$, Lemma 4.4.5 implies that X_i is for any $i \in \mathbb{Z}$ a centred $\boldsymbol{L}_1(\nu)$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

Setting $\mathcal{F}^0 := \sigma(Z_i : i \in \mathbb{Z} \setminus \mathbb{N})$ for the σ-algebra on Ω generated by the random variables Z_i, $i \in \mathbb{Z} \setminus \mathbb{N}$, and in view of

$$\begin{aligned}\int \|X_0(\,\cdot\,,t)\|_2\,\nu(dt) &= \int \big\|\mathbb{1}_{[Z_0(\cdot),\infty)}(t) - F(t)\big\|_2\,\nu(dt) \,=\, \int \sqrt{\mathbb{E}\big[\big(\mathbb{1}_{[Z_0(\cdot),\infty)}(t) - F(t)\big)^2\big]}\,\nu(dt)\\ &= \int \sqrt{\mathbb{E}[\mathbb{1}_{[Z_0(\cdot),\infty)}(t)] - 2\mathbb{E}[\mathbb{1}_{[Z_0(\cdot),\infty)}(t)]\,F(t) + F(t)^2}\,\nu(dt)\\ &= \int \sqrt{\mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t]\big\}\big] - 2\mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t]\big\}\big]\,F(t) + F(t)^2}\,\nu(dt)\\ &= \int \sqrt{F(t) - F(t)^2}\,\nu(dt) \,=\, \int \sqrt{F(t)(1-F(t))}\,\nu(dt) \,<\, \infty\end{aligned}$$

(by condition (a)) as well as (4.35), the sequence $(X_i)_{i\in\mathbb{Z}}$ of random variables satisfies conditions (2.1) and (2.5)–(2.6) in [29]. Here $\|\cdot\|_2$ refers to the L^2-norm on the usual $L^2 = L^2(\Omega,\mathcal{F},\mathbb{P})$ space on $(\Omega,\mathcal{F},\mathbb{P})$. Therefore the assumptions of Corollary 2.4 in [29] are fulfilled, and an application of this corollary entails that (4.34) holds.

To end the proof, it remains to show (4.32). At first, we observe for any $i \in \mathbb{Z}$ and $t_1, t_2 \in \mathbb{R}$

$$\begin{aligned}&\mathbb{C}\mathrm{ov}(X_0(\,\cdot\,,t_1), X_i(\,\cdot\,,t_2))\\ &\quad= \mathbb{C}\mathrm{ov}\big(\mathbb{1}_{[Z_0(\cdot),\infty)}(t_1) - F(t_1), \mathbb{1}_{[Z_i(\cdot),\infty)}(t_2) - F(t_2)\big) \,=\, \mathbb{C}\mathrm{ov}\big(\mathbb{1}_{[Z_0(\cdot),\infty)}(t_1), \mathbb{1}_{[Z_i(\cdot),\infty)}(t_2)\big)\\ &\quad= \mathbb{E}[\mathbb{1}_{[Z_0(\cdot),\infty)}(t_1)\mathbb{1}_{[Z_i(\cdot),\infty)}(t_2)] - \mathbb{E}[\mathbb{1}_{[Z_0(\cdot),\infty)}(t_1)]\,\mathbb{E}[\mathbb{1}_{[Z_i(\cdot),\infty)}(t_2)]\\ &\quad= \mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_1]\big\} \cap \big\{Z_i(\cdot) \in (-\infty,t_2]\big\}\big] - \mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_1]\big\}\big]\,\mathbb{P}\big[\big\{Z_i(\cdot) \in (-\infty,t_2]\big\}\big]\\ &\quad= \begin{cases} \mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_1]\big\}\big]\,\mathbb{P}\big[\big\{Z_i(\cdot) \in (-\infty,t_2]\big\}\big] & \\ \quad - \mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_1]\big\}\big]\,\mathbb{P}\big[\big\{Z_i(\cdot) \in (-\infty,t_2]\big\}\big] & ,\ i \neq 0\\ \mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_1 \wedge t_2]\big\}\big] - \mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_1]\big\}\big]\,\mathbb{P}\big[\big\{Z_0(\cdot) \in (-\infty,t_2]\big\}\big] & ,\ i = 0\end{cases}\\ &\quad= \begin{cases} 0 & ,\ i \neq 0\\ F(t_1 \wedge t_2) - F(t_1)F(t_2) & ,\ i = 0\end{cases}\\ &\quad= \begin{cases} 0 & ,\ i \neq 0\\ C_F(t_1,t_2) & ,\ i = 0\end{cases}\end{aligned}$$

because the sequence $(Z_i)_{i\in\mathbb{Z}}$ is independent and each Z_i has distribution function F (by construction), where the last "=" follows from (4.33). Therefore

$$\mathbb{C}\mathrm{ov}(X_0(\,\cdot\,,t_1), X_i(\,\cdot\,,t_2)) \,=\, C_F(t_1,t_2)\,\mathbb{1}_{\{i=0\}} \qquad \text{for all } i \in \mathbb{Z} \text{ and } t_1,t_2 \in \mathbb{R}. \tag{4.36}$$

Moreover since $h(X_i)$ corresponds for any $h \in \boldsymbol{L}_\infty(\nu)$ and $i \in \mathbb{Z}$ to a bounded linear functional, it follows from [20, Theorem 4.4.1] as well as Fubini's theorem that for all $h_1, h_2 \in \boldsymbol{L}_\infty(\nu)$ and $i \in \mathbb{Z}$

$$\begin{aligned}\mathbb{C}\mathrm{ov}(h_1(X_0), h_2(X_i)) &= \mathbb{E}\Big[\Big(\int h_1(t_1)\,X_0(\,\cdot\,,t_1)\,\nu(dt_1)\Big)\Big(\int h_2(t_2)\,X_i(\,\cdot\,,t_2)\,\nu(dt_2)\Big)\Big]\\ &\quad - \mathbb{E}\Big[\int h_1(t_1)\,X_0(\,\cdot\,,t_1)\,\nu(dt_1)\Big]\,\mathbb{E}\Big[\int h_2(t_2)\,X_i(\,\cdot\,,t_2)\,\nu(dt_2)\Big]\\ &= \int_{\mathbb{R}^2} h_1(t_1)\,h_2(t_2)\,\mathbb{E}\big[X_0(\,\cdot\,,t_1)X_i(\,\cdot\,,t_2)\big]\,(\nu\otimes\nu)(d(t_1,t_2))\\ &\quad - \int h_1(t_1)\,\mathbb{E}[X_0(\,\cdot\,,t_1)]\,\nu(dt_1)\int h_2(t_2)\,\mathbb{E}[X_i(\,\cdot\,,t_2)]\,\nu(dt_2)\end{aligned}$$

$$= \int_{\mathbb{R}^2} h_1(t_1)\, h_2(t_2)\, \mathbb{C}\mathrm{ov}(X_0(\,\cdot\,,t_1), X_i(\,\cdot\,,t_2))\, (\nu\otimes\nu)(d(t_1,t_2)).$$

Take into account that X_i is centred for any $i \in \mathbb{Z}$. Hence, in view of equation (2.4) in [29] as well as (4.36), we end up with

$$\begin{aligned}
\Gamma_{B_F,\nu}(h_1,h_2) &= \sum_{i\in\mathbb{Z}} \mathbb{C}\mathrm{ov}(h_1(X_0), h_2(X_i)) \\
&= \sum_{i\in\mathbb{Z}} \int_{\mathbb{R}^2} h_1(t_1)\, h_2(t_2)\, \mathbb{C}\mathrm{ov}(X_0(\,\cdot\,,t_1), X_i(\,\cdot\,,t_2))\, (\nu\otimes\nu)(d(t_1,t_2)) \\
&= \int_{\mathbb{R}^2} h_1(t_1)\, C_F(t_1,t_2)\, h_2(t_2)\, (\nu\otimes\nu)(d(t_1,t_2))
\end{aligned}$$

for every $h_1, h_2 \in \boldsymbol{L}_\infty(\nu)$. This shows (4.32). This completes the proof of Theorem 4.4.6. □

Now, we are in the position to verify the assertions in Theorem 4.4.4.

Proof of Theorem 4.4.4: We will only prove the claim in part (i). The assertion in part (ii) will follow with analogous arguments. Let $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$. First, part (i) of Theorem 4.3.8 ensures that the map $\mathcal{W}_n^{x_n;\pi}$ defined by (4.6) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}$ given by (4.16). Note that Theorem 4.3.8 is applicable because $F \in \boldsymbol{F}$ along with $\int \sqrt{F(1-F)}\, d\nu < \infty$ (by condition (a)) implies $F \in \boldsymbol{F}(\nu)$; see the discussion below of Theorem 4.4.6. Second, it follows from condition (c) that the expression $\sqrt{m}(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m) - \mathcal{W}_n^{x_n;\pi}(F))$ corresponds to an $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable map from Ω to $\mathbb{R}$ for every $m \in \mathbb{N}$. Thus, in view of Theorem 4.4.6, the functional delta-method in the form of Theorem B.3(i) in [59] implies that $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F)$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and that

$$\sqrt{m}\big(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m) - \mathcal{W}_n^{x_n;\pi}(F)\big) \;\rightsquigarrow\; \dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|),$$

where B_F is an $\boldsymbol{L}_1(\nu)$-valued centred Gaussian random variable on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\nu}$ given by (4.32). This completes the proof of Theorem 4.4.4. □

The following Remark 4.4.7 provides a criterion which ensures that the integrability condition $\int \sqrt{F(1-F)}\, d\nu < \infty$ in condition (a) of Theorem 4.4.4 is satisfied; see Subsection 4.4.3 for an application. Recall from Subsection 4.3.2 that a weight function ϕ is a continuous map $\phi : \mathbb{R} \to \mathbb{R}_{\ge 1}$ which is non-increasing on $\mathbb{R}_{\le 0}$ and non-decreasing on $\mathbb{R}_{\ge 0}$.

Remark 4.4.7 The integrability condition $\int \sqrt{F(1-F)}\, d\nu < \infty$ in condition (a) of Theorem 4.4.4 holds if $\int \phi^2\, dF < \infty$ for some weight function ϕ satisfying $\int 1/\phi\, d\nu < \infty$. In this case and under the first part of condition (a) of Theorem 4.4.4, we even have $F \in \boldsymbol{F}(\nu)$.

Proof First of all, note that the finiteness of the integral $\int \phi^2\, dF$ entails that we can find some finite constant $C > 0$ such that $1 - F(x) \le C\phi^{-2}(x)$ for all $x \in \mathbb{R}_{\ge 0}$ and $F(x) \le C\phi^{-2}(x)$ for all $x \in \mathbb{R}_{<0}$. Hence in view of $C_\phi := \int 1/\phi\, d\nu < \infty$, this implies

$$\int \sqrt{F(1-F)}\, d\nu \;=\; \int_{\mathbb{R}_{<0}} \sqrt{F(1-F)}\, d\nu + \int_{\mathbb{R}_{\ge 0}} \sqrt{F(1-F)}\, d\nu$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}_{<0}} \sqrt{F}\, d\nu + \int_{\mathbb{R}_{\geq 0}} \sqrt{1-F}\, d\nu \;\leq\; \sqrt{C} \int_{\mathbb{R}_{<0}} 1/\phi\, d\nu + \sqrt{C} \int_{\mathbb{R}_{\geq 0}} 1/\phi\, d\nu \\
&= \sqrt{C} \int 1/\phi\, d\nu \;=\; \sqrt{C} C_\phi \;<\; \infty.
\end{aligned}
$$

This shows the first assertion. It is discussed subsequent to Theorem 4.4.6 that the integrability condition $\int \sqrt{F(1-F)}\, d\nu < \infty$ entails that $\int_{\mathbb{R}_{<0}} F\, d\nu < \infty$ as well as $\int_{\mathbb{R}_{\geq 0}} (1-F)\, d\nu < \infty$. Hence the additional claim follows and completes the proof. $\diamond$

We conclude this subsection with the following remark.

Remark 4.4.8 The asymptotic error distribution of the sequences of estimators $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_{m\in\mathbb{N}}$ can also be obtained if, instead of Theorem 4.4.6, a central limit theorem for the empirical process in the (normed) space $(\boldsymbol{D}_\phi, \|\cdot\|_{1/\phi})$ (introduced in Example 4.3.11) is used; see, for instance, Example 4.3 in [15]. We note that it follows from the discussion in Example 4.3.11 that the maps $\mathcal{W}_n^{x_n;\pi}$ and $\mathcal{W}_n^{x_n}$ as defined in (4.6) are also quasi-Hadamard differentiable at any fixed $F \in \boldsymbol{F}(\nu)$ tangentially to $\boldsymbol{D}_\phi\langle\boldsymbol{D}_\phi\rangle$ with quasi-Hadamard derivatives $\dot{\mathcal{W}}_{n;F}^{x_n;\pi} : \boldsymbol{D}_\phi \to \mathbb{R}$ and $\dot{\mathcal{W}}_{n;F}^{x_n} : \boldsymbol{D}_\phi \to \mathbb{R}$ given by (4.16) and (4.17) restricted to $h \in \boldsymbol{D}_\phi$, respectively. $\diamond$

4.4.3 Bootstrap consistency

In this subsection we will present in Theorem 4.4.9 below a result concerning the bootstrap consistency (in probability) of the sequences of estimators $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_{m\in\mathbb{N}}$. For the latter asymptotic results we will use a functional delta method (for the bootstrap) in the form of [59] along with Theorem 4.3.8 and a bootstrap version of the central limit theorem for the empirical process in the Banach space $(\boldsymbol{L}_1(\nu), \|\cdot\|_{1,\nu})$ (see Theorem 4.4.10 ahead), where we assume throughout this subsection that the Borel measure ν is locally finite.

To explain the bootstrap method more explicitly, let $(W_{mi})_{m\in\mathbb{N},1\leq i\leq m}$ be a triangular array of nonnegative real-valued random variables on another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ such that one of the following settings is met.

- (B1) (Efron's bootstrap) The random vector $(W_{m1}, \ldots, W_{mm})$ is multinomially distributed according to the parameters m and $p_1 = \cdots = p_m = \frac{1}{m}$ for every $m \in \mathbb{N}$.
- (B2) (Bayesian bootstrap) $W_{mi} = Z_i/\overline{Z}_m$ for every $i = 1, \ldots, m$ and $m \in \mathbb{N}$, where $\overline{Z}_m := \frac{1}{m}\sum_{j=1}^m Z_j$ and $(Z_j)_{j\in\mathbb{N}}$ is any sequence of nonnegative i.i.d. random variables on $(\Omega', \mathcal{F}', \mathbb{P}')$ with common distribution function G satisfying $\int_{\mathbb{R}_{>0}} \sqrt{1-G}\, d\ell < \infty$, and whose standard deviation coincides with its mean and is strictly positive.

Now, extend the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the product space

$$
(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}'),
$$

and note that the sequences $(Y_i)_{i\in\mathbb{N}}$ and $(W_{mi})_{m\in\mathbb{N},1\leq i\leq m}$ regarded as families of random variables on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ are independent. Moreover define for every $m \in \mathbb{N}$

$$
\widehat{F}_m^*(\omega, \omega') := \frac{1}{m} \sum_{i=0}^m W_{mi}(\omega') \mathbb{1}_{[Y_i(\omega),\infty)}, \qquad (\omega, \omega') \in \overline{\Omega}. \tag{4.37}
$$

Note that $\widehat{F}_m^*$ is defined to be the distribution function of the empirical measure of $Y_1,\ldots,Y_m$, where the Dirac measure at point Y_i is weighted by the random variable W_{mi}. Since the mapping $t \mapsto \widehat{F}_m^*((\omega,\omega'),t)$ is clearly right-continuous, the bootstraped empirical distribution function $\widehat{F}_m^*$ can be seen as a real-valued measurable stochastic process on $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}})$. Hence (4.37) indeed defines a map $\widehat{F}_m^* : \overline{\Omega} \to \boldsymbol{F}$ which we will regard in the sequel as a bootstrap version of the empirical distribution function $\widehat{F}_m$ given by (4.31).

The assertion in part (ii) of Theorem 4.4.9 can be used to construct an asymptotic bootstrap confidence interval for the optimal value of the optimization problem (4.4); see Remark 4.4.13 below. Let d_{BL} be the bounded Lipschitz metric on $\mathcal{M}_1(\mathbb{R})$ as introduced in Example 2.1.4 (with $E := \mathbb{R}$). Finally, recall that for some given $F \in \boldsymbol{F}(\nu)$ the set $\Pi(F)$ consists of all optimal strategies w.r.t. F, and note that by $\check{\mathbb{P}}_\xi$ we mean the distribution of a random variable ξ on some probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$ under $\check{\mathbb{P}}$.

Theorem 4.4.9 (Bootstrap consistency of $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))$ and $(\mathcal{W}_n^{x_n}(\widehat{F}_m))$) *Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of real-valued random variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, and denote by F the common distribution function of the Y_i. Moreover let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1,\ldots,Y_m$ as defined in (4.31). Let $(W_{mi})_{m\in\mathbb{N},1\le i\le m}$ be a triangular array of nonnegative real-valued random variables on another probability space $(\Omega',\mathcal{F}',\mathbb{P}')$, and set $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}}) := (\Omega\times\Omega',\mathcal{F}\otimes\mathcal{F}',\mathbb{P}\otimes\mathbb{P}')$. Finally, let $\widehat{F}_m^*$ be for every $m \in \mathbb{N}$ given by (4.37), and assume that the following conditions hold.*

(a) *$F \in \boldsymbol{F}$, and $\int \phi^2\,dF < \infty$ for some weight function ϕ with $\int 1/\phi\,d\nu < \infty$ (in particular $F \in \boldsymbol{F}(\nu)$).*

(b) *$\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}(\nu)$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$.*

(c) *$\widehat{F}_m^*((\omega,\omega'),\,\cdot\,) \in \boldsymbol{F}(\nu)$ for every $(\omega,\omega') \in \overline{\Omega}$ and $m \in \mathbb{N}$.*

(d) *For every $x_n \in E$, $\pi \in \Pi$, $n = 0,\ldots,N$, and $m \in \mathbb{N}$, the estimators $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ and $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m)$ are $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable.*

(e) *For every $x_n \in E$, $\pi \in \Pi$, $n = 0,\ldots,N$, and $m \in \mathbb{N}$, the estimators $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m^*)$ and $\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m^*)$ are $(\overline{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable.*

(f) *For every $x \in E$, $\pi \in \Pi$, and $n = 0,\ldots,N-1$, the map $\dot{\Lambda}_{n;F}^{F;(\pi,x)} : \boldsymbol{L}_1(\nu) \to \mathbb{R}$ in condition (a) of Theorem 4.3.8 is linear.*

If one of the settings (B1)–(B2) is met, then under the assumptions of Theorem 4.3.8 the following two assertions hold.

(i) *For every $x_n \in E$, $\pi \in \Pi$, $n = 0,\ldots,N$, and $\delta > 0$, we have that $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and*

$$\lim_{m\to\infty} \mathbb{P}\big[\big\{\omega \in \Omega : d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m^*(\omega,\cdot))-\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m(\omega)))},\check{\mathbb{P}}_{\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F)}) \ge \delta\big\}\big] \;=\; 0, \qquad (4.38)$$

where B_F is an $\boldsymbol{L}_1(\nu)$-valued centred Gaussian random variable on some probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\nu}$ given by (4.32) and $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}$ is given by (4.16).

(ii) *If there exists a unique optimal strategy $\pi^F \in \Pi(F)$ w.r.t. F, then for every $x_n \in E$, $n =$*

$0,\ldots,N$, and $\delta > 0$, we have that $\dot{\mathcal{W}}^{x_n}_{n;F}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and

$$\lim_{m\to\infty} \mathbb{P}\big[\big\{\omega \in \Omega : d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}^{x_n}_n(\widehat{F}^*_m(\omega,\cdot))-\mathcal{W}^{x_n}_n(\widehat{F}_m(\omega)))}, \check{\mathbb{P}}_{\dot{\mathcal{W}}^{x_n}_{n;F}(B_F)}) \geq \delta\big\}\big] \;=\; 0,$$

where B_F is as in (i) and $\dot{\mathcal{W}}^{x_n}_{n;F}$ is given by (4.18).

For part (i) in Theorem 4.4.9 note that the mapping $\omega' \mapsto \sqrt{m}(\mathcal{W}^{x_n;\pi}_n(\widehat{F}^*_m(\omega,\omega')) - \mathcal{W}^{x_n;\pi}_n(\widehat{F}_m(\omega)))$ is $(\mathcal{F}',\mathcal{B}(\mathbb{R}))$-measurable for every fixed $\omega \in \Omega$ by conditions (d)–(e) of this theorem. Take into account that the latter conditions ensure that $\sqrt{m}(\mathcal{W}^{x_n;\pi}_n(\widehat{F}^*_m) - \mathcal{W}^{x_n;\pi}_n(\widehat{F}_m))$ is $(\overline{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable. That means that $\sqrt{m}(\mathcal{W}^{x_n;\pi}_n(\widehat{F}^*_m(\omega,\,\cdot\,)) - \mathcal{W}^{x_n;\pi}_n(\widehat{F}_m(\omega)))$ can be seen as a real-valued random variable on $(\Omega',\mathcal{F}',\mathbb{P}')$ for every fixed $\omega \in \Omega$. Therefore one can argue as in [15, p. 1186] that the mapping $\omega \mapsto d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}^{x_n;\pi}_n(\widehat{F}^*_m(\omega,\cdot))-\mathcal{W}^{x_n;\pi}_n(\widehat{F}_m(\omega)))}, \check{\mathbb{P}}_{\dot{\mathcal{W}}^{x_n;\pi}_{n;F}(B_F)})$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}_{\geq 0}))$-measurable. In particular, the expression in (4.38) is well-defined. Analogously, we may regard $\sqrt{m}(\mathcal{W}^{x_n}_n(\widehat{F}^*_m(\omega,\,\cdot\,)) - \mathcal{W}^{x_n}_n(\widehat{F}_m(\omega)))$ as a real-valued random variable on $(\Omega',\mathcal{F}',\mathbb{P}')$ for every fixed $\omega \in \Omega$, and that the mapping $\omega \mapsto d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}^{x_n}_n(\widehat{F}^*_m(\omega,\cdot))-\mathcal{W}^{x_n}_n(\widehat{F}_m(\omega)))}, \check{\mathbb{P}}_{\dot{\mathcal{W}}^{x_n}_{n;F}(B_F)})$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}_{\geq 0}))$-measurable. This matters for the formulation of part (ii) of the preceding theorem.

The proof of Theorem 4.4.9 avails the following theorem which is a consequence of Theorem 5.2 in [15]. Recall that $\mathcal{B}(\boldsymbol{L}_1(\nu))$ refers to the Borel σ-algebra on the separable Banach space $(\boldsymbol{L}_1(\nu), \|\cdot\|_{1,\nu})$.

Theorem 4.4.10 *With the notation and under the assumptions of Theorem 4.4.9 (except conditions (d)–(f) of Theorem 4.4.9) suppose that one of the settings (B1)–(B2) is met. Then*

$$\sqrt{m}\big(\widehat{F}^*_m(\omega,\cdot) - \widehat{F}_m(\omega)\big) \;\rightsquigarrow\; B_F \quad \text{in } (\boldsymbol{L}_1(\nu),\mathcal{B}(\boldsymbol{L}_1(\nu)),\|\cdot\|_{1,\nu}), \quad \mathbb{P}\text{-a.e. } \omega, \tag{4.39}$$

where B_F is as in Theorem 4.4.6.

Note that conditions (b)–(c) of Theorem 4.4.9 along with an analogue of Lemma 4.4.5 imply that the process $\sqrt{m}(\widehat{F}^*_m - \widehat{F}_m)$ is $(\overline{\mathcal{F}},\mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable. That is, the mapping $\omega' \mapsto \sqrt{m}(\widehat{F}^*_m(\omega,\omega') - \widehat{F}_m(\omega))$ is $(\mathcal{F}',\mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable for every fixed $\omega \in \Omega$. Hence we may regard $\sqrt{m}(\widehat{F}^*_m(\omega,\,\cdot\,) - \widehat{F}_m(\omega))$ as an $(\boldsymbol{L}_1(\nu)),\mathcal{B}(\boldsymbol{L}_1(\nu))))$-valued random element on $(\Omega',\mathcal{F}',\mathbb{P}')$ for every fixed $\omega \in \Omega$.

Proof of Theorem 4.4.10: Under the imposed assumptions, Theorem 5.2 in [15] shows that (4.39) holds with $\rightsquigarrow$ and $(\boldsymbol{L}_1(\nu),\mathcal{B}(\boldsymbol{L}_1(\nu)),\|\cdot\|_{1,\nu})$ replaced by $\rightsquigarrow^{\circ}$ and $(\boldsymbol{D}_\phi,\mathcal{D}_\phi,\|\cdot\|_{1/\phi})$, respectively. Here $\boldsymbol{D}_\phi$ is the space of all càdlàg functions $h \in \mathbb{R}^{\mathbb{R}}$ with $\|h\|_{1/\phi} = \|h\phi\|_\infty < \infty$ and $\lim_{|x|\to\infty}|h(x)| = 0$, $\mathcal{D}_\phi$ is the open-ball σ-algebra on $(\boldsymbol{D}_\phi,\|\cdot\|_{1/\phi})$ generated by the open balls of $\boldsymbol{D}_\phi$, and $\rightsquigarrow^{\circ}$ denotes convergence in distribution for the open-ball σ-algebra (see Appendix B in [15] for this concept). Note that $\|\cdot\|_{1,\nu} \leq C_\phi\|\cdot\|_{1/\phi}$ with $C_\phi := \int 1/\phi\,d\nu < \infty$ (by condition (a) of Theorem 4.4.9) and thus $\boldsymbol{D}_\phi \subseteq \boldsymbol{L}_1(\nu)$. Hence the embedding map $\boldsymbol{D}_\phi \to \boldsymbol{L}_1(\nu)$, $h \mapsto h$ is $(\|\cdot\|_{1/\phi},\|\cdot\|_{1,\nu})$-continuous, and the continuous mapping theorem (see, for example, [19, Theorem 6.4]) entails that the convergence in (4.39) holds. This completes the proof of Theorem 4.4.10. □

Let us turn to the proof of Theorem 4.4.9.

Proof of Theorem 4.4.9: Since the evidence of the claim in part (ii) will follow with similar arguments as the proof of the assertion in (i), we will focus only on the proof of part (i). Let

$x_n \in E$, $\pi \in \Pi$, $n = 0, \dots, N$, and $\delta > 0$. Under the imposed assumptions, part (i) of Theorem 4.3.8 shows that the map $\mathcal{W}_n^{x_n;\pi}$ defined by (4.6) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\nu)\langle\boldsymbol{L}_1(\nu)\rangle$ with linear quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}$ given by (4.16). Take into account that the latter theorem may be applied because in view of Remark 4.4.7 condition (a) implies that $F \in \boldsymbol{F}(\nu)$. It follows from conditions (d)–(e) that the expressions $\sqrt{m}(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m) - \mathcal{W}_n^{x_n;\pi}(F))$ as well as $\sqrt{m}(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m^*) - \mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))$ are for every $m \in \mathbb{N}$ real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, respectively. Moreover, in virtue of conditions (b)–(c), an analogue of Lemma 4.4.5 ensures that $\sqrt{m}(\widehat{F}_m^* - F)$ as well as $\sqrt{m}(\widehat{F}_m^* - \widehat{F}_m)$ are $\boldsymbol{L}_1(\nu)$-valued and $(\overline{\mathcal{F}}, \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable for every $m \in \mathbb{N}$.

Hence, in view of Theorems 4.4.6 and 4.4.10, the functional delta-method in the form of Theorem B.3(ii) in [59] entails that $\dot{\mathcal{W}}_{n;F}^{x_n;\pi}(B_F)$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and that (4.38) holds. Take into account that Theorem 4.4.6 is applicable because in view of Remark 4.4.7 the integrability condition $\int \sqrt{F(1-F)}\, d\nu < \infty$ is implied by the assumptions $\int \phi^2\, dF < \infty$ and $\int 1/\phi\, d\nu < \infty$. This completes the proof of Theorem 4.4.9. □

In view of part (i) (resp. (ii)) (and under the assumptions) of Theorems 4.4.4 and 4.4.9, the sequence of estimators $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m^*))_{m\in\mathbb{N}}$ (resp. $(\mathcal{W}_n^{x_n}(\widehat{F}_m^*))_{m\in\mathbb{N}}$) can be seen as a bootstrap version (in probability) of $(\mathcal{W}_n^{x_n;\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ (resp. $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_{m\in\mathbb{N}}$) in the sense of [15, Definition 2.3].

Remark 4.4.11 (i) Note that checking conditions (b)–(f) in Theorem 4.4.9 may be difficult in some situations. However, in the examples of Chapter 5 we will see that the latter conditions can be verified easily; see Subsections 5.1.3 and 5.2.3.

(ii) As already discussed in Remark 4.4.2, an optimal strategy w.r.t. the unknown distribution function F can be obtained approximately (in the setting of Theorem 4.4.9) by applying the Bellman equation (see part (i) of Theorem 1.3.3 in Section 1.3) to the transition function $\boldsymbol{P}_{\widehat{F}_m}$ w.r.t. the empirical distribution function $\widehat{F}_m$. However, this approach does *not* ensure that the resulting strategy $\pi^{\widehat{F}_m}$ solving the optimization problem (4.4) is unique. In some situations the uniqueness of an optimal strategy w.r.t. the unknown distribution function F is given; for an example, see Subsection 5.2.3. ◇

Remark 4.4.12 For any given $x_n \in E$ and $n = 0, \dots, N$, the statement in part (ii) of Theorem 4.4.9 can be improved in view of Theorem A.4 in [16] in the following way

$$\sqrt{m}\big(\mathcal{W}_n^{x_n}(\widehat{F}_m^*(\omega,\cdot)) - \mathcal{W}_n^{x_n}(\widehat{F}_m(\omega))\big) \rightsquigarrow \dot{\mathcal{W}}_{n;F}^{x_n}(B_F) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|), \quad \mathbb{P}\text{-a.e. } \omega, \tag{4.40}$$

if (under some additional assumptions) the value functional $\mathcal{W}_n^{x_n}$ defined by (4.6) is uniformly quasi-Hadamard differentiable at any fixed $F \in \boldsymbol{F}(\nu)$ in the sense of [16, Definition A.1]. However, this regularity property of the value functional $\mathcal{W}_n^{x_n}$ is probably not fulfilled, as our following considerations suggest. The proof of the quasi-Hadamard differentiability of $\mathcal{W}_n^{x_n}$ shown in part (ii) of Theorem 4.3.8 in Subsection 4.3.2 is based on the decomposition (4.19) and an appropriate chain rule, where the latter decomposition (4.19) involves the sup-functional Ψ as defined in (4.20). To the best of our knowledge, this sup-functional Ψ is *not* known to be uniformly Hadamard differentiable (in the sense of [16, Definition A.1]). Therefore we may not apply a chain rule for the uniform quasi-Hadamard differentiability in the form of Lemma A.1 in [16] to derive the uniform quasi-Hadamard differentiability of the value functional $\mathcal{W}_n^{x_n}$. As a consequence, (almost sure) bootstrap consistency of the sequence of estimators $(\mathcal{W}_n^{x_n}(\widehat{F}_m))_m$ in the sense of (4.40) does not apply. $\diamond$

Remark 4.4.13 For any fixed $x_0 \in E$ and under the assumptions of Theorems 4.4.4 and 4.4.9, part (ii) of the latter theorems reveals that

$$\mathbb{P} \circ \big\{\sqrt{m}\big(\mathcal{W}_0^{x_0}(\widehat{F}_m) - \mathcal{W}_0^{x_0;\pi^F}(F)\big)\big\}^{-1} \approx \check{\mathbb{P}} \circ \big\{\dot{\mathcal{W}}_{0;F}^{x_0;\pi^F}(B_F)\big\}^{-1}$$

as well as

$$\mathbb{P}' \circ \big\{\sqrt{m}\big(\mathcal{W}_0^{x_0}(\widehat{F}_m^*(\omega,\cdot)) - \mathcal{W}_0^{x_0}(\widehat{F}_m(\omega))\big)\big\}^{-1} \approx \check{\mathbb{P}} \circ \big\{\dot{\mathcal{W}}_{0;F}^{x_0;\pi^F}(B_F)\big\}^{-1}$$

for "large m" and every $\omega \in B$ for some event B with $\mathbb{P}[B]$ "large", where $\pi^F \in \Pi$ corresponds to the unique optimal strategy w.r.t. F. That is, informally

$$\mathbb{P} \circ \big\{\sqrt{m}\big(\mathcal{W}_0^{x_0}(\widehat{F}_m) - \mathcal{W}_0^{x_0;\pi^F}(F)\big)\big\}^{-1} \approx \mathbb{P}' \circ \big\{\sqrt{m}\big(\mathcal{W}_0^{x_0}(\widehat{F}_m^*(\omega,\cdot)) - \mathcal{W}_0^{x_0}(\widehat{F}_m(\omega))\big)\big\}^{-1} \quad (4.41)$$

for "large m" and every $\omega \in B$ for some event B with $\mathbb{P}[B]$ "large". Therefore, using the right-hand side of (4.41), we can approximate the asymptotic error distribution of $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_m$ without estimating the distribution of $\dot{\mathcal{W}}_{0;F}^{x_0;\pi^F}(B_F)$ in the unknown distribution function F. Especially when one is interested in an asymptotic bootstrap confidence interval for the optimal value $\mathcal{W}_0^{x_0;\pi^F}(F)$ of the optimization problem (4.4) at a given level $\kappa \in (0,1)$, this approximation can be beneficial. This is particularly the case when the estimated distribution of $\dot{\mathcal{W}}_{0;F}^{x_0;\pi^F}(B_F)$ based on the empirical distribution function $\widehat{F}_m$ depends on $\widehat{F}_m$ in a complex way. In the latter situation, however,

$$\Big[\mathcal{W}_0^{x_0}(\widehat{F}_m(\omega)) - \frac{1}{\sqrt{m}}\widehat{q}_{1-\kappa/2}^*(\omega), \mathcal{W}_0^{x_0}(\widehat{F}_m(\omega)) + \frac{1}{\sqrt{m}}\widehat{q}_{\kappa/2}^*(\omega)\Big] \quad (4.42)$$

can be seen for fixed $\omega \in \Omega$ as an asymptotic bootstrap confidence interval for $\mathcal{W}_0^{x_0;\pi^F}(F)$ at level $\kappa \in (0,1)$ which could have a better performance than an asymptotic confidence interval based on a nonparametric estimation of the distribution of $\dot{\mathcal{W}}_{0;F}^{x_0;\pi^F}(B_F)$, but probably at the expense of a higher computation effort. Here $\widehat{q}_t^*(\omega)$ refers to a t-quantile of (a Monte Carlo approximation of) the right-hand side of (4.41) for fixed $\omega \in \Omega$. In this thesis we will not investigate the performance of the asymptotic bootstrap interval in (4.42) for the optimal value. $\diamond$

4.5 Parametric estimation of $\mathcal{W}_0^{x_0}(F)$

In this section we will consider a parametric approach to estimate (the unknown distribution function F and thus) the (optimal) value $\mathcal{W}_0^{x_0}(F)$, and provide statistical quality criteria for the corresponding estimator, such as strong consistency and asymptotic error distribution. Here we will assume that the distribution function F generating the transition function of the MDM in (4.1) belongs to some class of distribution functions parametrized by some unknown parameter θ. To this end, we consider in the following a parametric statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}^\theta : \theta \in \Theta\})$, where the parameter set Θ is any subspace of $\mathbb{R}^d$ (with $d \in \mathbb{N}$ fixed), and let in the sequel $F_\theta \in \boldsymbol{F}(\nu)$ be for every $\theta \in \Theta$ a fixed distribution function. For any $m \in \mathbb{N}$, let $\widehat{\theta}_m : \Omega \to \Theta$ be a map which can be seen as an estimator for the unknown parameter θ. Therefore $\widehat{F}_m := F_{\widehat{\theta}_m}$ can be seen as an estimator for F_θ. In particular, this implies that for any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$ the (plug-in) estimator $\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m})$ (resp. $\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m})$) is a reasonable (point) estimator for the aspect $\mathcal{W}_n^{x_n;\pi}(F_\theta)$ (resp. $\mathcal{W}_n^{x_n}(F_\theta)$) if $\theta \in \Theta$.

4.5.1 Strong consistency

In the following Theorem 4.5.1 we show that the sequences of plug-in estimators $(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}))_{m \in \mathbb{N}}$ and $(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}))_{m \in \mathbb{N}}$ satisfy a strong law. These statements are guaranteed under a certain regularity assumption to the mapping $\theta \mapsto F_\theta$ (see condition (b) of Theorem 4.5.1) along with Theorem 4.3.3 if the sequence of estimators $(\widehat{\theta}_m)_{m \in \mathbb{N}}$ again satisfies a strong law. In what follows we equip the parameter set Θ with the usual Euclidean norm $\|\cdot\|$ on $\mathbb{R}^d$. Finally, let $\|\cdot\|_{1,\nu}$ be the norm introduced in (4.7).

Theorem 4.5.1 (Strong consistency of $(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}))$ and $(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}))$) *Let $\theta_0 \in \Theta$. Further let $\widehat{\theta}_m : \Omega \to \Theta$ be a map for every $m \in \mathbb{N}$, and assume that the following two conditions hold.*

(a) *The sequence of estimators $(\widehat{\theta}_m)_{m \in \mathbb{N}}$ is strongly consistent for θ_0 under $\mathbb{P}^{\theta_0}$ in the sense that $\|\widehat{\theta}_m - \theta_0\| \to 0 \quad \mathbb{P}^{\theta_0}$-a.s.*

(b) *The mapping $\theta \mapsto F_\theta$ from Θ to $\boldsymbol{F}(\nu)$ is continuous at θ_0 w.r.t. $(\|\cdot\|, \|\cdot\|_{1,\nu})$.*

Then under the assumptions of Theorem 4.3.3 (with F_{θ_0} in place of F) the following two assertions hold.

(i) *For any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$, the sequence of estimators $(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}))_{m \in \mathbb{N}}$ is strongly consistent for $\mathcal{W}_n^{x_n;\pi}(F_{\theta_0})$ under $\mathbb{P}^{\theta_0}$ in the sense that*

$$\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}) \to \mathcal{W}_n^{x_n;\pi}(F_{\theta_0}) \quad \mathbb{P}^{\theta_0}\text{-a.s.}$$

(ii) *For any $x_n \in E$ and $n = 0, \dots, N$, the sequence of estimators $(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}))_{m \in \mathbb{N}}$ is strongly consistent for $\mathcal{W}_n^{x_n}(F_{\theta_0})$ under $\mathbb{P}^{\theta_0}$ in the sense that*

$$\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}) \to \mathcal{W}_n^{x_n}(F_{\theta_0}) \quad \mathbb{P}^{\theta_0}\text{-a.s.}$$

Note that in Theorem 4.5.1 the estimators $\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m})$ and $\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m})$ need not to be measurable. If we find an optimal strategy $\pi^{F_{\theta_0}} \in \Pi$ w.r.t. F_{θ_0}, then it follows from part (ii) of Theorem 4.5.1 that under condition (b) of Theorem 4.5.1 and the assumptions of Theorem 4.3.3 the sequence of plug-in estimators $(\mathcal{W}_0^{x_0}(F_{\widehat{\theta}_m}))_{m \in \mathbb{N}}$ is strongly consistent (under $\mathbb{P}^{\theta_0}$) for the optimal value $\mathcal{W}_0^{x_0;\pi^{F_{\theta_0}}}(F_{\theta_0})$ of the optimization problem (4.4) (with F_{θ_0} playing the role of F) whenever the sequence of estimators $(\widehat{\theta}_m)_{m \in \mathbb{N}}$ is strongly consistent (under $\mathbb{P}^{\theta_0}$) for the (unknown) parameter θ_0 w.r.t. the norm $\|\cdot\|$. However, in some situations the validity of conditions (a)–(b) in Theorem 4.5.1 is not trivially satisfied. In the examples of Chapter 5 these conditions can be checked easily; see Subsections 5.1.4 and 5.2.4 for details.

The following Remark 4.5.2 shows how an optimal strategy can be achieved approximately by means of a statistical estimation of the unknown parameter.

Remark 4.5.2 In the parametric setting above an approximate optimal strategy can be achieved by make use of the Bellman equation in part (i) of Theorem 1.3.3 in Section 1.3 (applied to $\boldsymbol{P}_{F_\theta}$). Here the corresponding transition probabilities are computed by means of the estimated parametrized distribution function $F_{\widehat{\theta}_m}$ for some (suitable) estimator $\widehat{\theta}_m$ for the unknown parameter θ_0. Therefore, we obtain a strategy $\pi^{F_{\widehat{\theta}_m}} \in \Pi$ which is an approximate solution to the optimization problem (4.4) (with F replaced by F_{θ_0}) and can be regarded as an estimator for an exact (but unknown) optimal strategy $\pi^{F_{\theta_0}} \in \Pi$ w.r.t. F_{θ_0}. However, as already discussed in Remark 4.4.2, we are not in the position to derive asymptotic properties of the estimated strategy for the (true) optimal strategy $\pi^{F_{\theta_0}}$. ◇

Remark 4.5.3 (i) It is an immediate consequence of part (i) of Remark 4.3.4 that (under the assumptions of Theorem 4.5.1) the claim in part (i) of Theorem 4.5.1 holds even uniformly in $\pi \in \Pi$, that is, for every $x_n \in E$ and $n = 0, \dots, N$ we have $\sup_{\pi\in\Pi} |\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n;\pi}(F_{\theta_0})| \to 0$ $\mathbb{P}^{\theta_0}$-a.s.

(ii) If conditions (a) and (b) in Theorem 4.5.1 are replaced by the following two stronger conditions

(a') $m^r\|\theta_m - \theta_0\| \to 0 \quad \mathbb{P}^{\theta_0}$-a.s. for every $r < \frac{1}{2}$,

(b') The mapping $\theta \mapsto F_\theta$ from Θ to $\boldsymbol{F}(\nu)$ is Lipschitz continuous at θ_0 w.r.t. $(\|\cdot\|, \|\cdot\|_{1,\nu})$,

then we achieve even stronger results in parts (i) and (ii) of the latter theorem. In fact, it can be verified easily by means of Theorem 4.3.3 that under the assumptions of Theorem 4.5.1 with (a') and (b') in place of (a) and (b), respectively, the statements

(i') $m^r\big(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n;\pi}(F_{\theta_0})\big) \to 0 \quad \mathbb{P}^{\theta_0}$-a.s.

(ii') $m^r\big(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n}(F_{\theta_0})\big) \to 0 \quad \mathbb{P}^{\theta_0}$-a.s.

hold for any $x_n \in E$, $\pi \in \Pi$, $n = 0, \dots, N$, and $r < \frac{1}{2}$. ◇

4.5.2 Asymptotic error distribution

Throughout this subsection we assume that the parameter set Θ ($\subseteq \mathbb{R}^d$) is open, and that the measure ν on $\mathcal{B}(\mathbb{R})$ is locally finite. Theorem 4.5.4 below determines the asymptotic error distribution of the sequences of estimators $(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}))_{m\in\mathbb{N}}$ and $(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}))_{m\in\mathbb{N}}$. The central tool for this will be a specific functional delta-method in the form of [59]. Specifically, we will apply Theorem 4.3.8 along with this functional delta method to derive these asymptotic results from the asymptotic error distribution of the sequence of estimators $(\widehat{\theta}_m)_{m\in\mathbb{N}}$ for the unknown parameter θ_0 (provided in condition (a) of Theorem 4.5.4) and a suitable regularity property of the mapping $\theta \mapsto F_\theta$ (provided in condition (b) of Theorem 4.5.4).

In condition (b) of Theorem 4.5.4 we will assume that for any fixed $\theta_0 \in \Theta$ the map $\mathfrak{F} : \Theta \to \boldsymbol{F}(\nu)$ ($\subseteq \boldsymbol{L}_0(\nu)$) defined by

$$\mathfrak{F}(\theta) := F_\theta \tag{4.43}$$

is Hadamard differentiable at θ_0 with trace $\boldsymbol{L}_1(\nu)$ (in the sense of Definition A.1(ii) in Section A). That is, there exists a continuous map $\dot{\mathfrak{F}}_{\theta_0} : \mathbb{R}^d \to \boldsymbol{L}_1(\nu)$ ($\subseteq \boldsymbol{L}_0(\nu)$) (the Hadamard derivative) such that

$$\lim_{m\to\infty} \Big\| \frac{\mathfrak{F}(\theta_0 + \varepsilon_m \tau_m) - \mathfrak{F}(\theta_0)}{\varepsilon_m} - \dot{\mathfrak{F}}_{\theta_0}(\tau) \Big\|_{1,\nu} = 0$$

holds for each triplet $(\tau, (\tau_m), (\varepsilon_m)) \in \mathbb{R}^d \times (\mathbb{R}^d)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}_{>0}$ satisfying $\|\tau_m - \tau\| \to 0$, $\varepsilon_m \to 0$ as well as $(\theta_0 + \varepsilon_m \tau_m) \subseteq \Theta$. Recall that $F - G \in \boldsymbol{L}_1(\nu)$ holds for every $F, G \in \boldsymbol{F}(\nu)$. Finally, let $0_{\mathbb{R}^d}$ be the null in $\mathbb{R}^d$, and recall that $0_{\boldsymbol{L}_0(\nu)}$ stands for the null in $\boldsymbol{L}_0(\nu)$.

Theorem 4.5.4 (Asymptotic error distribution of $(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}))$ and $(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}))$) *Let $\theta_0 \in \Theta$ and $(c_m)_{m\in\mathbb{N}}$ be any sequence in $\mathbb{R}_{>0}$ tending to ∞. Moreover, let $\widehat{\theta}_m : \Omega \to \Theta$ be a map for every $m \in \mathbb{N}$, and assume that the following two conditions hold.*

(a) *$c_m(\widehat{\theta}_m - \theta_0)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^d))$-measurable for every $m \in \mathbb{N}$, and*

$$c_m(\widehat{\theta}_m - \theta_0) \rightsquigarrow Z_{\theta_0} \quad \text{in } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \|\cdot\|)$$

for some $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-valued random element Z_{θ_0} on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$.

(b) *The map $\mathfrak{F} : \Theta \to \boldsymbol{F}(\nu)$ defined by (4.43) is Hadamard differentiable at θ_0 with trace $\boldsymbol{L}_1(\nu)$ and Hadamard derivative $\dot{\mathfrak{F}}_{\theta_0} : \mathbb{R}^d \to \boldsymbol{L}_1(\nu)$ satisfying $\dot{\mathfrak{F}}_{\theta_0}(0_{\mathbb{R}^d}) = 0_{\boldsymbol{L}_0(\nu)}$.*

Then under the assumptions of Theorem 4.3.8 (with F_{θ_0} in place of F) the following two assertions hold.

(i) *For any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \ldots, N$, we have that $\dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n;\pi}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0}))$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and*

$$c_m\big(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n;\pi}(F_{\theta_0})\big) \rightsquigarrow \dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n;\pi}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0})) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|),$$

where $\dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n;\pi}$ is given by (4.16).

(ii) *For any $x_n \in E$ and $n = 0, \ldots, N$, we have that $\dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0}))$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and*

$$c_m\big(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n}(F_{\theta_0})\big) \rightsquigarrow \dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0})) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|),$$

where $\dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n}$ is given by (4.17).

Note that for part (i) of Theorem 4.5.4 the measurability of the mapping $\omega \mapsto c_m(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m(\omega)}) - \mathcal{W}_n^{x_n;\pi}(F_{\theta_0}))$ is ensured (under the assumptions of the latter theorem) by part (i) of Lemma 4.5.6 below. Similarly, it can be shown for (ii) of Theorem 4.5.4 that the mapping $\omega \mapsto c_m(\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m(\omega)}) - \mathcal{W}_n^{x_n}(F_{\theta_0}))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

If one is interested in an asymptotic confidence interval for the (optimal) value $\mathcal{W}_0^{x_0}(F_{\theta_0})$, part (ii) of Theorem 4.5.4 provides an approach. In fact, the examples of Chapter 5 will show that one can use the asymptotic error distribution of the corresponding plug-in estimator $\mathcal{W}_0^{x_0}(F_{\widehat{\theta}_m})$ to construct such an interval. Since in each of these examples we will see that $\dot{\mathcal{W}}_{0;F_{\theta_0}}^{x_0}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0}))$ depends on the unknown parameter θ_0 in a complex way, an analogous bootstrap result as in the nonparametric case could lead to a more efficient method than the method which is based on an estimation of the distribution of $\dot{\mathcal{W}}^{x_0}_{0;F_{\theta_0}}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0}))$ in the unknown parameter θ_0. However, in this thesis we will not deal with a parametric version of the bootstrap results from Theorem 4.4.9.

We now turn to the proof of Theorem 4.5.4. It needs the following two lemmas.

Lemma 4.5.5 *Let $\theta_0 \in \Theta$, and assume that condition (b) of Theorem 4.5.4 holds. Then the following two assertions hold.*

(i) *The mapping $\theta \mapsto F_\theta$ from Θ to $\boldsymbol{F}(\nu)$ is $(\|\cdot\|, \|\cdot\|_{1,\nu})$-continuous at θ_0.*

(ii) *The mapping $\theta \mapsto F_\theta - F_{\theta_0}$ from Θ to $\boldsymbol{L}_1(\nu)$ is $(\|\cdot\|, \|\cdot\|_{1,\nu})$-continuous at θ_0.*

If in addition the assumptions of Theorem 4.3.8 (or Theorem 4.3.3) are satisfied (with F_{θ_0} in place of F) then the following two assertions are valid.

(iii) *For any $x_n \in E$, $\pi \in \Pi$, and $n = 0, \ldots, N$, the mapping $\theta \mapsto \mathcal{W}^{x_n;\pi}_n(F_\theta)$ from Θ to $\mathbb{R}$ is $(\|\cdot\|, |\cdot|)$-continuous at θ_0.*

(iv) *For any $x_n \in E$ and $n = 0, \ldots, N$, the mapping $\theta \mapsto \mathcal{W}^{x_n}_n(F_\theta)$ from Θ to $\mathbb{R}$ is $(\|\cdot\|, |\cdot|)$-continuous at θ_0.*

Proof For part (i), let $(\theta_m)_{m\in\mathbb{N}}$ be any sequence in Θ with $\|\theta_m - \theta_0\| \to 0$. By assumption, the map $\mathfrak{F}$ defined by (4.43) is Hadamard differentiable at θ_0 with trace $\boldsymbol{L}_1(\nu)$ and Hadamard derivative $\dot{\mathfrak{F}}_{\theta_0} : \mathbb{R}^d \to \boldsymbol{L}_1(\nu)$ satisfying $\dot{\mathfrak{F}}_{\theta_0}(0_{\mathbb{R}^d}) = 0_{\boldsymbol{L}_0(\nu)}$. Then it follows from Lemma A.5 in Section A that the map $\mathfrak{F}$ is Lipschitz continuous at θ_0 with trace $\boldsymbol{L}_1(\nu)$ (in the sense of Definition A.3(ii)). This implies that there exists a finite constant $C > 0$ such that the expression

$$\|F_{\theta_m} - F_{\theta_0}\|_{1,\nu} \;=\; \|\mathfrak{F}(\theta_m) - \mathfrak{F}(\theta_0)\|_{1,\nu} \;=\; \|\mathfrak{F}(\theta_0 + (\theta_m - \theta_0)) - \mathfrak{F}(\theta_0)\|_{1,\nu}$$

is for every $m \in \mathbb{N}$ bounded from above by $C\|\theta_m - \theta_0\|$. Hence the assertion in part (i) follows. Moreover, the claim in part (ii) is an immediate consequence of part (i). Take into account that for every $\theta \in \Theta$ we have $F_\theta - F_{\theta_0} \in \boldsymbol{L}_1(\nu)$ because $F_\theta, F_{\theta_0} \in \boldsymbol{F}(\nu)$ (by assumption).

For the claims in parts (iii) and (iv) note at first that under the additional assumptions it follows from Remark 4.3.9(iv) that the assumptions of Theorem 4.3.3 (with F_{θ_0} playing the role of F) are satisfied. Then an application of parts (i) and (ii) of the latter theorem along with part (i) yields the assertions in (iii) and (iv), respectively. Note that the statements in (iii) and (iv) can also be derived directly from Theorem 4.3.3 using part (i). This completes the proof. □

In the sequel, we denote by $\mathcal{B}(\Theta)$ the Borel σ-algebra on $(\Theta, \|\cdot\|)$. Note that $\mathcal{B}(\Theta)$ coincides in view of [80, Problem 3.10(ii)] with the trace σ-algebra $\mathcal{B}(\mathbb{R}^d) \cap \Theta$ because Θ $(\subseteq \mathbb{R}^d)$ was assumed to be open. Recall that $\mathcal{B}(\boldsymbol{L}_1(\nu))$ refers to the Borel σ-algebra on the separable Banach space $(\boldsymbol{L}_1(\nu), \|\cdot\|_{1,\nu})$.

Lemma 4.5.6 *Let $\theta_0 \in \Theta$ and $\widehat{\theta}_m : \Omega \to \Theta$ be for every $m \in \mathbb{N}$ an $(\mathcal{F}, \mathcal{B}(\Theta))$-measurable map. Moreover let $(c_m)_{m\in\mathbb{N}}$ be any sequence in $\mathbb{R}_{>0}$, and assume that condition (b) of Theorem 4.5.4 holds. Then we have:*

(i) *$c_m(F_{\widehat{\theta}_m} - F_{\theta_0})$ takes values only in $\boldsymbol{L}_1(\nu)$ and is $(\mathcal{F}, \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable for every $m \in \mathbb{N}$.*

If in addition the assumptions of Theorem 4.3.8 (or Theorem 4.3.3) are satisfied (with F_{θ_0} in place of F) then the following two assertions hold.

(ii) *For any* $x_n \in E$, $\pi \in \Pi$, $n = 0, \dots, N$, *and* $m \in \mathbb{N}$, *the estimator* $\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m})$ *is* $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$*-measurable.*

(iii) *For any* $x_n \in E$, $n = 0, \dots, N$, *and* $m \in \mathbb{N}$, *the estimator* $\mathcal{W}_n^{x_n}(F_{\widehat{\theta}_m})$ *is* $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$*-measurable.*

Proof First, it follows from part (ii) of Lemma 4.5.5 that the mapping $\theta \mapsto F_\theta - F_{\theta_0}$ from Θ to $\boldsymbol{L}_1(\nu)$ is in particular $(\mathcal{B}(\Theta), \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable. Since $\widehat{\theta}_m$ is $(\mathcal{F}, \mathcal{B}(\Theta))$-measurable by assumption, the expression $c_m(F_{\widehat{\theta}_m} - F_{\theta_0})$ is indeed $(\mathcal{F}, \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable for every $m \in \mathbb{N}$. This shows the assertion in (i).

To prove (ii), let $x_n \in E$, $\pi \in \Pi$, $n = 0, \dots, N$, and $m \in \mathbb{N}$. Note at first that under the additional assumptions it follows from part (iii) of Lemma 4.5.5 that the mapping $\theta \mapsto \mathcal{W}_n^{x_n;\pi}(F_\theta)$ from Θ to $\mathbb{R}$ is $(\mathcal{B}(\Theta), \mathcal{B}(\mathbb{R}))$-measurable. In particular, in view of the $(\mathcal{F}, \mathcal{B}(\Theta))$-measurability of the map $\widehat{\theta}_m$ (by assumption), the estimator $\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m})$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. This shows (ii). Since the proof of the claim in (iii) can be obtained with analogous arguments, this completes the proof. □

Now we are in the position to prove Theorem 4.5.4.

Proof of Theorem 4.5.4: We intend to apply the functional delta-method in the form of Theorem B.3(i) in [59]. Since the proof of assertion (ii) can be carried out with analogous arguments as the proof of part (i), we will prove only the claim in part (i). Let $x_n \in E$, $\pi \in \Pi$, and $n = 0, \dots, N$. First, the map $\mathfrak{F}$ defined by (4.43) is Hadamard differentiable at θ_0 with trace $\boldsymbol{L}_1(\nu)$ and Hadamard derivative $\dot{\mathfrak{F}}_{\theta_0} : \mathbb{R}^d \to \boldsymbol{L}_1(\nu)$ satisfying $\dot{\mathfrak{F}}_{\theta_0}(0_{\mathbb{R}^d}) = 0_{\boldsymbol{L}_0(\nu)}$ (by condition (b)). Second, in view of condition (a), $c_m(\widehat{\theta}_m - \theta_0)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^d))$-measurable for every $m \in \mathbb{N}$ and satisfies $c_m(\widehat{\theta}_m - \theta_0) \rightsquigarrow Z_{\theta_0}$ under $\mathbb{P}^{\theta_0}$ for some $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-valued random element Z_{θ_0} on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$. Thus an application of part (i) of Theorem B.3 in [59] implies that $\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0})$ is $(\check{\mathcal{F}}, \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable and that

$$c_m(F_{\widehat{\theta}_m} - F_{\theta_0}) \;\rightsquigarrow\; \dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0}) \quad \text{in } (\boldsymbol{L}_1(\nu), \mathcal{B}(\boldsymbol{L}_1(\nu)), \|\cdot\|_{1,\nu}) \tag{4.44}$$

holds. Take into account that the latter theorem is applicable because $c_m(F_{\widehat{\theta}_m} - F_{\theta_0})$ takes values only in $\boldsymbol{L}_1(\nu)$ and is $(\mathcal{F}, \mathcal{B}(\boldsymbol{L}_1(\nu)))$-measurable for every $m \in \mathbb{N}$ by part (i) of Lemma 4.5.6.

Moreover, under the imposed assumptions, part (i) of Theorem 4.3.8 implies that the functional $\mathcal{W}_n^{x_n;\pi}$ defined by (4.6) is quasi-Hadamard differentiable at F_{θ_0} tangentially to $\boldsymbol{L}_1(\nu)\langle \boldsymbol{L}_1(\nu)\rangle$ (w.r.t. the norm $\|\cdot\|_{1,\nu}$) with quasi-Hadamard derivative $\dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n;\pi}$ given by (4.16). As a consequence of part (ii) of Lemma 4.5.6 as well as the first part of condition (a), the expression $c_m(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n;\pi}(F_{\theta_0}))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $m \in \mathbb{N}$. Take into account that the first part of condition (a) clearly implies the $(\mathcal{F}, \mathcal{B}(\Theta))$-measurability of the map $\widehat{\theta}_m : \Omega \to \Theta$ for every $m \in \mathbb{N}$. Thus the convergence in (4.44) and another application of the functional delta-method in the form of part (i) of Theorem B.3 in [59] yield that $\dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n;\pi}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0}))$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and that

$$c_m\big(\mathcal{W}_n^{x_n;\pi}(F_{\widehat{\theta}_m}) - \mathcal{W}_n^{x_n;\pi}(F_{\theta_0})\big) \;\rightsquigarrow\; \dot{\mathcal{W}}_{n;F_{\theta_0}}^{x_n;\pi}(\dot{\mathfrak{F}}_{\theta_0}(Z_{\theta_0})) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|).$$

This completes the proof of Theorem 4.5.4. □

In practice, condition (a) of Theorem 4.5.4 is easy to prove; we refer to Subsections 5.1.4 and 5.2.4 for a verification. In contrast, condition (b) of Theorem 4.5.4 is more difficult to verify. However, the following Lemma 4.5.7, which is a slight generalization of Lemma 4.6 in [59], provides a criterion based on the map $\mathfrak{f} : \Theta \times \mathbb{R} \to [0,1]$ defined by

$$\mathfrak{f}(\theta, t) := F_\theta(t) \tag{4.45}$$

which ensures that the Hadamard differentiability of the map $\mathfrak{F}$ defined by (4.43) in condition (b) of Theorem 4.5.4 holds. Its statement will be used in the examples of Chapter 5 to verify condition (b) of Theorem 4.5.4 in order to determine the asymptotic distribution of certain estimators for the optimal value of the MDM in (4.1) in the case of parametric statistical models; see Subsections 5.1.4 and 5.2.4.

Recall that Θ ($\subseteq \mathbb{R}^d$) is open, and note that $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product on $\mathbb{R}^d$. Further we denote by $\nabla_\theta \mathfrak{f}(\theta, t)$ the gradient of the map $\mathfrak{f}(\,\cdot\,, t)$ at some $\theta \in \Theta$ for every fixed $t \in \mathbb{R}$ (provided this expression exists).

Lemma 4.5.7 *Let $\theta_0 \in \Theta$ and $\widetilde{\Theta}$ be some open neighbourhood of θ_0 in Θ such that for every $t \in \mathbb{R}$ the map $\mathfrak{f}(\,\cdot\,, t)$ is continuously differentiable on $\widetilde{\Theta}$. Moreover let $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ be a ν-integrable map such that*

$$\sup_{\theta \in \widetilde{\Theta}} \|\nabla_\theta \mathfrak{f}(\theta, t)\| \le \mathfrak{h}(t) \qquad \text{for } \nu\text{-a.e. } t \in \mathbb{R}. \tag{4.46}$$

Then the map $\mathfrak{F} : \Theta \to \boldsymbol{F}(\nu)$ defined by (4.43) is Hadamard differentiable at θ_0 with trace $\boldsymbol{L}_1(\nu)$ and Hadamard derivative $\dot{\mathfrak{F}}_{\theta_0} : \mathbb{R}^d \to \boldsymbol{L}_1(\nu)$ given by

$$\dot{\mathfrak{F}}_{\theta_0}(\tau)(\,\cdot\,) := \big\langle \nabla_\theta \mathfrak{f}(\theta_0, \,\cdot\,), \tau \big\rangle, \qquad \tau \in \mathbb{R}^d. \tag{4.47}$$

In particular, we have $\dot{\mathfrak{F}}_{\theta_0}(0_{\mathbb{R}^d}) = 0_{\boldsymbol{L}_0(\nu)}$.

Proof We will adapt arguments of the proof of Lemma 4.6 in [59]. First of all, note that it follows from the estimate (4.48) below that $\dot{\mathfrak{F}}_{\theta_0}(\tau)(\,\cdot\,) \in \boldsymbol{L}_1(\nu)$ holds for every $\tau \in \mathbb{R}^d$. In particular, this implies that the map $\dot{\mathfrak{F}}_{\theta_0}$ defined by (4.47) is well-defined. Moreover the map $\dot{\mathfrak{F}}_{\theta_0}$ is $(\|\cdot\|, \|\cdot\|_{1,\nu})$-continuous because

$$\begin{aligned}
\big\|\dot{\mathfrak{F}}_{\theta_0}(\tau_1)(\,\cdot\,) - \dot{\mathfrak{F}}_{\theta_0}(\tau_2)(\,\cdot\,)\big\|_{1,\nu} &= \int \big|\big\langle \nabla_\theta \mathfrak{f}(\theta_0, t), \tau_1 - \tau_2 \big\rangle\big|\, \nu(dt) \\
&\le \int \sup_{\theta \in \widetilde{\Theta}} \|\nabla_\theta \mathfrak{f}(\theta, t)\| \cdot \|\tau_1 - \tau_2\|\, \nu(dt) \\
&\le \int \mathfrak{h}(t)\, \nu(dt) \cdot \|\tau_1 - \tau_2\| = C\, \|\tau_1 - \tau_2\|
\end{aligned} \tag{4.48}$$

holds for every $\tau_1, \tau_2 \in \mathbb{R}^d$, where $C := \int \mathfrak{h}(t)\, \nu(dt) < \infty$. Take into account that $\mathfrak{h}$ is ν-integrable by assumption.

Now, let $(\tau, (\tau_m), (\varepsilon_m)) \in \mathbb{R}^d \times (\mathbb{R}^d)^{\mathbb{N}} \times \mathbb{R}_{>0}^{\mathbb{N}}$ be a triplet with $\|\tau_m - \tau\| \to 0$, $\varepsilon_m \to 0$ as well as $(\theta_0 + \varepsilon_m \tau_m) \subseteq \Theta$. It remains to show that

$$\lim_{m \to \infty} \Big\| \frac{\mathfrak{F}(\theta_0 + \varepsilon_m \tau_m)(\,\cdot\,) - \mathfrak{F}(\theta_0)(\,\cdot\,)}{\varepsilon_m} - \dot{\mathfrak{F}}_{\theta_0}(\tau)(\,\cdot\,) \Big\|_{1,\nu} = 0,$$

that is,

$$\lim_{m \to \infty} \int \Big| \frac{\mathfrak{f}(\theta_0 + \varepsilon_m \tau_m, t) - \mathfrak{f}(\theta_0, t)}{\varepsilon_m} - \big\langle \nabla_\theta \mathfrak{f}(\theta_0, t), \tau \big\rangle \Big|\, \nu(dt) = 0. \tag{4.49}$$

Since the map $\mathfrak{f}(\,\cdot\,, t)$ is continuously differentiable on $\widetilde{\Theta}$ for every $x \in \mathbb{R}$ (by assumption), we obtain

$$\lim_{m\to\infty} \frac{\mathfrak{f}(\theta_0 + \varepsilon_m \tau_m, t) - \mathfrak{f}(\theta_0, t)}{\varepsilon_m} \;=\; \big\langle \nabla_\theta \mathfrak{f}(\theta_0, t), \tau \big\rangle$$

for every $t \in \mathbb{R}$. Note that we may assume without loss of generality that $\theta_0 + \varepsilon_m \tau_m \in \widetilde{\Theta}$ for every $m \in \mathbb{N}$ because $\widetilde{\Theta}$ ($\subseteq \mathbb{R}^d$) is open and contains θ_0. Further, assumption (4.46) along with the mean value theorem in several variables entail that

$$\Big| \frac{\mathfrak{f}(\theta_0 + \varepsilon_m \tau_m, t) - \mathfrak{f}(\theta_0, t)}{\varepsilon_m} \Big| \;\le\; \sup_{\theta \in \widetilde{\Theta}} \|\nabla_\theta \mathfrak{f}(\theta, t)\| \cdot \|\tau_m\| \;\le\; \mathfrak{h}(t) \cdot \sup_{k \in \mathbb{N}} \|\tau_k\|$$

for every $m \in \mathbb{N}$ and ν-a.e. $t \in \mathbb{R}$. Since $\sup_{k\in\mathbb{N}} \|\tau_k\| < \infty$ (recall $\|\tau_m - \tau\| \to 0$) and $\mathfrak{h}$ is ν-integrable by assumption, the dominated convergence theorem implies the convergence in (4.49). This shows the claimed Hadamard differentiability of the map $\mathfrak{F}$.

For the additional assertion note that it is easily seen that $\dot{\mathfrak{F}}_{\theta_0}(0_{\mathbb{R}^d})(t) = 0$ is valid for any $t \in \mathbb{R}$. Hence $\dot{\mathfrak{F}}_{\theta_0}(0_{\mathbb{R}^d}) = 0_{\boldsymbol{L}_0(\nu)}$. The proof is now complete. □

Chapter 5

Application to the Markov decision optimization problems from Chapter 3

This chapter is devoted to an application of the theory and results of Chapter 4 to the specific Markov decision optimization problems introduced in Chapter 3, where the corresponding transition probabilities are now determined by some single distribution function F. Further we will assume that the distribution function F is *unknown* and must be estimated by means of statistical methods. Therefore one could be interested how the estimation of the distribution function F effects the estimation of the optimal value of the corresponding reward maximization problems from Chapter 3.

5.1 Stochastic inventory control problem (revisited)

We consider again the stochastic inventory control problem introduced in Section 3.1, where the transition probabilities are now specified by the common distribution function F of the random variables describing the random commodity demand. Since in practice the future random demand is *not* known from the supplier's point of view, we will here assume the distribution function F is *unknown* and must be estimated. In Subsection 5.1.1 we reformulate the stochastic inventory control problem from Section 3.1 based on the above situation, and explain how the (adapted) inventory control model can be embedded into the setting of Section 1.5. Later on we will present in Subsection 5.1.2 regularity properties of the value function of the (adapted) stochastic inventory control problem. In Subsections 5.1.3–5.1.4 we will deal with the statistical estimation of the optimal value of the (adapted) stochastic inventory control problem in a nonparametric and a parametric statistical model.

5.1.1 Basic inventory control model, and the Markov decision model

We take up the setting of Section 3.1, that is, we consider an N-period inventory control system (with $N \in \mathbb{N}$ fixed) in which a supplier of a single product seeks optimal inventory level management to meet random demand in such a way that the expected total reward over the N periods is maximized. The random demand of the single product within the N periods will be modelled by $\mathbb{N}_0$-valued i.i.d. random variables $I_1, \ldots, I_N$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common but unknown distribution function F, where I_{n+1} corresponds to the random demand of the single product in the period between time n and $n+1$. Note that for any $n = 0, \dots, N-1$ the distribution function F induces the counting density of the random variable I_{n+1} through $\mathbb{P}[\{I_{n+1} = \cdot\,\}] = \mathfrak{p}_F(\cdot)$ for the map $\mathfrak{p}_F : \mathbb{R} \to [0,1]$ given by

$$\mathfrak{p}_F(x) \;:=\; \mathfrak{S}_F(x). \tag{5.1}$$

Here $\mathfrak{S}_h$ refers for any map $h \in \mathbb{R}^{\mathbb{R}}$ to the sum operator defined by

$$\mathfrak{S}_h(x) \;:=\; h(0)\mathbb{1}_{\{x=0\}} + \Delta_{x-1}^{x} h \mathbb{1}_{\{x\in\mathbb{N}\}}, \qquad x \in \mathbb{R} \tag{5.2}$$

with $\Delta_{x-1}^{x} h := h(x) - h(x-1)$. In particular, the distribution function F defines a probability measure $\mu_F \in \mathcal{M}_1(\mathbb{R}, \mathbb{N}_0)$ via

$$\mu_F[B] \;:=\; \int_B \mathfrak{p}_F(x)\,\zeta_{\mathbb{N}_0}(dx) \;=\; \sum_{\ell\in\mathbb{N}_0} \mathfrak{p}_F(\ell)\,\delta_\ell[B], \qquad B \in \mathcal{B}(\mathbb{R}), \tag{5.3}$$

where $\zeta_{\mathbb{N}_0} := \sum_{\ell\in\mathbb{N}_0} \delta_\ell$ is the (locally finite) counting measure on $\mathfrak{P}(\mathbb{N}_0)$ and δ_ℓ refers to the Dirac measure at point ℓ. Recall that $\mathcal{M}_1(\mathbb{R}, \mathbb{N}_0)$ stands for the set of all $\mu \in \mathcal{M}_1(\mathbb{R})$ satisfying $\mu[\mathbb{N}_0] = 1$. Note that that μ_F corresponds to the probability distribution of the random variables $I_1, \dots, I_N$. In view of $F(x) = F(\lfloor x \rfloor)$ for every $x \in \mathbb{R}$ (here $\lfloor\cdot\rfloor$ is the floor function), we may and do assume that F is an element of the subset $\boldsymbol{F}_0$ of all distribution functions on $\mathbb{R}$ whose discontinuity points all belong to $\mathbb{N}_0$ and which are supported on $\mathbb{R}_{\geq 0}$.

In the sequel, we will always assume that the inventory control model satisfies the following Assumption 5.1.1. This additional assumption compared to Subsection 3.1.1 is not very restrictive, as it only assumes that the expected random demand of the single product is finite, which is (often) true for distributions occurring in practice; see Subsection 5.1.4 for an example.

Assumption 5.1.1 $\int_{\mathbb{R}_{\geq 0}} y\,dF(y)\ \big(= \sum_{\ell\in\mathbb{N}_0} \ell\,\mathfrak{p}_F(\ell)\big) < \infty$.

Under Assumption 5.1.1 it follows from Lemma 5.1.9 below (applied to $M := 0$) along with (5.3) that F is even an element of $\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. Here $\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ refers to the set of all distribution functions $F \in \boldsymbol{F}_0$ which satisfy

$$\int_{\mathbb{R}_{\geq 0}} (1-F)\,d\zeta_{\mathbb{N}_0}\ \Big(= \sum_{\ell\in\mathbb{N}_0} (1 - F(\ell))\Big) \;<\; \infty.$$

Now, if $K \in \mathbb{N}$ is a fixed available inventory level of the single product within each period and if there is no backlogging of unsatisfied demand at the end of each period, the supplier intends to find for some given initial inventory level $y_0 \in \{0, \dots, K\}$ optimal order quantities according to an order strategy to maximize the expected profit. It follows from the discussion in Subsection 3.1.1 that the latter maximization problem can be modelled via a $\{0,\dots,K\}^2$-valued process $X^\varphi := (Y^\varphi, Z^\varphi)$, where $Y^\varphi := (Y_n^\varphi)_{n=0}^N$ and $Z^\varphi := (Z_n^\varphi)_{n=0}^N$ refer to the inventory and the sales process defined by (3.2) and (3.3), respectively, and $\varphi = (\varphi_n)_{n=0}^{N-1}$ is a $\{0,\dots,K\}$-valued stochastic process corresponding to a Markovian order strategy. The latter means that for any $n = 0, \dots, N-1$ there exists a map $f_n : \{0,\dots,K\}^2 \to \{0,\dots,K\}$ such that $\varphi_n = f_n(Y_n^\varphi, Z_n^\varphi)$.

Analogously to the elaborations in Subsection 3.1.1, the supplier is interested in those order strategies $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) for which the expectation of the expression in (3.5) under

$\mathbb{P}$ is maximized. Note that for given order strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) the process X^φ can be seen as a $\{0,\dots,K\}^2$-valued $(\mathcal{F}_n)$-Markov process (with $\mathcal{F}_n$ as in Subsection 3.1.1) whose one-step transition probability for the transition from state $x = (y,z) \in \{0,\dots,K\}^2$ at time $n \in \{0,\dots,N-1\}$ to state $x' = (y',z') \in \{0,\dots,K\}^2$ at time $n+1$ is given by

$$\mu_F \circ \eta^{-1}_{(y,f_n(x))}[\{z'\}] \cdot \mathbb{1}_{\{y'=y+f_n(x)-z'\}}$$

with $\eta_{(y,f_n(x))}$ defined as in (3.4).

Since the above optimization problem has a Markovian structure it can be modelled (similarly to the elaborations in Subsection 3.1.2) by a (finite horizon discrete time) MDM in the variant introduced in Section 1.5.

To this end, let E be as in (1.23) with $\mathfrak{e} := (K+1)^2$, and let $A_n(x_i)$ be given by (1.24) with $a_{n,i;k} := k-1$ and $\mathfrak{t}_{n,i} = \mathfrak{t}_i := K - y_i + 1$ for any $i = 1,\dots,\mathfrak{e}$ and $n = 0,\dots,N-1$. Set $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$, where $\mathbb{F}_n$ is equal to the set $\overline{\mathbb{F}}_n$ of all decision rules satisfying (3.6). Moreover let the components of the vector $\boldsymbol{r} = (r_n)_{n=0}^N$ be given by (3.10).

Let $\boldsymbol{p}$ be the vector given by (1.26) whose components $p_{n,i;a_{n,i;k}} = (p_{n,i;a_{n,i;k}}(1),\dots,p_{n,i;a_{n,i;k}}(\mathfrak{e}))$ are for any $i = 1,\dots,\mathfrak{e}$, $k = 1,\dots,\mathfrak{t}_i$, and $n = 0,\dots,N-1$ of the shape

$$p_{n,i;a_{n,i;k}}(j) \;:=\; \mu_F \circ \eta^{-1}_{(y_i,a_{n,i;k})}[\{z_j\}] \cdot \mathbb{1}_{\{y_j=y_i+a_{n,i;k}-z_j\}}, \qquad j = 1,\dots,\mathfrak{e} \tag{5.4}$$

for some $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$, where $\eta_{(y_i,a_{n,i;k})}$ is introduced in (3.4). Note that it is easily seen that $\boldsymbol{p}$ is an element of the set $\widetilde{\mathcal{P}}$ defined in (1.27).

Since any element $\boldsymbol{p}$ of $\widetilde{\mathcal{P}}$ is generated through (5.4) by some $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$, we write $\boldsymbol{p}_F$ for the vector defined as in (1.26) whose components $p^F_{n,i;a_{n,i;k}} = (p^F_{n,i;a_{n,i;k}}(1),\dots,p^F_{n,i;a_{n,i;k}}(\mathfrak{e}))$ are defined as on the right-hand side of (5.4). In virtue of

$$p^F_{n,i;a_{n,i;k}}(j) \;=\; \sum_{\ell \in \eta^{-1}_{(y_i,a_{n,i;k})}(z_j)} \mathfrak{p}_F(\ell) \cdot \mathbb{1}_{\{y_j=y_i+a_{n,i;k}-z_j\}} \tag{5.5}$$

for all $i,j = 1,\dots,\mathfrak{e}$, $k = 1,\dots,\mathfrak{t}_i$, and $n = 0,\dots,N-1$ (by (5.3)), we have immediately the following lemma.

Lemma 5.1.2 $\boldsymbol{p}_F \in \widetilde{\mathcal{P}}$ *for every* $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$

Note that it follows from the discussion in Section 1.5 that in the finite setting for any $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ the transition function $\boldsymbol{P}_F = (P_n^F)_{n=0}^{N-1}$ from $\overline{\mathcal{P}} = \overline{\mathcal{P}}_1$ (with $\overline{\mathcal{P}}$ given by (1.25)) can be identified with the vector $\boldsymbol{p}_F \in \widetilde{\mathcal{P}}$ as defined in (1.26) with (5.4). Also note that in the finite setting $\psi :\equiv 1$ provides a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for every $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. In particular, conditions (a)–(b) of Assumptions 4.2.1 are satisfied (with $\boldsymbol{F}(\nu)$ replaced by $\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$).

Thus, for every fixed $i_0 \in \{1,\dots,\mathfrak{e}\}$ and $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$, the inventory control problem above reads as

$$V_0^{F;\pi}(x_{i_0}) \longrightarrow \max \;(\text{in } \pi \in \Pi)! \tag{5.6}$$

where $V_0^{F;\pi}(x_{i_0}) := V_0^{\boldsymbol{p}_F;\pi}(x_{i_0})$ can be obtained from (1.29) (with $\boldsymbol{p}_F$ in place of $\boldsymbol{p}$) along with (3.10), and $x_{i_0} = (y_{i_0}, z_{i_0})$ refers to the initial state. A strategy $\pi^F \in \Pi$ is called an *optimal* *order strategy w.r.t. F* if it solves the maximization problem (5.6). Note that it follows from [73, Proposition 4.4.3] that in the finite setting there exists for every $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ an optimal order strategy $\pi^F \in \Pi$.

5.1.2 Regularity of the value function

Maintain the notation and terminology introduced in Subsection 5.1.1. In this subsection we will show that the value function of the stochastic inventory control problem (5.6) regarded as a real-valued functional defined on a set of distribution functions is 'Lipschitz continuous' and quasi-Hadamard differentiable in the sense of Definitions 4.3.1 and 4.3.7.

We emphasize that the regularity properties of the value function are *not* relevant (except the shape of the quasi-Hadamard derivatives in parts (i) and (ii) of Theorem 5.1.5 ahead) for the investigation of the asymptotics of certain estimators for the optimal value of the stochastic inventory control problem, which is part of Subsections 5.1.3–5.1.4. The purpose of the following elaborations is merely to illustrate the results presented in Section 4.3 in the context of the setting of Subsection 5.1.1. For a reader who is only interested in the statistical estimation of the optimal value of the stochastic inventory control problem, we recommend skipping this subsection and going to Subsections 5.1.3–5.1.4.

In the sequel, let the functionals $\mathcal{W}_0^{x_{i_0};\pi} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ and $\mathcal{W}_0^{x_{i_0}} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ be defined as in (4.13). Note that in the setting of Subsection 5.1.1 these functionals admit the representations

$$\mathcal{W}_0^{x_{i_0};\pi}(F) = V_0^{F;\pi}(x_{i_0}) \quad \text{and} \quad \mathcal{W}_0^{x_{i_0}}(F) = \max_{\pi\in\Pi} \mathcal{W}_0^{x_{i_0};\pi}(F) \tag{5.7}$$

for every $i_0 = 1, \dots, \mathfrak{e}$, $\pi \in \Pi$, and $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. Part (ii) of the following theorem shows that the value functional of the stochastic inventory control problem (5.6) is 'Lipschitz continuous' w.r.t. $(\|\cdot\|_{1,\zeta_{\mathbb{N}_0}}, |\cdot|)$. Note that it follows from Display (4.7) (with $\zeta_{\mathbb{N}_0}$ in place of ν) that the norm $\|\cdot\|_{1,\zeta_{\mathbb{N}_0}}$ admits the representation

$$\|h\|_{1,\zeta_{\mathbb{N}_0}} \Big(= \int |h(y)|\, \zeta_{\mathbb{N}_0}(dy) \Big) = \sum_{\ell\in\mathbb{N}_0} |h(\ell)| \quad \text{for all } h \in \boldsymbol{L}_1(\zeta_{\mathbb{N}_0}),$$

where $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ is defined as in Section 4.3.

Theorem 5.1.3 ('Lipschitz continuity' of $\mathcal{W}_0^{x_{i_0};\pi}$ and $\mathcal{W}_0^{x_{i_0}}$ in F) *In the setting above let $i_0 \in \{1, \dots, \mathfrak{e}\}$, $\pi \in \Pi$, and $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. Then the following two assertions hold.*

(i) *The map $\mathcal{W}_0^{x_{i_0};\pi} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ defined by (5.7) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\zeta_{\mathbb{N}_0}}, |\cdot|)$.*

(ii) *The map $\mathcal{W}_0^{x_{i_0}} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ defined by (5.7) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\zeta_{\mathbb{N}_0}}, |\cdot|)$.*

The statement in part (ii) of Theorem 5.1.3 will be used in Subsections 5.1.3 and 5.1.4 to derive asymptotic properties of plug-in estimators for the optimal value of the inventory control problem (5.6).

The proof of Theorem 5.1.3 needs the following Lemma 5.1.4. Recall that $\mathfrak{p}_F$ introduced in (5.1) refers for any $F \in \boldsymbol{F}_0$ to the counting density of the probability measure μ_F associated with F as defined in (5.3). Note that $F - G \in \boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ for any $F, G \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ (with $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ defined as in Section 4.3).

Lemma 5.1.4 $\|\mathfrak{p}_F - \mathfrak{p}_G\|_{1,\zeta_{\mathbb{N}_0}} \le 2\|F - G\|_{1,\zeta_{\mathbb{N}_0}}$ *for every* $F, G \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$.

Proof In view of (4.7) (with $\zeta_{\mathbb{N}_0}$ in place of ν) and (5.1)–(5.2) we obtain

$$\begin{aligned}
\|\mathfrak{p}_F - \mathfrak{p}_G\|_{1,\zeta_{\mathbb{N}_0}} &= \int |\mathfrak{p}_F(x) - \mathfrak{p}_G(x)|\,\zeta_{\mathbb{N}_0}(dx) \;=\; \sum_{\ell\in\mathbb{N}_0} \big|\mathfrak{p}_F(\ell) - \mathfrak{p}_G(\ell)\big| \\
&= \sum_{\ell\in\mathbb{N}_0} \big|(F(0) - G(0))\mathbb{1}_{\{\ell=0\}} + \Delta_{\ell-1}^{\ell}(F-G)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big| \\
&\le |F(0) - G(0)| + \sum_{\ell\in\mathbb{N}} \big|F(\ell) - G(\ell)\big| + \sum_{\ell\in\mathbb{N}} \big|F(\ell-1) - G(\ell-1)\big| \\
&= 2\sum_{\ell\in\mathbb{N}_0} \big|F(\ell) - G(\ell)\big| \;=\; 2\int |F(x) - G(x)|\,\zeta_{\mathbb{N}_0}(dx) \;=\; 2\|F - G\|_{1,\zeta_{\mathbb{N}_0}}
\end{aligned}$$

for every $F, G \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. □

Now, we are in the position to prove Theorem 5.1.3.

Proof of Theorem 5.1.3: We intend to apply Corollary 4.3.6. First, it follows from Lemma 5.1.2 that condition (a) of Corollary 4.3.6 holds. It remains to verify condition (b) of Corollary 4.3.6. That is, we will show in the sequel that for any $i = 1, \dots, \mathfrak{e}$, $\pi \in \Pi$, and $n = 0, \dots, N-1$ the map $\Lambda_n^{F;(\pi,x_i)} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ defined by (4.14) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\zeta_{\mathbb{N}_0}}, |\cdot|)$ (in the sense of Definition 4.3.1).

Now, let $(F_m)_{m\in\mathbb{N}}$ be any sequence in $\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ satisfying $\|F_m - F\|_{1,\zeta_{\mathbb{N}_0}} \to 0$. Note that the map $\Lambda_n^{F;(\pi,x_i)}$ admits in view of (5.5) the representation

$$\begin{aligned}
\Lambda_n^{F;(\pi,x_i)}(G) &= \sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j) \cdot p_{n,i;f_n(x_i)}^{G}(j) \\
&= \sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j) \cdot \sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)} \mathfrak{p}_G(\ell) \cdot \mathbb{1}_{\{y_j = y_i + f_n(x_i) - z_j\}} \qquad (5.8)
\end{aligned}$$

for all $i = 1, \dots, \mathfrak{e}$, $\pi = (f_n)_{n=0}^{N-1} \in \Pi$, $n = 0, \dots, N-1$, and $G \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$, where $\eta_{(y_i,f_n(x_i))}$ is given by (3.4). In virtue of (5.8), (1.29) (applied to $\boldsymbol{p}_F$), and Lemma 5.1.4, for every $n = 0, \dots, N-1$ there exists a finite constant $C_{F;n} > 0$ such that for any $i = 1, \dots, \mathfrak{e}$, $\pi \in \Pi$, $n = 0, \dots, N-1$, and $m \in \mathbb{N}$

$$\begin{aligned}
&\big|\Lambda_n^{F;(\pi,x_i)}(F_m) - \Lambda_n^{F;(\pi,x_i)}(F)\big| \\
&= \Big|\sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j) \cdot \sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)} \big(\mathfrak{p}_{F_m}(\ell) - \mathfrak{p}_F(\ell)\big)\Big| \cdot \mathbb{1}_{\{y_j = y_i + f_n(x_i) - z_j\}}
\end{aligned}$$

$$
\begin{aligned}
&\le\; C_{F;n}\cdot\sum_{j=1}^{\mathfrak{e}}\;\sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)}\big|\mathfrak{p}_{F_m}(\ell)-\mathfrak{p}_F(\ell)\big| \;\le\; C_{F;n}\,\mathfrak{e}\cdot\sum_{\ell\in\mathbb{N}_0}\big|\mathfrak{p}_{F_m}(\ell)-\mathfrak{p}_F(\ell)\big|\\
&=\; C_{F;n}\,\mathfrak{e}\cdot\|\mathfrak{p}_{F_m}-\mathfrak{p}_F\|_{1,\zeta_{\mathbb{N}_0}} \;\le\; 2C_{F;n}\,\mathfrak{e}\cdot\|F_m-F\|_{1,\zeta_{\mathbb{N}_0}} \;=\; C_{\Lambda,F;n}\,\|F_m-F\|_{1,\zeta_{\mathbb{N}_0}},
\end{aligned}
$$

where $C_{\Lambda,F;n} := 2C_{F;n}\,\mathfrak{e}\in\mathbb{R}_{>0}$ (is independent of i and π). Take into account that $\psi :\equiv 1$ is a bounding function for the MDM $(E,\boldsymbol{A},\boldsymbol{P}_F,\Pi,\boldsymbol{X},\boldsymbol{r})$. Thus

$$
\big|\Lambda_n^{F;(\pi,x_i)}(F_m)-\Lambda_n^{F;(\pi,x_i)}(F)\big| \;=\; \mathcal{O}\big(\|F_m-F\|_{1,\zeta_{\mathbb{N}_0}}\big)
$$

for every $i=1,\ldots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\ldots,N-1$. Hence condition (b) of Corollary 4.3.6 holds. Therefore, the assumptions of Corollary 4.3.6 are satisfied (for $\zeta_{\mathbb{N}_0}$ in place of ν), and the assertions in parts (i) and (ii) of Theorem 5.1.3 follow from parts (i) and (ii) of the latter corollary, respectively. This completes the proof of Theorem 5.1.3. □

The following Theorem 5.1.5 illustrates Corollary 4.3.15 in the setting of Subsection 5.1.1. Part (ii) of this theorem specifies the quasi-Hadamard derivative of the value functional of the stochastic inventory control problem (5.6). This derivative will be used later in Subsections 5.1.3 and 5.1.4 to establish the asymptotic error distribution of suitable estimators for the optimal value of the latter optimization problem. Recall from (5.2) the definition of the sum operator $\mathfrak{S}_h$ for some map $h\in\mathbb{R}^{\mathbb{R}}$, and note that $\Pi(F)$ contains all optimal strategies which solves the stochastic inventory control problem (5.6). Take into account that $\Pi(F)$ is non-empty (and finite) in the setting of Subsection 5.1.1.

Theorem 5.1.5 (Quasi-Hadamard differentiability of $\mathcal{W}_0^{x_i;\pi}$ and $\mathcal{W}_0^{x_i}$ in F) *In the setting above let $i_0\in\{1,\ldots,\mathfrak{e}\}$, $\pi=(f_n)_{n=0}^{N-1}\in\Pi$, and $F\in\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. Then the following two assertions hold.*

(i) *The map $\mathcal{W}_0^{x_{i_0};\pi}:\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ defined by (5.7) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}:\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ given by*

$$
\begin{aligned}
&\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(h)\\
&=\sum_{k=0}^{N-1}\sum_{i_1=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}}\sum_{i_{k+1}=1}^{\mathfrak{e}}V_{k+1}^{F;\pi}(x_{i_{k+1}})\cdot\sum_{\ell\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})}\mathfrak{S}_h(\ell)\cdot\mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}\\
&\qquad\cdot\,p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\cdots p^F_{0,i_0;f_0(x_{i_0})}(i_1),
\end{aligned}
\tag{5.9}
$$

where $\eta_{(y_{i_k},f_k(x_{i_k}))}$ is defined as in (3.4).

(ii) *The map $\mathcal{W}_0^{x_{i_0}}:\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ defined by (5.7) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_{i_0}}:\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ given by*

$$
\dot{\mathcal{W}}_{0;F}^{x_{i_0}}(h) \;=\; \max_{\pi\in\Pi(F)}\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(h).
\tag{5.10}
$$

Proof We intend to apply Corollary 4.3.15. First of all, condition (a) of Corollary 4.3.15 holds by Lemma 5.1.2. Therefore it suffices to verify condition (b) of Corollary 4.3.15. In the sequel, we will first show that for any $i=1,\dots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\dots,N-1$ the map $\Lambda_n^{F;(\pi,x_i)}:\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ defined by (4.14) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$ (in the sense of Definition 4.3.7) with quasi-Hadamard derivative $\dot\Lambda_{n;F}^{F;(\pi,x_i)}:\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ given by

$$\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h) := \sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j)\cdot\sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)}\mathfrak{S}_h(\ell)\cdot\mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}. \tag{5.11}$$

Let $(h,(h_m),(\varepsilon_m))\in\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\times\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}_{>0}$ be any triplet satisfying $\|h_m-h\|_{1,\zeta_{\mathbb{N}_0}}\to 0$, $\varepsilon_m\to 0$ as well as $(F+\varepsilon_m h_m)\subseteq\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$. Proceeding as in the proof of Lemma 5.1.4, we obtain by means of (5.11), (5.2), and (1.29) (applied to $\boldsymbol{p}_F$) for any $i=1,\dots,\mathfrak{e}$, $\pi=(f_n)_{n=0}^{N-1}\in\Pi$, $n=0,\dots,N-1$, and $m\in\mathbb{N}$

$$\begin{aligned}
&\big|\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h_m)-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\big|\\
&=\ \Big|\sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j)\cdot\sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)}\big((h_m-h)(0)\mathbb{1}_{\{\ell=0\}}+\Delta^{\ell}_{\ell-1}(h_m-h)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\cdot\mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}\Big|\\
&\le\ C_{F;n}\cdot\sum_{j=1}^{\mathfrak{e}}\sum_{\ell\in\mathbb{N}_0}\big|(h_m-h)(0)\mathbb{1}_{\{\ell=0\}}+\Delta^{\ell}_{\ell-1}(h_m-h)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big|\ \le\ 2C_{F;n}\,\mathfrak{e}\cdot\|h_m-h\|_{1,\zeta_{\mathbb{N}_0}},
\end{aligned}\tag{5.12}$$

where $C_{F;n}>0$ is a finite constant (depending on n). Recall that $\psi:\equiv 1$ is a bounding function for the MDM $(E,\boldsymbol{A},\boldsymbol{P}_F,\Pi,\boldsymbol{X},\boldsymbol{r})$. Thus

$$\lim_{m\to\infty}\big|\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h_m)-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\big|\ =\ 0 \tag{5.13}$$

for every $i=1,\dots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\dots,N-1$. In particular, for every $i=1,\dots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\dots,N-1$ the map $\dot\Lambda_{n;F}^{F;(\pi,x_i)}:\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ given by (5.11) is continuous w.r.t. $(\|\cdot\|_{1,\zeta_{\mathbb{N}_0}},|\cdot|)$. Similarly, we obtain in view of (5.8), (5.11) and (5.1)–(5.2) for every $i=1,\dots,\mathfrak{e}$, $\pi=(f_n)_{n=0}^{N-1}\in\Pi$, $n=0,\dots,N-1$, and $m\in\mathbb{N}$

$$\begin{aligned}
&\Big|\frac{\Lambda_n^{F;(\pi,x_i)}(F+\varepsilon_m h_m)-\Lambda_n^{F;(\pi,x_i)}(F)}{\varepsilon_m}-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\Big|\\
&=\ \Big|\sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j)\cdot\sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)}\frac{\mathfrak{p}_{F+\varepsilon_m h_m}(\ell)-\mathfrak{p}_F(\ell)}{\varepsilon_m}\cdot\mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\Big|\\
&=\ \Big|\sum_{j=1}^{\mathfrak{e}} V_{n+1}^{F;\pi}(x_j)\cdot\sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)}\big(h_m(0)\mathbb{1}_{\{\ell=0\}}+\Delta^{\ell}_{\ell-1}h_m\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\cdot\mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\Big|\\
&=\ \big|\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h_m)-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\big|.
\end{aligned}$$

Hence in view of (5.13)

$$\lim_{m\to\infty}\Big|\frac{\Lambda_n^{F;(\pi,x_i)}(F+\varepsilon_m h_m)-\Lambda_n^{F;(\pi,x_i)}(F)}{\varepsilon_m}-\dot\Lambda_{n;F}^{F;(\pi,x_i)}(h)\Big|\ =\ 0.$$

for any $i=1,\dots,\mathfrak{e}$, $\pi\in\Pi$, and $n=0,\dots,N-1$. This shows the quasi-Hadamard differentiability of the map $\Lambda_n^{F;(\pi,x_i)}:\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})\to\mathbb{R}$ defined by (4.14).

Moreover it follows from the representation (5.11) along with (5.2) that $\dot{\Lambda}_{n;F}^{F;(\pi,x_i)}(0_{\boldsymbol{L}_0(\zeta_{\mathbb{N}_0})}) = 0$ for all $i = 1, \ldots, \mathfrak{e}$, $\pi \in \Pi$, and $n = 0, \ldots, N-1$. Analogously to (5.12) one can show that $|\dot{\Lambda}_{n;F}^{F;(\pi,x_i)}(h)| \leq C_{\dot{\Lambda}}$ for all $i = 1, \ldots, \mathfrak{e}$, $\pi \in \Pi$, $n = 0, \ldots, N-1$, and $h \in \boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$, where $C_{\dot{\Lambda}} := 2C_{F;n}\,\mathfrak{e}\,\|h\|_{1,\zeta_{\mathbb{N}_0}}$ is a finite constant (depending on n and h). Thus, condition (b) of Corollary 4.3.15 holds too. In particular, we have verified the conditions of Corollary 4.3.15 for $\nu := \zeta_{\mathbb{N}_0}$.

(i): An application of part (i) of Corollary 4.3.15 entails that the map $\mathcal{W}_0^{x_{i_0};\pi} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ defined by (5.7) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$. The corresponding quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi} : \boldsymbol{L}_1(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ is in view of (4.30) as well as (5.11) given by

$$\begin{aligned}\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(h) \;&=\; \sum_{k=0}^{N-1}\sum_{i_1=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}} \dot{\Lambda}_{k;F}^{F;(\pi,x_{i_k})}(h)\cdot p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\,\cdots\, p^F_{0,i_0;f_0(x_{i_0})}(i_1)\\ &=\; \sum_{k=0}^{N-1}\sum_{i_1=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}}\sum_{i_{k+1}=1}^{\mathfrak{e}} V_{k+1}^{F;\pi}(x_{i_{k+1}})\cdot \sum_{\ell\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})} \mathfrak{S}_h(\ell)\cdot \mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}\\ &\qquad\cdot\, p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\,\cdots\, p^F_{0,i_0;f_0(x_{i_0})}(i_1).\end{aligned}$$

(ii): It follows from part (ii) of Corollary 4.3.15 that the map $\mathcal{W}_0^{x_{i_0}} : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ defined by (5.7) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_{i_0}} : \boldsymbol{L}_1(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ given by (5.10). □

The statement of the following remark is not relevant for the further investigations and is therefore only mentioned here in passing.

Remark 5.1.6 Using similar arguments as in the proof of Theorem 5.1.5 one can show that for any given $x \in \mathbb{R}$ the map $\mathcal{T}^x : \boldsymbol{F}_0(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ defined by

$$\mathcal{T}^x(F) \;:=\; \mathfrak{p}_F(x)$$

with $\mathfrak{p}_F$ given by (5.1) is quasi-Hadamard differentiable at any fixed $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{T}}_F^x : \boldsymbol{L}_1(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ given by

$$\dot{\mathcal{T}}_F^x(h) \;:=\; \mathfrak{S}_h(x), \tag{5.14}$$

where $\mathfrak{S}_h$ is introduced in (5.2). In particular, this means that the counting density $\mathfrak{p}_F$ of the probability measure μ_F given by (5.3) regarded as a real-valued map defined on $\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ is for any $x \in \mathbb{R}$ quasi-Hadamard differentiable at any fixed $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ tangentially to $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\langle\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})\rangle$ with quasi-Hadamard derivative given by the right-hand side of (5.14). Take into account that in view of (5.2) the latter derivative is independent of the distribution function F. ◇

5.1.3 Nonparametric estimation of the optimal value

This subsection is concerned with the nonparametric estimation of the optimal value of the inventory control problem (5.6) in the unknown distribution function F.

To this end, let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in $\mathbb{N}_0$, and denote by F the common distribution function of the random variables $Y_1,Y_2,\ldots$ which is assumed to be unknown. Thus we have $F\in\boldsymbol{F}_0$ (with $\boldsymbol{F}_0$ defined as in Subsection 5.1.1). The random variables Y_i can be seen as observed historical demands of the single product in the inventory control model from Subsection 5.1.1. Therefore, a natural choice for the estimator of F will be the empirical distribution function $\widehat{F}_m$ of $Y_1,\ldots,Y_m$ based on sample size $m\in\mathbb{N}$ as defined in (4.31).

Remark 5.1.7 (i) Note that the empirical distribution function $\widehat{F}_m$ of the $\mathbb{N}_0$-valued random variables $Y_1,\ldots,Y_m$ is a very simple object as it is nothing but a step function with support on $\mathbb{R}_{\geq 0}$ and jump discontinuities in $\mathbb{N}_0$.

(ii) In the setting of Subsection 5.1.1, the random transition mechanism of the MDP is determined by the (unknown) probability distribution μ_F (given by (5.3)) of the random demands $I_1,\ldots,I_N$. In view of (5.3), it would be more obvious to estimate the counting density $\mathfrak{p}_F$ (given by (5.1–5.2)) of μ_F than the distribution function F of μ_F. However, since in the discrete case there is a one-to-one correspondence between the distribution function and its counting density, we will estimate the (unknown) counting density $\mathfrak{p}_F$ (and thus the probability distribution μ_F) by the empirical distribution function $\widehat{F}_m$. $\diamond$

As a consequence, the expression $\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m)$ (resp. $\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m)$) can be seen as a reasonable (plug-in) estimator for $\mathcal{W}_0^{x_{i_0};\pi}(F)$ (resp. $\mathcal{W}_0^{x_{i_0}}(F)$) if $F\in\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$, where the functional $\mathcal{W}_0^{x_{i_0};\pi}$ (resp. $\mathcal{W}_0^{x_{i_0}}$) is defined as in (5.7). Take into account that it follows from Lemma 5.1.10(i) below that $\widehat{F}_m(\omega,\,\cdot\,)\in\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for every $\omega\in\Omega$ and $m\in\mathbb{N}$.

In the sequel, we will apply the regularity results from Subsection 5.1.2 to obtain consistency, asymptotic normality, and bootstrap consistency (in probability) of the nonparametric estimator $\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m)$ for the optimal value $\mathcal{W}_0^{x_{i_0}}(F)$ of the inventory control problem (5.6).

The following Theorem 5.1.8 illustrates Theorem 4.4.1 in the setting of Subsection 5.1.1.

Theorem 5.1.8 (Strong consistency of $(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m))$ and $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m))$) *In the setting of Subsection 5.1.1 let $i_0\in\{1,\ldots,\mathfrak{c}\}$ and $\pi\in\Pi$. Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathbb{N}_0$-valued random variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with common distribution function F, and suppose that $\int_{\mathbb{R}_{\geq 0}}(1-F)\,d\zeta_{\mathbb{N}_0}<\infty$ (in particular $F\in\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$). Moreover let $\widehat{F}_m$ be for every $m\in\mathbb{N}$ the empirical distribution function of $Y_1,\ldots,Y_m$ as defined in (4.31). Then the following two assertions hold.*

(i) *The sequence of estimators $(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_{i_0};\pi}(F)$ under $\mathbb{P}$ in the sense that*
$$\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m)\to\mathcal{W}_0^{x_{i_0};\pi}(F)\quad\mathbb{P}\text{-a.s.}$$

(ii) *The sequence of estimators $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_{i_0}}(F)$ under $\mathbb{P}$ in the sense that*
$$\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m)\to\mathcal{W}_0^{x_{i_0}}(F)\quad\mathbb{P}\text{-a.s.}$$

The statement in part (ii) of Theorem 5.1.8 can be interpreted in the following sense. If $\pi^F \in \Pi$ corresponds in the setting of Theorem 5.1.8 to an optimal order strategy w.r.t. F (the existence of such a strategy is ensured), then (under the assumptions of Theorem 5.1.8) part (ii) of the latter theorem implies that (for every initial inventory level $x_{i_0} = (y_{i_0}, \cdot\,) \in E$) the sequence of estimators $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent (under $\mathbb{P}$) for the optimal value $\mathcal{W}_0^{x_{i_0};\pi^F}(F)$ of the inventory control problem (5.6).

Let us turn to the proof of Theorem 5.1.8. It avails the following two Lemmas 5.1.9–5.1.10.

Lemma 5.1.9 *$\int_{[M,\infty)}(1-F(x))\,\zeta_{\mathbb{N}_0}(dx) = \int(y-M)\mathbb{1}_{\{y\geq M\}}\,dF(y)$ for every $M \in \mathbb{N}_0$ and $F \in \boldsymbol{F}_0$.*

Proof Note that Fubini's theorem entails that

$$\begin{aligned}
&\int_{[M,\infty)}(1-F(x))\,\zeta_{\mathbb{N}_0}(dx) \;=\; \sum_{\ell\in\mathbb{N}_0\cap[M,\infty)}(1-F(\ell))(\ell-(\ell-1)) \;=\; \int_{[M,\infty)}(1-F(x))\,\ell(dx)\\
&\quad=\; \int_{[M,\infty)}\int_{[x,\infty)}\mu_F(dy)\,\ell(dx) \;=\; \int_{[M,\infty)}\int_{[M,y)}\ell(dx)\,\mu_F(dy)\\
&\quad=\; \int_{[M,\infty)}(y-M)\,dF(y) \;=\; \int(y-M)\mathbb{1}_{\{y\geq M\}}\,dF(y)
\end{aligned}$$

for every $F \in \boldsymbol{F}_0$ and $M \in \mathbb{N}_0$, where we used for the second "=" the fact that F is in particular right-continuous and that the discontinuity points of F all belong to $\mathbb{N}_0$. □

Lemma 5.1.10 *With the notation of Theorem 5.1.8 the following two assertions hold.*

(i) *$\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$.*

(ii) *If $\int_{\mathbb{R}_{\geq 0}}(1-F)\,d\zeta_{\mathbb{N}_0} < \infty$, then $\|\widehat{F}_m - F\|_{1,\zeta_{\mathbb{N}_0}} \to 0$ $\mathbb{P}$-a.s.*

Proof Since trivially $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_0$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$, the claim in (i) is an immediate consequence of Lemma 5.1.9 (applied to $M := 0$) as well as the representation (4.31).

To prove part (ii), we observe at first

$$\begin{aligned}
\big\|\widehat{F}_m(\omega,\cdot)-F(\cdot)\big\|_{1,\zeta_{\mathbb{N}_0}} \;&=\; \int_{\mathbb{R}_{\geq 0}}\big|\widehat{F}_m(\omega,x)-F(x)\big|\,\zeta_{\mathbb{N}_0}(dx)\\
&\leq\; \int_{[0,M)}\big|\widehat{F}_m(\omega,x)-F(x)\big|\,\zeta_{\mathbb{N}_0}(dx) + \int_{[M,\infty)}(1-\widehat{F}_m(\omega,x))\,\zeta_{\mathbb{N}_0}(dx)\\
&\qquad+\int_{[M,\infty)}(1-F(x))\,\zeta_{\mathbb{N}_0}(dx)\\
&\leq\; M\cdot\big\|\widehat{F}_m(\omega,\cdot)-F(\cdot)\big\|_\infty + \int_{[M,\infty)}\big(1-\widehat{F}_m(\omega,x)\big)\,\zeta_{\mathbb{N}_0}(dx)\\
&\qquad+\int_{[M,\infty)}(1-F(x))\,\zeta_{\mathbb{N}_0}(dx)\\
&=:\; S_1(m,\omega,M)+S_2(m,\omega,M)+S_3(M)
\end{aligned}$$

for every $m \in \mathbb{N}$, $\omega \in \Omega$, and $M \in \mathbb{N}$. Take into account that $\widehat{F}_m(\omega,\,\cdot\,) - F(\cdot) \in \boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$ by part (i). In view of the classical Glivenko–Cantelli theorem (e.g. in the form of [86, Theorem 19.1]), we have $\lim_{m\to\infty} S_1(m,\omega,M) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $M \in \mathbb{N}$.

Moreover, in view of Lemma 5.1.9, we get

$$\int_{[M,\infty)} \big(1 - \widehat{F}_m(\omega, x)\big)\, \zeta_{\mathbb{N}_0}(dx) \;=\; \frac{1}{m}\sum_{k=1}^{m}(Y_k(\omega) - M)\mathbb{1}_{\{Y_k(\omega)\geq M\}}$$

for every $m \in \mathbb{N}$, $\omega \in \Omega$, and $M \in \mathbb{N}$. Hence the strong law of large numbers yields that $\lim_{m\to\infty} S_2(m, \omega, M) = \mathbb{E}[(Y_1 - M)\mathbb{1}_{\{Y_1\geq M\}}]$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $M \in \mathbb{N}$. Take into account that $\mathbb{E}[|Y_1|] < \infty$ by the integrability assumption $\int_{\mathbb{R}_{\geq 0}}(1 - F)\, d\zeta_{\mathbb{N}_0} < \infty$, Lemma 5.1.9 (applied to $M := 0$), and the fact that F is supported on $\mathbb{R}_{\geq 0}$.

Similarly, we obtain

$$\int_{[M,\infty)} (1 - F(x))\, \zeta_{\mathbb{N}_0}(dx) \;=\; \int_{\mathbb{R}_{\geq 0}} (y - M)\mathbb{1}_{\{y\geq M\}}\, dF(y)$$

and thus $S_3(M) = \mathbb{E}[(Y_1 - M)\mathbb{1}_{\{Y_1\geq M\}}]$ for every $M \in \mathbb{N}$ by Lemma 5.1.9. Therefore we have shown that for $\mathbb{P}$-a.e. ω and every $M \in \mathbb{N}$

$$\limsup_{m\to\infty} \|\widehat{F}_m(\omega, \cdot) - F(\cdot)\|_{1,\zeta_{\mathbb{N}_0}} \;\leq\; 2\,\mathbb{E}[(Y_1 - M)\mathbb{1}_{\{Y_1\geq M\}}]. \tag{5.15}$$

Hence the assertion follows by letting $M \to \infty$ in (5.15) (recall $\mathbb{E}[|Y_1|] < \infty$). This shows (ii). □

Now, we are able to prove Theorem 5.1.8.

Proof of Theorem 5.1.8: We intend to apply Theorem 4.4.1. First, it is discussed in the proof of Theorem 5.1.3 that the assumptions of Corollary 4.3.6 are satisfied. Note that the assumptions of Corollary 4.3.6 matches the assumptions of Theorem 4.3.3 in the finite setting. Second, it follows from part (i) of Lemma 5.1.10 that $\widehat{F}_m(\omega, \cdot\,) \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for any $\omega \in \Omega$ and $m \in \mathbb{N}$. Third, part (ii) of Lemma 5.1.10 entails that $\|\widehat{F}_m - F\|_{1,\zeta_{\mathbb{N}_0}} \to 0$ $\mathbb{P}$-a.s.

Thus the assumptions of Theorem 4.4.1 hold (with $\zeta_{\mathbb{N}_0}$ playing the role of ν), and an application of parts (i) and (ii) of the latter theorem yield the claims in (i) and (ii) of Theorem 5.1.8, respectively. This completes the proof of Theorem 5.1.8. □

The following Theorem 5.1.11 illustrates Theorem 4.4.4 in the setting of Subsection 5.1.1. Part (ii) of this theorem can be used to construct an asymptotic confidence interval for the optimal value of inventory control problem (5.6); see Remark 5.1.14 below. Note that N_{0,s^2} refers to the normal distribution with zero mean and variance s^2, and that $\xi \sim \mathrm{N}_{0,s^2}$ means that the random variable ξ has law N_{0,s^2}. Recall that $\rightsquigarrow$ refers to the convergence in distribution.

Theorem 5.1.11 (Asymptotic error distribution of $(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m))$ and $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m))$) *In the setting of Subsection 5.1.1 let $i_0 \in \{1, \ldots, \mathfrak{e}\}$ and $\pi = (f_n)_{n=0}^{N-1} \in \Pi$. Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathbb{N}_0$-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F, and suppose that $\int \sqrt{F(1 - F)}\, d\zeta_{\mathbb{N}_0} < \infty$ (in particular $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$). Moreover let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1, \ldots, Y_m$ as defined in (4.31). Then the following two assertions hold.*

(i) *We have*

$$\sqrt{m}\big(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m) - \mathcal{W}_0^{x_{i_0};\pi}(F)\big) \;\rightsquigarrow\; Z_{F;i_0,\pi} \qquad \textit{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|) \tag{5.16}$$

for $Z_{F;i_0,\pi} \sim \mathrm{N}_{0,s^2}$ *with*

$$s^2 \;=\; s^2_{F;i_0,\pi} \;:=\; \int_{\mathbb{R}^2} h_F^{i_0,\pi}(t_1)\, C_F(t_1,t_2)\, h_F^{i_0,\pi}(t_2)\, (\zeta_{\mathbb{N}_0}\otimes\zeta_{\mathbb{N}_0})(d(t_1,t_2)), \tag{5.17}$$

where

$$h_F^{i_0,\pi}(t) \;:=\; \widetilde{h}_F^{i_0,\pi}(t) - \widetilde{h}_F^{i_0,\pi}(t+1), \qquad t\in\mathbb{R} \tag{5.18}$$

for

$$\begin{aligned}\widetilde{h}_F^{i_0,\pi}(t) \;\;:=\;\; & \sum_{k=0}^{N-1}\Big(\sum_{i_1=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}}\sum_{i_{k+1}=1}^{\mathfrak{e}} V_{k+1}^{F;\pi}(x_{i_{k+1}})\cdot\mathbb{1}_{\{t\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})\}} \\ & \cdot\mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}\cdot p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\,\cdots\, p^F_{0,i_0;f_0(x_{i_0})}(i_1)\Big),\end{aligned} \tag{5.19}$$

and C_F *is given by (4.33).*

(ii) *If there exists a unique optimal order strategy* $\pi^F\in\Pi$ *w.r.t.* F*, then*

$$\sqrt{m}\big(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m) - \mathcal{W}_0^{x_{i_0}}(F)\big) \;\rightsquigarrow\; Z_{F;i_0} \qquad \textit{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|) \tag{5.20}$$

for $Z_{F;i_0}\sim\mathrm{N}_{0,s^2}$ *with* $s^2 = s^2_{F;i_0,\pi^F}$ *given by (5.17) (with* π *replaced by* π^F*).*

The proof of Theorem 5.1.11 requires the following lemma.

Lemma 5.1.12 *With the notation and under the assumptions of Theorem 5.1.11 the following two assertions hold for any* $i_0 = 1,\ldots,\mathfrak{e}$*,* $\pi\in\Pi$*, and* $m\in\mathbb{N}$*.*

(i) *The estimator* $\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m)$ *is* $(\mathcal{F},\mathcal{B}(\mathbb{R}))$*-measurable.*

(ii) *The estimator* $\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m)$ *is* $(\mathcal{F},\mathcal{B}(\mathbb{R}))$*-measurable.*

Proof For (i), note at first that the backward induction scheme in Display (1.29) admits in the setting of Subsection 5.1.1 in view of (5.7), (5.5), and (5.1)–(5.2) the representation

$$\mathcal{W}_N^{x_i;\pi}(\widehat{F}_m(\omega,\cdot\,)) \;=\; r_N(x_i)$$

and

$$\begin{aligned}\mathcal{W}_n^{x_i;\pi}(\widehat{F}_m(\omega,\cdot\,)) \;&=\; r_n(x_i,f_n(x_i)) + \sum_{j=1}^{\mathfrak{e}}\mathcal{W}_{n+1}^{x_j;\pi}(\widehat{F}_m(\omega,\cdot\,))\cdot p^{\widehat{F}_m(\omega,\cdot\,)}_{n,i;f_n(x_i)}(j) \\ \;&=\; r_n(x_i,f_n(x_i)) + \sum_{j=1}^{\mathfrak{e}}\mathcal{W}_{n+1}^{x_j;\pi}(\widehat{F}_m(\omega,\cdot\,))\cdot\Big(\sum_{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)}\mathfrak{p}_{\widehat{F}_m(\omega,\cdot\,)}(\ell)\cdot\mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}\Big) \\ \;&=\; r_n(x_i,f_n(x_i)) + \sum_{j=1}^{\mathfrak{e}}\mathcal{W}_{n+1}^{x_j;\pi}(\widehat{F}_m(\omega,\cdot\,)) \\ &\quad\cdot\Big(\sum_{\ell\in\mathbb{N}_0}\big(\widehat{F}_m(\omega,0)\mathbb{1}_{\{\ell=0\}}+\Delta^{\ell}_{\ell-1}\widehat{F}_m(\omega,\cdot\,)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\cdot\mathbb{1}_{\{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)\}}\cdot\mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}\Big)\end{aligned}$$

$$= r_n(x_i, f_n(x_i)) + \sum_{j=1}^{\mathfrak{c}} \mathcal{W}_{n+1}^{x_j;\pi}(\widehat{F}_m(\omega,\,\cdot\,))$$
$$\cdot \Big(\lim_{M\to\infty} \Big\{ \sum_{\ell=0}^{M} \big(\widehat{F}_m(\omega,0)\mathbb{1}_{\{\ell=0\}} + \Delta_{\ell-1}^{\ell}\widehat{F}_m(\omega,\,\cdot\,)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\cdot \mathbb{1}_{\{\ell\in\eta^{-1}_{(y_i,f_n(x_i))}(z_j)\}}\Big\}\cdot \mathbb{1}_{\{y_j=y_i+f_n(x_i)-z_j\}}\Big)$$

for every $i = 1,\ldots,\mathfrak{c}$, $n = 0,\ldots,N-1$, and $\omega \in \Omega$. Take into account that $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for every $\omega \in \Omega$ by Lemma 5.1.10(i). Thus it can be verified by (backward) induction on n that the mapping

$$\omega \mapsto \mathcal{W}_n^{x_i;\pi}(\widehat{F}_m(\omega,\,\cdot\,))$$

is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable for any $i = 1,\ldots,\mathfrak{c}$ and $n = 0,\ldots,N$ because $\omega \mapsto \widehat{F}_m(\omega,t)$ is clearly $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable for all $t \in \mathbb{R}$. In particular this implies that $\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m)$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable. This shows (i).

The assertion in (ii) follows from part (i) along with the representation (5.7). Note that in the setting of Subsection 5.1.1 the set Π (of all admissible strategies) is finite. □

Let us turn to the proof of Theorem 5.1.11.

Proof of Theorem 5.1.11: We intend to apply Theorem 4.4.4. At first, part (i) of Lemma 5.1.10 entails that $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$. Therefore condition (b) of Theorem 4.4.4 holds. Moreover, Lemma 5.1.12 ensures that condition (c) of Theorem 4.4.4 is satisfied. Since it follows from the proof of Theorem 5.1.5 that the assumptions of Corollary 4.3.15 hold, we have verified the assumptions of Theorem 4.4.4 (with $\zeta_{\mathbb{N}_0}$ in place of ν). Take into account that in the finite setting of Subsection 5.1.1 the assumptions of Corollary 4.3.15 imply the assumptions of Theorem 4.3.8 (with $\zeta_{\mathbb{N}_0}$ in place of ν).

(i): It follows from part (i) of Theorem 4.4.4 that $\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and

$$\sqrt{m}\big(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m) - \mathcal{W}_0^{x_{i_0};\pi}(F)\big) \;\rightsquigarrow\; \dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F) \quad \text{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|), \tag{5.21}$$

where $\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}$ is given by (5.9) and B_F is an $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$-valued centred Gaussian random variable on some probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\zeta_{\mathbb{N}_0}}$ given by (4.32). Further it is easily seen that the right-hand side of (5.21) admits in view of (5.9) and (5.2) the representation

$$\begin{aligned}
&\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F)\\
&= \sum_{k=0}^{N-1}\sum_{i_1=1}^{\mathfrak{c}}\cdots\sum_{i_k=1}^{\mathfrak{c}}\sum_{i_{k+1}=1}^{\mathfrak{c}} V_{k+1}^{F;\pi}(x_{i_{k+1}})\cdot\sum_{\ell\in\mathbb{N}_0}\big(B_F(\,\cdot\,,0)\mathbb{1}_{\{\ell=0\}} + \Delta_{\ell-1}^{\ell}B_F(\,\cdot\,,\bullet)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\\
&\qquad \cdot\mathbb{1}_{\{\ell\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})\}}\cdot\mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}\cdot p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\cdots p^F_{0,i_0;f_0(x_{i_0})}(i_1)\\
&= \sum_{\ell\in\mathbb{N}_0}\widetilde{h}_F^{i_0,\pi}(\ell)\cdot\big(B_F(\,\cdot\,,0)\mathbb{1}_{\{\ell=0\}} + \Delta_{\ell-1}^{\ell}B_F(\,\cdot\,,\bullet)\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\\
&= \sum_{\ell\in\mathbb{N}_0}\widetilde{h}_F^{i_0,\pi}(\ell)\,B_F(\,\cdot\,,\ell) - \sum_{\ell\in\mathbb{N}}\widetilde{h}_F^{i_0,\pi}(\ell)\,B_F(\,\cdot\,,\ell-1) \;=\; \sum_{\ell\in\mathbb{N}_0}\big(\widetilde{h}_F^{i_0,\pi}(\ell)-\widetilde{h}_F^{i_0,\pi}(\ell+1)\big)\,B_F(\,\cdot\,,\ell)\\
&= \sum_{\ell\in\mathbb{N}_0} h_F^{i_0,\pi}(\ell)\,B_F(\,\cdot\,,\ell) \;=\; \int h_F^{i_0,\pi}(t)\,B_F(\,\cdot\,,t)\,\zeta_{\mathbb{N}_0}(dt)
\end{aligned} \tag{5.22}$$

with $\widetilde{h}_F^{i_0,\pi}$ and $h_F^{i_0,\pi}$ given by (5.19) and (5.18), respectively. Hence, since $h_F^{i_0,\pi}(\cdot) \in \boldsymbol{L}_\infty(\zeta_{\mathbb{N}_0})$ and since B_F is a centred Gaussian random variable with values in $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$, the real-valued random variable $Z_{F;i_0,\pi} := \dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F)$ is normally distributed with mean

$$\check{\mathbb{E}}\big[\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F)\big] \;=\; \int h_F^{i_0,\pi}(t)\,\check{\mathbb{E}}[B_F(\,\cdot\,,t)]\,\zeta_{\mathbb{N}_0}(dt) \;=\; 0$$

(by Fubini's theorem) and variance

$$\begin{aligned}
\check{\mathbb{V}}\mathrm{ar}\big[\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F)\big] \;&=\; \check{\mathbb{E}}\big[\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F)^2\big]\\
&=\; \check{\mathbb{E}}\Big[\Big(\int h_F^{i_0,\pi}(t_1)B_F(\,\cdot\,,t_1)\,\zeta_{\mathbb{N}_0}(dt_1)\Big)\Big(\int h_F^{i_0,\pi}(t_2)B_F(\,\cdot\,,t_2)\,\zeta_{\mathbb{N}_0}(dt_2)\Big)\Big]\\
&=\; \Gamma_{B_F,\zeta_{\mathbb{N}_0}}(h_F^{i_0,\pi},h_F^{i_0,\pi}),
\end{aligned}$$

where the latter expression is in view of Theorem 4.4.6 (with ν and $\boldsymbol{F}$ replaced by $\zeta_{\mathbb{N}_0}$ and $\boldsymbol{F}_0$, respectively) equal to the right-hand side of (5.17). This shows (5.16).

(ii): By part (ii) of Theorem 4.4.4 we obtain that $\dot{\mathcal{W}}_{0;F}^{x_{i_0}}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable as well as

$$\sqrt{m}\big(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m)-\mathcal{W}_0^{x_{i_0}}(F)\big) \;\rightsquigarrow\; \dot{\mathcal{W}}_{0;F}^{x_{i_0}}(B_F) \quad \text{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|), \tag{5.23}$$

where $\dot{\mathcal{W}}_{0;F}^{x_{i_0}}$ is given by (5.10) and B_F is as in (i). If in addition there exists a unique optimal order strategy $\pi^F \in \Pi$ w.r.t. F, then $\Pi(F)=\{\pi^F\}$ and the right-hand side of (5.23) admits in view of (5.10) and (5.22) the representation

$$\dot{\mathcal{W}}_{0;F}^{x_{i_0}}(B_F) \;=\; \max_{\pi\in\Pi(F)}\dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi}(B_F) \;=\; \dot{\mathcal{W}}_{0;F}^{x_{i_0};\pi^F}(B_F) \;=\; \int h_F^{i_0;\pi^F}(t)B_F(\,\cdot\,,t)\,\zeta_{\mathbb{N}_0}(dt).$$

Hence, using similar arguments as in the proof of (i), one can show that the real-valued random variable $Z_{F;i_0} := \dot{\mathcal{W}}_{0;F}^{x_{i_0}}(B_F)$ is normally distributed with zero mean and variance $\check{\mathbb{V}}\mathrm{ar}[\dot{\mathcal{W}}_{0;F}^{x_{i_0}}(B_F)] = \Gamma_{B_F,\zeta_{\mathbb{N}_0}}(h_F^{i_0;\pi^F},h_F^{i_0;\pi^F})$, where the latter is equal to the right-hand side of (5.17) (with π replaced by π^F). In particular, this shows (5.20) and completes the proof of Theorem 5.1.11. □

Remark 5.1.13 An easy computation shows that in the setting (and under the assumptions) of Theorem 5.1.11 the variance $s^2_{F;i_0,\pi}$ in (5.17) (and thus the variance $s^2_{F;i_0,\pi^F}$ in part (ii) of Theorem 5.1.11) admits the representation

$$\begin{aligned}
s^2_{F;i_0,\pi} \;=\;& \sum_{k=0}^{N-1}\Big(\sum_{i_1,j_1=1}^{\mathfrak{e}}\cdots\sum_{i_k,j_k=1}^{\mathfrak{e}}\;\sum_{i_{k+1},j_{k+1}=1}^{\mathfrak{e}} V_{k+1}^{F;\pi}(x_{i_{k+1}})V_{k+1}^{F;\pi}(x_{j_{k+1}})\\
&\cdot\sum_{t_{i_k}\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})}\;\sum_{t_{j_k}\in\eta^{-1}_{(x_{j_k},f_k(x_{j_k}))}(\{z_{j_{k+1}}\})}\mathcal{C}_F(t_{i_k},t_{j_k})\cdot\mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}\\
&\cdot\mathbb{1}_{\{y_{j_{k+1}}=y_{j_k}+f_k(x_{j_k})-z_{j_{k+1}}\}}\cdot p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\,p^F_{k-1,j_{k-1};f_{k-1}(x_{j_{k-1}})}(j_k)\\
&\cdots p^F_{0,i_0;f_0(x_{i_0})}(i_1)\,p^F_{0,i_0;f_0(x_{i_0})}(j_1)\Big)\\
&+\sum_{\substack{k,\ell=0\\k\neq\ell}}^{N-1}\Big(\sum_{i_1=1}^{\mathfrak{e}}\sum_{j_1=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}}\sum_{j_\ell=1}^{\mathfrak{e}}\sum_{i_{k+1}=1}^{\mathfrak{e}}\sum_{j_{\ell+1}=1}^{\mathfrak{e}} V_{k+1}^{F;\pi}(x_{i_{k+1}})V_{\ell+1}^{F;\pi}(x_{j_{\ell+1}})
\end{aligned}$$

$$\cdot \sum_{t_{i_k}\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})} \sum_{t_{j_\ell}\in\eta^{-1}_{(x_{j_\ell},f_\ell(x_{j_\ell}))}(\{z_{j_{\ell+1}}\})} \mathcal{C}_F(t_{i_k},t_{j_\ell})\cdot \mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}$$
$$\cdot \mathbb{1}_{\{y_{j_{\ell+1}}=y_{j_\ell}+f_\ell(x_{j_\ell})-z_{j_{\ell+1}}\}}\cdot p^F_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\, p^F_{\ell-1,j_{\ell-1};f_{\ell-1}(x_{j_{\ell-1}})}(j_\ell)$$
$$\cdots p^F_{0,i_0;f_0(x_{i_0})}(i_1)\, p^F_{0,i_0;f_0(x_{i_0})}(j_1)\Big)$$

for any $i_0 = 1,\ldots,\mathfrak{e}$ and $\pi\in\Pi$, where

$$\mathcal{C}_F(t_i,t_j) := \begin{cases} C_F(0,0) & , \quad t_i,t_j=0 \\ C_F(0,t_j)-C_F(0,t_j-1) & , \quad t_i=0,\ t_j\in\mathbb{N} \\ C_F(t_i,0)-C_F(t_i-1,0) & , \quad t_i\in\mathbb{N},\ t_j=0 \\ C_F(t_i,t_j)-C_F(t_i,t_j-1) & \\ -\,C_F(t_i-1,t_j)+C_F(t_i-1,t_j-1) & , \quad t_i\in\mathbb{N},\ t_j\in\mathbb{N} \end{cases}$$

with C_F given by (4.33). ◇

In the following Remark 5.1.14 we will discuss the significance of the statement in part (ii) of Theorem 5.1.11 for the estimation of the optimal value of the inventory control problem (5.6).

Remark 5.1.14 In view of part (ii) of Theorem 5.1.11 we can derive (under the assumptions of the latter theorem and the additional assumption that there exists a unique optimal order strategy $\pi^F\in\Pi$ w.r.t. F) from equation (5.20) an asymptotic confidence interval at a given level $\kappa\in(0,1)$ for the optimal value $\mathcal{W}_0^{x_{i_0};\pi^F}(F)$ of the inventory control problem (5.6). In this case, however, one has to perform a nonparametric estimation of the variance $s^2 = s^2_{F;i_0;\pi^F}$ in (5.17) (with π replaced by π^F) which is of the form

$$\widehat{s}^{\,2}_m \;=\; s^2_{\widehat{F}_m;i_0,\pi^{\widehat{F}_m}} \;:=\; \int_{\mathbb{R}^2} h^{i_0,\pi^{\widehat{F}_m}}_{\widehat{F}_m}(s)\, C_{\widehat{F}_m}(s,t)\, h^{i_0,\pi^{\widehat{F}_m}}_{\widehat{F}_m}(t)\, (\zeta_{\mathbb{N}_0}\otimes\zeta_{\mathbb{N}_0})(d(t_1,t_2)) \tag{5.24}$$

with $h^{i_0,\pi^{\widehat{F}_m}}_{\widehat{F}_m}$ and $C_{\widehat{F}_m}$ defined as in (5.18)–(5.19) and (4.33), respectively. Here $\pi^{\widehat{F}_m}\in\Pi$ corresponds to an optimal order strategy w.r.t. $\widehat{F}_m$ (computed via [73, p. 92]). Since the estimator $\widehat{s}^{\,2}_m$ in (5.24) for s^2 depends on $\widehat{F}_m$ in a quite complex manner, it is not clear how good the performance of the asymptotic confidence interval based on $\widehat{s}^{\,2}_m$ is. In order to handle this problem, we will show in the next theorem a bootstrap result (in probability) with its help we are able to derive a so-called bootstrap confidence interval for the optimal value $\mathcal{W}_0^{x_{i_0};\pi^F}(F)$; see Remark 4.4.13. ◇

Part (ii) of the following Theorem 5.1.15 shows that the sequence $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}^*_m))_{m\in\mathbb{N}}$ is a bootstrap version (in probability) of the sequence of estimators $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m))_{m\in\mathbb{N}}$ for the optimal value $\mathcal{W}_0^{x_{i_0}}(F)$ of the inventory control problem (5.6). Recall that d_{BL} introduced in Example 2.1.4 (with $E:=\mathbb{R}$) refers to the bounded Lipschitz-metric on $\mathcal{M}_1(\mathbb{R})$. Also note that a weight function ϕ is continuous on $\mathbb{R}$ as well as non-decreasing on $\mathbb{R}_{\geq 0}$ and non-increasing on $\mathbb{R}_{\leq 0}$.

Theorem 5.1.15 (Bootstrap consistency of $(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m))$ and $(\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m))$) *In the setting of Subsection 5.1.1 let $i_0\in\{1,\ldots,\mathfrak{e}\}$ and $\pi\in\Pi$. Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathbb{N}_0$-valued random* *variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F, and assume that $\int \phi^2\, dF < \infty$ for some weight function ϕ satisfying $\int 1/\phi\, d\zeta_{\mathbb{N}_0} < \infty$ (in particular $F \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$). Let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1, \ldots, Y_m$ as defined in (4.31). Moreover let $(W_{mi})_{m\in\mathbb{N},1\le i\le m}$ be a triangular array of nonnegative real-valued random variables on another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, set $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\otimes\mathbb{P}')$, and let $\widehat{F}^*_m$ be defined as in (4.37). If one of the settings (B1)–(B2) in Subsection 4.4.3 is met, then the following two assertions hold.*

(i) *For every $\delta > 0$ we have*

$$\lim_{m\to\infty} \mathbb{P}\big[\big\{\omega\in\Omega : d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}^*_m(\omega,\cdot))-\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m(\omega)))}, \mathrm{N}_{0,s^2}) \ge \delta\big\}\big] \;=\; 0, \tag{5.25}$$

where $s^2 = s^2_{F;i_0,\pi}$ is given by (5.17).

(ii) *If there exists a unique optimal order strategy $\pi^F \in \Pi$ w.r.t. F, then for any $\delta > 0$ we have*

$$\lim_{m\to\infty} \mathbb{P}\big[\big\{\omega\in\Omega : d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}_0^{x_{i_0}}(\widehat{F}^*_m(\omega,\cdot))-\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m(\omega)))}, \mathrm{N}_{0,s^2}) \ge \delta\big\}\big] \;=\; 0, \tag{5.26}$$

where $s^2 = s^2_{F;i_0,\pi^F}$ is given by (5.17) (with π replaced by π^F).

Proof We intend to apply Theorem 4.4.9. First, Lemma 5.1.10(i) ensures that $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$. Second, it follows from Lemma 5.1.9 (applied to $M := 0$) along with the representation (4.37) that $\widehat{F}^*_m((\omega,\omega'),\,\cdot\,) \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ for every $(\omega,\omega') \in \overline{\Omega}$ and $m \in \mathbb{N}$. Third, Lemma 5.1.12 implies that the estimators $\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}_m)$ and $\mathcal{W}_0^{x_{i_0}}(\widehat{F}_m)$ are $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $m \in \mathbb{N}$. Using similar arguments as in the proof of Lemma 5.1.12 it is easily seen that the estimators $\mathcal{W}_0^{x_{i_0};\pi}(\widehat{F}^*_m)$ as well as $\mathcal{W}_0^{x_{i_0}}(\widehat{F}^*_m)$ are $(\overline{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable for any $m \in \mathbb{N}$. Moreover, it follows from the proof of Theorem 5.1.5 and Display (5.2) that the map $\dot{\Lambda}^{F;(\pi,x_i)}_{n;F} : \boldsymbol{L}_1(\zeta_{\mathbb{N}_0}) \to \mathbb{R}$ given by (5.11) is linear for any $i = 1, \ldots, \mathfrak{e}$ and $n = 0, \ldots, N-1$, and satisfies condition (a) of Theorem 4.3.8. That is, in the setting of Subsection 5.1.1 conditions (b)–(f) of Theorem 4.4.9 are satisfied, and condition (a) of Theorem 4.4.9 holds by assumption. Finally, note that it is discussed in the proof of Theorem 5.1.11 that in the above setting the assumptions of Theorem 4.3.8 hold. Hence this implies that the assumptions of Theorem 4.4.9 are satisfied for $\nu := \zeta_{\mathbb{N}_0}$. In particular, it follows from the discussion subsequent to Theorem 4.4.9 that the expressions in (5.25) and (5.26) are well-defined.

(i): Part (i) of Theorems 4.4.9 and 5.1.5 entails that (5.25) holds for every $\delta > 0$ with N_{0,s^2} replaced by $\check{\mathbb{P}}_{\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F}(B_F)}$, where $\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F}$ is given by (5.9) and B_F refers to an $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$-valued centred Gaussian random variable on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\zeta_{\mathbb{N}_0}}$ defined as in (4.32). Since $Z_{F;i_0;\pi} := \dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F}(B_F) \sim \mathrm{N}_{0,s^2}$ (under $\check{\mathbb{P}}$) with $s^2 = s^2_{F;i_0,\pi}$ defined in (5.17) by (the proof of) part (i) of Theorem 5.1.11, the assertion in (5.25) follows. Take into account that we may apply Theorem 5.1.11 because $\int \sqrt{F(1-F)}\, d\zeta_{\mathbb{N}_0} < \infty$ is implied by the assumptions $\int \phi^2\, dF < \infty$ and $\int 1/\phi\, d\zeta_{\mathbb{N}_0} < \infty$; see Remark 4.4.7 (applied to $\nu := \zeta_{\mathbb{N}_0}$).

(ii): If there exists a unique optimal strategy $\pi^F \in \Pi$ w.r.t. F, then part (ii) of Theorem 4.4.9 along with Theorem 5.1.5 imply that (5.26) holds for every $\delta > 0$ with N_{0,s^2} replaced by $\check{\mathbb{P}}_{\dot{\mathcal{W}}^{x_{i_0}}_{0;F}(B_F)}$, where $\dot{\mathcal{W}}^{x_{i_0}}_{0;F}$ is given by (5.10) and B_F is as in (i). Since $\Pi(F) = \{\pi^F\}$ and thus $\dot{\mathcal{W}}^{x_{i_0}}_{0;F} = \dot{\mathcal{W}}^{x_{i_0};\pi^F}_{0;F}$

(by (5.10)), we obtain as in the proof of Theorem 5.1.11 that $\dot{\mathcal{W}}_{0;F}^{x_{i_0}}(B_F) \sim \mathrm{N}_{0,s^2}$ (under $\check{\mathbb{P}}$) with $s^2 = s^2_{F;i_0,\pi^F}$ defined in (5.17) (with π replaced by π^F). Therefore the assertion in (5.26) follows. □

5.1.4 Parametric estimation of the optimal value

The objective of this subsection is the parametric estimation of the optimal value of the inventory control problem (5.6) in which the distribution function F describing the random demands of the single product within each period is unknown. Here we assume that the distribution of the (i.i.d.) random demands of the single product is drawn from a Poisson distribution whose parameter is not known. Note that in practice the Poisson distribution is an appropriate choice to model the random demand. In this case the distribution function F corresponds to the distribution function of the Poisson distribution with unknown parameter.

To this end, we consider the parametric statistical infinite product model

$$(\Omega, \mathcal{F}, \{\mathbb{P}^\theta : \theta \in \Theta\}) := \big(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \{\mathbb{P}^\lambda := \mathrm{Poiss}_\lambda^{\otimes \mathbb{N}} : \lambda \in \Theta\}\big) \tag{5.27}$$

for the (open) parameter set $\Theta := \mathbb{R}_{>0}$ ($\subseteq \mathbb{R}$), where the Poisson distribution Poiss_λ with parameter $\lambda \in \Theta$ is given by the standard counting density

$$\mathfrak{p}_\lambda(x) := \begin{cases} \frac{\lambda^x}{x!}\, e^{-\lambda} & , \quad x \in \mathbb{N}_0 \\ 0 & , \quad \text{otherwise} \end{cases}. \tag{5.28}$$

Moreover let F_λ be for every $\lambda \in \Theta$ the distribution function of the Poisson distribution Poiss_λ. Note that F_λ admits in view of (5.28) the representation

$$F_\lambda(x) = \sum_{\ell=0}^{\lfloor x \rfloor} \mathfrak{p}_\lambda(\ell) \quad \text{for all } x \in \mathbb{R} \text{ and } \lambda \in \Theta. \tag{5.29}$$

Using this along with Lemma 5.1.9 we immediately obtain the following lemma.

Lemma 5.1.16 $F_\lambda \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ *for every* $\lambda \in \Theta$.

Now, a reasonable estimator for the parameter $\lambda \in \Theta$ based on sample size $m \in \mathbb{N}$ will be the map $\widehat{\lambda}_m : \Omega \to \Theta$ defined by

$$\widehat{\lambda}_m(y_1, y_2, \ldots) = \widehat{\lambda}_m(y_1, \ldots, y_m) := \begin{cases} \frac{1}{m} \sum_{i=1}^m y_i & , \quad y_1, \ldots, y_m \in \mathbb{N}_0 \\ \lambda' & , \quad \text{otherwise} \end{cases} \tag{5.30}$$

for some fixed $\lambda' \in \Theta$. We note that the case differentiation in (5.30) guarantees that the estimator $\widehat{\lambda}_m$ takes values only in the parameter set Θ.

As a consequence of Lemma 5.1.16, the expression $\mathcal{W}_0^{x_{i_0};\pi}(F_{\widehat{\lambda}_m})$ (resp. $\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m})$ can be regarded as a suitable (plug-in) estimator for $\mathcal{W}_0^{x_{i_0};\pi}(F_\lambda)$ (resp. $\mathcal{W}_0^{x_{i_0}}(F_\lambda)$). In the rest of this subsection we will investigate the asymptotics of the latter estimators.

The following Theorem 5.1.17 illustrates Theorem 4.5.1 in the setting of Subsection 5.1.1.

Theorem 5.1.17 (Strong consistency of $(\mathcal{W}_0^{x_{i_0};\pi}(F_{\widehat{\lambda}_m}))$ and $(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}))$) *In the setting of Subsection 5.1.1 let $i_0 \in \{1, \dots, \mathfrak{e}\}$, $\pi \in \Pi$, and $\lambda_0 \in \Theta$. Then the following two assertions hold.*

(i) *The sequence of estimators $(\mathcal{W}_0^{x_{i_0};\pi}(F_{\widehat{\lambda}_m}))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_{i_0};\pi}(F_{\lambda_0})$ under $\mathbb{P}^{\lambda_0}$ in the sense that*

$$\mathcal{W}_0^{x_{i_0};\pi}(F_{\widehat{\lambda}_m}) \to \mathcal{W}_0^{x_{i_0};\pi}(F_{\lambda_0}) \quad \mathbb{P}^{\lambda_0}\text{-a.s.}$$

(ii) *The sequence of estimators $(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_{i_0}}(F_{\lambda_0})$ under $\mathbb{P}^{\lambda_0}$ in the sense that*

$$\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}) \to \mathcal{W}_0^{x_{i_0}}(F_{\lambda_0}) \quad \mathbb{P}^{\lambda_0}\text{-a.s.}$$

If $\pi^{F_{\lambda_0}} \in \Pi$ is in the setting of Subsection 5.1.1 an optimal order strategy w.r.t. F_{λ_0} (such a strategy exists), then it follows from part (ii) of Theorem 5.1.17 that (for any initial inventory level $x_{i_0} = (y_{i_0}, \cdot\,) \in E$) the sequence of plug-in estimators $(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}))_{m\in\mathbb{N}}$ is strongly consistent (under $\mathbb{P}^{\lambda_0}$) for the optimal value $\mathcal{W}_0^{x_{i_0};\pi^{F_{\lambda_0}}}(F_{\lambda_0})$ of the inventory control problem (5.6) (with F_{λ_0} in place of F).

We now devote ourselves to the proof of Theorem 5.1.17.

Proof of Theorem 5.1.17: For the proof of parts (i) and (ii) we intend to apply Theorem 4.5.1. First of all, in the infinite product model $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \{\mathbb{P}^\lambda := \mathrm{Poiss}_\lambda^{\otimes\mathbb{N}} : \lambda \in \Theta\})$ (see (5.27)), the sequence of estimators $(\widehat{\lambda}_m)_{m\in\mathbb{N}}$ given by (5.30) is in view of the ordinary strong law of large numbers strongly consistent for λ_0 under $\mathbb{P}^{\lambda_0}$ (w.r.t. $|\cdot|$). Thus condition (a) of Theorem 4.5.1 is satisfied.

In the following, we will verify that the mapping $\lambda \to F_\lambda$ from Θ to $\boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ defined by (5.28)–(5.29) is continuous at λ_0 w.r.t. $(|\cdot|, \|\cdot\|_{1,\zeta_{\mathbb{N}_0}})$, where $\|\cdot\|_{1,\zeta_{\mathbb{N}_0}}$ is defined as in (4.7) (with $\zeta_{\mathbb{N}_0}$ in place of ν).

Now, let $(\lambda_m)_{m\in\mathbb{N}}$ be any sequence in Θ with $\lambda_m \to \lambda_0$. Set $\underline{\lambda} := \inf_{m\in\mathbb{N}} \lambda_m$ and $\overline{\lambda} := \sup_{m\in\mathbb{N}} \lambda_m$, and note that $0 < \underline{\lambda} < \overline{\lambda} < \infty$. At first, in view of (5.28), we observe for any $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$

$$\begin{aligned}\big|\mathfrak{p}_{\lambda_m}(k) - \mathfrak{p}_{\lambda_0}(k)\big| &= \frac{1}{k!}\Big|\frac{\lambda_m^k}{e^{\lambda_m}} - \frac{\lambda_0^k}{e^{\lambda_0}}\Big| = \frac{1}{k!}\big|\big((\lambda_m^k - \lambda_0^k) + \lambda_0^k\big)\,e^{\lambda_0} - \lambda_0^k e^{\lambda_m}\big| \cdot e^{-(\lambda_m+\lambda_0)} \\ &\le |\lambda_m^k - \lambda_0^k|\,e^{-\lambda_m} + |e^{\lambda_m} - e^{\lambda_0}|\,\lambda_0^k\,e^{-(\lambda_m+\lambda_0)} \le |\lambda_m^k - \lambda_0^k|\,e^{-\underline{\lambda}} + |e^{\lambda_m} - e^{\lambda_0}|\,\lambda_0^k\,e^{-(\underline{\lambda}+\lambda_0)}.\end{aligned}$$

Thus $\mathfrak{p}_{\lambda_m}(k) \to \mathfrak{p}_{\lambda_0}(k)$ for every $k \in \mathbb{N}_0$. In view of (5.28)–(5.29), this implies

$$1 - F_{\lambda_m}(x) \to 1 - F_{\lambda_0}(x) \quad \text{for every } x \in \mathbb{R}_{\ge 0}.$$

Moreover, we obtain by means of (5.28)–(5.29)

$$\begin{aligned}1 - F_{\lambda_m}(x) &= 1 - \sum_{k=0}^{\lfloor x\rfloor} \mathfrak{p}_{\lambda_m}(k) = 1 - \sum_{k=0}^{\lfloor x\rfloor} \frac{\lambda_m^k}{k!}e^{-\lambda_m} = 1 + (-e^{-\lambda_m}) \cdot \sum_{k=0}^{\lfloor x\rfloor} \frac{\lambda_m^k}{k!} \\ &\le 1 + (-e^{-\overline{\lambda}}) \cdot \sum_{k=0}^{\lfloor x\rfloor} \frac{\overline{\lambda}^k}{k!} = 1 - \sum_{k=0}^{\lfloor x\rfloor} \frac{\overline{\lambda}^k}{k!}e^{-\overline{\lambda}} = 1 - F_{\overline{\lambda}}(x)\end{aligned}$$

for any $x \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{N}$. Hence the map $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathfrak{h}(x) := \begin{cases} 1 - F_{\overline{\lambda}}(x) & , \quad x \in \mathbb{R}_{\geq 0} \\ 1 & , \quad \text{otherwise} \end{cases}$$

is in view of

$$\|\mathfrak{h}\|_{1,\zeta_{\mathbb{N}_0}} = \int_{\mathbb{R}_{\geq 0}} \big(1 - F_{\overline{\lambda}}(x)\big)\, \zeta_{\mathbb{N}_0}(dx) = \int_{\mathbb{R}_{\geq 0}} y\, dF_{\overline{\lambda}}(y) = \sum_{\ell \in \mathbb{N}_0} \ell\, \mathfrak{p}_{\overline{\lambda}}(\ell) = \overline{\lambda} < \infty$$

(by Lemma 5.1.9) a $\zeta_{\mathbb{N}_0}$-integrable majorant. Then an application of the dominated convergence theorem entails that

$$\begin{aligned} \lim_{m\to\infty} \|F_{\lambda_m} - F_{\lambda_0}\|_{1,\zeta_{\mathbb{N}_0}} &= \lim_{m\to\infty} \big\|(1 - F_{\lambda_m}) - (1 - F_{\lambda_0})\big\|_{1,\zeta_{\mathbb{N}_0}} \\ &= \lim_{m\to\infty} \int_{\mathbb{R}_{\geq 0}} \big|(1 - F_{\lambda_m}(x)) - (1 - F_{\lambda_0}(x))\big|\, \zeta_{\mathbb{N}_0}(dx) = 0. \end{aligned}$$

Thus the mapping $\lambda \to F_\lambda$ defined by (5.28)–(5.29) is indeed continuous at λ_0 w.r.t. $(|\cdot|, \|\cdot\|_{1,\zeta_{\mathbb{N}_0}})$. In particular, condition (b) of Theorem 4.5.1 holds, too.

Further it follows from the proof of Theorem 5.1.3 that the assumptions of Corollary 4.3.6 and thus of Theorem 4.3.3 (with $\zeta_{\mathbb{N}_0}$ and F_{λ_0} instead of ν and F, respectively) hold in the finite setting of Subsection 5.1.1. Take into account that $F_{\lambda_0} \in \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ by Lemma 5.1.16. As a consequence the assumptions of Theorem 4.5.1 are satisfied (for $\zeta_{\mathbb{N}_0}$ in place of ν), and the claims in (i) and (ii) of Theorem 5.1.17 are immediate consequences of parts (i) and (ii) of the Theorem 4.5.1, respectively. This completes the proof of Theorem 5.1.17. □

Part (ii) of Theorem 5.1.18 provides the asymptotic error distribution of the sequence of plug-in estimators $(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}))_{m\in\mathbb{N}}$ for the optimal value $\mathcal{W}_0^{x_{i_0}}(F_{\lambda_0})$ of the inventory control problem (5.6) (with F_{λ_0} playing the role of F). Recall that in the setting of Subsection 5.1.1 the set $\Pi(F_{\lambda_0})$ consisting for some given $\lambda_0 \in \Theta$ of all optimal order strategies $\pi^{F_{\lambda_0}} \in \Pi$ w.r.t. F_{λ_0} is non-empty and finite.

Theorem 5.1.18 (Asymptotic error distribution of $(\mathcal{W}_0^{x_{i_0};\pi}(F_{\widehat{\lambda}_m}))$ and $(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}))$) *In the setting of Subsection 5.1.1 let $i_0 \in \{1, \ldots, \mathfrak{c}\}$, $\pi \in \Pi$, and $\lambda_0 \in \Theta$. Then the following two assertions hold.*

(i) *We have*

$$\sqrt{m}\big(\mathcal{W}_0^{x_{i_0};\pi}(F_{\widehat{\lambda}_m}) - \mathcal{W}_0^{x_{i_0};\pi}(F_{\lambda_0})\big) \rightsquigarrow Z_{\lambda_0;i_0,\pi} \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|)$$

for $Z_{\lambda_0;i_0,\pi} \sim \mathrm{N}_{0,s^2}$, where

$$s^2 = s^2_{\lambda_0;i_0,\pi} = \lambda_0^2 \cdot \big(\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\mathfrak{p}_{\lambda_0}(\cdot))\big)^2 \tag{5.31}$$

with $\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}$ and $\mathfrak{p}_{\lambda_0}$ given by (5.9) and (5.28), respectively.

(ii) *We have*

$$\sqrt{m}\big(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m}) - \mathcal{W}_0^{x_{i_0}}(F_{\lambda_0})\big) \rightsquigarrow Z_{\lambda_0;i_0} \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|)$$

for $Z_{\lambda_0;i_0} \sim \mathrm{N}_{0,s^2}$*, where*

$$s^2 \;=\; s^2_{\lambda_0;i_0} \;:=\; \lambda_0^2 \cdot \big(\min_{\pi \in \Pi(F_{\lambda_0})} \dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\mathfrak{p}_{\lambda_0}(\cdot)) \big)^2 \tag{5.32}$$

with $\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}$ *and* $\mathfrak{p}_{\lambda_0}$ *given by (5.9) and (5.28), respectively.*

For parts (i) and (ii) of the preceding theorem note that for any fixed $\lambda_0 \in \Theta$ we clearly have $\mathfrak{p}_{\lambda_0}(\cdot) \in \boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ by (4.7) (with ν replaced by $\zeta_{\mathbb{N}_0}$) and the shape of (the counting density) $\mathfrak{p}_{\lambda_0}$ defined in (5.28). Therefore, the expressions on the right-hand side of (5.31) and (5.32) are well-defined.

Proof of Theorem 5.1.18: We intend to apply Theorem 4.5.4. First, the expression $\sqrt{m}(\widehat{\lambda}_m - \lambda_0)$ is clearly $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $m \in \mathbb{N}$. Moreover it is easily seen that the family $\{\mathrm{Poiss}_\lambda : \lambda \in \Theta\}$ fulfils the assumptions of Theorem 6.5.1 in [63]. Therefore, the latter theorem implies that the estimator $\widehat{\lambda}_m$ given by (5.30) in the corresponding infinite product model $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \{\mathbb{P}^\lambda := \mathrm{Poiss}_\lambda^{\otimes \mathbb{N}} : \lambda \in \Theta\})$ (see also (5.27)) satisfies

$$\sqrt{m}(\widehat{\lambda}_m - \lambda_0) \;\rightsquigarrow\; \widetilde{Z}_{\lambda_0} \quad \text{ in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|),$$

where $\widetilde{Z}_{\lambda_0}$ corresponds to a normally distributed random variable on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ with zero mean and variance $\mathcal{I}(\lambda_0)^{-1}$. Note that $\mathcal{I}(\lambda_0)$ refers to the Fisher information matrix at λ_0, and it is easily seen that $\mathcal{I}(\lambda_0)^{-1} = \lambda_0$. Thus condition (a) of Theorem 4.5.4 holds.

In the next step we will verify condition (b) of Theorem 4.5.4. To this end, we consider the map $\mathfrak{F} : \Theta \to \boldsymbol{F}_0(\zeta_{\mathbb{N}_0})$ defined by

$$\mathfrak{F}(\lambda) \;:=\; F_\lambda. \tag{5.33}$$

Take into account that the latter map is well-defined by Lemma 5.1.16. In the sequel, we will prove by means of Lemma 4.5.7 that the map $\mathfrak{F}$ defined by (5.33) is Hadamard differentiable at λ_0 with trace $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ (in the sense of Definition A.1(ii) in Section A) and Hadamard derivative $\dot{\mathfrak{F}}_{\lambda_0} : \mathbb{R} \to \boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ given by

$$\dot{\mathfrak{F}}_{\lambda_0}(\tau)(x) \;:=\; \begin{cases} -\mathfrak{p}_{\lambda_0}(x) \cdot \tau & , \quad x \in \mathbb{N}_0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{5.34}$$

with $\mathfrak{p}_{\lambda_0}$ as in (5.28).

In order to apply Lemma 4.5.7, let the map $\mathfrak{f} : \Theta \times \mathbb{R} \to \mathbb{R}$ from (4.45) be defined by

$$\mathfrak{f}(\lambda, x) \;:=\; F_\lambda(x).$$

For any fixed $x \in \mathbb{R}$, it is easily seen that in view of (5.28)–(5.29) the map $\lambda \mapsto \mathfrak{f}(\lambda, x)$ is continuously differentiable on Θ with gradient

$$\nabla_\lambda \mathfrak{f}(\lambda, x) \;=\; \begin{cases} -\mathfrak{p}_\lambda(x) & , \quad x \in \mathbb{N}_0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{5.35}$$

for all $\lambda \in \Theta$.

Now, let $\lambda_0 \in \Theta$ be fixed, and define for some given $\delta > 0$ a map $\mathfrak{h}_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{h}_\delta(x) := \begin{cases} \frac{(\lambda_0+\delta)^x}{x!} e^{-(\lambda_0-\delta)} & , \quad x \in \mathbb{N}_0 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

In the sequel, we claim that

$$|\nabla_\lambda \mathfrak{f}(\lambda, x)| \le \mathfrak{h}_\delta(x) \quad \text{ for all } (\lambda, x) \in (\lambda_0 - \delta, \lambda_0 + \delta) \times \mathbb{R} \tag{5.36}$$

for some sufficiently small $\delta > 0$. For any $(\lambda, x) \in (\lambda_0 - \delta, \lambda_0 + \delta) \times \mathbb{R}$, we get

$$|\nabla_\lambda \mathfrak{f}(\lambda, x)| = \mathfrak{p}_\lambda(x) = \frac{\lambda^x}{x!} e^{-\lambda} \mathbb{1}_{\{x \in \mathbb{N}_0\}} \le \frac{(\lambda_0 + \delta)^x}{x!} e^{-(\lambda_0 - \delta)} \mathbb{1}_{\{x \in \mathbb{N}_0\}} = \mathfrak{h}_\delta(x).$$

Hence the map $\mathfrak{h}_\delta$ satisfies (5.36). Moreover since

$$\|\mathfrak{h}_\delta\|_{1,\zeta_{\mathbb{N}_0}} = \sum_{\ell \in \mathbb{N}_0} |\mathfrak{h}_\delta(\ell)| = \sum_{\ell \in \mathbb{N}_0} \frac{(\lambda_0 + \delta)^\ell}{\ell!} e^{-(\lambda_0 - \delta)} = e^{\lambda_0 + \delta} e^{-(\lambda_0 - \delta)} = e^{2\delta} < \infty,$$

the map $\mathfrak{h}_\delta$ is $\zeta_{\mathbb{N}_0}$-integrable.

Thus the assumptions of Lemma 4.5.7 are satisfied and an application of this lemma along with (5.35) ensure that the map $\mathfrak{F}$ given by (5.33) is Hadamard differentiable at λ_0 with trace $\boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ and Hadamard derivative $\dot{\mathfrak{F}}_{\lambda_0} : \mathbb{R} \to \boldsymbol{L}_1(\zeta_{\mathbb{N}_0})$ given by (5.34). In particular, condition (b) of Theorem 4.5.4 holds.

Moreover it follows from the proofs of Theorem 5.1.5 and Corollary 4.3.15 along with Lemma 5.1.16 that the assumptions of Theorem 4.3.8 are fulfilled (with $\nu := \zeta_{\mathbb{N}_0}$ and F replaced by F_{λ_0}). Therefore we have verified the assumptions of Theorem 4.5.4.

(i): Part (i) of Theorem 4.5.4 entails that $\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\widetilde{Z}_{\lambda_0}))$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and that

$$\sqrt{m}\big(\mathcal{W}^{x_{i_0};\pi}_0(F_{\widehat{\lambda}_m}) - \mathcal{W}^{x_{i_0};\pi}_0(F_{\lambda_0})\big) \rightsquigarrow \dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\widetilde{Z}_{\lambda_0})) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|),$$

where $\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}$ is defined as in (5.9). Since in view of (5.2), (5.9), and (5.34)

$$\begin{aligned}
&\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\tau)) \\
&= \sum_{k=0}^{N-1} \sum_{i_1=1}^{\mathfrak{e}} \cdots \sum_{i_k=1}^{\mathfrak{e}} \sum_{i_{k+1}=1}^{\mathfrak{e}} V^{F_{\lambda_0};\pi}_{k+1}(x_{i_{k+1}}) \cdot \sum_{\ell \in \eta^{-1}_{(y_{i_k}, f_k(x_{i_k}))}(z_{i_{k+1}})} \big(\dot{\mathfrak{F}}_{\lambda_0}(\tau)(0) \mathbb{1}_{\{\ell = 0\}} \\
&\quad + \Delta^\ell_{\ell-1} \dot{\mathfrak{F}}_{\lambda_0}(\tau)(\cdot) \mathbb{1}_{\{\ell \in \mathbb{N}\}}\big) \cdot \mathbb{1}_{\{y_{i_{k+1}} = y_{i_k} + f_k(x_{i_k}) - z_{i_{k+1}}\}} \cdot p^{F_{\lambda_0}}_{k-1, i_{k-1}; f_{k-1}(x_{i_{k-1}})}(i_k) \cdots p^{F_{\lambda_0}}_{0, i_0; f_0(x_{i_0})}(i_1) \\
&= \sum_{k=0}^{N-1} \sum_{i_1=1}^{\mathfrak{e}} \cdots \sum_{i_k=1}^{\mathfrak{e}} \sum_{i_{k+1}=1}^{\mathfrak{e}} V^{F_{\lambda_0};\pi}_{k+1}(x_{i_{k+1}}) \cdot \sum_{\ell \in \eta^{-1}_{(y_{i_k}, f_k(x_{i_k}))}(z_{i_{k+1}})} \big((-\mathfrak{p}_{\lambda_0}(0) \cdot \tau) \mathbb{1}_{\{\ell = 0\}} \\
&\quad + \Delta^\ell_{\ell-1}(-\mathfrak{p}_{\lambda_0} \cdot \tau) \mathbb{1}_{\{\ell \in \mathbb{N}\}}\big) \cdot \mathbb{1}_{\{y_{i_{k+1}} = y_{i_k} + f_k(x_{i_k}) - z_{i_{k+1}}\}} \cdot p^{F_{\lambda_0}}_{k-1, i_{k-1}; f_{k-1}(x_{i_{k-1}})}(i_k) \\
&\quad \cdots p^{F_{\lambda_0}}_{0, i_0; f_0(x_{i_0})}(i_1)
\end{aligned}$$

$$
\begin{aligned}
&= (-\tau)\cdot\sum_{k=0}^{N-1}\sum_{i_1=1}^{\mathfrak{e}}\cdots\sum_{i_k=1}^{\mathfrak{e}}\sum_{i_{k+1}=1}^{\mathfrak{e}} V_{k+1}^{F_{\lambda_0};\pi}(x_{i_{k+1}})\cdot\sum_{\ell\in\eta^{-1}_{(y_{i_k},f_k(x_{i_k}))}(z_{i_{k+1}})}\big(\mathfrak{p}_{\lambda_0}(0)\mathbb{1}_{\{\ell=0\}}\\
&\quad+\Delta_{\ell-1}^{\ell}\mathfrak{p}_{\lambda_0}\mathbb{1}_{\{\ell\in\mathbb{N}\}}\big)\cdot\mathbb{1}_{\{y_{i_{k+1}}=y_{i_k}+f_k(x_{i_k})-z_{i_{k+1}}\}}\cdot p^{F_{\lambda_0}}_{k-1,i_{k-1};f_{k-1}(x_{i_{k-1}})}(i_k)\cdots p^{F_{\lambda_0}}_{0,i_0;f_0(x_{i_0})}(i_1)\\
&= (-\tau)\cdot\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\mathfrak{p}_{\lambda_0}(\cdot))
\end{aligned}
\tag{5.37}
$$

for any $\tau\in\mathbb{R}$, the real-valued random variable $Z_{\lambda_0;i_0,\pi}:=\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\widetilde{Z}_{\lambda_0}))$ is normally distributed with zero mean and variance as in (5.31).

(ii): In view of part (ii) of Theorem 4.5.4 we have that $\dot{\mathcal{W}}^{x_{i_0}}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(Z_{\lambda_0}))$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and

$$
\sqrt{m}\big(\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m})-\mathcal{W}_0^{x_{i_0}}(F_{\lambda_0})\big)\ \rightsquigarrow\ \dot{\mathcal{W}}^{x_{i_0}}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\widetilde{Z}_{\lambda_0}))\quad\text{in }(\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|),
$$

where $\dot{\mathcal{W}}^{x_{i_0}}_{0;F_{\lambda_0}}$ is defined as in (5.10). By (5.10) and (5.37) we get

$$
\begin{aligned}
\dot{\mathcal{W}}^{x_{i_0}}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\tau)) &= \max_{\pi\in\Pi(F_{\lambda_0})}\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\tau)) = \max_{\pi\in\Pi(F_{\lambda_0})}(-\tau)\cdot\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\mathfrak{p}_{\lambda_0}(\cdot))\\
&= (-\tau)\cdot\min_{\pi\in\Pi(F_{\lambda_0})}\dot{\mathcal{W}}^{x_{i_0};\pi}_{0;F_{\lambda_0}}(\mathfrak{p}_{\lambda_0}(\cdot))
\end{aligned}
$$

for every $\tau\in\mathbb{R}$. Thus the real-valued random variable $Z_{\lambda_0;i_0}:=\dot{\mathcal{W}}^{x_{i_0}}_{0;F_{\lambda_0}}(\dot{\mathfrak{F}}_{\lambda_0}(\widetilde{Z}_{\lambda_0}))$ is normally distributed with zero mean and variance as in (5.32). This completes the proof of Theorem 5.1.18. □

The following remark concludes this subsection. It discusses a conclusion of the statement in part (ii) of Theorem 5.1.18 with regard to the statistical estimation of the optimal value of the inventory control problem.

Remark 5.1.19 Part (ii) of Theorem 5.1.18 can be used to construct an asymptotic confidence interval at a given level $\kappa\in(0,1)$ for the optimal value $\mathcal{W}_0^{x_{i_0}}(F_{\lambda_0})$ of the inventory control problem (5.6) (with F replaced by F_{λ_0}). For this construction, however, we have to estimate the variance $s^2=s^2_{\lambda_0;i_0}$ in (5.32) in the unknown parameter λ_0. Since $\widehat{\lambda}_m$ given by (5.30) provides a suitable estimator for λ_0, the expression $\widehat{s}^2_m$ given by

$$
\widehat{s}^2_m = \mathfrak{s}^2_{\widehat{\lambda}_m;i_0} := \widehat{\lambda}^2_m\cdot\big(\min_{\pi\in\Pi(F_{\widehat{\lambda}_m})}\mathcal{W}^{x_{i_0};\pi}_{0;F_{\widehat{\lambda}_m}}(\mathfrak{p}_{\widehat{\lambda}_m}(\cdot))\big)^2
$$

can be regarded as an estimator for the variance s^2. However, this estimator depends on $\widehat{\lambda}_m$ in a quite complex manner so that the actual performance of the asymptotic confidence interval based on $\widehat{s}^2_m$ is not clear. A parametric bootstrap technique for the asymptotic error distribution of $\mathcal{W}_0^{x_{i_0}}(F_{\widehat{\lambda}_m})$, which we will not discuss in this thesis, could probably lead to an improvement. ◇

5.2 Terminal wealth optimization problem (revisited)

In this section we will again look at the terminal wealth problem which was introduced in Section 3.2. Here we will assume that the transition probabilities of the portfolio process are now governed by some single distribution function F which describes the dynamics of the asset. Since the asset returns are not predictable for the controller we will suppose that the distribution F is *unknown* and must be estimated by means of statistical methods. At first, we will reformulate in Subsection 5.2.1 the (adapted) terminal wealth problem from Section 3.2 in the new setting and embed subsequently the market model into the MDM from Section 4.1. Thereafter we establish in Subsection 5.2.2 regularity properties of the value functional of the (adapted) terminal wealth problem. In Subsections 5.2.3–5.2.4 we will perform a nonparametric as well as a parametric estimation of the optimal value of the (adapted) terminal wealth problem.

5.2.1 Basic financial market model, and the Markov decision model

Let us take up the setting of Section 3.2, that is, we consider an N-period financial market (with $N \in \mathbb{N}$ fixed) consisting of one riskless bond $S^0 = (S_0^0, \dots, S_N^0)$ and one risky asset $S = (S_0, \dots, S_N)$. Here we assume that the value of the bond evolves deterministically according to

$$S_0^0 = 1, \qquad S_{n+1}^0 = \mathfrak{r}_{n+1} S_n^0, \qquad n = 0, \dots, N-1$$

for some fixed constants $\mathfrak{r}_1, \dots, \mathfrak{r}_N \in \mathbb{R}_{\geq 1}$, and that the value of the asset evolves stochastically according to

$$S_0 = s_0, \qquad S_{n+1} = \mathfrak{R}_{n+1} S_n, \qquad n = 0, \dots, N-1$$

for some fixed constant $s_0 \in \mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$-valued i.i.d. random variables $\mathfrak{R}_1, \dots, \mathfrak{R}_N$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common but unknown distribution function F.

In the sequel, we will always assume that the financial market satisfies conditions (b)–(c) of Assumption 3.2.1 as well as the following Assumption 5.2.1. It can be deduced from Examples 3.2.7 and 3.2.8 that the respective financial market models satisfy conditions (b)–(c) of Assumption 3.2.1 and Assumption 5.2.1. Recall from (3.16) the definition of the power utility function u_α, where $\alpha \in (0, 1)$ is fixed.

Assumption 5.2.1 $\int_{\mathbb{R}_{\geq 0}} u_\alpha \, dF < \infty$.

Under condition (b) of Assumption 3.2.1 we may assume without loss of generality that F belongs to the subset $\boldsymbol{F}_{>0}$ of all distribution functions on $\mathbb{R}$ which are supported on $\mathbb{R}_{>0}$. Moreover Assumption 5.2.1 along with part (i) of Lemma 5.2.4 below ensure that F is even an element of $\boldsymbol{F}_{>0}(\mu_{u_\alpha})$. Here $\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ denotes the set of all $F \in \boldsymbol{F}_{>0}$ satisfying

$$\int_{\mathbb{R}_{\geq 0}} (1 - F) \, d\mu_{u_\alpha} < \infty,$$

and μ_{u_α} refers to the (locally finite) Stieltjes measure w.r.t. u_α on $\mathcal{B}(\mathbb{R}_{\geq 0})$ (see Proposition B.1). Take into account that the power utility function u_α is clearly non-decreasing and right-continuous, and that the measure μ_{u_α} can be considered as a (locally finite) measure defined on $\mathcal{B}(\mathbb{R})$ which is supported on $\mathbb{R}_{\geq 0}$.

Now, an agent intends to find for some given initial amount of capital $x_0 \in \mathbb{R}_{\geq 0}$ a self-financing trading strategy w.r.t. x_0 in such a way that the expected utility of the discounted terminal wealth over N periods is maximized. It is discussed in the elaborations of Subsection 3.2.1 that the latter optimization problem can be modelled via an $\mathbb{R}_{\geq 0}$-valued stochastic process $X^\varphi = (X_n^\varphi)_{n=0}^N$ defined as in (3.13), where $\varphi = (\varphi_n)_{n=0}^{N-1}$ regarded as an $\mathbb{R}_{\geq 0}$-valued stochastic process corresponds to a self-financing trading strategy w.r.t. x_0 which is Markovian. The latter property means that we may find for any $n = 0, \dots, N-1$ some Borel measurable map $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\varphi_n = f_n(X_n^\varphi)$.

If the agent's attitude towards risk is described by the power utility u_α, then objective of the agent is to find a self-financing trading strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) w.r.t. x_0 for which the expectation of $u_\alpha(X_N^\varphi / S_N^0)$ under $\mathbb{P}$ is maximized. Note that for given trading strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ (resp. $\pi = (f_n)_{n=0}^{N-1}$) the process X^φ can be seen as an $\mathbb{R}_{\geq 0}$-valued $(\mathcal{F}_n)$-Markov process (with $\mathcal{F}_n$ as in Subsection 3.2.1) whose one-step transition probability at time $n \in \{0, \dots, N-1\}$ given state $x_n \in \mathbb{R}_{\geq 0}$ is given by

$$\mu_F \circ \eta_{n,(x_n, f_n(x_n))}^{-1},$$

where μ_F corresponds to the Stieltjes measure w.r.t. F on $\mathcal{B}(\mathbb{R})$ (see Remark B.2) and $\eta_{n,(x_n,f_n(x_n))}$ is given by (3.15).

Since the above optimization problem has a Markovian structure it can be modelled (similarly to the elaborations in Subsection 3.2.2) via a (finite horizon discrete time) MDM in the sense of Display (4.1) in Section 4.1.

Let $(E, \mathcal{E}) := (\mathbb{R}_{\geq 0}, \mathcal{B}(\mathbb{R}_{\geq 0}))$ and $A_n(x) := [0, x]$ for any $x \in \mathbb{R}_{\geq 0}$ and $n = 0, \dots, N-1$. Hence $A_n = \mathbb{R}_{\geq 0}$ and $D_n = D := \{(x, a) \in \mathbb{R}_{\geq 0}^2 : a \in [0, x]\}$. Set $\mathcal{A}_n := \mathcal{B}(\mathbb{R}_{\geq 0})$ and note that $\mathcal{D}_n = \mathcal{B}(\mathbb{R}_{\geq 0}^2) \cap D$. Further let $\Pi := \mathbb{F}_0 \times \cdots \times \mathbb{F}_{N-1}$, where the set $\mathbb{F}_n$ of all admissible decision rules is equal to the set $\overline{\mathbb{F}}_n$ of all decision rules at time n consisting of all Borel measurable maps $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $f_n(x) \in [0, x]$ for all $x \in \mathbb{R}_{\geq 0}$ (in particular $\overline{\mathbb{F}}_n$ is independent of n).

Moreover let the components of the vector $\boldsymbol{r} = (r_n)_{n=0}^N$ be given by (3.19), and let the gauge function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ be as in (3.17). Let $\mathcal{P}_\psi$ be the set of all transition functions $\boldsymbol{P} = (P_n)_{n=0}^{N-1} \in \overline{\mathcal{P}}$ consisting of transition kernels of the shape

$$P_n\big((x, a), \bullet\big) := \mu_F \circ \eta_{n,(x,a)}^{-1}[\bullet], \qquad (x, a) \in D_n,\ n = 0, \dots, N-1 \tag{5.38}$$

for some $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, where the map $\eta_{n,(x,a)}$ is defined as in (3.15). Note that $\mu_F \in \mathcal{M}_1^\alpha(\mathbb{R}, \mathbb{R}_{\geq 0})$ for every $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ by Assumption 5.2.1 and part (i) of Lemma 5.2.4 below, where the set $\mathcal{M}_1^\alpha(\mathbb{R}, \mathbb{R}_{\geq 0})$ is defined as in Subsection 3.2.2. Also note that it is easily seen that $\mathcal{P}_\psi \subseteq \overline{\mathcal{P}}_\psi$ with $\overline{\mathcal{P}}_\psi$ defined as in Subsection 2.1.2.

Since any $\boldsymbol{P} \in \mathcal{P}_\psi$ is generated through (5.38) by some $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, we write $\boldsymbol{P}_F = (P_n^F)_{n=0}^{N-1}$ for the transition function whose transition kernels are defined by the right-hand side of (5.38). Thus $\boldsymbol{P}_F \in \mathcal{P}_\psi$ for any $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, and it follows from Lemma 3.2.10(i) that ψ given by (3.17) provides a bounding function for the MDM $(E, \boldsymbol{A}, \boldsymbol{P}_F, \Pi, \boldsymbol{X}, \boldsymbol{r})$ for every $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$. Take into account that $\boldsymbol{X}$ plays the role of the stochastic process X^φ and that for some fixed $x_0 \in \mathbb{R}_{\geq 0}$ any self-financing Markovian trading strategy $\varphi = (\varphi_n)_{n=0}^{N-1}$ w.r.t. x_0 may be identified with some $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ via $\varphi_n = f_n(X_n^\varphi)$. In particular, the conditions in Assumption 4.2.1 hold (with ν replaced by μ_{u_α}).

Therefore, for every fixed $x_0 \in \mathbb{R}_{\geq 0}$ and $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, the terminal wealth problem above reads as

$$\mathbb{E}^{x_0, \boldsymbol{P}_F; \pi}[r_N(X_N)] \longrightarrow \max \ \text{(in } \pi \in \Pi)! \tag{5.39}$$

A strategy $\pi^F \in \Pi$ is called an *optimal (self-financing) trading strategy w.r.t. F (and x_0)* if it solves the maximization problem (5.39). It follows from part (i) of Theorem 3.2.5 (applied to $\boldsymbol{P}_F$) that the strategy $\pi^F := (f_n^F)_{n=0}^{N-1} \in \Pi$ defined by $f_n^F(x) := \gamma_n^F x$, $x \in \mathbb{R}_{\geq 0}$, forms an optimal trading strategy w.r.t. F, where γ_n^F is in view of Lemma 3.2.4 (with $\boldsymbol{P}_F$ in place of $\boldsymbol{P}$) the unique solution of the optimization problem

$$v_n^{F;\gamma} := \int_{\mathbb{R}_{\geq 0}} (u_\alpha \circ \eta_n^\gamma)(y)\, \mu_F(dy) \longrightarrow \max \ \ (\text{in } \gamma \in [0,1])! \tag{5.40}$$

with η_n^γ defined as in (3.22); see also (3.21) (with $\boldsymbol{P}$ and $\mathfrak{m}_{n+1}^{\boldsymbol{P}}$ replaced by $\boldsymbol{P}_F$ and μ_F, respectively). Note that it can be deduced from the second part of Theorem 3.2.5(ii) (applied to $\boldsymbol{P}_F$) that the optimal trading strategy π^F belongs to Π_{lin} and is unique among all $\pi \in \Pi_{\text{lin}}(F)$. Recall that Π_{lin} refers to the set of all linear trading strategies $\pi = (f_n)_{n=0}^{N-1} \in \Pi$ defined by (3.26).

5.2.2 Regularity of the value function

Maintain the notation and terminology introduced in Subsection 5.2.1. In this subsection we will investigate the value function of the terminal wealth problem (5.39) regarded as a real-valued functional defined on a set of distribution functions for 'Lipschitz continuity' and quasi-Hadamard differentiability in the sense of Definitions 4.3.1 and 4.3.7. Recall that $\alpha \in (0,1)$ introduced in (3.16) as well as $\mathfrak{r}_1, \ldots, \mathfrak{r}_N \in \mathbb{R}_{\geq 1}$ are fixed.

We point out that the regularity results of the value function are *not* relevant (except the shape of the quasi-Hadamard derivatives in parts (i) and (ii) of Theorem 5.2.6 ahead) for the investigation of the asymptotics of certain estimators for the optimal value of the terminal wealth problem (5.39), which is discussed in Subsections 5.2.3–5.2.4. The purpose of the following elaborations is merely to illustrate the results presented in Section 4.3 in the context of the setting of Subsection 5.2.1. For a reader who is only interested in the statistical estimation of the optimal value of the terminal wealth problem, we recommend skipping this subsection and going to Subsections 5.2.3–5.2.4.

For the formulation of Theorems 5.2.2 and 5.2.6 below recall from (4.6) the definitions of the functionals $\mathcal{W}_0^{x_0;\pi} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ and $\mathcal{W}_0^{x_0} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$, where the maps $V_0^{F;\pi}$ and V_0^F are given by (4.3) and (4.5), respectively. In the setting of Subsection 5.2.1 these functionals admit the representations

$$\mathcal{W}_0^{x_0;\pi}(F) = V_0^{F;\pi}(x_0) = \mathbb{E}^{x_0,\boldsymbol{P}_F;\pi}[r_N(X_N)] \quad \text{and} \quad \mathcal{W}_0^{x_0}(F) = \sup_{\pi\in\Pi} \mathcal{W}_0^{x_0;\pi}(F) \tag{5.41}$$

for any $x_0 \in \mathbb{R}_{\geq 0}$, $\pi \in \Pi$, and $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$.

The following Theorem 5.2.2 illustrates Theorem 4.3.3 in the setting of Subsection 5.2.1. Part (ii) of this theorem will be used in Subsections 5.2.3–5.2.4 to show strong consistency of plug-in estimators for the optimal value of the terminal wealth problem (5.39). Note that any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol{\gamma}} := (f_n^{\boldsymbol{\gamma}})_{n=0}^{N-1} \in \Pi_{\text{lin}}$ through (3.26). Further let the norm $\|\cdot\|_{1,\mu_{u_\alpha}}$ be defined as in (4.7) (with μ_{u_α} instead of ν). Note that $\|\cdot\|_{1,\mu_{u_\alpha}}$ can be represented as

$$\|h\|_{1,\mu_{u_\alpha}} = \int_{\mathbb{R}_{\geq 0}} |h(y)|\, \mu_{u_\alpha}(dy) \quad \text{ for all } h \in \boldsymbol{L}_1(\mu_{u_\alpha}),$$

where $\boldsymbol{L}_1(\mu_{u_\alpha})$ is defined as in Section 4.3. Take into account that μ_{u_α} can be considered as a measure on $\mathcal{B}(\mathbb{R})$ that is supported on $\mathbb{R}_{\geq 0}$. Finally recall Definition 4.3.1.

Theorem 5.2.2 ('Lipschitz continuity' of $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ and $\mathcal{W}_0^{x_0}$ in F) *In the setting above let $x_0 \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, and $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$. Then the following two assertions hold.*

(i) *The map $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ defined by (5.41) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\mu_{u_\alpha}}, |\cdot|)$.*

(ii) *The map $\mathcal{W}_0^{x_0} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ defined by (5.41) is 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\mu_{u_\alpha}}, |\cdot|)$.*

The proof of Theorem 5.2.2 relies on the following three lemmas. Let the map $\eta_n^{\gamma_n}$ be defined as on the right-hand side of (3.22), and set $\eta_n^{\boldsymbol{\gamma}} := \eta_n^{\gamma_n}$ for any $n = 0, \ldots, N-1$. Here and elsewhere we denote by $(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'$ the first derivative of the (continuously differentiable) map $(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})(\cdot)$.

Lemma 5.2.3 *Let $h \in \boldsymbol{L}_1(\mu_{u_\alpha})$, $\boldsymbol{\gamma} \in [0,1]^N$, and $n = 0, \ldots, N-1$. Then*

$$\int_{\mathbb{R}_{\geq 0}} \big|h(y)(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\big|\,\ell(dy) \;\leq\; \bar{\mathfrak{r}}^{1-\alpha}\,\|h\|_{1,\mu_{u_\alpha}},$$

where $\bar{\mathfrak{r}} := \max_{k=0,\ldots,N-1} \mathfrak{r}_{k+1} \in \mathbb{R}_{\geq 1}$.

Proof In view of (3.16), (3.22), and (4.7) (with ν replaced by μ_{u_α}), we get

$$\begin{aligned}
&\int_{\mathbb{R}_{\geq 0}} \big|h(y)(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\big|\,\ell(dy) \;=\; \int_{\mathbb{R}_{\geq 0}} |h(y)|\,\alpha\,(\gamma_n/\mathfrak{r}_{n+1}) \cdot u_{\alpha-1}(\eta_n^{\boldsymbol{\gamma}}(y))\,\ell(dy)\\
&\leq\; \int_{\mathbb{R}_{\geq 0}} |h(y)|\,\alpha\,\gamma_n \cdot \big((1-\gamma_n) + \gamma_n(y/\mathfrak{r}_{n+1})\big)^{\alpha-1}\,\ell(dy) \;\leq\; \int_{\mathbb{R}_{\geq 0}} |h(y)|\,\alpha\,\gamma_n^\alpha\,\mathfrak{r}_{n+1}^{1-\alpha}\,y^{\alpha-1}\,\ell(dy)\\
&\leq\; \bar{\mathfrak{r}}^{1-\alpha} \int_{\mathbb{R}_{\geq 0}} |h(y)|\,u_\alpha'(y)\,\ell(dy) \;=\; \bar{\mathfrak{r}}^{1-\alpha} \int_{\mathbb{R}_{\geq 0}} |h(y)|\,\mu_{u_\alpha}(dy) \;=\; \bar{\mathfrak{r}}^{1-\alpha}\,\|h\|_{1,\mu_{u_\alpha}}.
\end{aligned} \tag{5.42}$$

Take into account that for the second last "=" in (5.42) we have used Lemma B.3 (which may be applied because $h \in \boldsymbol{L}_1(\mu_{u_\alpha})$) and the fact that μ_{u_α} can be taken as a measure on $\mathcal{B}(\mathbb{R})$ which is supported on $\mathbb{R}_{\geq 0}$. □

Recall for the next lemma the definition of the gauge function ψ given by (3.17).

Lemma 5.2.4 *The following two assertions hold.*

(i) $\int_{\mathbb{R}_{\geq 0}} u_\alpha\,dF = \int_{\mathbb{R}_{\geq 0}} (1-F)\,d\mu_{u_\alpha}$ *for every $F \in \boldsymbol{F}_{>0}$.*

(ii) *For every $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ we have* $\lim_{x\to\infty}(1-F(x))\psi(x) = 0$.

Proof For the claim in (i), we observe by means of Fubini's theorem and Lemma B.3

$$\begin{aligned}
\int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\,dF(y) \;&=\; \int_{[0,\infty)}\int_{[0,u_\alpha(y))} \ell(dx)\,\mu_F(dy) \;=\; \int_{[0,\infty)}\int_{[0,y)} u_\alpha'(z)\,\ell(dz)\,\mu_F(dy)\\
&=\; \int_{[0,\infty)}\int_{[0,y)} \mu_{u_\alpha}(dz)\,\mu_F(dy) \;=\; \int_{[0,\infty)}\int_{[z,\infty)} \mu_F(dy)\,\mu_{u_\alpha}(dz) \;=\; \int_{\mathbb{R}_{\geq 0}} (1-F(z))\,\mu_{u_\alpha}(dz)
\end{aligned}$$

for any $F \in \boldsymbol{F}_{>0}$. This shows (i).

To prove part (ii), let $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ be arbitrary but fixed. Note at first that for any $x \in \mathbb{R}_{>0}$

$$\begin{aligned}\int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, dF(y) \;&=\; \int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, \mu_F(dy) \;=\; \int_{[0,x)} u_\alpha(y)\, \mu_F(dy) + \int_{[x,\infty)} u_\alpha(y)\, \mu_F(dy) \\ &\geq\; \int_{[0,x)} u_\alpha(y)\, \mu_F(dy) + u_\alpha(x) \cdot \int_{[x,\infty)} \mu_F(dy) \;=\; \int_{[0,x)} u_\alpha(y)\, \mu_F(dy) + u_\alpha(x)\,(1 - F(x)).\end{aligned} \tag{5.43}$$

Recall that μ_F refers to the Stieltjes measure w.r.t. F on $\mathcal{B}(\mathbb{R})$ (see Remark B.2). By rearranging the expressions in (5.43), we obtain

$$0 \;\leq\; (1 - F(x))\, u_\alpha(x) \;\leq\; \int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, \mu_F(dy) - \int_{[0,x)} u_\alpha(y)\, \mu_F(dy) \tag{5.44}$$

for any $x \in \mathbb{R}_{>0}$. Since $\int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, \mu_F(dy)$ $(= \int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, dF(y)) \;<\; \infty$ (by assumption), it follows from the continuity from below of the finite measure $\int_{\cdot} u_\alpha(y)\, \mu_F(dy)$ on $\mathbb{R}_{\geq 0}$ that the right-hand side in (5.44) converges to 0 as $x \to \infty$. Thus $\lim_{x\to\infty}(1 - F(x))u_\alpha(x) = 0$. In view of the shape of the gauge function ψ, this implies the assertion in (ii). □

Lemma 5.2.5 *Let $F, G \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, $\boldsymbol{\gamma} \in [0,1]^N$, and $n = 0, \ldots, N-1$. Then we have*

$$\int_{\mathbb{R}_{\geq 0}} u_\alpha \circ \eta_n^{\boldsymbol{\gamma}}\, d(F - G) \;=\; -\int_{\mathbb{R}_{\geq 0}} (F - G)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'\, d\ell.$$

Proof Since in view of Lemma B.3

$$\int_{\mathbb{R}_{\geq 0}} (F - G)\; d(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}}) \;=\; \int_{\mathbb{R}_{\geq 0}} (F - G)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'\, d\ell,$$

it suffices for the claim to show that

$$\int_{\mathbb{R}_{\geq 0}} u_\alpha \circ \eta_n^{\boldsymbol{\gamma}}\, d(F - G)|_{\mathbb{R}_{\geq 0}} \;=\; -\int_{\mathbb{R}_{\geq 0}} (F - G)|_{\mathbb{R}_{\geq 0}}\; d(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}}), \tag{5.45}$$

where $(F - G)|_{\mathbb{R}_{\geq 0}}$ denotes the restriction of $F - G$ to $\mathbb{R}_{\geq 0}$. Note that $F - G$ coincides on $\mathbb{R}_{\geq 0}$ with $(F - G)|_{\mathbb{R}_{\geq 0}}$ (by definition), and that the latter lemma may be applied because in view of Lemma 5.2.3 we have

$$\int_{\mathbb{R}_{\geq 0}} \big|(F - G)(y)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\big|\, \ell(dy) \;\leq\; \bar{\mathfrak{r}}^{1-\alpha}\, \|F - G\|_{1,\mu_{u_\alpha}} \;<\; \infty, \tag{5.46}$$

where $\bar{\mathfrak{r}} := \max_{k=0,\ldots,N-1} \mathfrak{r}_{k+1} \in \mathbb{R}_{\geq 1}$. Take into account that $F - G \in \boldsymbol{L}_1(\mu_{u_\alpha})$.

Now, we will verify (5.45). For this reason, we intend to apply the integration-by-parts formula in the form of Lemma B.5 in Section B.1. At first, it is easily seen that $(F - G)|_{\mathbb{R}_{\geq 0}}(\cdot) \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\geq 0})$ as well as $u_\alpha \circ \eta_n^{\boldsymbol{\gamma}}(\cdot) \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\geq 0})$. Recall that $\mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\geq 0})$ stands for the (linear) space of all right-continuous maps $v \in \mathbb{R}^{\mathbb{R}_{\geq 0}}$ that are of locally bounded variation.

Since $[(F-G)|_{\mathbb{R}_{\geq 0}}] = F+G$ on $\mathbb{R}_{\geq 0}$ (with $[\,\cdot\,]$ defined as in Section B.1), we observe in view of part (i) of Lemma 5.2.4

$$\begin{aligned}
&\int_{\mathbb{R}_{\geq 0}} \big|(u_\alpha \circ \eta_n^\gamma)(y)\big|\, d[(F-G)|_{\mathbb{R}_{\geq 0}}](y) \;=\; \int_{\mathbb{R}_{\geq 0}} (u_\alpha \circ \eta_n^\gamma)(y)\, d(F+G)(y)\\
&\quad= \int_{\mathbb{R}_{\geq 0}} u_\alpha\big((1-\gamma_n)+\gamma_n(y/\mathfrak{r}_{k+1})\big)\, dF(y) + \int_{\mathbb{R}_{\geq 0}} u_\alpha\big((1-\gamma_n)+\gamma_n(y/\mathfrak{r}_{k+1})\big)\, dG(y)\\
&\quad\leq \int_{\mathbb{R}_{\geq 0}} u_\alpha(1+y)\, dF(y) + \int_{\mathbb{R}_{\geq 0}} u_\alpha(1+y)\, dG(y) \;\leq\; 2 + \int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, dF(y) + \int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, dG(y)\\
&\quad=\; 2 + \int_{\mathbb{R}_{\geq 0}} (1-F(y))\, \mu_{u_\alpha}(dy) + \int_{\mathbb{R}_{\geq 0}} (1-G(y))\, \mu_{u_\alpha}(dy) \;<\; \infty.
\end{aligned}$$

Moreover, in view of (5.46), we get

$$\begin{aligned}
&\int_{\mathbb{R}_{\geq 0}} \big|(F-G)|_{\mathbb{R}_{\geq 0}}(y)\big|\, d[u_\alpha \circ \eta_n^\gamma](y) \;=\; \int_{\mathbb{R}_{\geq 0}} \big|(F-G)(y)\big|\, du_\alpha \circ \eta_n^\gamma(y)\\
&\quad=\; \int_{\mathbb{R}_{\geq 0}} \big|(F-G)(y)\big|\, (u_\alpha \circ \eta_n^\gamma)'(y)\, \ell(dy) \;\leq\; \bar{\mathfrak{r}}^{1-\alpha}\, \|F-G\|_{1,\mu_{u_\alpha}} \;<\; \infty,
\end{aligned}$$

where we used Lemma B.3 (which may be applied in view of (5.46)) for the last "=". Since F and G are right-continuous as well as supported on $\mathbb{R}_{>0}$, we have

$$\lim_{x \searrow 0} (F-G)|_{\mathbb{R}_{\geq 0}}(x) \cdot (u_\alpha \circ \eta_n^\gamma)(x) \;=\; \big(F(0)-G(0)\big) \cdot (u_\alpha \circ \eta_n^\gamma)(0) \;=\; 0.$$

Moreover, in view of

$$\big|(F-G)|_{\mathbb{R}_{\geq 0}}(x) \cdot (u_\alpha \circ \eta_n^\gamma)(x)\big| \;\leq\; \big|(F-G)(x)\big| \cdot u_\alpha(1+x) \;\leq\; (1-F(x))\psi(x) + (1-G(x))\psi(x)$$

for any $x \in \mathbb{R}_{\geq 0}$, we obtain by means of part (ii) of Lemma 5.2.4 (applied to F and G) that the latter bound converges to 0 as $x \to \infty$. Thus an application of the integration-by-parts formula in Lemma B.5 yields (5.45). □

We are now in the position to prove Theorem 5.2.2. Let $v_n^{F;\gamma_n}$ be defined as on the left-hand side of (5.40), and set $v_n^{F;\gamma} := v_n^{F;\gamma_n}$ for any $n = 0, \ldots, N-1$.

Proof of Theorem 5.2.2: We intend to apply Theorem 4.3.3. At first, it is discussed in Subsection 5.2.1 that conditions (a)–(b) of Assumption 4.2.1 hold (with μ_{u_α} in place of ν). Further note that the value functional $\mathcal{W}_0^{x_0}$ given by (5.41) can be represented in view of the first assertion in Theorem 3.2.5(i) (applied to $\boldsymbol{P}_F$) along with (4.6) by

$$\mathcal{W}_0^{x_0}(F) \;=\; \sup_{\pi \in \Pi_{\mathrm{lin}}} \mathcal{W}_0^{x_0;\pi}(F). \tag{5.47}$$

Recall that Π_{lin} stands for the set of all linear trading strategies. Hence, in view of Remark 4.3.5, we only have to ensure that the assumptions of Theorem 4.3.3 hold for Π_{lin} instead of Π.

In the remainder of the proof we will verify conditions (a) and (b) of Theorem 4.3.3 (with Π_{lin} replaced by Π). That is, we will show that for any $n = 0, \ldots, N-1$ the maps $\boldsymbol{\Lambda}_n^F : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to$

$\ell^\infty_\psi(\Pi_{\rm lin}\times E)$ and $\boldsymbol{\Phi}_n : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \ell^\infty_\psi(\Pi_{\rm lin}\times E)$ defined by (4.10) are 'Lipschitz continuous' at F w.r.t. $(\|\cdot\|_{1,\mu_{u_\alpha}}, \|\cdot\|_{\infty,\psi})$ (in the sense of Definition 4.3.1). Here the norm $\|\cdot\|_{\infty,\psi}$ is introduced in (4.11).

Now, let $(F_m)_{m\in\mathbb{N}}$ be any sequence in $\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ with $\|F_m - F\|_{1,\mu_{u_\alpha}} \to 0$. Since any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol{\gamma}} := (f_n^{\boldsymbol{\gamma}})_{n=0}^{N-1} \in \Pi_{\rm lin}$ through (3.26) it suffices for the 'Lipschitz continuity' of the maps $\boldsymbol{\Lambda}_n^F$ and $\boldsymbol{\Phi}_n$ in view of Definition 4.3.1 and (4.11) to show that for any $n = 0, \ldots, N-1$

$$\sup_{\boldsymbol{\gamma}\in[0,1]^N} \big\|\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F_m) - \Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F)\big\|_\psi \;=\; \mathcal{O}\big(\|F_m - F\|_{1,\mu_{u_\alpha}}\big) \tag{5.48}$$

as well as

$$\sup_{\boldsymbol{\gamma}\in[0,1]^N} \big\|\Phi_n^{(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F_m) - \Phi_n^{(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F)\big\|_\psi \;=\; \mathcal{O}\big(\|F_m - F\|_{1,\mu_{u_\alpha}}\big), \tag{5.49}$$

where the maps $\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ and $\Phi_n^{(\pi_{\boldsymbol{\gamma}},x)} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ are defined as in (4.8).

First of all, note that the maps $\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}$ and $\Phi_n^{(\pi_{\boldsymbol{\gamma}},x)}$ given by (4.8) admit for any $x \in \mathbb{R}_{\ge 0}$, $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$, $G \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, and $n = 0, \ldots, N-1$ in view of Lemma 3.2.6 (applied to $\boldsymbol{P}_F$), equations (5.38), (3.26), and (3.22) as well as the shape of the bounding function $\psi := 1 + u_\alpha$ (see (3.17)) the representations

$$\begin{aligned}
\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(G) &= \int_{\mathbb{R}_{\ge 0}} V_{n+1}^{F;\pi_{\boldsymbol{\gamma}}}(y)\, P_n^G\big((x, f_n^{\boldsymbol{\gamma}}(x)), dy\big) \\
&= \int_{\mathbb{R}_{\ge 0}} \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(y/S_{n+1}^0)\, \mu_G \circ \eta^{-1}_{n,(x,f_n^{\boldsymbol{\gamma}}(x))}(dy) \\
&= \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x + f_n^{\boldsymbol{\gamma}}(x)(y - \mathfrak{r}_{n+1})}{\mathfrak{r}_{n+1}S_n^0}\Big)\, \mu_G(dy) \\
&= \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot \int_{\mathbb{R}_{\ge 0}} u_\alpha\Big(\frac{\mathfrak{r}_{n+1}x + \gamma_n\, x\cdot(y - \mathfrak{r}_{n+1})}{\mathfrak{r}_{n+1}S_n^0}\Big)\, \mu_G(dy) \\
&= \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot \int_{\mathbb{R}_{\ge 0}} u_\alpha(x/S_n^0)\cdot u_\alpha\Big(1 + \gamma_n\Big(\frac{y}{\mathfrak{r}_{n+1}} - 1\Big)\Big)\, \mu_G(dy) \\
&= \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(x/S_n^0) \cdot \int_{\mathbb{R}_{\ge 0}} (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})(y)\, dG(y)
\end{aligned} \tag{5.50}$$

and

$$\begin{aligned}
\Phi_n^{(\pi_{\boldsymbol{\gamma}},x)}(G) &= \int_{\mathbb{R}_{\ge 0}} \psi(y)\, P_n^G\big((x, f_n^{\boldsymbol{\gamma}}(x)), dy\big) \\
&= 1 + \int_{\mathbb{R}_{\ge 0}} u_\alpha(y)\, \mu_G \circ \eta^{-1}_{n,(x,f_n^{\boldsymbol{\gamma}}(x))}(dy) \\
&= 1 + \int_{\mathbb{R}_{\ge 0}} u_\alpha\big(\mathfrak{r}_{n+1}x + f_n^{\boldsymbol{\gamma}}(x)(y - \mathfrak{r}_{n+1})\big)\, \mu_G(dy) \\
&= 1 + \int_{\mathbb{R}_{\ge 0}} u_\alpha\big(\mathfrak{r}_{n+1}x + \gamma_n\, x\cdot(y - \mathfrak{r}_{n+1})\big)\, \mu_G(dy)
\end{aligned}$$

$$
\begin{aligned}
&= 1+\int_{\mathbb{R}_{\geq 0}} u_\alpha(\mathfrak{r}_{n+1})\,u_\alpha(x)\cdot u_\alpha\Big(1+\gamma_n\Big(\frac{y}{\mathfrak{r}_{n+1}}-1\Big)\Big)\,\mu_G(dy)\\
&= 1+u_\alpha(\mathfrak{r}_{n+1})\,u_\alpha(x)\cdot\int_{\mathbb{R}_{\geq 0}}(u_\alpha\circ\eta_n^{\gamma})(y)\,dG(y),
\end{aligned}
\tag{5.51}
$$

respectively.

In virtue of (5.50), Lemmas 5.2.3 and 5.2.5, and Displays (3.23)–(3.24) (with $\boldsymbol{P}_F$ in place of $\boldsymbol{P}$), we observe for every $x\in\mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$, $n=0,\dots,N-1$, and $m\in\mathbb{N}$

$$
\begin{aligned}
&\frac{1}{\psi(x)}\cdot\big|\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F_m)-\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F)\big|\\
&= \frac{1}{\psi(x)}\cdot\Big|\prod_{j=n+1}^{N-1}v_j^{F;\boldsymbol{\gamma}}\cdot u_\alpha(x/S_n^0)\cdot\int_{\mathbb{R}_{\geq 0}}(u_\alpha\circ\eta_n^{\gamma})(y)\,d(F_m-F)(y)\Big|\\
&= \frac{1}{\psi(x)}\cdot\Big|\prod_{j=n+1}^{N-1}v_j^{F;\boldsymbol{\gamma}}\cdot u_\alpha(x/S_n^0)\cdot\int_{\mathbb{R}_{\geq 0}}(F_m-F)(y)\,(u_\alpha\circ\eta_n^{\gamma})'(y)\,\ell(dy)\Big|\\
&= \frac{u_\alpha(x)}{\psi(x)}\cdot\prod_{j=n+1}^{N-1}v_j^{F;\boldsymbol{\gamma}}\cdot u_\alpha(1/S_n^0)\cdot\Big|\int_{\mathbb{R}_{\geq 0}}(F_m-F)(y)\,(u_\alpha\circ\eta_n^{\gamma})'(y)\,\ell(dy)\Big|\\
&\leq (1+\overline{\mathfrak{m}}_F)^{N-n-1}\cdot\int_{\mathbb{R}_{\geq 0}}\big|(F_m-F)(y)\,(u_\alpha\circ\eta_n^{\gamma})'(y)\big|\,\ell(dy)\\
&\leq (1+\overline{\mathfrak{m}}_F)^{N-n-1}\cdot\overline{\mathfrak{r}}^{1-\alpha}\,\|F_m-F\|_{1,\mu_{u_\alpha}},
\end{aligned}
$$

where $\overline{\mathfrak{m}}_F:=\int_{\mathbb{R}_{\geq 0}}u_\alpha\,dF\in\mathbb{R}_{>0}$ (by condition (b) of Assumption 3.2.1 and Assumption 5.2.1) and $\overline{\mathfrak{r}}:=\max_{k=0,\dots,N-1}\mathfrak{r}_{k+1}\in\mathbb{R}_{\geq 1}$. Take into account that $F_m-F\in\boldsymbol{L}_1(\mu_{u_\alpha})$ for any $m\in\mathbb{N}$. Hence

$$
\begin{aligned}
\sup_{\boldsymbol{\gamma}\in[0,1]^N}\big\|\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F_m)-\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F)\big\|_\psi &= \sup_{\boldsymbol{\gamma}\in[0,1]^N}\sup_{x\in\mathbb{R}_{\geq 0}}\frac{1}{\psi(x)}\cdot\big|\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F_m)-\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F)\big|\\
&\leq C_{\boldsymbol{\Lambda}}\|F_m-F\|_{1,\mu_{u_\alpha}}
\end{aligned}
$$

for every $n=0,\dots,N-1$ and $m\in\mathbb{N}$ (by (1.18)), where $C_{\boldsymbol{\Lambda}}:=(1+\overline{\mathfrak{m}}_F)^{N-n-1}\cdot\overline{\mathfrak{r}}^{1-\alpha}\in\mathbb{R}_{\geq 1}$ (is independent of x and $\boldsymbol{\gamma}$). In particular, this shows (5.48) and thus condition (a) of Theorem 4.3.3 (with Π_{lin} playing the role of Π).

To prove (5.49), note that it follows from (5.51) as well as Lemmas 5.2.3 and 5.2.5

$$
\begin{aligned}
&\frac{1}{\psi(x)}\cdot\big|\Phi_n^{(\pi_{\boldsymbol{\gamma}},x)}(F_m)-\Phi_n^{(\pi_{\boldsymbol{\gamma}},x)}(F)\big|\\
&= \frac{1}{\psi(x)}\cdot\Big|u_\alpha(\mathfrak{r}_{n+1})\,u_\alpha(x)\cdot\int_{\mathbb{R}_{\geq 0}}(u_\alpha\circ\eta_n^{\gamma})(y)\,d(F_m-F)(y)\Big|\\
&= \frac{u_\alpha(x)}{\psi(x)}\,u_\alpha(\mathfrak{r}_{n+1})\cdot\Big|\int_{\mathbb{R}_{\geq 0}}(F_m-F)(y)\,(u_\alpha\circ\eta_n^{\gamma})'(y)\,\ell(dy)\Big|\\
&\leq \overline{\mathfrak{r}}^{\alpha}\cdot\int_{\mathbb{R}_{\geq 0}}\big|(F_m-F)(y)\,(u_\alpha\circ\eta_n^{\gamma})'(y)\big|\,\ell(dy)\\
&\leq \overline{\mathfrak{r}}^{\alpha}\cdot\overline{\mathfrak{r}}^{1-\alpha}\,\|F_m-F\|_{1,\mu_{u_\alpha}} = \overline{\mathfrak{r}}\,\|F_m-F\|_{1,\mu_{u_\alpha}}
\end{aligned}
$$

for any $x \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, $n = 0, \ldots, N-1$, and $m \in \mathbb{N}$. Then

$$\begin{aligned}\sup_{\boldsymbol{\gamma}\in[0,1]^N} \big\|\Phi_n^{(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F_m) - \Phi_n^{(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F)\big\|_\psi &= \sup_{\boldsymbol{\gamma}\in[0,1]^N}\ \sup_{x\in\mathbb{R}_{\geq 0}} \frac{1}{\psi(x)} \cdot \big|\Phi_n^{(\pi_{\boldsymbol{\gamma}},x)}(F) - \Phi_n^{(\pi_{\boldsymbol{\gamma}},x)}(G)\big| \\ &\leq C_{\boldsymbol{\Phi}} \|F_m - F\|_{1,\mu_{u_\alpha}}\end{aligned}$$

for $n = 0, \ldots, N-1$ and $m \in \mathbb{N}$ (by (1.18)), where $C_{\boldsymbol{\Phi}} := \bar{\mathfrak{r}}$ ($\in \mathbb{R}_{\geq 1}$) (is independent of x and $\boldsymbol{\gamma}$). Thus (5.49) holds and we have shown that condition (b) of Theorem 4.3.3 (with Π_{lin} in place of Π) holds, too.

In particular, the assumptions of Theorem 4.3.3 are satisfied (with Π and ν replaced by Π_{lin} and μ_{u_α}, respectively), and an application of parts (i) and (ii) of the latter theorem yields the assertions in (i) and (ii) of Theorem 5.2.2, respectively. This completes the proof of Theorem 5.2.2. □

Part (ii) of the following Theorem 5.2.6 uses Theorem 4.3.8 to specify the quasi-Hadamard derivative of the value functional of the terminal wealth problem (5.39). In Subsections 5.2.3–5.2.4 this derivative plays a key role to derive the asymptotic error distribution of suitable estimators for the optimal value of the terminal wealth problem (5.39). For the formulation of the following theorem, note that the component γ_n^F of $\boldsymbol{\gamma}^F := (\gamma_n^F)_{n=0}^{N-1}$ corresponds to the unique solution to the reduced optimization problem (5.40). Take into account that it follows from Lemma 3.2.4 (applied to $\boldsymbol{P}_F$) that there exists a unique solution of the latter optimization problem. Recall that $v_n^{F;\boldsymbol{\gamma}} := v_n^{F;\gamma_n}$, $n = 0, \ldots, N-1$, for $v_n^{F;\gamma_n}$ be defined as on the left-hand side of (5.40), and that $(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'$ refers to the first derivative of the (continuously differentiable) map $(u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})(\cdot)$.

Theorem 5.2.6 (Quasi-Hadamard differentiability of $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ and $\mathcal{W}_0^{x_0}$ in F) *In the setting of Subsection 5.2.1 let $x_0 \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, and $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$. Then the following two assertions hold.*

(i) *The map $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ defined by (5.41) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\mu_{u_\alpha})\langle\boldsymbol{L}_1(\mu_{u_\alpha})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}} : \boldsymbol{L}_1(\mu_{u_\alpha}) \to \mathbb{R}$ given by*

$$\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(h) = -\sum_{k=0}^{N-1} \prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{F;\boldsymbol{\gamma}}\, u_\alpha(x_0) \cdot \int_{\mathbb{R}_{\geq 0}} h(y)(u_\alpha \circ \eta_k^{\boldsymbol{\gamma}})'(y)\, \ell(dy). \tag{5.52}$$

(ii) *The map $\mathcal{W}_0^{x_0} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ defined by (5.41) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\mu_{u_\alpha})\langle\boldsymbol{L}_1(\mu_{u_\alpha})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0} : \boldsymbol{L}_1(\mu_{u_\alpha}) \to \mathbb{R}$ given by*

$$\dot{\mathcal{W}}_{0;F}^{x_0}(h) = \sup_{\pi\in\Pi_{\mathrm{lin}}(F)} \dot{\mathcal{W}}_{0;F}^{x_0;\pi}(h) = \dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}^F}}(h). \tag{5.53}$$

Note that the first "=" in (5.53) is ensured by Theorem 4.3.8 (with $\Pi(F)$ replaced by $\Pi_{\mathrm{lin}}(F)$) along with the representation (5.47) of the value functional $\mathcal{W}_0^{x_0}$. The validity of the second "=" in (5.53) will be carried out in the proof of (part (ii) of) Theorem 5.2.6.

Proof of Theorem 5.2.6: We intend to apply Theorem 4.3.8. First of all, as already discussed in Subsection 5.2.1, Assumption 4.2.1 is satisfied (with μ_{u_α} playing the role of ν). Moreover in view of (5.47) and Remark 4.3.10 it suffices to verify the assumptions of Theorem 4.3.8 for $\Pi_{\rm lin}$ instead of Π. Recall that $\Pi_{\rm lin}$ consists of all linear trading strategies given by (3.26).

In the sequel, we will show that conditions (a)–(c) of Theorem 4.3.8 hold (with $\Pi_{\rm lin}$ in place of Π). Since condition (c) of Theorem 4.3.8 is nothing but condition (b) of Theorem 4.3.3, and since we have already shown in the proof of Theorem 5.2.2 that condition (b) of Theorem 4.3.3 is fulfilled (with $\Pi_{\rm lin}$ in place of Π), it remains to show that conditions (a)–(b) of Theorem 4.3.8 hold for $\Pi_{\rm lin}$ instead of Π.

To verify condition (a), let $\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},x)}:\boldsymbol{L}_1(\mu_{u_\alpha})\to\mathbb{R}$ be for any $x\in\mathbb{R}_{\ge0}$, $\boldsymbol\gamma\in[0,1]^N$, and $n=0,\dots,N-1$ a map defined by

$$\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},x)}(h)\;:=\;-\prod_{j=n+1}^{N-1}v_j^{F;\boldsymbol\gamma}\cdot u_\alpha(x/S_n^0)\cdot\int_{\mathbb{R}_{\ge0}}h(y)\,(u_\alpha\circ\eta_n^{\boldsymbol\gamma})'(y)\,\ell(dy). \tag{5.54}$$

Note that for any $x\in\mathbb{R}_{\ge0}$, $\boldsymbol\gamma\in[0,1]^N$, and $n=0,\dots,N-1$ the map $\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},x)}$ is well-defined by equations (3.23)–(3.24) (applied to $\boldsymbol{P}_F$) along with Lemma 5.2.3. Also note that in view of (5.54) we clearly have $\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},x)}(0_{\boldsymbol{L}_0(\mu_{u_\alpha})})=0$ as well as $\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(h)\in\mathbb{M}(\mathbb{R}_{\ge0})$ for all $x\in\mathbb{R}_{\ge0}$, $\boldsymbol\gamma\in[0,1]^N$, $n=0,\dots,N-1$, and $h\in\boldsymbol{L}_1(\mu_{u_\alpha})$. Proceeding as in Display (5.58) below we get $\sup_{\boldsymbol\gamma\in[0,1]^N}\|\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(h)\|_\psi\le C_{\dot\Lambda}$ for every $h\in\boldsymbol{L}_1(\mu_{u_\alpha})$ and $n=0,\dots,N-1$ (by (1.18)), where $C_{\dot\Lambda}:=(1+\overline{\mathfrak{m}}_F)^{N-n-1}\,\overline{\mathfrak{r}}^{1-\alpha}\,\|h\|_{1,\mu_{u_\alpha}}$ (with $\overline{\mathfrak{m}}_F$ and $\overline{\mathfrak{r}}$ as in the proof of Theorem 5.2.2). Since any $\boldsymbol\gamma=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol\gamma}:=(f_n^{\boldsymbol\gamma})_{n=0}^{N-1}\in\Pi_{\rm lin}$ through (3.26), this shows that condition (a) of Theorem 4.3.8 holds (with $\Pi_{\rm lin}$ in place of Π).

In the remainder of the proof we will verify condition (b) of Theorem 4.3.8 for $\Pi_{\rm lin}$ instead of Π. That is, we will prove that for any $n=0,\dots,N-1$ the map $\boldsymbol\Lambda_n^F:\boldsymbol{F}_{>0}(\mu_{u_\alpha})\to\ell_\psi^\infty(\Pi_{\rm lin}\times E)$ defined by (4.10) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\mu_{u_\alpha})\langle\boldsymbol{L}_1(\mu_{u_\alpha})\rangle$ (in the sense of Definition 4.3.7) with quasi-Hadamard derivative $\dot{\boldsymbol\Lambda}_{n;F}^F:\boldsymbol{L}_1(\mu_{u_\alpha})\to\ell_\psi^\infty(\Pi_{\rm lin}\times E)$ given by

$$\dot{\boldsymbol\Lambda}_{n;F}^F(h)\;:=\;\big(\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},x)}\big)_{(\boldsymbol\gamma,x)\in[0,1]^N\times E}. \tag{5.55}$$

with $\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},x)}$ as in (5.54). Take into account that (as discussed above) any linear trading strategy $\pi\in\Pi_{\rm lin}$ can be identified with some $\boldsymbol\gamma\in[0,1]^N$.

Now, let $(h,(h_m),(\varepsilon_m))\in\boldsymbol{L}_1(\mu_{u_\alpha})\times\boldsymbol{L}_1(\mu_{u_\alpha})^{\mathbb{N}}\times\mathbb{R}_{>0}^{\mathbb{N}}$ be any triplet satisfying $\|h_m-h\|_{1,\mu_{u_\alpha}}\to0$, $\varepsilon_m\to0$ as well as $(F+\varepsilon_mh_m)\subseteq\boldsymbol{F}_{>0}(\mu_{u_\alpha})$. Since any $\boldsymbol\gamma=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol\gamma}:=(f_n^{\boldsymbol\gamma})_{n=0}^{N-1}\in\Pi_{\rm lin}$ through (3.26) it suffices for the quasi-Hadamard differentiability of the map $\boldsymbol\Lambda_n^F$ in view of Definition 4.3.7, (4.11), and (5.55) to show that for any $n=0,\dots,N-1$

$$\lim_{m\to\infty}\sup_{\boldsymbol\gamma\in[0,1]^N}\big\|\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(h_m)-\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(h)\big\|_\psi\;=\;0 \tag{5.56}$$

as well as

$$\lim_{m\to\infty}\sup_{\boldsymbol\gamma\in[0,1]^N}\Big\|\frac{\Lambda_n^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(F+\varepsilon_mh_m)-\Lambda_n^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(F)}{\varepsilon_m}-\dot\Lambda_{n;F}^{F;(\pi_{\boldsymbol\gamma},\,\cdot\,)}(h)\Big\|_\psi\;=\;0, \tag{5.57}$$

where the map $\Lambda_n^{F;(\pi_{\boldsymbol\gamma},x)}:\boldsymbol{F}_{>0}(\mu_{u_\alpha})\to\mathbb{R}$ is defined as in (4.8).

First, in view of (5.54), Lemma 5.2.3, and Displays (3.23)–(3.24) (applied to $\boldsymbol{P}_F$), we get for any $x \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$, $n = 0, \ldots, N-1$, and $m \in \mathbb{N}$

$$
\begin{aligned}
&\frac{1}{\psi(x)} \cdot \big|\dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h)\big| \\
&= \frac{1}{\psi(x)} \cdot \Big| \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(x/S_n^0) \cdot \int_{\mathbb{R}_{\geq 0}} (h_m - h)(y)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\, \ell(dy) \Big| \\
&= \frac{u_\alpha(x)}{\psi(x)} \cdot \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(1/S_n^0) \cdot \Big| \int_{\mathbb{R}_{\geq 0}} (h_m - h)(y)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\, \ell(dy) \Big| \\
&\leq (1 + \overline{\mathfrak{m}}_F)^{N-n-1} \cdot \int_{\mathbb{R}_{\geq 0}} \big|(h_m - h)(y)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\big|\, \ell(dy) \\
&\leq (1 + \overline{\mathfrak{m}}_F)^{N-n-1} \cdot \overline{\mathfrak{r}}^{1-\alpha}\, \|h_m - h\|_{1,\mu_{u_\alpha}}. \qquad (5.58)
\end{aligned}
$$

By (1.18) this implies

$$
\begin{aligned}
\sup_{\boldsymbol{\gamma} \in [0,1]^N} \big\|\dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(h)\big\|_\psi &= \sup_{\boldsymbol{\gamma} \in [0,1]^N} \sup_{x \in \mathbb{R}_{\geq 0}} \frac{1}{\psi(x)} \cdot \big|\dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h)\big| \\
&\leq C_{\dot{\Lambda},1}\, \|h_m - h\|_{1,\mu_{u_\alpha}}
\end{aligned}
$$

for any $n = 0, \ldots, N-1$ and $m \in \mathbb{N}$, where $C_{\dot{\Lambda},1} := (1 + \overline{\mathfrak{m}}_F)^{N-n-1}\, \overline{\mathfrak{r}}^{1-\alpha} \in \mathbb{R}_{\geq 1}$ (is independent of x and $\boldsymbol{\gamma}$). Thus (5.56) follows.

To prove (5.57), we observe at first in virtue of (5.50), (5.54) as well as Lemma 5.2.5

$$
\begin{aligned}
&\frac{1}{\psi(x)} \cdot \Big| \frac{\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F + \varepsilon_m h_m) - \Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F)}{\varepsilon_m} - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h) \Big| \\
&= \frac{1}{\psi(x)} \cdot \Big| \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(x/S_n^0)\, \frac{1}{\varepsilon_m} \cdot \int_{\mathbb{R}_{\geq 0}} (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})(y)\, d\big((F + \varepsilon_m h_m) - F\big)(y) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h) \Big| \\
&= \frac{1}{\psi(x)} \cdot \Big| - \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(x/S_n^0)\, \frac{1}{\varepsilon_m} \cdot \int_{\mathbb{R}_{\geq 0}} \big((F + \varepsilon_m h_m) - F\big)(y)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\, \ell(dy) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h) \Big| \\
&= \frac{1}{\psi(x)} \cdot \Big| \Big(- \prod_{j=n+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(x/S_n^0) \cdot \int_{\mathbb{R}_{\geq 0}} h_m(y)\, (u_\alpha \circ \eta_n^{\boldsymbol{\gamma}})'(y)\, \ell(dy) \Big) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h) \Big| \\
&= \frac{1}{\psi(x)} \cdot \big|\dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h)\big|
\end{aligned}
$$

for every $x \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, $n = 0, \ldots, N-1$, and $m \in \mathbb{N}$. Thus

$$
\begin{aligned}
&\sup_{\boldsymbol{\gamma} \in [0,1]^N} \Big\| \frac{\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F + \varepsilon_m h_m) - \Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(F)}{\varepsilon_m} - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},\,\cdot\,)}(h) \Big\|_\psi \\
&= \sup_{\boldsymbol{\gamma} \in [0,1]^N} \sup_{x \in \mathbb{R}_{\geq 0}} \frac{1}{\psi(x)} \cdot \Big| \frac{\Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F + \varepsilon_m h_m) - \Lambda_n^{F;(\pi_{\boldsymbol{\gamma}},x)}(F)}{\varepsilon_m} - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h) \Big| \\
&= \sup_{\boldsymbol{\gamma} \in [0,1]^N} \sup_{x \in \mathbb{R}_{\geq 0}} \frac{1}{\psi(x)} \cdot \big|\dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h_m) - \dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)}(h)\big| \;\leq\; C_{\dot{\Lambda},1}\, \|h_m - h\|_{1,\mu_{u_\alpha}}
\end{aligned}
$$

for any $n = 0, \dots, N-1$ and $m \in \mathbb{N}$ by (1.18) and (5.58). Therefore we arrive at (5.57). This shows condition (b) of Theorem 4.3.8 (with Π_{lin} in place of Π).

In particular, we have verified the assumptions of Theorem 4.3.8 for $\nu := \mu_{u_\alpha}$ and Π_{lin} instead of Π.

(i): It follows from part (i) of Theorem 4.3.8 that the map $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}} : \boldsymbol{F}_{>0}(\mu_{u_\alpha}) \to \mathbb{R}$ defined as in (4.6) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\mu_{u_\alpha})\langle \boldsymbol{L}_1(\mu_{u_\alpha})\rangle$. The related quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}$ of $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ at F is in view of (4.16), (5.54), and (5.38) given by (recall that $S_0^0 = 1$)

$$\begin{aligned}
\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(h) &= \sum_{k=0}^{N-1} \int_{\mathbb{R}_{\geq 0}} \cdots \int_{\mathbb{R}_{\geq 0}} \dot{\Lambda}_{k;F}^{F;(\pi_{\boldsymbol{\gamma}},y_k)}(h)\, P_{k-1}^F\big((y_{k-1}, f_{k-1}^{\boldsymbol{\gamma}}(y_{k-1})), dy_k\big) \cdots P_0^F\big((x_0, f_0^{\boldsymbol{\gamma}}(x_0)), dy_1\big) \\
&= -\sum_{k=0}^{N-1} \int_{\mathbb{R}_{\geq 0}} \cdots \int_{\mathbb{R}_{\geq 0}} \prod_{j=k+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(y_k/B_k) \cdot \int_{\mathbb{R}_{\geq 0}} h(y_{k+1})\, (u_\alpha \circ \eta_k^{\boldsymbol{\gamma}})'(y_{k+1})\, \ell(dy_{k+1}) \\
&\qquad P_{k-1}^F\big((y_{k-1}, f_{k-1}^{\boldsymbol{\gamma}}(y_{k-1})), dy_k\big) \cdots P_0^F\big((x_0, f_0^{\boldsymbol{\gamma}}(x_0)), dy_1\big) \\
&= -\sum_{k=0}^{N-1} \prod_{j=k+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot \int_{\mathbb{R}_{\geq 0}} h(y_{k+1})\, (u_\alpha \circ \eta_k^{\boldsymbol{\gamma}})'(y_{k+1})\, \ell(dy_{k+1}) \\
&\qquad \cdot \int_{\mathbb{R}_{\geq 0}} \cdots \int_{\mathbb{R}_{\geq 0}} u_\alpha(y_k/B_k)\, P_{k-1}^F\big((y_{k-1}, f_{k-1}^{\boldsymbol{\gamma}}(y_{k-1})), dy_k\big) \cdots P_0^F\big((x_0, f_0^{\boldsymbol{\gamma}}(x_0)), dy_1\big) \\
&= -\sum_{k=0}^{N-1} \prod_{j=k+1}^{N-1} v_j^{F;\boldsymbol{\gamma}} \cdot \int_{\mathbb{R}_{\geq 0}} h(y_{k+1})\, (u_\alpha \circ \eta_k^{\boldsymbol{\gamma}})'(y_{k+1})\, \ell(dy_{k+1}) \\
&\qquad \cdot \prod_{j=0}^{k-1} v_j^{F;\boldsymbol{\gamma}} \cdot u_\alpha(x_0/S_0^0) \\
&= -\sum_{k=0}^{N-1} \prod_{\substack{j=0 \\ j\neq k}}^{N-1} v_j^{F;\boldsymbol{\gamma}}\, u_\alpha(x_0) \cdot \int_{\mathbb{R}_{\geq 0}} h(y_{k+1})\, (u_\alpha \circ \eta_k^{\boldsymbol{\gamma}})'(y_{k+1})\, \ell(dy_{k+1})
\end{aligned}$$

for all $h \in \boldsymbol{L}_1(\mu_{u_\alpha})$.

(ii): For every $n = 0, \dots, N-1$, let $\gamma_n^F \in [0,1]$ be the unique solution to the (reduced) optimization problem (5.40). Set $\boldsymbol{\gamma}^F := (\gamma_n^F)_{n=0}^{N-1} \in [0,1]^N$ and note that it follows from the first assertion in part (ii) of Theorem 3.2.5 (applied to $\boldsymbol{P}_F$) that the linear trading strategy $\pi^F = \pi_{\boldsymbol{\gamma}^F} := (f_n^{\gamma^F})_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ defined as in (3.26) is optimal w.r.t. F. Take into account that $\boldsymbol{P}_F \in \mathcal{P}_\psi$. Also note that the value functional $\mathcal{W}_0^{x_0}$ admits the representation (5.47). Therefore, an application of part (ii) of Theorem 4.3.8 entails that the value functional $\mathcal{W}_0^{x_0}$ given by (5.47) is quasi-Hadamard differentiable at F tangentially to $\boldsymbol{L}_1(\mu_{u_\alpha})\langle \boldsymbol{L}_1(\mu_{u_\alpha})\rangle$ with quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0}$ given by

$$\dot{\mathcal{W}}_{0;F}^{x_0}(h) = \sup_{\pi \in \Pi_{\mathrm{lin}}(F)} \dot{\mathcal{W}}_{0;F}^{x_0;\pi}(h) \tag{5.59}$$

for every $h \in \boldsymbol{L}_1(\mu_{u_\alpha})$. Since $\Pi_{\mathrm{lin}}(F) = \{\pi_{\boldsymbol{\gamma}^F}\}$ by the second assertion in part (ii) of Theorem 3.2.5 (applied to $\boldsymbol{P}_F$), the representation of the quasi-Hadamard derivative $\dot{\mathcal{W}}_{0;F}^{x_0}$ in (5.59) simplifies to

$$\dot{\mathcal{W}}_{0;F}^{x_0}(h) = \sup_{\pi \in \Pi_{\mathrm{lin}}(F)} \dot{\mathcal{W}}_{0;F}^{x_0;\pi}(h) = \dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}^F}}(h)$$

for any $h \in \boldsymbol{L}_1(\mu_{u_\alpha})$. This completes the proof of Theorem 5.2.6. $\square$

5.2.3 Nonparametric estimation of the optimal value

In this subsection we will deal with the nonparametric estimation of the optimal value of the terminal wealth problem (5.39) in the unknown distribution function F.

For this reason, let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}_{>0}$, and denote by F the common distribution function of the random variables $Y_1, Y_2, \dots$ which is supposed to be unknown. Therefore we have $F \in \boldsymbol{F}_{>0}$ (with $\boldsymbol{F}_{>0}$ defined as in Subsection 5.2.1). Note that the random variables Y_i can be regarded as observed historical (or simulated) asset returns in the setting of the financial market model from Subsection 5.2.1. As a consequence, a canonical choice for an estimator for F will be the empirical distribution function $\widehat{F}_m$ of $Y_1, \dots, Y_m$ based on sample size $m \in \mathbb{N}$ as defined in (4.31). Therefore, the expression $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m)$ (resp. $\mathcal{W}_0^{x_0}(\widehat{F}_m)$) is a reasonable (plug-in) estimator for $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F)$ (resp. $\mathcal{W}_0^{x_0}(F)$) if $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$, where the functional $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}$ (resp. $\mathcal{W}_0^{x_0}$) is defined as in (5.41). Note that $\widehat{F}_m(\omega, \cdot\,) \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$ by part (i) of Lemma 5.2.8 below. Throughout this subsection we fix $\alpha \in (0,1)$ as well as the constants $\mathfrak{r}_1, \dots, \mathfrak{r}_N \in \mathbb{R}_{\geq 1}$. Recall that α determines the degree of risk aversion of the agent.

In the sequel, we present based on the elaborations from Subsection 5.2.2 asymptotic properties, such as consistency, asymptotic normality, and bootstrap consistency (in probability) of the nonparametric estimator $\mathcal{W}_0^{x_0}(\widehat{F}_m)$ for the optimal value $\mathcal{W}_0^{x_0}(F)$ of the terminal wealth problem (5.39).

At first, Theorem 5.2.7 illustrates Theorem 4.4.1 in the setting of Subsection 5.2.1. Recall that any $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$ induces a linear trading strategy $\pi_{\boldsymbol{\gamma}} := (f_n^{\boldsymbol{\gamma}})_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ through (3.26), and let μ_{u_α} be the locally finite Stieltjes measure w.r.t. u_α. Note that μ_{u_α} can be considered as a (locally finite) measure on $\mathcal{B}(\mathbb{R})$ which is supported on $\mathbb{R}_{\geq 0}$.

Theorem 5.2.7 (Strong consistency of $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m))$ and $(\mathcal{W}_0^{x_0}(\widehat{F}_m))$) *In the setting of Subsection 5.2.1 let $x_0 \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{\gamma} \in [0,1]^N$. Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathbb{R}_{>0}$-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F, and suppose that $\int_{\mathbb{R}_{\geq 0}}(1-F)\,d\mu_{u_\alpha} < \infty$ (in particular $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$). Moreover let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1, \dots, Y_m$ as defined in (4.31). Then the following two assertions hold.*

(i) *The sequence of estimators $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F)$ under $\mathbb{P}$ in the sense that*
$$\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m) \to \mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F) \quad \mathbb{P}\text{-a.s.}$$

(ii) *The sequence of estimators $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_0}(F)$ under $\mathbb{P}$ in the sense that*
$$\mathcal{W}_0^{x_0}(\widehat{F}_m) \to \mathcal{W}_0^{x_0}(F) \quad \mathbb{P}\text{-a.s.}$$

Part (ii) of the latter theorem provides the following information. If $\pi_{\boldsymbol{\gamma}^F} := (f_n^{\boldsymbol{\gamma}^F})_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ corresponds in the setting of Theorem 5.2.7 to the optimal (linear) trading strategy w.r.t. F (the existence is ensured by part (ii) of Theorem 3.2.5 (applied to $\boldsymbol{P}_F$)), then (under the assumptions of Theorem 5.2.7) the assertion (ii) of this theorem implies that for any initial amount of capital $x_0 \in \mathbb{R}_{\geq 0}$ the sequence of estimators $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_{m\in\mathbb{N}}$ is strongly consistent (under $\mathbb{P}$) for the optimal value $\mathcal{W}_0^{x_0;\pi_{\gamma^F}}(F)$ of the terminal wealth problem (5.39).

The proof of Theorem 5.2.7 avails the following lemma.

Lemma 5.2.8 *With the notation of Theorem 5.2.7 the following two assertions hold.*

(i) $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ *for every* $\omega \in \Omega$ *and* $m \in \mathbb{N}$.

(ii) *If* $\int_{\mathbb{R}_{\geq 0}}(1-F)\,d\mu_{u_\alpha} < \infty$ *then* $\|\widehat{F}_m - F\|_{1,\mu_{u_\alpha}} \to 0 \quad \mathbb{P}$*-a.s.*

Proof For part (i), note that at first that $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_{>0}$ clearly holds for every $\omega \in \Omega$ and $m \in \mathbb{N}$. Then the assertion can be deduced from part (i) of Lemma 5.2.4 as well as the representation (4.31).

To show (ii), let $X_i : \Omega \times \mathbb{R} \to \mathbb{R}$ be for any $i \in \mathbb{N}$ a real-valued stochastic process defined by

$$X_i(\omega,t) \;:=\; \mathbb{1}_{[Y_i(\omega),\infty)}(t) - F(t).$$

Note that it is easily seen that the process X_i is measurable for any $i \in \mathbb{N}$. Since $\int_{\mathbb{R}_{\geq 0}}(1-F)\,d\mu_{u_\alpha} < \infty$ (by assumption), we get in view of equations (4.7) and (B.1)

$$\begin{aligned}
\|X_i(\omega,\,\cdot\,)\|_{1,\mu_{u_\alpha}} &= \int_{\mathbb{R}_{\geq 0}} \big|\mathbb{1}_{[Y_i(\omega),\infty)}(t) - F(t)\big|\,\mu_{u_\alpha}(dt)\\
&\leq \int_{\mathbb{R}_{\geq 0}} \big(1-\mathbb{1}_{[Y_i(\omega),\infty)}(t)\big)\,\mu_{u_\alpha}(dt) + \int_{\mathbb{R}_{\geq 0}} (1-F(t))\,\mu_{u_\alpha}(dt)\\
&= \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{[0,Y_i(\omega))}(t)\,\mu_{u_\alpha}(dt) + \int_{\mathbb{R}_{\geq 0}} (1-F(t))\,\mu_{u_\alpha}(dt)\\
&= \mu_{u_\alpha}\big[(0,Y_i(\omega))\big] + \int_{\mathbb{R}_{\geq 0}} (1-F(t))\,\mu_{u_\alpha}(dt)\\
&= u_\alpha(Y_i(\omega)) + \int_{\mathbb{R}_{\geq 0}} (1-F(t))\,\mu_{u_\alpha}(dt) \;<\; \infty
\end{aligned}$$

for any $i \in \mathbb{N}$ and $\omega \in \Omega$. Take into account that $\mu_{u_\alpha}[\{t\}] = 0$ for every $t \in \mathbb{R}_{\geq 0}$. Hence $X_i(\omega,\cdot) \in \boldsymbol{L}_1(\mu_{u_\alpha})$ for every $i \in \mathbb{N}$ and $\omega \in \Omega$. Therefore Lemma 4.4.5 ensures that X_i can be seen for any $i \in \mathbb{N}$ as an $(\boldsymbol{L}_1(\mu_{u_\alpha}), \mathcal{B}(\boldsymbol{L}_1(\mu_{u_\alpha})))$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

Moreover, we observe

$$\mathbb{E}[X_i(\,\bullet\,,t)] \;=\; \mathbb{E}\big[\mathbb{1}_{[Y_i(\bullet),\infty)}(t)\big] - F(t) \;=\; \mathbb{P}\big[\big\{Y_i(\bullet) \in (\infty,t]\big\}\big] - F(t) \;=\; F(t) - F(t) \;=\; 0$$

for any $i \in \mathbb{N}$ and $t \in \mathbb{R}$ as well as

$$\begin{aligned}
\mathbb{E}\big[\|X_i(\,\bullet\,,\,\cdot\,)\|_{1,\mu_{u_\alpha}}\big] &= \mathbb{E}\Big[\int_{\mathbb{R}_{\geq 0}} \big|\mathbb{1}_{[Y_i(\bullet),\infty)}(t) - F(t)\big|\,\mu_{u_\alpha}(dt)\Big]\\
&\leq \mathbb{E}\Big[\int_{\mathbb{R}_{\geq 0}} \Big(\big(1-\mathbb{1}_{[Y_i(\bullet),\infty)}(t)\big) + (1-F(t))\Big)\,\mu_{u_\alpha}(dt)\Big]\\
&= \mathbb{E}\Big[\int_{\mathbb{R}_{\geq 0}} \big(1-\mathbb{1}_{[Y_i(\bullet),\infty)}(t)\big)\,\mu_{u_\alpha}(dt)\Big] + \int_{\mathbb{R}_{\geq 0}} (1-F(t))\,\mu_{u_\alpha}(dt)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_{\geq 0}} \big(1 - \mathbb{E}\big[\mathbb{1}_{[Y_i(\bullet),\infty)}(t)\big]\big)\,\mu_{u_\alpha}(dt) + \int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\mu_{u_\alpha}(dt)\\
&= 2\int_{\mathbb{R}_{\geq 0}} (1 - F(t))\,\mu_{u_\alpha}(dt) \;<\; \infty
\end{aligned}
$$

for every $i \in \mathbb{N}$ (by Fubini's theorem) because $\int_{\mathbb{R}_{\geq 0}}(1-F)\,d\mu_{u_\alpha} < \infty$ (by assumption). Then, in view of (4.31), an application of Corollary 7.10 in [62] yields

$$
\big\|\widehat{F}_m(\omega,\,\cdot\,) - F(\cdot)\big\|_{1,\mu_{u_\alpha}} \;=\; \Big\|\frac{1}{m}\sum_{i=1}^{m} X_i(\omega,\,\cdot\,)\Big\|_{1,\mu_{u_\alpha}} \;\to 0
$$

for $\mathbb{P}$-a.e. ω. This shows (ii). □

Now, let us turn to the proof of Theorem 5.2.7.

Proof of Theorem 5.2.7: We intend to apply Theorem 4.4.1. Note at first that it follows from the proof of Theorem 5.2.2 that the assumptions of Theorem 4.3.3 are fulfilled (with Π_{lin} playing the role of Π). Moreover, part (i) of Lemma 5.2.8 implies that $\widehat{F}_m(\omega,\,\cdot\,) \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$. Finally, part (ii) of Lemma 5.2.8 ensures that $\|\widehat{F}_m - F\|_{1,\mu_{u_\alpha}} \to 0$ $\mathbb{P}$-a.s.

In particular, this shows that the assumptions of Theorem 4.4.1 are satisfied for $\nu := \mu_{u_\alpha}$ and Π_{lin} in place of Π, and an application of parts (i) and (ii) of the latter theorem ensures the assertions in (i) and (ii) of Theorem 5.2.7, respectively. This completes the proof of Theorem 5.2.7. □

Next, the assertion in part (ii) of the following Theorem 5.2.9 determines the asymptotic error distribution of the sequence of estimators $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_{m\in\mathbb{N}}$ which can be used to construct an asymptotic confidence interval for the optimal value $\mathcal{W}_0^{x_0}(F)$ of the terminal wealth problem (5.39); see Remark 5.2.12 ahead. Note that N_{0,s^2} refers to the normal distribution with zero mean and variance s^2, and that $\xi \sim \mathrm{N}_{0,s^2}$ refers to any random variable ξ with distribution N_{0,s^2}. Set $\boldsymbol{\gamma}^F := (\gamma_n^F)_{n=0}^{N-1} \in [0,1]^N$ and note that γ_n^F corresponds in view of Lemma 3.2.4 (applied to $\boldsymbol{P}_F$) to the unique solution to the reduced optimization problem (5.40). Finally, set as before $v_n^{F;\boldsymbol{\gamma}} := v_n^{F;\gamma_n}$ for any $n = 0,\dots,N-1$ (with $v_n^{F;\gamma_n}$ defined as on the left-hand side of (5.40), and recall that $\rightsquigarrow$ stands for the convergence in distribution.

Theorem 5.2.9 (Asymptotic error distribution of $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m))$ and $(\mathcal{W}_0^{x_0}(\widehat{F}_m))$) *In the setting of Subsection 5.2.1 let $x_0 \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$. Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathbb{R}_{>0}$-valued random variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with common distribution function F, and suppose that $\int \sqrt{F(1-F)}\,d\mu_{u_\alpha} < \infty$ (in particular $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$). Moreover let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1,\dots,Y_m$ as defined in (4.31). Then the following two assertions hold.*

(i) *We have*

$$
\sqrt{m}\big(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m) - \mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F)\big) \;\rightsquigarrow\; Z_{F;x_0,\boldsymbol{\gamma}} \quad \textit{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|)
$$

for $Z_{F;x_0,\boldsymbol{\gamma}} \sim \mathrm{N}_{0,s^2}$ with

$$
s^2 \;=\; s^2_{F;x_0,\boldsymbol{\gamma}} \;:=\; \int_{\mathbb{R}^2} h^{x_0,\boldsymbol{\gamma}}_{\alpha,F}(t_1)\,C_F(t_1,t_2)\,h^{x_0,\boldsymbol{\gamma}}_{\alpha,F}(t_2)\,(\mu_{u_\alpha}\otimes\mu_{u_\alpha})(d(t_1,t_2)), \tag{5.60}
$$

where

$$h^{x_0;\gamma}_{\alpha,F}(t) \;:=\; -\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{F;\gamma} u_\alpha(x_0)\cdot\Big(\frac{\gamma_k}{\mathfrak{r}_{k+1}}\Big)^{\alpha+1}\mathbb{1}_{\left[\mathfrak{r}_{k+1}\frac{\gamma_k-1}{\gamma_k},\infty\right)}(t)\,\mathbb{1}_{\{\gamma_k\neq 0\}},\qquad t\in\mathbb{R} \tag{5.61}$$

and C_F is given by (4.33).

(ii) *We have*

$$\sqrt{m}\big(\mathcal{W}_0^{x_0}(\widehat{F}_m)-\mathcal{W}_0^{x_0}(F)\big)\;\rightsquigarrow\; Z_{F;x_0}\quad \textit{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|)$$

for $Z_{F;x_0}\sim \mathrm{N}_{0,s^2}$ with $s^2=s^2_{F;x_0,\boldsymbol{\gamma}^F}$ given by (5.60) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}^F$).

The proof of Theorem 5.2.9 relies on the following lemma.

Lemma 5.2.10 *With the notation and under the assumptions of Theorem 5.2.9 the following two assertions hold for every $x_0\in\mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma}\in[0,1]^N$, and $m\in\mathbb{N}$.*

(i) *The estimator $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m)$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable.*

(ii) *The estimator $\mathcal{W}_0^{x_0}(\widehat{F}_m)$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable.*

Proof We will only prove the claim in (i). The verification of the assertion in (ii) will follow with analogous arguments. At first it can be verified easily that the mapping $\omega\to\widehat{F}_m(\omega,\cdot\,)$ from Ω to $\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ is $(\mathcal{F},\mathcal{B}(\boldsymbol{F}_{>0}(\mu_{u_\alpha})))$-measurable for the Borel σ-algebra $\mathcal{B}(\boldsymbol{F}_{>0}(\mu_{u_\alpha}))$ on $(\boldsymbol{F}_{>0}(\mu_{u_\alpha}),\|\cdot\|_{1,\mu_{u_\alpha}})$, where $\|\cdot\|_{1,\mu_{u_\alpha}}$ refers to the (separable) norm defined as in (4.7) (with ν replaced by μ_{u_α}). Note here that the mapping $\mathbb{R}^m_{>0}\to\boldsymbol{F}_{>0}(\mu_{u_\alpha})$, $(y_1,\dots,y_m)\mapsto\frac{1}{m}\sum_{i=1}^m\mathbb{1}_{[y_i,\infty)}$ is easily seen to be $(\|\cdot\|,\|\cdot\|_{1,\mu_{u_\alpha}})$-continuous. As a consequence of part (i) of Theorem 5.2.2, the functional $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}:\boldsymbol{F}_{>0}(\mu_{u_\alpha})\to\mathbb{R}$ given by (5.41) is $(\|\cdot\|_{1,\mu_{u_\alpha}},|\cdot|)$-continuous and thus in particular $(\mathcal{B}(\boldsymbol{F}_{>0}(\mu_{u_\alpha})),\mathcal{B}(\mathbb{R}))$-measurable. Therefore the estimator $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m)$ is a real-valued random variable on $(\Omega,\mathcal{F},\mathbb{P})$. □

Thus we are in the position to prove Theorem 5.2.9.

Proof of Theorem 5.2.9: We intend to apply Theorem 4.4.4. First, we note that in view of Lemma 5.2.8(i) we have $\widehat{F}_m(\omega,\cdot\,)\in\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ for every $\omega\in\Omega$ and $m\in\mathbb{N}$. Second, it follows from the proof of Theorem 5.2.6 that the assumptions of Theorem 4.3.8 are satisfied for Π_{lin} in place of Π. Thus, in view of Lemma 5.2.10, the assumptions of Theorem 4.4.4 hold for $\nu:=\mu_{u_\alpha}$ and Π_{lin} instead of Π. Note in the following that any $\pi\in\Pi_{\mathrm{lin}}$ is induced by some $\boldsymbol{\gamma}\in[0,1]^N$ through (3.26).

(i): As a consequence of part (i) of Theorem 4.4.4 we have that $\dot{\mathcal{W}}^{x_0;\pi_{\boldsymbol{\gamma}}}_{0;F}(B_F)$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and

$$\sqrt{m}\big(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m)-\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F)\big)\;\rightsquigarrow\;\dot{\mathcal{W}}^{x_0;\pi_{\boldsymbol{\gamma}}}_{0;F}(B_F)\quad\text{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|), \tag{5.62}$$

where $\dot{\mathcal{W}}^{x_0;\pi_{\boldsymbol{\gamma}}}_{0;F}$ is given by (5.52) and B_F is an $\boldsymbol{L}_1(\mu_{u_\alpha})$-valued centred Gaussian random variable on some probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\mu_{u_\alpha}}$ defined as in (4.32).

Similarly to (5.42) we get for every $n=0,\dots,N-1$ and $\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$ with $\gamma_n=0$

$$\int_{\mathbb{R}_{\geq 0}} B_F(\,\cdot\,,y)(u_\alpha\circ\eta_n^{\boldsymbol{\gamma}})'(y)\,\ell(dy)\;=\;\int_{\mathbb{R}_{\geq 0}} B_F(\,\cdot\,,y)\,\alpha\,(\gamma_n/\mathfrak{r}_{n+1})\cdot u_{\alpha-1}(\eta_n^{\boldsymbol{\gamma}}(y))\,\ell(dy)\;=\;0 \tag{5.63}$$

by (3.22). Moreover, in view of Lemmas B.3 and B.4, the change of variable formula yields

$$\begin{aligned}\int_{\mathbb{R}_{\geq 0}} B_F(\,\cdot\,,y)(u_\alpha\circ\eta_n^{\boldsymbol{\gamma}})'(y)\,\ell(dy) &= \int_{\mathbb{R}_{\geq 0}} B_F(\,\cdot\,,y)\,d(u_\alpha\circ\eta_n^{\boldsymbol{\gamma}})(y) \;=\; \int_{\mathbb{R}_{\geq 0}} B_F\circ(\eta_n^{\boldsymbol{\gamma}})^{-1}(\,\cdot\,,y)\,du_\alpha(y)\\
&= \int_{\mathbb{R}_{\geq 0}} B_F\Big(\,\cdot\,,\frac{\mathfrak{r}_{n+1}}{\gamma_n}\big(y+(\gamma_n-1)\big)\Big)\,u_\alpha'(y)\,\ell(dy)\\
&= \int_{\mathbb{R}_{\geq 0}} B_F\Big(\,\cdot\,,t+\frac{\mathfrak{r}_{n+1}}{\gamma_n}(\gamma_n-1)\Big)\,u_\alpha'\Big(\frac{\gamma_n}{\mathfrak{r}_{n+1}}t\Big)\cdot\frac{\gamma_n}{\mathfrak{r}_{n+1}}\,\ell(dt)\\
&= \int_{\mathbb{R}_{\geq 0}} \Big(\frac{\gamma_n}{\mathfrak{r}_{n+1}}\Big)^{\alpha+1} B_F\Big(\,\cdot\,,t+\mathfrak{r}_{n+1}\frac{\gamma_n-1}{\gamma_n}\Big)u_\alpha'(t)\,\ell(dt)\\
&= \int_{\mathbb{R}_{\geq 0}} \Big(\frac{\gamma_n}{\mathfrak{r}_{n+1}}\Big)^{\alpha+1} B_F\Big(\,\cdot\,,t+\mathfrak{r}_{n+1}\frac{\gamma_n-1}{\gamma_n}\Big)du_\alpha(t)\\
&= \int \Big(\frac{\gamma_n}{\mathfrak{r}_{n+1}}\Big)^{\alpha+1} \mathbb{1}_{\left[\mathfrak{r}_{n+1}\frac{\gamma_n-1}{\gamma_n},\infty\right)}(t)\,B_F(\,\cdot\,,t)\,\mu_{u_\alpha}(dt) \qquad (5.64)\end{aligned}$$

for every $n=0,\ldots,N-1$ and $\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$ with $\gamma_n\neq 0$. Take into account that in this case Lemmas B.3 and B.4 may be applied by Lemma 5.2.3 as well as the facts that B_F is an $\boldsymbol{L}_1(\mu_{u_\alpha})$-valued random element and that $\eta_n^{\boldsymbol{\gamma}}(\cdot)$ is a strictly increasing, (right-) continuous map. Using (5.52) and (5.63)–(5.64), it is easily seen that the right-hand side of (5.62) admits the representation

$$\begin{aligned}\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F) &= -\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{F;\boldsymbol{\gamma}}\,u_\alpha(x_0)\cdot\int_{\mathbb{R}_{\geq 0}} B_F(\,\cdot\,,y)(u_\alpha\circ\eta_k^{\boldsymbol{\gamma}})'(y)\,\ell(dy)\\
&= -\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{F;\boldsymbol{\gamma}}u_\alpha(x_0)\cdot\int\Big(\frac{\gamma_k}{\mathfrak{r}_{k+1}}\Big)^{\alpha+1}\mathbb{1}_{\left[\mathfrak{r}_{k+1}\frac{\gamma_k-1}{\gamma_k},\infty\right)}(t)\,\mathbb{1}_{\{\gamma_k\neq 0\}}\,B_F(\,\cdot\,,t)\,\mu_{u_\alpha}(dt)\\
&= \int h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}(t)B_F(\,\cdot\,,t)\,\mu_{u_\alpha}(dt) \qquad (5.65)\end{aligned}$$

for any $\boldsymbol{\gamma}=(\gamma_n)_{n=0}^{N-1}\in[0,1]^N$ with $h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}$ given by (5.61). Therefore, since $h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}(\cdot)\in\boldsymbol{L}_\infty(\mu_{u_\alpha})$ for every $\boldsymbol{\gamma}\in[0,1]^N$ (by equations (3.23)–(3.24) (applied to $\boldsymbol{P}_F$)) and since B_F is a centred Gaussian random element in $\boldsymbol{L}_1(\mu_{u_\alpha})$, the real-valued random variable $Z_{F;x_0,\boldsymbol{\gamma}}:=\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F)$ is for any $\boldsymbol{\gamma}\in[0,1]^N$ normally distributed with mean

$$\check{\mathbb{E}}\big[\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F)\big] \;=\; \int h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}(t)\check{\mathbb{E}}[B_F(\,\cdot\,,t)]\,\mu_{u_\alpha}(dt) \;=\; 0$$

(by Fubini's theorem) and variance

$$\begin{aligned}&\check{\mathbb{V}\mathrm{ar}}\big[\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F)\big] \;=\; \check{\mathbb{E}}\big[\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F)^2\big]\\
&= \check{\mathbb{E}}\Big[\Big(\int h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}(t_1)B_F(\,\cdot\,,t_1)\,\mu_{u_\alpha}(dt_1)\Big)\Big(\int h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}(t_2)B_F(\,\cdot\,,t_2)\,\mu_{u_\alpha}(dt_2)\Big)\Big]\\
&= \Gamma_{B_F,\mu_{u_\alpha}}(h_{\alpha,F}^{x_0,\boldsymbol{\gamma}},h_{\alpha,F}^{x_0,\boldsymbol{\gamma}}),\end{aligned}$$

where the latter expression is in view of Theorem 4.4.6 (with ν replaced by μ_{u_α}) equal to the right-hand side of (5.60).

(ii): It follows from part (ii) of Theorem 4.4.4 that $\dot{\mathcal{W}}^{x_0}_{0;F}(B_F)$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and

$$\sqrt{m}\big(\mathcal{W}^{x_0}_0(\widehat{F}_m) - \mathcal{W}^{x_0}_0(F)\big) \;\rightsquigarrow\; \dot{\mathcal{W}}^{x_0}_{0;F}(B_F) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|) \tag{5.66}$$

with $\dot{\mathcal{W}}^{x_0}_{0;F}$ given by (5.53) and B_F as above. Now, the right-hand side of (5.66) admits in view of (5.53) and (5.65) the representation

$$\dot{\mathcal{W}}^{x_0}_{0;F}(B_F) \;=\; \dot{\mathcal{W}}^{x_0;\pi_{\boldsymbol{\gamma}^F}}_{0;F}(B_F) \;=\; \int h^{x_0,\boldsymbol{\gamma}^F}_{\alpha,F}(t) B_F(t)\, \mu_{u_\alpha}(dt),$$

where $\boldsymbol{\gamma}^F := (\gamma^F_n)_{n=0}^{N-1} \in [0,1]^N$ consists in view of Lemma 3.2.4 (with $\boldsymbol{P}$ replaced by $\boldsymbol{P}_F$) of all unique solutions γ^F_n to the optimization problem (5.40) and $h^{x_0,\boldsymbol{\gamma}^F}_{\alpha,F}$ is defined as in (5.61) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}^F$). Therefore it can be verified easily with similar arguments as in the proof of (i) that the real-valued random variable $Z_{F;x_0} := \dot{\mathcal{W}}^{x_0}_{0;F}(B_F)$ is normally distributed with zero mean and variance $\check{\mathbb{V}}\mathrm{ar}[\dot{\mathcal{W}}^{x_0}_{0;F}(B_F)] = \Gamma_{B_F,\mu_{u_\alpha}}(h^{x_0,\boldsymbol{\gamma}^F}_{\alpha,F}, h^{x_0,\boldsymbol{\gamma}^F}_{\alpha,F})$, where the latter is equal to the right-hand side of (5.60) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}^F$). The proof of Theorem 5.2.9 is now complete. □

Remark 5.2.11 An easy computation shows that in the setting (and under the assumptions) of Theorem 5.2.9 the variance $s^2_{F;x_0,\boldsymbol{\gamma}}$ in (5.60) (and thus the variance $s^2_{F;x_0,\boldsymbol{\gamma}^F}$ in part (ii) of Theorem 5.2.9) admits the representation

$$\begin{aligned} s^2_{F;x_0,\boldsymbol{\gamma}} \;&=\; \sum_{k=0}^{N-1} \int_{\mathbb{R}^2_{\geq 0}} h^{F;\boldsymbol{\gamma}}_{\alpha;k}(s)\, C_F(t_1,t_2)\, h^{F;\boldsymbol{\gamma}}_{\alpha;k}(t)\, (\ell\otimes\ell)(d(t_1,t_2)) \\ &\quad + \sum_{\substack{i,k=0\\ i\neq k}}^{N-1} \int_{\mathbb{R}^2_{\geq 0}} h^{F;\boldsymbol{\gamma}}_{\alpha;i}(s)\, C_F(t_1,t_2)\, h^{F;\boldsymbol{\gamma}}_{\alpha;k}(t)\, (\ell\otimes\ell)(d(t_1,t_2)) \end{aligned}$$

for any $x_0 \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{\gamma} = (\gamma_n)_{n=0}^{N-1} \in [0,1]^N$ with

$$h^{F;\boldsymbol{\gamma}}_{\alpha;\cdot}(\bullet) \;:=\; \prod_{\substack{j=0\\ j\neq\cdot}}^{N-1} v^{F;\boldsymbol{\gamma}}_j u_\alpha(x_0)\, (u_\alpha \circ \eta^{\boldsymbol{\gamma}}_k)'(\bullet),$$

where $\eta^{\boldsymbol{\gamma}}_n$ is defined as in (3.22) and C_F is given by (4.33). ◇

In Remark 5.2.12 we discuss the utility of the statement in part (ii) of Theorem 5.1.18 with regard to the statistical estimation of the optimal value of the terminal wealth problem.

Remark 5.2.12 In view of part (ii) of Theorem 5.2.9 we can derive (under the assumptions of the latter theorem) from equation (5.2.9) an asymptotic confidence interval at a given level $\kappa \in (0,1)$ for the optimal value $\mathcal{W}^{x_0;\pi_{\boldsymbol{\gamma}^F}}_0(F)$ of the terminal wealth problem (5.39). In this case, however, one has to perform a nonparametric estimation of the variance $s^2 = s^2_{F;x_0;\boldsymbol{\gamma}^F}$ in (5.60) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}^F$) which is of the form

$$\widehat{s}^2_m \;=\; s^2_{\widehat{F}_m;x_0,\boldsymbol{\gamma}^{\widehat{F}_m}} \;:=\; \int_{\mathbb{R}^2} h^{x_0,\boldsymbol{\gamma}^{\widehat{F}_m}}_{\alpha,\widehat{F}_m}(s)\, C_{\widehat{F}_m}(s,t)\, h^{x_0,\boldsymbol{\gamma}^{\widehat{F}_m}}_{\alpha,\widehat{F}_m}(t)\, (\mu_{u_\alpha}\otimes\mu_{u_\alpha})(d(t_1,t_2)) \tag{5.67}$$

with $h^{x_0;\gamma^{\widehat{F}_m}}_{\alpha,\widehat{F}_m}$ and $C_{\widehat{F}_m}$ defined as in (5.61) and (4.33). Here the vector $\boldsymbol{\gamma}^{\widehat{F}_m} = (\gamma_n^{\widehat{F}_m})_{n=0}^{N-1} \in [0,1]^N$ consists of components $\gamma_n^{\widehat{F}_m}$ which are the solutions to the reduced optimization problem (5.40) with F replaced by $\widehat{F}_m$ (the existence is ensured by Lemma 3.2.4 (with $\boldsymbol{P}$ replaced by $\boldsymbol{P}_{\widehat{F}_m}$)). As the estimator $\widehat{s}_m^2$ in (5.67) for s^2 depends on $\widehat{F}_m$ in a quite complex way, it is not clear how good the performance of the asymptotic confidence interval based on $\widehat{s}_m^2$ is. To get around this problem, we will present in the following Theorem 5.2.13 a bootstrap result (in probability) with its help we are able to construct a so-called bootstrap confidence interval (see Remark 4.4.13) for the optimal value $\mathcal{W}_0^{x_0;\pi_{\gamma^F}}(F)$ of the terminal wealth problem (5.39). $\diamond$

Next, Theorem 5.2.13 illustrates Theorem 4.4.9 in the setting of Subsection 5.2.1. Part (ii) of this theorem reveals that the sequence $(\mathcal{W}_0^{x_0}(\widehat{F}_m^*))_{m\in\mathbb{N}}$ is a bootstrap version (in probability) of the sequence of estimators $(\mathcal{W}_0^{x_0}(\widehat{F}_m))_{m\in\mathbb{N}}$ for the optimal value of the terminal wealth problem (5.39). Note that d_{BL} introduced in Example 2.1.4 (with $E := \mathbb{R}$) stands for the bounded Lipschitz-metric on $\mathcal{M}_1(\mathbb{R})$. Finally, recall from Subsection 4.3.2 that a map $\phi : \mathbb{R} \to \mathbb{R}_{\geq 1}$ is said to be a weight function if it is continuous as well as non-decreasing on $\mathbb{R}_{\geq 0}$ and non-increasing on $\mathbb{R}_{\leq 0}$.

Theorem 5.2.13 (Bootstrap consistency of $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m))$ and $(\mathcal{W}_0^{x_0}(\widehat{F}_m))$) *In the setting of Subsection 5.2.1 let $x_0 \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{\gamma} \in [0,1]^N$. Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathbb{R}_{>0}$-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F, and assume that $\int \phi^2\, dF < \infty$ for some weight function ϕ satisfying $\int 1/\phi\, d\mu_{u_\alpha} < \infty$ (in particular $F \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$). Let $\widehat{F}_m$ be for every $m \in \mathbb{N}$ the empirical distribution function of $Y_1, \ldots, Y_m$ as defined in (4.31). Moreover let $(W_{mi})_{m\in\mathbb{N},1\leq i\leq m}$ be a triangular array of nonnegative real-valued random variables on another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, set $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$, and let $\widehat{F}_m^*$ be defined as in (4.37). If one of the settings (B1)–(B2) in Subsection 4.4.3 is met, then the following two assertions hold.*

(i) *For every $\delta > 0$ we have*

$$\lim_{m\to\infty} \mathbb{P}\big[\big\{\omega \in \Omega : d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m^*(\omega,\cdot)) - \mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m(\omega)))}, \mathrm{N}_{0,s^2}) \geq \delta\big\}\big] = 0, \tag{5.68}$$

where $s^2 = s^2_{F;x_0,\boldsymbol{\gamma}}$ is given by (5.60).

(ii) *For every $\delta > 0$ we have*

$$\lim_{m\to\infty} \mathbb{P}\big[\big\{\omega \in \Omega : d_{\mathrm{BL}}(\mathbb{P}'_{\sqrt{m}(\mathcal{W}_0^{x_0}(\widehat{F}_m^*(\omega,\cdot)) - \mathcal{W}_0^{x_0}(\widehat{F}_m(\omega)))}, \mathrm{N}_{0,s^2}) \geq \delta\big\}\big] = 0, \tag{5.69}$$

where $s^2 = s^2_{F;x_0;\boldsymbol{\gamma}^F}$ is given by (5.60) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}^F$).

Proof For the proof we intend to apply Theorem 4.4.9. At first, Lemma 5.2.8(i) ensures that $\widehat{F}_m(\omega, \cdot) \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$. It follows from part (i) of Lemma 5.2.4 along with the representation (4.37) that $\widehat{F}_m^*((\omega,\omega'), \cdot) \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ for every $(\omega,\omega') \in \overline{\Omega}$ and $m \in \mathbb{N}$. Moreover Lemma 5.2.10 implies that the estimators $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m)$ and $\mathcal{W}_0^{x_0}(\widehat{F}_m)$ are $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $m \in \mathbb{N}$. Using similar arguments as in the proof of Lemma 5.2.10 it is easily seen that the estimators $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(\widehat{F}_m^*)$ as well as $\mathcal{W}_0^{x_0}(\widehat{F}_m^*)$ are $(\overline{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable for any $m \in \mathbb{N}$. By the proof of Theorem 5.2.6 the map $\dot{\Lambda}_{n;F}^{F;(\pi_{\boldsymbol{\gamma}},x)} : \boldsymbol{L}_1(\mu_{u_\alpha}) \to \mathbb{R}$ given by (5.54) is linear for any

$x \in \mathbb{R}_{\geq 0}$ and $n = 0, \ldots, N-1$ and satisfies condition (a) of Theorem 4.3.8 (with Π replaced by Π_{lin}). Finally, note that in view of proof of Theorem 5.2.6 conditions (b)–(c) of Theorem 4.3.8 hold with ψ given by (3.17) and Π_{lin} playing the role of Π. Thus, we have verified the assumptions of Theorem 4.4.9 for Π_{lin} in place of Π. Take into account that any $\boldsymbol{\gamma} \in [0,1]^N$ induces a linear trading strategy $\pi \in \Pi_{\mathrm{lin}}$ by (3.26). In particular, it follows from the discussion subsequent to Theorem 4.4.9 that the expressions in (5.68) and (5.69) are well-defined.

(i): In view of part (i) of Theorem 4.4.9 and part (i) of Theorem 5.2.6, Display (5.68) holds for every $\delta > 0$ with N_{0,s^2} replaced by $\check{\mathbb{P}}_{\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F)}$, where $\dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}$ is defined in (5.52) and B_F refers to an $\boldsymbol{L}_1(\mu_{u_\alpha})$-valued centred Gaussian random variable on some probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ with covariance operator $\Gamma_{B_F,\mu_{u_\alpha}}$ given by (4.32) (with ν replaced by μ_{u_α}). Since by (the proof of) part (i) of Theorem 5.2.9 we have $Z_{F;x_0,\boldsymbol{\gamma}} := \dot{\mathcal{W}}_{0;F}^{x_0;\pi_{\boldsymbol{\gamma}}}(B_F) \sim \mathrm{N}_{0,s^2}$ (under $\check{\mathbb{P}}$) with $s^2 = s^2_{F;x_0,\boldsymbol{\gamma}}$ given by (5.60), the claim in (5.68) follows. We note that Theorem 5.2.9 is applicable because in view of Remark 4.4.7 (applied to $\nu := \mu_{u_\alpha}$) the assumptions $\int \phi^2 \, dF < \infty$ and $\int 1/\phi \, d\mu_{u_\alpha} < \infty$ ensure that $\int \sqrt{F(1-F)} \, d\mu_{u_\alpha} < \infty$.

(ii): By parts (ii) of Theorems 4.4.9 and 5.2.6, the claim in (5.69) holds for every $\delta > 0$ with N_{0,s^2} replaced by $\check{\mathbb{P}}_{\dot{\mathcal{W}}_{0;F}^{x_0}(B_F)}$, where $\dot{\mathcal{W}}_{0;F}^{x_0}$ is given by (5.53) and B_F is as in (i). In view of (the proof of) part (ii) of Theorem 5.2.9 we have $Z_{F;x_0} := \dot{\mathcal{W}}_{0;F}^{x_0} \sim \mathrm{N}_{0,s^2}$ (under $\check{\mathbb{P}}$) with $s^2 = s^2_{F;x_0,\boldsymbol{\gamma}^F}$ given by (5.60) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}^F$). Therefore the assertion in (5.69) holds. □

5.2.4 Parametric estimation of the optimal value

In the following we deal with a parametric estimation of the optimal value of the terminal wealth problem (5.39) in which the distribution function F describing the dynamics of the risky asset is not known. Throughout this section we will assume that the distribution of the (i.i.d.) asset returns follow a log-normal distribution with unknown parameters. This setting is motivated by the standard (time discretized) Black–Scholes–Merton model which is discussed in Example 3.2.8 in Subsection 3.2.3. Therefore the distribution function F corresponds to the distribution function of the log-normal distribution.

To this end, we consider the parametric statistical infinite product model

$$(\Omega, \mathcal{F}, \{\mathbb{P}^\theta : \theta \in \Theta\}) := \big(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \{\mathbb{P}^{(\mathfrak{m},\mathfrak{s}^2)} := \mathrm{LN}_{(\mathfrak{m},\mathfrak{s}^2)}^{\otimes \mathbb{N}} : (\mathfrak{m}, \mathfrak{s}^2) \in \Theta\}\big), \tag{5.70}$$

for the (open) parameter set $\Theta := \mathbb{R} \times \mathbb{R}_{>0}$ ($\subseteq \mathbb{R}^2$), where the log-normal distribution $\mathrm{LN}_{(\mathfrak{m},\mathfrak{s}^2)}$ with parameter $(\mathfrak{m}, \mathfrak{s}^2) \in \Theta$ is given by the standard Lebesgue density

$$\varphi^{\mathrm{LN}}_{(\mathfrak{m},\mathfrak{s}^2)}(x) := \begin{cases} (2\pi\mathfrak{s}^2)^{-1/2} x^{-1} e^{-(\log(x)-\mathfrak{m})^2/(2\mathfrak{s}^2)} & , \ x \in \mathbb{R}_{>0} \\ 0 & , \ \text{otherwise} \end{cases}. \tag{5.71}$$

Further, let $F_{(\mathfrak{m},\mathfrak{s}^2)}$ be for every $(\mathfrak{m}, \mathfrak{s}^2) \in \Theta$ the distribution function of the log-normal distribution $\mathrm{LN}_{(\mathfrak{m},\mathfrak{s}^2)}$. It is known that $F_{(\mathfrak{m},\mathfrak{s}^2)}$ can be represented as

$$F_{(\mathfrak{m},\mathfrak{s}^2)}(x) = \begin{cases} \Phi_{0,1}((\log(x) - \mathfrak{m})/\mathfrak{s}) & , \ x \in \mathbb{R}_{>0} \\ 0 & , \ \text{otherwise} \end{cases}, \tag{5.72}$$

for every $(\mathfrak{m}, \mathfrak{s}^2) \in \Theta$, where $\Phi_{0,1}$ stands for the distribution function of the standard normal distribution. Thus the following lemma is a direct conclusion of Lemma 5.2.4(i).

Lemma 5.2.14 *$F_{(\mathfrak{m},\mathfrak{s}^2)} \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ for every $(\mathfrak{m}, \mathfrak{s}^2) \in \Theta$.*

Now, a suitable estimator for the parameter $(\mathfrak{m}, \mathfrak{s}^2) \in \Theta$ based on sample size $m \in \mathbb{N}$ will be the map $(\widehat{\mathfrak{m}}_m, \widehat{\mathfrak{s}}^2_m) : \Omega \to \Theta$ defined by

$$\begin{aligned} \widehat{\mathfrak{m}}_m(y_1, y_2, \ldots) &= \widehat{\mathfrak{m}}_m(y_1, \ldots, y_m) \\ &:= \begin{cases} \frac{1}{m}\sum_{i=1}^m \log(y_i) & , \quad \min_{i=1,\ldots,m} y_i > 0 \\ \overline{\mathfrak{m}} & , \quad \text{otherwise} \end{cases}, \\ \widehat{\mathfrak{s}}^2_m(y_1, y_2, \ldots) &= \widehat{\mathfrak{s}}^2_m(y_1, \ldots, y_m) \\ &:= \begin{cases} \frac{1}{m}\sum_{i=1}^m \big(\log(y_i) - \widehat{\mathfrak{m}}_m(y_1, \ldots, y_m)\big)^2 & , \quad \min_{i=1,\ldots,m} y_i > 0 \\ \overline{\mathfrak{s}}^2 & , \quad \text{otherwise} \end{cases} \end{aligned} \tag{5.73}$$

for some fixed $(\overline{\mathfrak{m}}, \overline{\mathfrak{s}}^2) \in \Theta$. Take into account that the case differentiations in (5.73) ensure that the estimator $(\widehat{\mathfrak{m}}_m, \widehat{\mathfrak{s}}^2_m)$ takes values only in Θ.

By Lemma 5.2.14, we have that $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)})$ (resp. $\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)})$) can be seen as a reasonable (plug-in) estimator for $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\mathfrak{m},\mathfrak{s}^2)})$ (resp. $\mathcal{W}_0^{x_0}(F_{(\mathfrak{m},\mathfrak{s}^2)})$). In what follows we will investigate the sequences of estimators $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))_{m\in\mathbb{N}}$ and $(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))_{m\in\mathbb{N}}$ for strong consistency and asymptotic normality.

The following Theorem 5.2.15 illustrates Theorem 4.5.1 in the setting of Subsection 5.2.1.

Theorem 5.2.15 (Strong consistency of $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))$ and $(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))$) *In the setting of Subsection 5.2.1 let $x_0 \in \mathbb{R}_{\geq 0}$, $\boldsymbol{\gamma} \in [0,1]^N$, and $(\mathfrak{m}_0, \mathfrak{s}^2_0) \in \Theta$. Then the following two assertions hold.*

(i) *The sequence of estimators $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)})$ under $\mathbb{P}^{(\mathfrak{m}_0,\mathfrak{s}^2_0)}$ in the sense that*

$$\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}) \to \mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}) \quad \mathbb{P}^{(\mathfrak{m}_0,\mathfrak{s}^2_0)}\text{-a.s.}$$

(ii) *The sequence of estimators $(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))_{m\in\mathbb{N}}$ is strongly consistent for $\mathcal{W}_0^{x_0}(F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)})$ under $\mathbb{P}^{(\mathfrak{m}_0,\mathfrak{s}^2_0)}$ in the sense that*

$$\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}) \to \mathcal{W}_0^{x_0}(F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}) \quad \mathbb{P}^{(\mathfrak{m}_0,\mathfrak{s}^2_0)}\text{-a.s.}$$

If $\pi_{(\mathfrak{m}_0,\mathfrak{s}^2_0)} := (f_n^{\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}})_{n=0}^{N-1} \in \Pi_{\mathrm{lin}}$ corresponds to an optimal (linear) trading strategy w.r.t. $F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}$ (the existence is ensured by part (ii) of Theorem 3.2.5 (applied to $\boldsymbol{P}_{F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}}$), then part (ii) of the latter theorem yields that (for any initial amount of capital $x_0 \in \mathbb{R}_{\geq 0}$) the sequence of estimators $(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}))_{m\in\mathbb{N}}$ is strongly consistent (under $\mathbb{P}^{(\mathfrak{m}_0,\mathfrak{s}^2_0)}$) for the optimal $\mathcal{W}_0^{x_0;\pi_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}}(F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)})$ of the terminal wealth problem (5.39) (with F replaced by $F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}$). Take into account that the component $\gamma_n^{F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}}$ of the vector $\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}^2_0)} := (\gamma_n^{F_{(\mathfrak{m}_0,\mathfrak{s}^2_0)}})_{n=0}^{N-1}$ corresponds to the unique solution to the reduced optimization problem (5.40) with F replaced by $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$. Note that it follows from Lemma 3.2.4 (applied to $\boldsymbol{P}_{F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}$) that the latter optimization problem admits a unique solution.

Proof of Theorem 5.2.15: For the claims in (i) and (ii) we intend to apply Theorem 4.5.1. At first, in the corresponding infinite product model $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \{\mathbb{P}^{(\mathfrak{m},\mathfrak{s}^2)} := \mathrm{LN}_{(\mathfrak{m},\mathfrak{s}^2)}^{\otimes\mathbb{N}} : (\mathfrak{m},\mathfrak{s}^2) \in \Theta\})$ (see (5.70)), we obtain by means of the ordinary strong law of large numbers that the sequence of estimators $((\widehat{\mathfrak{m}}_m, \widehat{\mathfrak{s}}_m^2))_{m\in\mathbb{N}}$ given by (5.73) is strongly consistent for $(\mathfrak{m}_0, \mathfrak{s}_0^2)$ under $\mathbb{P}^{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$ (w.r.t. $\|\cdot\|$). Hence condition (a) of Theorem 4.5.1 holds.

Next, we will show that the mapping $(\mathfrak{m},\mathfrak{s}^2) \mapsto F_{(\mathfrak{m},\mathfrak{s}^2)}$ from Θ to $\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ is continuous at $(\mathfrak{m}_0, \mathfrak{s}_0^2)$ w.r.t. $(\|\cdot\|, \|\cdot\|_{1,\mu_{u_\alpha}})$, where the norm $\|\cdot\|_{1,\mu_{u_\alpha}}$ is introduced in (4.7) (with μ_{u_α} playing the role of ν).

To this end, let $(\mathfrak{m}_m, \mathfrak{s}_m^2)_{m\in\mathbb{N}}$ be any sequence in Θ with $\|(\mathfrak{m}_m, \mathfrak{s}_m^2) - (\mathfrak{m}_0, \mathfrak{s}_0^2)\| \to 0$. In particular this implies $\mathfrak{m}_m \to \mathfrak{m}_0$ and $\mathfrak{s}_m \to \mathfrak{s}_0$. Set $\overline{\mathfrak{m}} := \sup_{m\in\mathbb{N}} \mathfrak{m}_m$, $\overline{\mathfrak{s}} := \sup_{m\in\mathbb{N}} \mathfrak{s}_m$, and note that $\overline{\mathfrak{m}} < \infty$ as well as $0 < \overline{\mathfrak{s}} < \infty$. In view of (5.72) and the continuity of $x \mapsto \Phi_{0,1}(x)$ we observe

$$1 - F_{(\mathfrak{m}_m,\mathfrak{s}_m^2)}(x) \;=\; 1 - \Phi_{0,1}((\log(x) - \mathfrak{m}_m)/\mathfrak{s}_m) \;\longrightarrow\; 1 - \Phi_{0,1}((\log(x) - \mathfrak{m}_0)/\mathfrak{s}_0) \;=\; 1 - F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(x)$$

for every $x \in \mathbb{R}_{>0}$. Using (5.72) and the monotonicity of the mapping $x \mapsto \Phi_{0,1}(x)$ we obtain

$$1 - F_{(\mathfrak{m}_m,\mathfrak{s}_m^2)}(x) \;=\; 1 - \Phi_{0,1}((\log(x) - \mathfrak{m}_m)/\mathfrak{s}_m) \;\leq\; 1 - \Phi_{0,1}((\log(x) - \overline{\mathfrak{m}})/\overline{\mathfrak{s}}) \;=\; 1 - F_{(\overline{\mathfrak{m}},\overline{\mathfrak{s}}^2)}(x)$$

for any $x \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}$. Thus the mapping $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathfrak{h}(x) \;:=\; \begin{cases} 1 - F_{(\overline{\mathfrak{m}},\overline{\mathfrak{s}}^2)}(x) & , \quad x \in \mathbb{R}_{>0} \\ 1 & , \quad \text{otherwise} \end{cases}$$

is in view of

$$\|\mathfrak{h}\|_{1,\mu_{u_\alpha}} \;=\; \int_{\mathbb{R}_{\geq 0}} \big(1 - F_{(\overline{\mathfrak{m}},\overline{\mathfrak{s}}^2)}(x)\big)\, \mu_{u_\alpha}(dx) \;=\; \int_{\mathbb{R}_{\geq 0}} u_\alpha(y)\, dF_{(\overline{\mathfrak{m}},\overline{\mathfrak{s}}^2)}(y) \;=\; e^{\alpha\overline{\mathfrak{m}} + (\alpha\overline{\mathfrak{s}})^2/2} \;<\; \infty$$

(by Lemma 5.2.4(i)) a μ_{u_α}-integrable majorant. Hence an application of the dominated convergence theorem yields

$$\begin{aligned}\lim_{m\to\infty} \|F_{(\mathfrak{m}_m,\mathfrak{s}_m^2)} - F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}\|_{1,\mu_{u_\alpha}} \;&=\; \lim_{m\to\infty} \big\|(1 - F_{(\mathfrak{m}_m,\mathfrak{s}_m^2)}) - (1 - F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})\big\|_{1,\mu_{u_\alpha}} \\ &=\; \lim_{m\to\infty} \int_{\mathbb{R}_{>0}} \big|(1 - F_{(\mathfrak{m}_m,\mathfrak{s}_m^2)}(x)) - (1 - F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(x))\big|\, \mu_{u_\alpha}(dx) \;=\; 0.\end{aligned}$$

Therefore the mapping $(\mathfrak{m},\mathfrak{s}^2) \mapsto F_{(\mathfrak{m},\mathfrak{s}^2)}$ from Θ to $\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ is continuous at $(\mathfrak{m}_0, \mathfrak{s}_0^2)$ w.r.t. $(\|\cdot\|, \|\cdot\|_{1,\mu_{u_\alpha}})$. Thus condition (b) of Theorem 4.5.1 is also satisfied.

Finally, it follows from the proof of Theorem 5.2.2 that the assumptions of Theorem 4.3.3 are fulfilled with $\nu := \mu_{u_\alpha}$, $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$ in place of F, and Π replaced by Π_{lin}. Take into account that $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)} \in \boldsymbol{F}_{>0}(\mu_{u_\alpha})$ by Lemma 5.2.14. In particular, we have verified the assumptions of Theorem 4.5.1 for $\nu := \mu_{u_\alpha}$ and Π_{lin} playing the role of Π. Thus the assertions in (i) and (ii) of Theorem 5.2.15 are direct consequences of parts (i) and (ii) of Theorem 4.5.1, respectively. This completes the proof of Theorem 5.2.15. □

Part (ii) of Theorem 5.2.16 indicates the asymptotic error distribution of the sequence of estimators $(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}_m^2)}))_{m\in\mathbb{N}}$ for the optimal value $\mathcal{W}_0^{x_0}(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})$ of the terminal wealth problem (5.39) (with $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$ in place of F). For the formulation of this theorem we set as before $v_n^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma} := v_n^{F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)};\gamma_n}$ for any $n = 0,\ldots,N-1$ with $v_n^{F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)};\gamma_n}$ defined as on the left-hand side of (5.40), and recall that $(u_\alpha\circ\eta_n^\gamma)'$ denotes the first derivative of the (continuously differentiable) map $(u_\alpha\circ\eta_n^\gamma)(\cdot)$. Finally set as above $\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)} := (\gamma_n^{F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}})_{n=0}^{N-1}$ for the vector whose components are the unique solutions to the reduced optimization problem (5.40) (with F replaced by $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$).

Theorem 5.2.16 (Asymptotic error distribution of $(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}_m^2)}))$ and $(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}_m^2)}))$) *In the setting of Subsection 5.2.1 let $x_0\in\mathbb{R}_{\ge 0}$, $\boldsymbol{\gamma}\in[0,1]^N$, and $(\mathfrak{m}_0,\mathfrak{s}_0^2)\in\Theta$. Then the following two assertions hold.*

(i) *We have*

$$\sqrt{m}\big(\mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}_m^2)}) - \mathcal{W}_0^{x_0;\pi_{\boldsymbol{\gamma}}}(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})\big) \;\rightsquigarrow\; Z_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\boldsymbol{\gamma}} \qquad \text{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|)$$

for $Z_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\boldsymbol{\gamma}}\sim \mathrm{N}_{0,s^2}$, where

$$s^2 \;=\; s^2_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\boldsymbol{\gamma}} \;:=\; \Bigg(\mathfrak{s}_0^2\cdot\Big\{\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\cdot I_{1;k}^{\boldsymbol{\gamma}}\Big\}^2 + 2(\mathfrak{s}_0^2)^2\cdot\Big\{\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\cdot I_{2;k}^{\boldsymbol{\gamma}}\Big\}^2\Bigg)\cdot u_\alpha(x_0)^2 \tag{5.74}$$

with

$$\begin{aligned} I_{1;k}^{\boldsymbol{\gamma}} &:= \int_{\mathbb{R}_{\ge 0}} y\,(u_\alpha\circ\eta_k^\gamma)'(y)\,\varphi^{\mathrm{LN}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(y)\,\ell(dy),\\ I_{2;k}^{\boldsymbol{\gamma}} &:= \int_{\mathbb{R}_{\ge 0}} y\,(\log(y)-\mathfrak{m}_0)/(2\mathfrak{s}_0^2)\,(u_\alpha\circ\eta_k^\gamma)'(y)\,\varphi^{\mathrm{LN}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(y)\,\ell(dy). \end{aligned} \tag{5.75}$$

(ii) *We have*

$$\sqrt{m}\big(\mathcal{W}_0^{x_0}(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}_m^2)}) - \mathcal{W}_0^{x_0}(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})\big) \;\rightsquigarrow\; Z_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0} \qquad \text{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|)$$

for $Z_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0}\sim\mathrm{N}_{0,s^2}$, where $s^2 := s^2_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}$ is given by (5.74) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$).

Proof We intend to apply Theorem 4.5.4. At first, it can be verified easily that the family $\{\mathrm{LN}_{(\mathfrak{m},\mathfrak{s}^2)} : (\mathfrak{m},\mathfrak{s}^2)\in\Theta\}$ satisfies the assumptions of Theorem 6.5.1 in [63]. Hence an application of the latter theorem ensures that the estimator $(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}_m^2)$ given by (5.73) in the corresponding infinite product model $(\mathbb{R}^{\mathbb{N}},\mathcal{B}(\mathbb{R}^{\mathbb{N}}),\{\mathbb{P}^{(\mathfrak{m},\mathfrak{s}^2)} := \mathrm{LN}^{\otimes\mathbb{N}}_{(\mathfrak{m},\mathfrak{s}^2)} : (\mathfrak{m},\mathfrak{s}^2)\in\Theta\})$ satisfies

$$\sqrt{m}\left(\begin{bmatrix}\widehat{\mathfrak{m}}_m\\ \widehat{\mathfrak{s}}_m^2\end{bmatrix} - \begin{bmatrix}\mathfrak{m}_0\\ \mathfrak{s}_0^2\end{bmatrix}\right) \;\rightsquigarrow\; \widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)} \qquad \text{in } (\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2),\|\cdot\|), \tag{5.76}$$

where $\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$ is a bivariate normally distributed random variable with zero mean vector and covariance matrix $\mathcal{I}(\mathfrak{m}_0,\mathfrak{s}_0^2)^{-1}$. Here, $\mathcal{I}(\mathfrak{m}_0,\mathfrak{s}_0^2)$ denotes the Fisher information matrix at $(\mathfrak{m}_0,\mathfrak{s}_0^2)$, and it can be easily shown that

$$\mathcal{I}(\mathfrak{m}_0,\mathfrak{s}_0^2)^{-1} = \begin{bmatrix} \mathfrak{s}_0^2 & 0 \\ 0 & 2(\mathfrak{s}_0^2)^2 \end{bmatrix}.$$

In particular, condition (a) of Theorem 4.5.4 is satisfied. Take into account that the expression on the left-hand side in (5.76) is clearly $(\mathcal{F},\mathcal{B}(\mathbb{R}^2))$-measurable.

Next, we will verify condition (b) of Theorem 4.5.4. Consider the map $\mathfrak{F}:\Theta\to\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ defined by

$$\mathfrak{F}(\mathfrak{m},\mathfrak{s}^2) := F_{(\mathfrak{m},\mathfrak{s}^2)}. \tag{5.77}$$

note that the map $\mathfrak{F}$ is well-defined by Lemma 5.2.14. In the following we will show (using Lemma 4.5.7) that the map $\mathfrak{F}:\Theta\to\boldsymbol{F}_{>0}(\mu_{u_\alpha})$ defined by (5.77) is Hadamard differentiable at $(\mathfrak{m}_0,\mathfrak{s}_0^2)$ with trace $\boldsymbol{L}_1(\mu_{u_\alpha})$ (in the sense of Definition A.1(ii) in Section A) and Hadamard derivative $\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}:\mathbb{R}^2\to\boldsymbol{L}_1(\mu_{u_\alpha})$ given by

$$\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\tau_1,\tau_2)(x) := \begin{cases} -\big(\frac{\tau_1}{\mathfrak{s}_0}+\frac{\tau_2(\log(x)-\mathfrak{m}_0)}{2\mathfrak{s}_0^3}\big)\,\varphi_{0,1}^{\mathrm{N}}\big(\frac{\log(x)-\mathfrak{m}_0}{\mathfrak{s}_0}\big) & ,\ x\in\mathbb{R}_{>0} \\ 0 & ,\ \text{otherwise} \end{cases}, \tag{5.78}$$

where $\varphi_{0,1}^{\mathrm{N}}$ refers to the standard Lebesgue density of the standard normal distribution $\mathrm{N}_{0,1}$. For this proof, we will adapt arguments of the proof of Example 4.7 in [59].

For an application of Lemma 4.5.7, let the map $\mathfrak{f}:\Theta\times\mathbb{R}\to\mathbb{R}$ from (4.45) be defined by

$$\mathfrak{f}((\mathfrak{m},\mathfrak{s}^2),x) := F_{(\mathfrak{m},\mathfrak{s}^2)}(x). \tag{5.79}$$

From (5.72) we observe that for any fixed $x\in\mathbb{R}$ the map $\mathfrak{f}(\,\cdot\,,x)$ given by (5.79) is continuously differentiable on Θ with gradient

$$\nabla_{(\mathfrak{m},\mathfrak{s}^2)}\mathfrak{f}((\mathfrak{m},\mathfrak{s}^2),x) = \begin{cases} -\big(\frac{1}{\mathfrak{s}},\frac{\log(x)-\mathfrak{m}}{2\mathfrak{s}^3}\big)\,\varphi_{0,1}^{\mathrm{N}}\big(\frac{\log(x)-\mathfrak{m}}{\mathfrak{s}}\big) & ,\ x\in\mathbb{R}_{>0} \\ (0,0) & ,\ \text{otherwise} \end{cases}$$

for all $(\mathfrak{m},\mathfrak{s}^2)\in\Theta$.

Now, let $(\mathfrak{m}_0,\mathfrak{s}_0^2)\in\Theta$ be fixed, and define a map $\mathfrak{h}_\delta:\mathbb{R}\to\mathbb{R}$ by

$$\mathfrak{h}_\delta(x) := \begin{cases} 0 & ,\ x\in\mathbb{R}_{\le 0} \\ \frac{1}{\sqrt{\pi(\mathfrak{s}_0^2-\delta)}}\big(1+\frac{2}{\delta}\big) & ,\ x\in(0,e^{\mathfrak{m}_0-2\delta}) \\ \frac{1}{\sqrt{\pi}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{C_\delta+|\mathfrak{m}_0|+\delta}{\sqrt{\mathfrak{s}_0^2-\delta}^3}\Big) & ,\ x\in[e^{\mathfrak{m}_0-2\delta},e^{\mathfrak{m}_0+\delta}] \\ \frac{1}{\sqrt{\pi}}\,e^{-\frac{(\log(x)-\mathfrak{m}_0-\delta)^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\log(x)-\mathfrak{m}_0+\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^3}\Big) & ,\ x\in(e^{\mathfrak{m}_0+\delta},\infty) \end{cases}, \tag{5.80}$$

where $C_\delta := \sup_{x\in[e^{\mathfrak{m}_0-2\delta},e^{\mathfrak{m}_0+\delta}]}|\log(x)|\in\mathbb{R}_{\ge0}$, and $\delta>0$ is chosen such that $\mathfrak{s}_0^2-\delta>0$ (see also [59, p. 444]). Using the same line of argumentation as in the proof of Example 4.7 in [59] we get

$$\|\nabla_{(\mathfrak{m},\mathfrak{s}^2)}\mathfrak{f}((\mathfrak{m},\mathfrak{s}^2),x)\| \le \mathfrak{h}_\delta(x) \ \text{ for all } ((\mathfrak{m},\mathfrak{s}^2),x)\in((\mathfrak{m}_0-\delta,\mathfrak{m}_0+\delta)\times(\mathfrak{s}_0^2-\delta,\mathfrak{s}_0^2+\delta))\times\mathbb{R}.$$

Next, we show that the map $\mathfrak{h}_\delta$ is μ_{u_α}-integrable. Note that the Stieltjes measure μ_{u_α} w.r.t. u_α has Lebesgue density u'_α (by Lemma B.3). At first, we have

$$\int_{(0,e^{\mathfrak{m}_0-2\delta})} |\mathfrak{h}_\delta(y)|\,\mu_{u_\alpha}(dy) \;=\; \int_{(0,e^{\mathfrak{m}_0-2\delta})} \frac{1}{\sqrt{\pi(\mathfrak{s}_0^2-\delta)}}\Big(1+\frac{2}{\delta}\Big)\,u'_\alpha(y)\,\ell(dy)$$
$$=\; \frac{1}{\sqrt{\pi(\mathfrak{s}_0^2-\delta)}}\Big(1+\frac{2}{\delta}\Big)\cdot\int_0^{e^{\mathfrak{m}_0-2\delta}} u'_\alpha(y)\,dy \;=\; \frac{1}{\sqrt{\pi(\mathfrak{s}_0^2-\delta)}}\Big(1+\frac{2}{\delta}\Big)\,u_\alpha(e^{\mathfrak{m}_0-2\delta}) \;<\;\infty.$$

Further

$$\begin{aligned}
&\int_{[e^{\mathfrak{m}_0-2\delta},e^{\mathfrak{m}_0+\delta}]} |\mathfrak{h}_\delta(y)|\,\mu_{u_\alpha}(dy)\\
&=\; \int_{[e^{\mathfrak{m}_0-2\delta},e^{\mathfrak{m}_0+\delta}]} \frac{1}{\sqrt{\pi}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{C_\delta+|\mathfrak{m}_0|+\delta}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,u'_\alpha(y)\,\ell(dy)\\
&=\; \frac{1}{\sqrt{\pi}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{C_\delta+|\mathfrak{m}_0|+\delta}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\cdot\int_{e^{\mathfrak{m}_0-2\delta}}^{e^{\mathfrak{m}_0+\delta}} u'_\alpha(y)\,dy\\
&=\; \frac{1}{\sqrt{\pi}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{C_\delta+|\mathfrak{m}_0|+\delta}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\cdot\big(u_\alpha(e^{\mathfrak{m}_0+\delta})-u_\alpha(e^{\mathfrak{m}_0-2\delta})\big)\;<\;\infty.
\end{aligned}$$

For any $\lambda > e^{\mathfrak{m}_0+\delta}$, the change of variable formula entails that (recall that $\alpha\in(0,1)$ is fixed)

$$\begin{aligned}
&\int_{(e^{\mathfrak{m}_0+\delta},\lambda)} |\mathfrak{h}_\delta(y)|\,\mu_{u_\alpha}(dy)\\
&=\; \int_{(e^{\mathfrak{m}_0+\delta},\lambda)} \frac{1}{\sqrt{\pi}}\,e^{-\frac{(\log(y)-\mathfrak{m}_0-\delta)^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\log(y)-\mathfrak{m}_0+\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,u'_\alpha(y)\,\ell(dy)\\
&=\; \int_{e^{\mathfrak{m}_0+\delta}}^{\lambda} \frac{1}{\sqrt{\pi}}\,e^{-\frac{(\log(y)-\mathfrak{m}_0-\delta)^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\log(y)-\mathfrak{m}_0+\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,u'_\alpha(y)\,dy\\
&=\; \int_{\mathfrak{m}_0+\delta}^{\log(\lambda)} \frac{1}{\sqrt{\pi}}\,e^{-\frac{(z-\mathfrak{m}_0-\delta)^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|z-\mathfrak{m}_0+\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,u'_\alpha(e^z)\,e^z\,dz\\
&=\; \int_0^{\log(\lambda)-\mathfrak{m}_0-\delta} \frac{1}{\sqrt{\pi}}\,e^{-\frac{\widetilde{z}^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\widetilde{z}+2\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,\alpha\,e^{\alpha(\widetilde{z}+\mathfrak{m}_0+\delta)}\,d\widetilde{z}\\
&\le\; \int_0^{\log(\lambda)-\mathfrak{m}_0-\delta} \frac{1}{\sqrt{\pi}}\,e^{-\frac{\widetilde{z}^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\widetilde{z}+2\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,e^{\widetilde{z}+\mathfrak{m}_0+\delta}\,d\widetilde{z}\\
&=\; e^{(2\mathfrak{m}_0+\mathfrak{s}_0^2+\delta)/2}\cdot\int_0^{\log(\lambda)-\mathfrak{m}_0-\delta} \frac{1}{\sqrt{\pi}}\,e^{-\frac{(\widetilde{z}-\mathfrak{s}_0^2+\delta)^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\widetilde{z}+2\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,d\widetilde{z}.
\end{aligned}$$

Then proceeding as in [59, p. 446] we get

$$\begin{aligned}
&\int_{(e^{\mathfrak{m}_0+\delta},\infty)} |\mathfrak{h}_\delta(y)|\,\mu_{u_\alpha}(dy)\\
&=\; \lim_{\lambda\to\infty}\int_{e^{\mathfrak{m}_0+\delta}}^{\lambda} \frac{1}{\sqrt{\pi}}\,e^{-\frac{(\log(y)-\mathfrak{m}_0-\delta)^2}{2(\mathfrak{s}_0^2+\delta)}}\Big(\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}}+\frac{|\log(y)-\mathfrak{m}_0+\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^{\,3}}\Big)\,u'_\alpha(y)\,dy
\end{aligned}$$

$$\leq \sqrt{2(\mathfrak{s}_0^2+\delta)}\, e^{(2\mathfrak{m}_0+\mathfrak{s}_0^2+\delta)/2}\, \mathbb{E}\Big[\frac{1}{\sqrt{\mathfrak{s}_0^2-\delta}} + \frac{|Z+2\delta|}{\sqrt{\mathfrak{s}_0^2-\delta}^3}\Big] < \infty,$$

where Z denotes any normally distributed random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with mean $\mathfrak{s}_0^2 - \delta$ and variance $\mathfrak{s}_0^2 + \delta$. Hence $\mathfrak{h}_\delta$ given by (5.80) is indeed μ_{u_α}-integrable.

Therefore, the assumptions of Lemma 4.5.7 are satisfied and an application of this lemma yields that the map $\mathfrak{F}$ given by (5.77) is Hadamard differentiable at $(\mathfrak{m}_0, \mathfrak{s}_0^2)$ with trace $\boldsymbol{L}_1(\mu_{u_\alpha})$ and Hadamard derivative $\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)} : \mathbb{R}^2 \to \boldsymbol{L}_1(\mu_{u_\alpha})$ given by (5.78). Hence, condition (b) of Theorem 4.5.4 holds, too.

Since it follows from the proof of Theorem 5.2.6 as well as Lemma 5.2.14 that the assumptions of Theorem 4.3.8 are satisfied for $\nu := \mu_{u_\alpha}$, $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$ in place of F, Π_{lin} instead of Π, we have verified the assumptions of Theorem 4.5.4.

(i): As a consequence of part (i) of Theorem 4.5.4, $\dot{\mathcal{W}}^{x_0;\pi_\gamma}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}))$ is $(\check{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$-measurable and

$$\sqrt{m}\big(\mathcal{W}^{x_0;\pi_\gamma}_0(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}) - \mathcal{W}^{x_0;\pi_\gamma}_0(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})\big) \rightsquigarrow \dot{\mathcal{W}}^{x_0;\pi_\gamma}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|),$$

where $\dot{\mathcal{W}}^{x_0;\pi_\gamma}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}$ is defined as in (5.52). Further in view of (5.52) and (5.78) we observe

$$\begin{aligned}
&\dot{\mathcal{W}}^{x_0;\pi_\gamma}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\tau)) \\
&= -\sum_{k=0}^{N-1}\Bigg(\int_{\mathbb{R}_{\geq 0}} -\Big\{\frac{\tau_1}{\mathfrak{s}_0} + \frac{\tau_2(\log(y)-\mathfrak{m}_0)}{2\mathfrak{s}_0^3}\Big\}\,\varphi^{\mathrm{N}}_{0,1}\Big(\frac{\log(y)-\mathfrak{m}_0}{\mathfrak{s}_0}\Big)\,(u_\alpha\circ\eta^\gamma_k)'(y)\,\ell(dy) \\
&\qquad \cdot \prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\Bigg)\cdot u_\alpha(x_0) \\
&= \sum_{k=0}^{N-1}\Bigg(\Big\{\tau_1\cdot\int_{\mathbb{R}_{\geq 0}}\frac{1}{\mathfrak{s}_0}\,(u_\alpha\circ\eta^\gamma_k)'(y)\,\varphi^{\mathrm{N}}_{0,1}\Big(\frac{\log(y)-\mathfrak{m}_0}{\mathfrak{s}_0}\Big)\,\ell(dy) \\
&\qquad + \tau_2\cdot\int_{\mathbb{R}_{\geq 0}}\frac{\log(y)-\mathfrak{m}_0}{2\mathfrak{s}_0^3}\,(u_\alpha\circ\eta^\gamma_k)'(y)\,\varphi^{\mathrm{N}}_{0,1}\Big(\frac{\log(y)-\mathfrak{m}_0}{\mathfrak{s}_0}\Big)\,\ell(dy)\Big\} \\
&\qquad \cdot \prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\Bigg)\cdot u_\alpha(x_0) \\
&= \sum_{k=0}^{N-1}\Bigg(\Big\{\tau_1\cdot I^\gamma_{1;k} + \tau_2\cdot I^\gamma_{2;k}\Big\}\cdot\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\Bigg)\cdot u_\alpha(x_0) \\
&= \Bigg(\tau_1\cdot\Big\{\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\cdot I^\gamma_{1;k}\Big\} + \tau_2\cdot\Big\{\sum_{k=0}^{N-1}\prod_{\substack{j=0\\ j\neq k}}^{N-1} v_j^{(\mathfrak{m}_0,\mathfrak{s}_0^2);\gamma}\cdot I^\gamma_{2;k}\Big\}\Bigg)\cdot u_\alpha(x_0) \qquad (5.81)
\end{aligned}$$

for any $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$ with $I^\gamma_{1;k}$ and $I^\gamma_{2;k}$ given by (5.75). Take into account that we clearly have

$$\varphi^{\mathrm{N}}_{0,1}\Big(\frac{\log(y)-\mathfrak{m}_0}{\mathfrak{s}_0}\Big) = y\,\mathfrak{s}_0\,\varphi^{\mathrm{LN}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(y)$$

for every $y \in \mathbb{R}_{>0}$. Thus it is easily seen that the real-valued random variable

$$Z_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\gamma} := \dot{\mathcal{W}}^{x_0;\pi_\gamma}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}))$$

is centred normal with variance as in (5.74).

(ii): By part (ii) of Theorem 4.5.4 we get that that $\dot{\mathcal{W}}^{x_0}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})$ is $(\check{\mathcal{F}},\mathcal{B}(\mathbb{R}))$-measurable and

$$\sqrt{m}\big(\mathcal{W}^{x_0}_0(F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}) - \mathcal{W}^{x_0}_0(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})\big) \rightsquigarrow \dot{\mathcal{W}}^{x_0}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})) \quad \text{in } (\mathbb{R},\mathcal{B}(\mathbb{R}),|\cdot|),$$

where $\dot{\mathcal{W}}^{x_0}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}$ is defined as in (5.53). Hence it follows from (5.53) and (5.81) that the real-valued random variable

$$Z_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0} := \dot{\mathcal{W}}^{x_0}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})) = \dot{\mathcal{W}}^{x_0;\pi_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}_{0;F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(\dot{\mathfrak{F}}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}(\widetilde{Z}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}))$$

is centred normal with variance $s^2 := s^2_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\gamma_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}$ given by (5.74) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$). This completes the proof. □

The following remark concludes this subsection.

Remark 5.2.17 Part (ii) of Theorem 5.2.16 can be used to construct an asymptotic confidence interval at a given level $\kappa \in (0,1)$ for the optimal value $\mathcal{W}_0^{x_0;\pi_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})$ of the terminal wealth problem (5.39) (with $F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$ playing the role of F). However, for this construction the variance $s^2 := s^2_{(\mathfrak{m}_0,\mathfrak{s}_0^2);x_0,\gamma_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}}$ given by (5.74) (with $\boldsymbol{\gamma}$ replaced by $\boldsymbol{\gamma}_{(\mathfrak{m}_0,\mathfrak{s}_0^2)}$) must be estimated. As seen above, $(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)$ given by (5.73) provides a reasonable estimator for $(\mathfrak{m}_0,\mathfrak{s}_0^2)$. Thus the expression $\widehat{s}^2_m$ given by

$$\widehat{s}^2_m := s^2_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m);x_0,\gamma_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}}$$

defined as in (5.74) can be seen as a suitable estimator for the unknown variance s^2. Note that the vector $\boldsymbol{\gamma}_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)} = (\gamma_n^{F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}})_{n=0}^{N-1} \in [0,1]^N$ consists of components $\gamma_n^{F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}}$ which are the solutions to the reduced optimization problem (5.40) with F replaced by $F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}$ (the existence is ensured by Lemma 3.2.4 (with $\boldsymbol{P}$ replaced by $\boldsymbol{P}_{F_{(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)}}$). This estimator depends on $(\widehat{\mathfrak{m}}_m,\widehat{\mathfrak{s}}^2_m)$ in a rather complex manner so that the actual performance of the asymptotic confidence interval based on $\widehat{s}^2_m$ is not known. We note that a parametric bootstrap technique for the asymptotic error distribution of $\mathcal{W}_0^{x_0}(F_{(\mathfrak{m}_0,\mathfrak{s}_0^2)})$, which we will not discuss in this thesis, could improve the performance. ◇

Part II

Statistical inference for risk measures of collective risks in an individual model

Chapter 6

Foundations of risk measures and risk functionals

Risk measurement is a major task of risk management in financial institutions, such as insurance companies, banks and others. Therefore, risk measurement techniques are becoming increasingly important for the process of managing and assessing financial risks in practice. In most cases, it is useful to assess risks with a real number that can interpreted as an amount of capital which is required to financially secure these risks. A popular tool that maps a financial risks expressed by a random variable to a capital requirement expressed by a real number is a risk measure. Here and below we will only consider risks in an insurance context, that is, a risk corresponds to a financial claim resulting from an insurance contract.

In this chapter we will first give a short introduction into the theory of risk measures and associated risk functionals. A formal definition of risk measures and associated risk functionals is part of Section 6.1. Subsequently, we study in Section 6.2 so-called distortion risk measures as an important class of risk measures. In Section 4.3 we will present regularity properties of risk functionals associated with a large class of risk measures, and Section 6.4 is devoted to examples of risk measures used in actuarial practice.

6.1 Formal definition of risk measures and risk functionals

In order to give a formal definition of risk measures, we let in the following $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space in the sense of Definition A.26 in [35]. Let L^0 be the space of all finite-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ modulo the equivalence relation of $\mathbb{P}$-a.s. identity, and let $\mathbb{X} \subseteq L^0$ be a fixed vector space containing the constants. Note that the L^p-space (construed as the space of all p-fold integrable random elements from L^0) is for some given $p \in \mathbb{R}_{\geq 1}$ a standard example for $\mathbb{X}$. In the sequel, any element X of the space $\mathbb{X}$ will be interpreted as a financial risk (i.e. a possible claim) resulting from an insurance contract.

Now, let $\rho : \mathbb{X} \to \mathbb{R}$ be a map, referred to as *risk measure*. Note that in actuarial practice the expression $\rho(X)$ specifies the amount of capital (i.e. premium) needed for covering the risk $X \in \mathbb{X}$. The assessment of financial risks with regard to an appropriate risk measure is of crucial importance, especially for insurers. In particular from an insurer's point of view, one could therefore ask what characteristics a risk measure should have in order to adequately quantify these risks. The following terminologies in (i)–(iv) below has proved to be appropriate.

A risk measure $\rho : \mathbb{X} \to \mathbb{R}$ is said to be

(i) *monotone*, if $\rho(X_1) \leq \rho(X_2)$ for every $X_1, X_2 \in \mathbb{X}$ with $X_1 \leq X_2$,

(ii) *cash additive*, if $\rho(X+m) = \rho(X) + m$ for every $X \in \mathbb{X}$ and $m \in \mathbb{R}$,

(iii) *subadditive*, if $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for every $X_1, X_2 \in \mathbb{X}$,

(iv) *positively homogeneous*, if $\rho(\lambda X) = \lambda\,\rho(X)$ for every $X \in \mathbb{X}$ and $\lambda \geq 0$.

Following Definition 2.4 in [3], we will say that ρ is *coherent* if it satisfies conditions (i)–(iv).

For statistical investigations it is favorable to assess the risk $X \in \mathbb{X}$ in terms of its distribution (under $\mathbb{P}$) using a so-called risk functional associated with a suitable risk measure. In this case, however, one has to ensure that risks with the same distribution always have the same capital requirement measured by the corresponding risk measure. For this reason, we will restrict ourselves in the following to so-called law-invariant risk measures. Formally, a risk measure ρ is said to be *law-invariant* if $\rho(X_1) = \rho(X_2)$ whenever the elements X_1 and X_2 of $\mathbb{X}$ have the same law under $\mathbb{P}$. Hence, we may and do associate with any law-invariant risk measure ρ a *risk functional* $\mathcal{R}_\rho : \mathcal{M}(\mathbb{X}) \to \mathbb{R}$ through

$$\mathcal{R}_\rho(\mu) \;:=\; \rho(X_\mu). \tag{6.1}$$

Here $\mathcal{M}(\mathbb{X})$ stands for the set of all distributions of elements of $\mathbb{X}$, and X_μ is any random variable from $\mathbb{X}$ with law μ.

6.2 Distortion risk measures and the Kusuoka representation

In this section we will introduce with the distortion risk measure a typical example of a risk measure which is widely used in theory and applications; see, for example, [7, 31, 54, 56, 57, 60, 90] and references cited therein. In the second part of this section we will present the so-called Kusuoka representation which says that general law-invariant coherent risk measures can be expressed by distortion risk measures in a certain way.

Let $g : [0,1] \to [0,1]$ be a right-continuous distortion function, that is, a right-continuous and non-decreasing function satisfying $g(0) = 0$ and $g(1) = 1$. Since the left-sided limits of any monotonic function exist, a right-continuous distortion function is even càdlàg. Now, the *distortion risk measure associated with* g is the map $\rho_g : \mathbb{X}_g \to \mathbb{R}$ defined by

$$\rho_g(X) \;:=\; \int_{-\infty}^{\infty} y\, d(g \circ F_X)(y), \tag{6.2}$$

where $\mathbb{X}_g$ denotes the set of all real-valued random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$ for which $\int_{-\infty}^{\infty} |y|\, d(g \circ F_X)(y) < \infty$, and F_X corresponds to the distribution function of X. Note that $\mathbb{X}_g \subseteq L^0$ is a linear subspace of L^1. In particular, $\mathcal{M}(\mathbb{X}_g) \subseteq \mathcal{M}(L^1)$.

The value $\rho_g(X)$ can be seen for some given $X \in \mathbb{X}_g$ as the expectation w.r.t. the distorted distribution function $g \circ F_X$. Thus it is easily seen that the right-hand side of (6.2) admits the representation

$$\rho_g(X) \;=\; -\int_{-\infty}^{0} g(F_X(y))\, dy + \int_0^{\infty} \big(1 - g(F_X(y))\big)\, dy \quad \text{ for all } X \in \mathbb{X}_g. \tag{6.3}$$

It can be deduced from [90] that ρ_g is a law-invariant coherent risk measure on $\mathbb{X}_g$ if and only if g is convex. Note that any convex distortion function g is continuous on $[0,1)$ and might jump at 1.

Moreover it follows from [35, Theorem 4.70] that for some convex distortion function g the corresponding distortion risk measure ρ_g admits the representation

$$\rho_g(X) \;=\; \int_0^1 F_X^{\leftarrow}(y)\, g'_+(y)\, dy \qquad \text{for all } X \in \mathbb{X}_g, \tag{6.4}$$

where g'_+ refers to the right-sided derivative of g, and $F_X^{\leftarrow}$ denotes the left-continuous inverse of F_X defined by $F_X^{\leftarrow}(\cdot) := \inf\{z \in \mathbb{R} : F_X(z) \geq \cdot\}$. Note that every convex distortion function g admits a right-sided derivative g'_+; see, for example, Proposition A.4 in [35]. As a consequence of Proposition 2.22 in [56] and for convex g, the space L^p is for any $p > 1$ contained in $\mathbb{X}_g$ if and only if $\int_0^1 g'_+(y)^{p/(p-1)}\, dy < \infty$. In this case we have $\mathcal{M}(L^p) \subseteq \mathcal{M}(\mathbb{X}_g)$ for every $p > 1$.

Next we will present in part (i) of Theorem 6.2.1 below the so-called Kusuoka representation. This representation goes back to the pioneer work of [60], where the author showed that distortion risk measures ρ_g w.r.t. convex distortion functions g build blocks of general law-invariant risk measures ρ on $\mathbb{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ denoting the space of all bounded random variables from L^0. This result was extended to law-invariant coherent risk measures on more general spaces $\mathbb{X}$ under some additional (technical) assumptions; see [54].

The first part and the statement in part (i) of the following Theorem 6.2.1 is, in a special case, already known from [54, Theorem 2.12] for concave distortion functions. Part (ii) of this theorem, which can be deduced from Theorem 2.2(iii) in [57], involves for some given law-invariant coherent risk measure ρ the map $g_\rho : [0,1] \to [0,1]$ defined by

$$g_\rho(y) \;:=\; 1 - \rho(B_{1-y}), \tag{6.5}$$

where B_{1-y} corresponds to any Bernoulli distributed random variable with expectation $1-y$. Note that g_ρ is clearly a distortion function, and we will refer to g_ρ as the *distortion function associated with* ρ. Finally, recall that g'_+ denotes the right-sided derivative of g.

Theorem 6.2.1 (Kusuoka representation) *Fix $p \in \mathbb{R}_{\geq 1}$, and let $\rho : L^p \to \mathbb{R}$ be a law-invariant coherent risk measure. Then there exists a set $\mathcal{G}_\rho$ of continuous convex distortion functions such that the following two assertions hold:*

(i) $\rho(X) \;=\; \sup_{g \in \mathcal{G}_\rho} \rho_g(X)$ *for all $X \in L^p$.*

(ii) $\sup_{g \in \mathcal{G}_\rho} g'_+(y) \;\leq\; (1 - g_\rho(1 - \gamma y))/(\gamma y)$ *for all $\gamma, y \in (0,1)$.*

The statements in part (i) and (ii) of Theorem 6.2.1 will be used in the next section to verify regularity properties of risk functionals associated with certain risk measures.

6.3 Regularity of risk functionals

In the sequel, we will show regularity properties of certain risk functionals w.r.t. the so-called L^p-Wasserstein metric $d_{\mathrm{Wass},p}$ introduced in (6.7) below. Theorem 6.3.1 ahead recalls a statement from [56] which says that for a large class of risk measures on L^p the associated risk functional is continuous w.r.t. $d_{\mathrm{Wass},p}$. The main concern in this section is to verify that for a suitable class of risk measures on some $\mathbb{X}$ the associated risk functional is even Lipschitz continuous w.r.t. $d_{\mathrm{Wass},p}$; see Theorems 6.3.2 and 6.3.3 below.

Now, consider for some given $p \in \mathbb{R}_{\geq 1}$ the gauge function $\psi_p : \mathbb{R} \to \mathbb{R}_{\geq 1}$ defined by

$$\psi_p(y) := 1 + |y|^p. \tag{6.6}$$

In this case, the set $\mathcal{M}_1^{\psi_p}(\mathbb{R})$ (defined as in Subsection 2.1.1) can be identified with the set $\mathcal{M}(L^p)$ of all distributions of random variables from L^p (and vice versa). For brevity, we will write in the following $\mathcal{M}_1^p(\mathbb{R})$ instead of $\mathcal{M}_1^{\psi_p}(\mathbb{R})$. Moreover, for any given $\mu \in \mathcal{M}_1(\mathbb{R})$, we denote by $F_\mu^{-1}(\cdot) := \inf\{y \in \mathbb{R} : F_\mu(y) \geq \cdot\}$ the generalized inverse of the distribution function F_μ of μ. Then for some given $p \in \mathbb{R}_{\geq 1}$, the *$L^p$-Wasserstein distance* $d_{\mathrm{Wass},p}$ between $\mu, \nu \in \mathcal{M}_1^p(\mathbb{R})$ is defined by

$$d_{\mathrm{Wass},p}(\mu,\nu) := \Big(\int_0^1 \big|F_\mu^{-1}(y) - F_\nu^{-1}(y)\big|^p\,dy\Big)^{1/p}. \tag{6.7}$$

It is known from [28, 65] that $d_{\mathrm{Wass},p}$ admits for any $\mu, \nu \in \mathcal{M}_1^p(\mathbb{R})$ the following equivalent representation

$$d_{\mathrm{Wass},p}(\mu,\nu) = \inf_{(X_1,X_2)\in\Xi(\mu,\nu)} \mathbb{E}\big[|X_1 - X_2|^p\big]^{1/p}, \tag{6.8}$$

where $\Xi(\mu,\nu)$ denotes the set of all vectors (Z_1, Z_2) of real-valued random variables Z_1 and Z_2 on $(\Omega, \mathcal{F}, \mathbb{P})$ having distributions μ and ν, respectively. Therefore, it follows from [17, Lemma 8.1] that (6.7) defines a map $d_{\mathrm{Wass},p} : \mathcal{M}_1^p(\mathbb{R}) \times \mathcal{M}_1^p(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ which provides a metric on $\mathcal{M}_1^p(\mathbb{R})$. Moreover, it is shown in [17, Lemma 8.3] that $d_{\mathrm{Wass},p}$ generates the ψ_p-weak topology $\mathcal{O}_{\mathrm{w}}^{\psi_p}$ on $\mathcal{M}_1^p(\mathbb{R})$. Recall from Subsection 2.1.1 that the latter topology is defined to be the coarsest topology on $\mathcal{M}_1^p(\mathbb{R})$ such that all mappings $\mu \mapsto \int h\,d\mu$, $h \in \mathbb{C}_{\psi_p}(\mathbb{R})$, are continuous. Here $\mathbb{C}_{\psi_p}(\mathbb{R})$ stands for the set of all continuous maps $h \in \mathbb{M}_{\psi_p}(\mathbb{R})$ (with $\mathbb{M}_{\psi_p}(\mathbb{R})$ defined as in Section 1.4).

Further note that the L^1-Wasserstein metric admits for any $\mu, \nu \in \mathcal{M}_1^1(\mathbb{R})$ the representation

$$d_{\mathrm{Wass},1}(\mu,\nu) = \int_{-\infty}^{\infty} \big|F_\mu(y) - F_\nu(y)\big|\,dy. \tag{6.9}$$

Recall from Example 2.6 that the L^1-Wasserstein metric $d_{\mathrm{Wass},1}$ coincides with the Kantorovich metric d_{Kant} on $\mathcal{M}_1^1(\mathbb{R})$ given by (2.6).

The statement of the following theorem is an immediate consequence of Theorem 2.8 along with Remark 2.9 in [56].

Theorem 6.3.1 *Let $p \in \mathbb{R}_{\geq 1}$. Moreover let $\rho : L^p \to \mathbb{R}$ be a law-invariant coherent risk measure, and let $\mathcal{R}_\rho : \mathcal{M}(L^p) \to \mathbb{R}$ be the associated risk functional introduced in (6.1). Then $\mathcal{R}_\rho$ is continuous w.r.t. $(d_{\mathrm{Wass},p}, |\cdot|)$. In particular, $\mathcal{R}_\rho$ is even continuous for the ψ_p-weak topology $\mathcal{O}_{\mathrm{w}}^{\psi_p}$ on $\mathcal{M}_1^p(\mathbb{R})$.*

The following result gives sufficient conditions for a convex distortion function g under which the risk functional $\mathcal{R}_{\rho_g}$ associated with the (law-invariant coherent) distortion risk measure ρ_g is even Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},p}, |\cdot|)$. Note that this regularity result was already shown in [93, Lemma 3.1] (under different assumptions) for some weighted sup-norm in place of $d_{\mathrm{Wass},p}$. Recall that g'_+ refers to the right-sided derivative of g.

Theorem 6.3.2 *Let g be a convex distortion function. Moreover let $\rho_g : \mathbb{X}_g \to \mathbb{R}$ be the distortion risk measure associated with g as defined in (6.2), and let $\mathcal{R}_{\rho_g} : \mathcal{M}(\mathbb{X}_g) \to \mathbb{R}$ be the associated risk functional introduced in (6.1). Finally, suppose that there exists a finite constant $K > 0$ such that*

$$1 - g(y) \;\le\; K(1-y) \quad \textit{for all } y \in [0,1]. \tag{6.10}$$

Then the following two assertions hold.

(i) *$\mathcal{R}_{\rho_g}$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},1}, |\cdot|)$.*

(ii) *If in addition $\int_0^1 g'_+(y)^{p/(p-1)}\,dy < \infty$ for every $p > 1$, then the restriction of $\mathcal{R}_{\rho_g}$ to $\mathcal{M}(L^p)$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},p}, |\cdot|)$ for any $p > 1$.*

Proof We first prove the assertion in (i). By condition (6.10) and the convexity of the distortion function g, we observe that $|g(y) - g(y')| \le K|y - y'|$ for all $y, y' \in [0,1]$. Using this as well as Display (6.1) as well as the representations (6.3) and (6.9), we obtain

$$\begin{aligned}
&\big|\mathcal{R}_{\rho_g}(\mu) - \mathcal{R}_{\rho_g}(\nu)\big| \\
&= \Big|\int_{-\infty}^{0} \big(g(F_\nu(y)) - g(F_\mu(y))\big)\,dy + \int_0^{\infty} \big(g(F_\nu(y)) - g(F_\mu(y))\big)\,dy\Big| \\
&\le \int_{-\infty}^{0} \big|g(F_\mu(y)) - g(F_\nu(y))\big|\,dy + \int_0^{\infty} \big|g(F_\mu(y)) - g(F_\nu(y))\big|\,dy \\
&= \int_{-\infty}^{\infty} \big|g(F_\mu(y)) - g(F_\nu(y))\big|\,dy \;\le\; \int_{-\infty}^{\infty} K\big|F_\mu(y) - F_\nu(y)\big|\,dy \;=\; K\,d_{\mathrm{Wass},1}(\mu,\nu)
\end{aligned} \tag{6.11}$$

for every $\mu, \nu \in \mathcal{M}(\mathbb{X}_g)$ ($\subseteq \mathcal{M}(L^1)$). Thus $\mathcal{R}_{\rho_g}$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},1}, |\cdot|)$ (with Lipschitz constant K). This shows (i).

To prove (ii), note at first that it is discussed below of Display (6.4) that $\mathcal{M}(L^p) \subseteq \mathcal{M}(\mathbb{X}_g)$ holds for every $p > 1$ if (and only if) $\int_0^1 g'_+(y)^{p/(p-1)}\,dy < \infty$. Therefore, the claim in (ii) can be deduced from (6.11) along with the fact that $d_{\mathrm{Wass},1} \le d_{\mathrm{Wass},\lambda}$ for every $\lambda \ge 1$; see, for instance, [78, p. 163]. □

There are some popular law-invariant coherent risk measures which are not distortion risk measures as, for instance, the Expectile-based risk measure discussed in Example 6.4.5 in Section 6.4. In particular, Theorem 6.3.2 can *not* be used to verify the Lipschitz continuity (w.r.t. the Wasserstein metric) of the risk functional $\mathcal{R}_\rho$ for general law-invariant coherent risk measures ρ. For this reason, the following Theorem 6.3.3 will give a general device if ρ is not a distortion risk measure. Recall from (6.5) the definition of distortion function g_ρ associated with some (law-invariant) coherent risk measure ρ.

Theorem 6.3.3 *Let $p \in \mathbb{R}_{\ge 1}$. Moreover let $\rho : L^p \to \infty$ be a law-invariant coherent risk measure, and let $\mathcal{R}_\rho : \mathcal{M}(L^p) \to \mathbb{R}$ be the associated risk functional introduced in (6.1). Finally, suppose that there exist finite constants $K, \beta > 0$ such that*

$$1 - g_\rho(y) \;\le\; K(1-y)^\beta \quad \textit{for all } y \in [0,1]. \tag{6.12}$$

Then the restriction of $\mathcal{R}_\rho$ to $\mathcal{M}(L^\lambda)$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},\lambda}, |\cdot|)$ for every $\lambda > p$ with $\lambda\beta > 1$.

Proof Let $\lambda > p$ with $\lambda\beta > 1$ be arbitrary but fixed. Since ρ is a law-invariant coherent risk measure, there exists in view of Theorem 6.2.1 a set $\mathcal{G}_\rho$ of continuous convex distortion functions such that in view of part (ii) of this theorem and (6.12) we obtain

$$\begin{aligned}
\int_0^1 g'_+(y)^{\frac{\lambda}{\lambda-1}}\,dy \;&\le\; \int_0^1 \big(\sup_{g\in\mathcal{G}_\rho} g'_+(y)\big)^{\frac{\lambda}{\lambda-1}}\,dy \;\le\; \int_0^1 \Big(\frac{1-g_\rho(1-\gamma y)}{\gamma y}\Big)^{\frac{\lambda}{\lambda-1}}\,dy\\
&\le\; \int_0^1 \Big(\frac{K(1-(1-\gamma y))^\beta}{\gamma y}\Big)^{\frac{\lambda}{\lambda-1}}\,dy \;=\; K^{\frac{\lambda}{\lambda-1}}\cdot\gamma^{\frac{(\beta-1)\lambda}{\lambda-1}}\cdot\int_0^1 y^{\frac{(\beta-1)\lambda}{\lambda-1}}\,dy \;=\; C_\lambda \qquad (6.13)
\end{aligned}$$

for some fixed $\gamma \in (0,1)$, where $C_\lambda := K^{\lambda/(\lambda-1)}\cdot\gamma^{(\beta-1)\lambda/(\lambda-1)}\cdot\frac{\lambda-1}{\lambda\beta-1} > 0$ is clearly finite. In particular, this implies $\mathcal{M}(L^\lambda)\subseteq\mathcal{M}(\mathbb{X}_g)$. Hence and in view of part (i) of Theorem 6.2.1, Hölder's inequality as well as Displays (6.1), (6.4), (6.7), and (6.13) we have

$$\begin{aligned}
\big|\mathcal{R}_\rho(\mu)-\mathcal{R}_\rho(\nu)\big| \;&=\; \big|\sup_{g\in\mathcal{G}_\rho}\mathcal{R}_{\rho_g}(\mu)-\sup_{g\in\mathcal{G}_\rho}\mathcal{R}_{\rho_g}(\nu)\big|\\
&\le\; \sup_{g\in\mathcal{G}_\rho}\big|\mathcal{R}_{\rho_g}(\mu)-\mathcal{R}_{\rho_g}(\nu)\big| \;\le\; \sup_{g\in\mathcal{G}_\rho}\int_0^1 \big|F^{\leftarrow}_{X_\mu}(y)-F^{\leftarrow}_{X_\nu}(y)\big|\,g'_+(y)\,dy\\
&\le\; \sup_{g\in\mathcal{G}_\rho}\Big\{\Big(\int_0^1 \big|F^{\leftarrow}_{X_\mu}(y)-F^{\leftarrow}_{X_\nu}(y)\big|^\lambda\,dy\Big)^{1/\lambda}\cdot\Big(\int_0^1 g'_+(y)^{\frac{\lambda}{\lambda-1}}\,dy\Big)^{\frac{\lambda-1}{\lambda}}\Big\}\\
&\le\; C_\lambda^{\frac{\lambda-1}{\lambda}}\cdot d_{\mathrm{Wass},\lambda}(\mu,\nu)
\end{aligned}$$

for every $\mu,\nu\in\mathcal{M}(L^\lambda)$ ($\subseteq\mathcal{M}(\mathbb{X}_g)$). Consequently, the restriction of $\mathcal{R}_\rho$ to $\mathcal{M}(L^\lambda)$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},\lambda},|\cdot|)$ (with Lipschitz constant $C_\lambda^{(\lambda-1)/\lambda}$). □

Note that if ρ is a distortion risk measure associated with a convex distortion function g, then $g_\rho = g$ and condition (6.12) (with $\beta = 1$) boils down to condition (6.10).

6.4 Examples of risk measures used in practice

In this section we present several examples of risk measures which are widely used in economic practice and illustrate with these examples the terminologies and results from Sections 6.1–6.3.

Example 6.4.1 introduces the so-called mean value-based risk measure which can be interpreted in the insurance context as a net risk premium for future claims.

Example 6.4.1 (Mean value-based risk measure) The *mean value-based risk measure* is the map $\mathrm{MV}: L^1\to\mathbb{R}$ defined by

$$\mathrm{MV}(X) \;:=\; \mathbb{E}[X].$$

Clearly, MV is law-invariant and easily seen to be coherent. Moreover, it can be deduced from (6.2) that MV coincides with the distortion risk measure $\rho_{g_{\mathrm{MV}}}$ associated with the (convex) distortion function $g_{\mathrm{MV}} := \mathrm{Id}$, where Id refers to the identity map on $[0,1]$.

Since $\mathbb{X}_{g_{\mathrm{MV}}} = L^1$ and g_{MV} satisfied condition (6.10) for $K := 1$, it follows from part (i) of Theorem 6.3.2 that the risk functional $\mathcal{R}_{\mathrm{MV}}:\mathcal{M}(L^1)\to\mathbb{R}$ associated with MV defined as in (6.1) is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},1},|\cdot|)$. Further, in view of $(g_{\mathrm{MV}})'_+\equiv 1$, we observe $\int_0^1 g'_+(y)^{p/(p-1)}\,dy =$

$1 < \infty$ for every $p > 1$. Hence part (ii) of Theorem 6.3.2 ensures that the restriction of $\mathcal{R}_{\text{MV}}$ to $\mathcal{M}(L^p)$ $(\subseteq \mathcal{M}(L^1))$ is Lipschitz continuous w.r.t. $(d_{\text{Wass},p}, |\cdot|)$ for any $p > 1$. ◇

Note that for insurance companies the downside risk of future claims is highly relevant for determining a risk premium. In contrast to the mean value-based risk measure, this risk is taken into account by the so-called Value-at-Risk at level $\alpha \in (0,1)$ which is frequently used in actuarial practice. In particular, the premium based on the Value-at-Risk is sufficient to cover losses from future claims in (1-α)·100 percent of the cases.

Example 6.4.2 (Value-at-Risk) The *Value-at-Risk at level* $\alpha \in (0,1)$ is the map $\text{V@R}_\alpha : L^0 \to \mathbb{R}$ defined by

$$\text{V@R}_\alpha(X) := F_X^{\leftarrow}(\alpha) = \inf\{y \in \mathbb{R} : F_X(y) \geq \alpha\}.$$

Clearly, V@R_α is law-invariant, and it can be verified easily that it is monotone, cash additive, and positively homogeneous. According to [3, p. 216] V@R_α is *not* subadditive in general and therefore *not* coherent. It is known that V@R_α corresponds to the distortion risk measure ρ_{g_α} associated with the distortion function $g_\alpha := \mathbb{1}_{[\alpha,1]}$; see, for instance, [31, p. 590]. Since g_α does *not* satisfy condition (6.10), we can *not* apply Theorem 6.3.2 to ensure that the risk functional $\mathcal{R}_{\text{V@R}_\alpha} : \mathcal{M}(L^0) \to \mathbb{R}$ associated with V@R_α defined as in (6.1) is Lipschitz continuous w.r.t. the Wasserstein metric.

However, it follows from (6.7) (for $p = 1$) as well as the estimate $d_{\text{Wass},1} \leq d_{\text{Wass},p}$ for every $p \in \mathbb{R}_{\geq 1}$ (see, for example, [78, p. 163]) that the restriction of $\mathcal{R}_{\text{V@R}_\alpha}$ to $\mathcal{M}(L^p)$ $(\subseteq \mathcal{M}(L^0))$ is Lipschitz continuous w.r.t. $(d_{\text{Wass},p}, |\cdot|)$ for any $p \in \mathbb{R}_{\geq 1}$. ◇

The Value-at-Risk is often criticized for a number of two reasons. On the one hand, its lack of subadditivity penalizes diversification effects of risks. On the other hand, the Value-at-Risk completely ignores the severity of losses in the far tail of the claim distribution. In order to solve these issues, the so-called Average Value-at-Risk at level $\alpha \in (0,1)$ was introduced. It is sometimes also referred to as Tail Value-at-Risk.

Example 6.4.3 (Average Value-at-Risk) The *Average Value-at-Risk at level* $\alpha \in (0,1)$ is the map $\text{AV@R}_\alpha : L^1 \to \mathbb{R}$ defined by

$$\text{AV@R}_\alpha(X) := \frac{1}{1-\alpha} \int_\alpha^1 \text{V@R}_y(X)\, dy.$$

Clearly, AV@R_α is law-invariant, and it follows from Propositions 3.1 and 3.2 in [1] that AV@R_α is coherent. If F_X is continuous at $\text{V@R}_\alpha(X)$, then it can be deduced from [1, Corollary 5.3] that AV@R_α admits the representation

$$\text{AV@R}_\alpha(X) = \mathbb{E}[X \,|\, X \geq \text{V@R}_\alpha(X)] \quad \text{for all } X \in L^1.$$

Moreover it is known that AV@R_α corresponds to the distortion risk measure ρ_{g_α} associated with the (convex) distortion function $g_\alpha(\cdot) := \frac{1}{1-\alpha} \max\{\,\cdot - \alpha; 0\}$; see, for instance, [31, p. 591].

Since $\mathbb{X}_{g_\alpha} = L^1$ and g_α satisfies condition (6.10) for $K := \frac{1}{1-\alpha}$, it follows from part (i) of Theorem 6.3.2 that the risk functional $\mathcal{R}_{\text{AV@R}_\alpha} : \mathcal{M}(L^1) \to \mathbb{R}$ associated with AV@R_α defined as in (6.1) is Lipschitz continuous w.r.t. $(d_{\rm Wass,1}, |\cdot|)$. Further, in view of $(g_\alpha)'_+ = \frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]}$, it is easily seen that $\int_0^1 (g_\alpha)'_+(y)^{p/(p-1)}\,dy = (1-\alpha)^{1/(1-p)} < \infty$ for every $p > 1$. Thus an application of Theorem 6.3.2(ii) yields that the restriction of $\mathcal{R}_{\rm AV@R_\alpha}$ to $\mathcal{M}(L^p)$ $(\subseteq \mathcal{M}(L^1))$ is Lipschitz continuous w.r.t. $(d_{{\rm Wass},p}, |\cdot|)$ for every $p > 1$. ◇

It is known in the insurance context that the premium for future claims based on the mean value-based risk measure introduced in Example 6.4.1 is not suitable because it does not take into account fluctuations in the risks. The risk measure presented in the following example is more appropriate in this respect.

Example 6.4.4 (One-sided pth moment risk measure) The *one-sided pth moment risk measure associated with* $p \in \mathbb{R}_{\ge 1}$ *and* $\alpha \in (0,1]$ is the map ${\rm OM}_{p,\alpha} : L^p \to \mathbb{R}$ defined by

$$\mathrm{OM}_{p,\alpha}(X) \;:=\; \mathbb{E}[X] + \alpha\,\mathbb{E}\big[\big((X-\mathbb{E}[X])^+\big)^p\big]^{1/p},$$

where $y^+ := \max\{y, 0\}$, $y \in \mathbb{R}$. Clearly, ${\rm OM}_{p,\alpha}$ is law-invariant, and it follows from Lemma 4.1 in [34] that ${\rm OM}_{p,\alpha}$ is also coherent. However, ${\rm OM}_{p,\alpha}$ is not a distortion risk measure according to Lemma A.5 in [54]. Moreover it is easily seen that the distortion function $g_{{\rm OM}_{p,\alpha}}$ associated with ${\rm OM}_{p,\alpha}$ as defined in (6.5) can be represented as $g_{{\rm OM}_{p,\alpha}}(y) = y - \alpha y(1-y)^{1/p}$ for every $y \in [0,1]$.

Since $g_{{\rm OM}_{p,\alpha}}$ satisfies condition (6.12) for $K := 1+\alpha$ and $\beta := \frac{1}{p}$, it follows from Theorem 6.3.3 that restriction of the risk functional $\mathcal{R}_{{\rm OM}_{p,\alpha}} : \mathcal{M}(L^p) \to \mathbb{R}$ associated with ${\rm OM}_{p,\alpha}$ defined as in (6.1) to $\mathcal{M}(L^\lambda)$ $(\subseteq \mathcal{M}(L^p))$ is Lipschitz continuous w.r.t. $(d_{{\rm Wass},\lambda}, |\cdot|)$ for every $\lambda > p$. ◇

We conclude this section with the following Example 6.4.5. It introduces the so-called expectile-based risk measure (defined on L^2) which is increasingly finding interest in actuarial practice.

Example 6.4.5 (Expectile-based risk measure) The *expectile-based risk measure associated with* $\alpha \in [1/2, 1)$ is the map ${\rm E}_\alpha : L^2 \to \mathbb{R}$ defined by

$$\mathrm{E}_\alpha(X) \;:=\; \mathrm{argmin}_{z\in\mathbb{R}}\big\{\alpha\,\mathbb{E}\big[\big((X-z)^+\big)^2\big] + (1-\alpha)\,\mathbb{E}\big[\big((z-X)^+\big)^2\big]\big\}.$$

Clearly, ${\rm E}_\alpha$ is law-invariant, and it is shown in [8, Proposition 6] that E_α is coherent. It follows from Theorem 8 in [30] that ${\rm E}_\alpha$ is *not* a distortion risk measure unless $\alpha = \frac{1}{2}$. In the latter case, however, we even get ${\rm E}_\alpha = {\rm MV}$, where MV refers to the mean value-based risk measure introduced in Example 6.4.1. Moreover it is easily seen that the distortion function $g_{{\rm E}_\alpha}$ associated with ${\rm E}_\alpha$ as defined in (6.5) can be represented as $g_{{\rm E}_\alpha}(y) = \frac{(1-\alpha)y}{1-\alpha+(1-y)(2\alpha-1)}$ for every $y \in [0,1]$.

Since $g_{{\rm E}_\alpha}$ satisfies condition (6.12) for $K := \frac{\alpha}{1-\alpha}$ and $\beta := 1$, it follows from Theorem 6.3.3 that the restriction of the risk functional $\mathcal{R}_{{\rm E}_\alpha} : \mathcal{M}(L^2) \to \mathbb{R}$ associated with ${\rm E}_\alpha$ defined as in (6.1) to $\mathcal{M}(L^\lambda)$ $(\subseteq \mathcal{M}(L^2))$ is Lipschitz continuous w.r.t. $(d_{{\rm Wass},\lambda}, |\cdot|)$ for every $\lambda > 2$. ◇

Chapter 7

Nonparametric estimation of risk measures of collective risks in the individual model

In this chapter we deal (using the notation and terminology introduced in Section 6.1) with the statistical estimation of an appropriate individual premium for an insurance contract from a not necessarily homogeneous insurance collective using observed historical claims. As mentioned in the main introduction, the consideration of such insurance collectives is motivated by actuarial practice. In the following we will present candidates for the estimator of the individual premium based on a fixed number of observations of past claims which coincides with the collective size, and investigate their performance for a sufficiently large collective size in terms of consistency, asymptotic normality, and qualitative robustness. The main focus for constructing such estimators will be on suitable estimates of the distribution of the total claim of the insurance collective. If these estimates are plugged into a suitable risk functional associated with a risk measure (determined by the insurer), we derive candidates for the estimator of the (collective and) individual premium in the insurance collective.

Throughout this chapter we consider a so-called non-homogeneous individual risk model in the course of non-life insurance mathematics, where the involved random variables describing the individual risks are independent but not necessarily identically distributed. For the sake of simplicity, we will assume that the size $m \in \mathbb{N}$ of an insurance collective coincides with number of observed historically claims. Since in actuarial practice there can be deviations in the distributions of the observed claims despite collective formation, we will *not* assume that the corresponding risks resulting from these contracts are identically distributed according to some common law. However, we will suppose that these risks are independent since insurers often use this assumption in their risk models. As a consequence, the estimators for the distribution of the total claim (and thus for the individual premium) will be therefore based on independent but non identically distributed random variables. This approach should be of interest from insurer's point of view and extends the setting in [54, 61].

In Section 7.1 we will take up the nonparametric setting in [54, 61] to establish two candidates for the estimator of the sought (but unknown) total claim distribution and thus for the individual premium for each insurance contract for the next insurance period. Afterwards, in Section 7.2, we will present asymptotic properties of the corresponding estimators for the individual premium, such as strong consistency and asymptotic normality, which are in line with some results in [54, 61]. Finally, Section 7.3 is devoted to the so-called qualitative robustness of the sequence of estimators for individual premium which are based on the convolution of the empirical measure. The investigations in this section will justify (under certain assumptions) the choice of the latter estimator for the individual premium in a 'slightly' non-homogeneous insurance collective when the insurer assumes a homogeneous individual risk model for the computation of future single premiums, a procedure which is common in actuarial practice.

7.1 Nonparametric estimators for the individual premium

As already mentioned in the introduction above, we deal in the sequel with a non-homogeneous individual model in the context of non-life insurance mathematics. That is, we let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent real-valued random variables on a common probability space, where each random variable X_i has some distribution μ_i, and set for every $m \in \mathbb{N}$

$$S_m := \sum_{i=1}^{m} X_i.$$

Note that it is easily seen that the distribution of S_m is given by the m-fold convolution $*_{i=1}^{m}\,\mu_i$ of $\mu_1, \dots, \mu_m$. In actuarial practice, the random variables $X_1, \dots, X_m$ can be interpreted as the m individual risks in a (non-homogeneous) insurance collective of size m with single claim distributions $\mu_1, \dots, \mu_m$. In particular, the random variable S_m refers to the total claim of this collective and $*_{i=1}^{m}\,\mu_i$ corresponds to the total claim distribution. Thus $\mathcal{R}_\rho(*_{i=1}^{m}\,\mu_i)$ corresponds to the total premium for the insurance collective for an appropriate choice of a law-invariant risk measure ρ (in the sense of Section 6.1) describing the insurer's risk position. If we divide the total premium by the collective size m, then

$$\mathcal{R}_m := \frac{1}{m}\mathcal{R}_\rho(*_{i=1}^{m}\,\mu_i) \tag{7.1}$$

can be considered as a candidate for the premium of each insurance contract. Note that the expression on the right-hand side of (7.1) is in any case justified as a premium for each individual risk in a homogeneous insurance collective, where all single claim distributions $\mu_1, \dots, \mu_m$ coincide. However, due to the small data base, we are *not* in the position to estimate the single claim distributions $\mu_1, \dots, \mu_m$ individually but only approximately their convolution $*_{i=1}^{m}\,\mu_i$. As a consequence, no capital allocation is possible, and the only feasible individual premium in the non-homogeneous insurance collective can be obtained by dividing the total premium $\mathcal{R}_\rho(*_{i=1}^{m}\,\mu_i)$ equally by the collective size m. We note that Display (7.1) reflects the so-called balancing of risks in 'large' insurance collectives because the quantity $\frac{1}{m}\mathcal{R}_\rho(*_{i=1}^{m}\,\mu_i)$ is frequently essentially smaller than $\mathcal{R}_\rho(\mu_i)$, $i = 1, \dots, m$.

Motivated by the studies in [54, 61], we will present below to possibilities to estimate the individual premium $\mathcal{R}_m$ for future claims in the insurance collective based on observed data of size $m \in \mathbb{N}$ from the previous insurance period(s). In view of the right-hand side of (7.1), a suitable estimator for the individual premium $\mathcal{R}_m$ will be based on a statistical estimation of the total claim distribution $*_{i=1}^{m}\,\mu_i$. To explain our approach more explicitly, let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where each Y_i has law μ_i under $\mathbb{P}$. The random variable Y_i can be interpreted as the observed historical claim of the ith insurance contract from the previous insurance period(s).

At first, one way to construct an estimator for the total claim distribution $\ast_{i=1}^{m} \mu_i$ is to apply (under a suitable moment assumption) a central limit theorem to the total claim S_m; see, for example, Corollary 7.2.11(i) below. In fact, as the latter result implies that random variable S_m is asymptotically normally distributed, we could approximate its distribution by a normal distribution with the same mean and the same variance as S_m. Therefore, the normal distribution $\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}$ with estimated parameters can be seen as a nonparametric estimator for the total claim distribution $\ast_{i=1}^{m} \mu_i$. Here $\widehat{\mathfrak{m}}_m$ and $\widehat{\mathfrak{s}}_m^2$ correspond to the empirical mean and the empirical variance based on the observed historical claims $Y_1, \ldots, Y_m$; see Display (7.6) ahead. Note that the expressions $m\widehat{\mathfrak{m}}_m$ and $m\widehat{\mathfrak{s}}_m^2$ can only be considered (under a suitable moment assumption) as reasonable estimators for the mean and the variance of S_m in 'large' insurance collectives, respectively (this can be deduced from parts (i)–(ii) of Lemma 7.2.12 in Section 7.2). Consequently, the plug-in estimator

$$\widehat{\mathcal{R}}_m^{\mathtt{NA}} := \frac{1}{m} \mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) \tag{7.2}$$

can (asymptotically) be regarded as a reasonable nonparametric estimator for the individual premium $\mathcal{R}_m = \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^{m} \mu_i)$. In the sequel, we refer to $\widehat{\mathcal{R}}_m^{\mathtt{NA}}$ as the *normal approximation estimator*. This estimation approach of the individual premium was already discussed in the works of [54, 61] in the case of i.i.d. observed historical claims.

Besides this, the total claim distribution can be estimated directly by using the convolution of the nonparametric estimators for the single claim distributions $\mu_1, \ldots, \mu_m$. In actuarial practice, there is insufficient data on observed historical claims for some individual risks within the insurance collective. For this reason, we will use in the following the empirical probability measure $\widehat{\mu}_m$ based on all observed claims $Y_1, \ldots, Y_m$ given by

$$\widehat{\mu}_m := \frac{1}{m} \sum_{i=1}^{m} \delta_{Y_i} \tag{7.3}$$

as a nonparametric estimator for *each* of the single claim distributions $\mu_1, \ldots, \mu_m$. Note that $\widehat{\mu}_m$ is generally *not* an obvious choice for an estimator for each single claim distribution μ_i because it is based on data from all observed historical claims. In 'large' insurance collectives, however, this choice is justified (this can be deduced from Lemma 7.3.5 in Section 7.3). Thus the m-fold convolution

$$\widehat{\mu}_m^{\ast m} := (\widehat{\mu}_m)^{\ast m} \tag{7.4}$$

of $\widehat{\mu}_m$ can be seen as an estimator for the total claim distribution $\ast_{i=1}^{m} \mu_i$. In this case, the plug-in estimator

$$\widehat{\mathcal{R}}_m^{\mathtt{CE}} := \frac{1}{m} \mathcal{R}_\rho(\widehat{\mu}_m^{\ast m}), \tag{7.5}$$

which we refer to in the following as *empirical convolution estimator*, provides (asymptotically) a reasonable nonparametric estimator for the individual premium $\mathcal{R}_m = \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^{m} \mu_i)$. This approach for estimating the total claim distribution is already known from [61] in the case of i.i.d. claims.

In comparison to the normal approximation estimator $\widehat{\mathcal{R}}_m^{\mathtt{NA}}$ in (7.2), which is basically based on an evaluation of a normal distribution with estimated parameters, the determination of the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ in (7.5) is (significantly) more complex. This is due to the fact that the computation of the m-fold convolution $\widehat{\mu}_m^{*m}$ of the empirical measure $\widehat{\mu}_m$ can *not* be carried out exactly but only approximately, for instance, by an iteration scheme based on the so-called Panjer recursion (see [70]). We refer to the Appendix A in [61] for an iteration scheme for $\widehat{\mu}_m^{*m}$ in the case of i.i.d. claims.

In Section 7.2 we will show (under some assumptions) in Theorems 7.2.5 and 7.2.6 several asymptotic properties of the normal approximation estimator $\widehat{\mathcal{R}}_m^{\mathtt{NA}}$ and the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$, such as strong consistency and asymptotic normality. In contrast to [54, 61], where the results concerning the asymptotic behaviour of these estimators are based on (i.i.d. claims and) a regularity assumption for a large class of risk measures ρ w.r.t. some nonuniform weighted sup-norm, our results provide similar asymptotic properties for the respective estimators under (slightly) different assumptions.

Moreover, we will present in Displays (7.8) and (7.9) below asymptotic confidence intervals for the individual premium based on the nonparametric estimators $\widehat{\mathcal{R}}_m^{\mathtt{NA}}$ and $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$. Moreover we will see in Remark 7.2.8(ii) in Section 7.2 that both the estimated individual premiums $\widehat{\mathcal{R}}_m^{\mathtt{NA}}$ and $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ as well as the exact individual premium $\mathcal{R}_m$ can be approximated on the basis of a premium principle, which corresponds to a standard deviation principle.

7.2 Strong consistency and asymptotic error distribution for the individual premium estimators

In this section, we assume that $(Y_i)_{i\in\mathbb{N}}$ is a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that each Y_i has law μ_i. Recall that the random variables $Y_1, \ldots, Y_m$ represents the observed historical claims in the insurance collective of size m from the last period(s). Moreover let the estimators $\widehat{\mu}_m$ and $\widehat{\mu}_m^{*m}$ be defined as in (7.3) and (7.4), respectively, and set

$$\overline{\mathfrak{m}}_m := \frac{1}{m}\sum_{i=1}^m \mathbb{E}[Y_i] \quad \text{and} \quad \overline{\mathfrak{s}}_m^2 := \frac{1}{m}\sum_{i=1}^m \mathbb{V}\mathrm{ar}[Y_i].$$

Note that the latter expressions are well-defined under condition (a) of Assumption 7.2.1 below. The corresponding canonical nonparametric estimators for $\overline{\mathfrak{m}}_m$ and $\overline{\mathfrak{s}}_m^2$ are of the form

$$\widehat{\mathfrak{m}}_m := \frac{1}{m}\sum_{i=1}^m Y_i \quad \text{and} \quad \widehat{\mathfrak{s}}_m^2 := \frac{1}{m}\sum_{i=1}^m (Y_i - \widehat{\mathfrak{m}}_m)^2 \tag{7.6}$$

respectively.

Theorems 7.2.5 and 7.2.6 below show that under the following Assumption 7.2.1 the normal approximation estimator $\widehat{\mathcal{R}}_m^{\mathtt{NA}} = \frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2})$ and the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}} = \frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{*m})$ for the individual premium $\mathcal{R}_m = \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)$ are strongly consistent and asymptotically normal.

Condition (a) of Assumption 7.2.1 assumes that the sequence $(Y_i)_{i\in\mathbb{N}}$ is $L^{2\lambda}$-*bounded* for some $\lambda > 2$ in the sense that

$$\sup_{i\in\mathbb{N}} \mathbb{E}[|Y_i|^{2\lambda}] < \infty$$

(for a motivation of this condition, see part (i) of Remark 7.2.2 ahead). In particular, we note that the sequence $(Y_i)_{i\in\mathbb{N}}$ is $L^{2\lambda}$-bounded if and only if $(\mu_i) \in \mathcal{M}_\infty((L^{2\lambda})^{\mathbb{N}})$, where $\mathcal{M}_\infty((L^{2\lambda})^{\mathbb{N}})$ stands for the set of the distributions of the elements $(Z_i) \in (L^{2\lambda})^{\mathbb{N}}$ satisfying $\sup_{i\in\mathbb{N}} \int |z|^{2\lambda}\, \mathbb{P}_{Z_i}(dz) < \infty$.

For the formulation of the remaining conditions in Assumption 7.2.1, recall the notation and terminology introduced in Section 6.1. Let $d_{\mathrm{Wass},p}$ again be the L^p-Wasserstein metric on $\mathcal{M}_1^p(\mathbb{R})$ introduced in (6.7), and note that $\mathcal{M}_1^p(\mathbb{R})$ can be identified for any $p \in \mathbb{R}_{\geq 1}$ with the set $\mathcal{M}(L^p)$ of all distributions of random variables from L^p (and vice versa). Finally, let $\mathbb{X} \subseteq L^0$ be a fixed vector space containing the constants, and recall that $\mathrm{N}_{0,1}$ refers to the standard normal distribution.

Assumption 7.2.1 *Let $\rho : \mathbb{X} \to \mathbb{R}$ be a law-invariant risk measure. Moreover let $\mathcal{R}_\rho : \mathcal{M}(\mathbb{X}) \to \mathbb{R}$ be the associated risk functional introduced in (6.1), and assume that the following four conditions hold for some $\lambda > 2$.*

(a) $(\mu_i) \in \mathcal{M}_\infty((L^{2\lambda})^{\mathbb{N}})$*, that is, the sequence* $(Y_i)_{i\in\mathbb{N}}$ *is* $L^{2\lambda}$*-bounded.*

(b) $\overline{\mathfrak{s}}_m^2 > 0$ *for every* $m \in \mathbb{N}$*, and* $\lim_{m\to\infty} \overline{\mathfrak{s}}_m = \mathfrak{s}$ *for some* $\mathfrak{s} \in \mathbb{R}_{>0}$*.*

(c) ρ *is cash additive and positively homogeneous, and* $\mathcal{M}(L^\lambda) \subseteq \mathcal{M}(\mathbb{X})$*.*

(d) *The restriction of* $\mathcal{R}_\rho$ *to* $\mathcal{M}(L^\lambda)$ *is* $(d_{\mathrm{Wass},\lambda}, |\cdot|)$*-continuous at* $\mathrm{N}_{0,1}$*.*

The following Remark 7.2.2 as well as Examples 7.2.3–7.2.4 discuss and illustrate conditions (a)–(d) of Assumption 7.2.1.

Remark 7.2.2 (i) The proofs of Theorems 7.2.5 and 7.2.6 ahead reveal that the assertions of the latter theorems can be verified if we apply an appropriate strong law of large numbers and a suitable central limit theorem to the sequence $(Y_i)_{i\in\mathbb{N}}$; see Lemma 7.2.12 and Corollary 7.2.11 ahead. However, since the random variables $Y_1, Y_2, \ldots$ are assumed to be independent but *not* identically distributed, it turns out that for an application of these asymptotic results condition (a) of Assumption 7.2.1 is sufficient for $(Y_i)_{i\in\mathbb{N}}$.

(ii) If the random variables $Y_1, Y_2, \ldots$ are additionally identically distributed according to some common law μ, then it can be deduced from the proofs of Theorems 7.2.5 and 7.2.6 below that condition (a) of Assumption 7.2.1 can be replaced by the condition $\mu \in \mathcal{M}(L^\lambda)$, that is $\mathbb{E}[|Y_1|^\lambda] < \infty$. ◇

Example 7.2.3 (i) If the random variables $Y_1, Y_2, \ldots$ are additionally identically distributed according to some common law μ such that $\mathbb{V}\mathrm{ar}[Y_1] > 0$, then condition (b) of Assumption 7.2.1 is always fulfilled for $\mathfrak{s} := \mathbb{V}\mathrm{ar}[Y_1]^{1/2}$.

(ii) If $\mathbb{V}\mathrm{ar}[Y_i] > 0$ for any $i = 1, \ldots, m$ and $(k_m)_{m\in\mathbb{N}}$ is any sequence satisfying $k_m = o(m)$, $\overline{\mathfrak{s}}_{k_m} = \mathcal{O}(1)$ as well as $(\frac{1}{m}\sum_{i=k_m+1}^m \mathbb{V}\mathrm{ar}[Y_i])^{1/2} \to \mathfrak{s}$ for some $\mathfrak{s} \in \mathbb{R}_{>0}$, then condition (b) of Assumption 7.2.1 holds (with this $\mathfrak{s}$). Note that the last condition is satisfied, for example, if $\mathbb{V}\mathrm{ar}[Y_{k_m+1}] = \cdots = \mathbb{V}\mathrm{ar}[Y_m] =: \sigma^2$ with $\mathfrak{s} := \sigma > 0$. ◇

In view of Remark 7.2.3(ii) the convergence of $\overline{\mathfrak{s}}_m$ in the second part of condition (b) of Assumption 7.2.1 is ensured even if there is a subsequence $(Y_{k_m})_{m\in\mathbb{N}}$ of observed single claims with $k_m/m \to 0$ whose (cumulated) variances are bounded but which may assume large values. In particular, the setting allows (under some assumptions) a finite number of (extreme) outliers in the observed historical claims whose variances may differ significantly from the variances of the other observed historical claims.

Example 7.2.4 (i) If in the setting of Assumption 7.2.1 the risk measure ρ defined on $\mathbb{X} = L^p$ is additionally coherent, then condition (c) holds trivially and condition (d) of Assumption 7.2.1 is satisfied by Theorem 6.3.1.

(ii) Note that conditions (c)–(d) of Assumption 7.2.1 are not very restrictive. Indeed, it is discussed in the examples of Section 6.4 that there are some popular law-invariant risk measures which satisfy the corresponding conditions. We refer by way of example to the Value-at-Risk, the Average Value-at-Risk, and the Expectile-based risk measure introduced in Examples 6.4.2, 6.4.3, and 6.4.5, respectively. ◇

Theorem 7.2.5 below shows the asymptotic behaviour of the normal approximation estimator $\widehat{\mathcal{R}}_m^{\mathtt{NA}} = \frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2})$ for the individual premium $\mathcal{R}_m = \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)$. Its statements are basically known from [54, 61] in the case of i.i.d. observed claims. Note for part (v) of this theorem that the estimator $\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2})$ is clearly $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for any $m \in \mathbb{N}$ due to the representation in equation (7.7) below. Recall that $\stackrel{\mathrm{w}}{\longrightarrow}$ refers to the weak convergence of probability measures. Here and in the sequel, $o_{\mathbb{P}\text{-}a.s.}(m^{-1/2})$ refers to any sequence $(\xi_m)_{m\in\mathbb{N}}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ for which the expression $m^{1/2}\xi_m$ converges $\mathbb{P}$-a.s. to 0.

Theorem 7.2.5 (Asymptotics of $\widehat{\mathcal{R}}_m^{\mathtt{NA}}$) *Suppose that Assumption 7.2.1 holds for some $\lambda > 2$. Then the following assertions hold.*

(i) $\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2}) = (\widehat{\mathfrak{m}}_m - \overline{\mathfrak{m}}_m) + o_{\mathbb{P}\text{-}a.s.}(m^{-1/2})$.

(ii) $\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i) = o(m^{-1/2})$.

(iii) $\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i) = (\widehat{\mathfrak{m}}_m - \overline{\mathfrak{m}}_m) + o_{\mathbb{P}\text{-}a.s.}(m^{-1/2})$.

(iv) $m^r\big(\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big) \longrightarrow 0$ $\mathbb{P}$-*a.s.* *for every* $r < \frac{1}{2}$.

(v) $\mathbb{P} \circ \big\{\sqrt{m}\big(\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big)\big\}^{-1} \stackrel{\mathrm{w}}{\longrightarrow} \mathrm{N}_{0,\mathfrak{s}^2}$.

The following theorem gives analogue asymptotic results for the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}} = \frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m})$ introduced in (7.5) for the individual premium $\mathcal{R}_m = \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)$ given by (7.1). It extends in some way Theorem 2.3 in [61] for the case of non identically distributed observed claims.

Theorem 7.2.6 (Asymptotics of $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$) *Suppose that Assumption 7.2.1 holds for some $\lambda > 2$ and that $\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m})$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $m \in \mathbb{N}$. Then the following assertions hold.*

(i) $\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m}) = o_{\mathbb{P}\text{-}a.s.}(m^{-1/2})$.

(ii) $\frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i) = (\widehat{\mathfrak{m}}_m - \overline{\mathfrak{m}}_m) + o_{\mathbb{P}\text{-}a.s.}(m^{-1/2})$.

(iii) $m^r\big(\frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big) \longrightarrow 0$ $\mathbb{P}$-*a.s.* *for every* $r < \frac{1}{2}$.

(iv) $\mathbb{P} \circ \big\{\sqrt{m}\big(\frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big)\big\}^{-1} \stackrel{\mathrm{w}}{\longrightarrow} \mathrm{N}_{0,\mathfrak{s}^2}$.

The following remark shows that the measurability of the estimator $\mathcal{R}_\rho(\widehat{\mu}_m^{\ast m})$ assumed in Theorem 7.2.6 is not very restrictive.

Remark 7.2.7 It can be verified easily that for ρ equals the Value at Risk (see Example 6.4.2) or the distortion risk measure (see Section 6.2) the estimator $\mathcal{R}_\rho(\widehat{\mu}_m^{*m})$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $m \in \mathbb{N}$. Moreover, it follows from Remark 2.4 in [61] that the measurability also holds when ρ is a law-invariant coherent risk measures on $\mathbb{X} = L^p$ for some given $p \in \mathbb{R}_{\geq 1}$. ◇

Remark 7.2.8 (i) Under Assumption 7.2.1, part (iii) of Theorem 7.2.5 and part (ii) of Theorem 7.2.6 reveal that the asymptotic behaviour of the normal approximation estimator $\widehat{\mathcal{R}}_m^{\texttt{NA}} = \frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2})$ and the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\texttt{CE}} = \frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{*m})$ for the individual premium $\mathcal{R}_m = \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)$ is exactly the same and determined by the asymptotics of the empirical and the average mean, regardless of the underlying law-invariant risk measure ρ.

(ii) As a consequence of (the proofs of) Theorems 7.2.5 and 7.2.6, the nonparametric estimators $\widehat{\mathcal{R}}_m^{\texttt{NA}}$ and $\widehat{\mathcal{R}}_m^{\texttt{CE}}$ given by (7.2) and (7.5) admit the following more useful representations

$$\begin{aligned}\widehat{\mathcal{R}}_m^{\texttt{NA}} &= \widehat{\mathfrak{m}}_m + \frac{1}{\sqrt{m}}\mathcal{R}_\rho(\mathrm{N}_{0,1})\widehat{\mathfrak{s}}_m, \\ \widehat{\mathcal{R}}_m^{\texttt{CE}} &= \widehat{\mathfrak{m}}_m + \frac{1}{\sqrt{m}}\mathcal{R}_\rho(\mathrm{N}_{0,1})\widehat{\mathfrak{s}}_m + o_{\mathbb{P}\text{-}a.s.}(m^{-1/2}),\end{aligned} \tag{7.7}$$

respectively. Moreover, the individual premium can be represented as

$$\mathcal{R}_m = \overline{\mathfrak{m}}_m + \frac{1}{\sqrt{m}}\mathcal{R}_\rho(\mathrm{N}_{0,1})\overline{\mathfrak{s}}_m + o(m^{-1/2}).$$

Hence, these identities show that for a large collective size the individual premium can be approximated by the premium (with estimated parameters) which is determined according to the standard deviation principle with safety loading $\frac{1}{\sqrt{m}}\mathcal{R}_\rho(\mathrm{N}_{0,1})$. Note that the latter expression reflects the balancing of risks in 'large' insurance collectives. ◇

Remark 7.2.9 Under Assumption 7.2.1, we derive by means of part (ii) of Corollary 7.2.11, Lemma 7.2.12(iii) as well as Slutzky's lemma from part (iii) of Theorem 7.2.5 as well as part (ii) of Theorem 7.2.6 the following asymptotic confidence intervals at level $\kappa \in (0, 1)$ for the individual premium $\mathcal{R}_m$ in (7.1):

$$\Big[\widehat{\mathcal{R}}_m^{\texttt{NA}} - \frac{\widehat{\mathfrak{s}}_m}{\sqrt{m}}\Phi_{0,1}^{-1}\Big(1 - \frac{\kappa}{2}\Big), \widehat{\mathcal{R}}_m^{\texttt{NA}} + \frac{\widehat{\mathfrak{s}}_m}{\sqrt{m}}\Phi_{0,1}^{-1}\Big(1 - \frac{\kappa}{2}\Big)\Big] \tag{7.8}$$

and

$$\Big[\widehat{\mathcal{R}}_m^{\texttt{CE}} - \frac{\widehat{\mathfrak{s}}_m}{\sqrt{m}}\Phi_{0,1}^{-1}\Big(1 - \frac{\kappa}{2}\Big), \widehat{\mathcal{R}}_m^{\texttt{CE}} + \frac{\widehat{\mathfrak{s}}_m}{\sqrt{m}}\Phi_{0,1}^{-1}\Big(1 - \frac{\kappa}{2}\Big)\Big], \tag{7.9}$$

where $\Phi_{0,1}$ refers to the distribution function of the standard normal distribution. ◇

Let us turn to the proofs of Theorems 7.2.5 and 7.2.6. In addition to Corollary 7.2.11 as well as Lemma 7.2.12 below, they rely on the following Wasserstein inequality in Display (7.10). It provides an upper bound for the distance between a suitable centred sum of random variables and the standard normal distribution w.r.t. the Wasserstein metric $d_{\mathrm{Wass},\lambda}$. The proof of this inequality is basically based on an invariance principle in the form of Theorem 5 in [77]; see also Theorem 1 in [37].

Theorem 7.2.10 (Wasserstein inequality) *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\sup_{i\in\mathbb{N}} \mathbb{E}[|X_i|^\lambda] < \infty$ for some $\lambda > 2$ and $\mathfrak{s}_m^2 := \sum_{i=1}^m \mathbb{V}\mathrm{ar}[X_i] > 0$ for any $m \in \mathbb{N}$. Moreover assume that $\lim_{m\to\infty} \mathfrak{s}_m/\sqrt{m} = \mathfrak{s}$ for some $\mathfrak{s} \in \mathbb{R}_{>0}$, and let for any $m \in \mathbb{N}$*

$$Z_m := \frac{\sum_{i=1}^m (X_i - \mathbb{E}[X_i])}{\sqrt{\sum_{i=1}^m \mathbb{V}\mathrm{ar}[X_i]}}.$$

Then there exists a finite constant $C_\lambda > 0$ such that

$$d_{\mathrm{Wass},\lambda}(\mathbb{P}_{Z_m}, \mathrm{N}_{0,1}) \le C_\lambda\, m^{1/\lambda - 1/2} \quad \text{for all } m \in \mathbb{N}. \tag{7.10}$$

Proof At first, for any fixed $m \in \mathbb{N}$, set $Z_{m;i} := (X_i - \mathbb{E}[X_i])/\mathfrak{s}_m$, $i = 1, \ldots, m$. Note that the random variables $Z_{m;1}, \ldots, Z_{m;m}$ are independent and satisfy $\mathbb{E}[Z_{m;i}] = 0$ as well as

$$\mathbb{E}[Z_{m;i}^2] = \frac{\mathbb{E}\big[(X_i - \mathbb{E}[X_i])^2\big]}{\mathfrak{s}_m^2} = \frac{\mathbb{V}\mathrm{ar}[X_i]}{\mathfrak{s}_m^2} \le 1 \; (< \infty) \tag{7.11}$$

for every $i = 1, \ldots, m$. Moreover, by assumption we find some $c_{\lambda,1} \in \mathbb{R}_{>0}$ such that $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^\lambda] \le c_{\lambda,1}$ for all $i \in \mathbb{N}$. In view of $\lim_{m\to\infty} \mathfrak{s}_m/\sqrt{m} = \mathfrak{s}$ for some $\mathfrak{s} \in \mathbb{R}_{>0}$ (by assumption), there exists a constant $c_{\lambda,2} \in \mathbb{R}_{>0}$ such that

$$\sum_{i=1}^m \mathbb{E}\big[|Z_{m;i}|^\lambda\big] = \sum_{i=1}^m \frac{\mathbb{E}\big[|X_i - \mathbb{E}[X_i]|^\lambda\big]}{\mathfrak{s}_m^\lambda} \le c_{\lambda,1} \cdot m^{1-\lambda/2} \cdot (m^{-1/2}\mathfrak{s}_m)^{-\lambda} \le c_{\lambda,1} c_{\lambda,2} \cdot m^{1-\lambda/2} \tag{7.12}$$

for any $m \in \mathbb{N}$. Note that the latter bound is clearly finite. Therefore it can be deduced from Theorem 1 in [37] that there exist independent real-valued random variables $\xi_1, \ldots, \xi_m$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}_{\xi_i} = \mathbb{P}_{Z_{m;i}}$, $i = 1, \ldots, m$, and independent real-valued random variables $\eta_1, \ldots, \eta_m$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}_{\eta_i} = \mathrm{N}_{0,\mathbb{E}[Z_{m;i}^2]}$, $i = 1, \ldots, m$, such that

$$\mathbb{E}\Big[\Big(\max_{k=1,\ldots,m} \Big|\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i\Big|\Big)^\lambda\Big] \le c_{\lambda,3} \sum_{i=1}^m \mathbb{E}\big[|Z_{m;i}|^\lambda\big], \tag{7.13}$$

where $c_{\lambda,3} \in \mathbb{R}_{>0}$ is a constant depending only on λ. Take into account that $\mathbb{E}[\eta_i] = 0$ and $\mathbb{E}[\xi_i^2] = \mathbb{E}[Z_{m;i}^2] = \mathbb{E}[\eta_i^2]$ for every $i = 1, \ldots, m$. For any fixed $m \in \mathbb{N}$, let $\widetilde{Z}_m := \sum_{i=1}^m \eta_i$, and note that $\mathbb{P}_{\widetilde{Z}_m} = \mathrm{N}_{0,1}$ because

$$\mathbb{V}\mathrm{ar}[\widetilde{Z}_m] = \sum_{i=1}^m \mathbb{V}\mathrm{ar}[\eta_i] = \sum_{i=1}^m \mathbb{E}[Z_{m;i}^2] = \sum_{i=1}^m \frac{\mathbb{V}\mathrm{ar}[X_i]}{\mathfrak{s}_m^2} = 1$$

by (7.11). Hence, in virtue of $Z_m = \sum_{i=1}^m Z_{m;i} \overset{\mathrm{d}}{=} \sum_{i=1}^m \xi_i$, we obtain by means of the representation (6.8) of the Wasserstein metric $d_{\mathrm{Wass},\lambda}$ as well as the estimates in (7.12)–(7.13)

$$\begin{aligned} d_{\mathrm{Wass},\lambda}(\mathbb{P}_{Z_m}, \mathrm{N}_{0,1})^\lambda &\le \mathbb{E}\Big[\Big|\sum_{i=1}^m \xi_i - \sum_{i=1}^m \eta_i\Big|^\lambda\Big] \le \mathbb{E}\Big[\Big(\max_{k=1,\ldots,m}\Big|\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i\Big|\Big)^\lambda\Big] \\ &\le c_{\lambda,3} \sum_{i=1}^m \mathbb{E}\big[|Z_{m;i}|^\lambda\big] \le c_{\lambda,3}\, c_{\lambda,1} c_{\lambda,2} \cdot m^{1-\lambda/2} \end{aligned} \tag{7.14}$$

for any $m \in \mathbb{N}$. Thus the assertion follows. □

Part (i) of the following Corollary 7.2.11 provides a central limit theorem for independent random variables satisfying a suitable moment assumption. The statement in (i) can also be deduced from the Lyapunov central limit theorem (see, for instance, [18, Theorem 27.3]), and part (ii) will be used to verify the assertions in Theorem 7.2.5(v) as well as Theorem 7.2.6(iv).

Corollary 7.2.11 *With the notation and under the assumptions of Theorem 7.2.10 the following two assertions hold.*

(i) $\mathbb{P} \circ \big\{\frac{1}{\mathfrak{s}_m} \sum_{i=1}^m (X_i - \mathbb{E}[X_i])\big\}^{-1} \stackrel{\mathrm{w}}{\longrightarrow} \mathrm{N}_{0,1}$.

(ii) $\mathbb{P} \circ \big\{\sqrt{m}\frac{1}{m} \sum_{i=1}^m (X_i - \mathbb{E}[X_i])\big\}^{-1} \stackrel{\mathrm{w}}{\longrightarrow} \mathrm{N}_{0,\mathfrak{s}^2}$.

Proof The claim in (i) is an immediate consequence of Theorem 7.2.10 and Lemma 2.1.1(ii)⇒(i). Take into account that the L^λ-Wasserstein metric $d_{\mathrm{Wass},\lambda}$ defined as in (6.7) generates in view of [17, Lemma 8.3] the ψ_λ-weak topology $\mathcal{O}_{\mathrm{w}}^{\psi_\lambda}$ on $\mathcal{M}_1^\lambda(\mathbb{R})$, where ψ_λ is given by (6.6). This shows (i). For (ii), note at first that

$$\sqrt{m}\frac{1}{m}\sum_{i=1}^m (X_i - \mathbb{E}[X_i]) \;=\; \mathfrak{s}_m m^{-1/2}\frac{1}{\mathfrak{s}_m}\sum_{i=1}^m (X_i - \mathbb{E}[X_i])$$

holds for any $m \in \mathbb{N}$. Thus, in virtue of $\lim_{m\to\infty} \mathfrak{s}_m/\sqrt{m} = \mathfrak{s}$ for some $\mathfrak{s} \in \mathbb{R}_{>0}$ (by assumption), the assertion in (ii) can be deduced from Slutzky's lemma as well as part (i). □

To verify Theorems 7.2.5 and 7.2.6, we finally need the following Lemma 7.2.12. Here and in the sequel, we set

$$\overline{\mathfrak{m}}_{\lambda,m} \;:=\; \frac{1}{m}\sum_{i=1}^m \mathbb{E}[|Y_i|^\lambda],$$

and consider

$$\widehat{\mathfrak{m}}_{\lambda,m} \;:=\; \frac{1}{m}\sum_{i=1}^m |Y_i|^\lambda$$

as the corresponding nonparametric estimator. Note that under condition (a) of Assumption 7.2.1 (for some $\lambda > 2$) the expectation $\mathbb{E}[|Y_m|^{\widetilde{\lambda}}]$ (and thus $\overline{\mathfrak{m}}_{\widetilde{\lambda},m}$) is clearly finite for any $\widetilde{\lambda} \in [1, 2\lambda]$ and $m \in \mathbb{N}$. Part (i) of the following lemma provides a strong law for the sequence of estimators $(\widehat{\mathfrak{m}}_m)_{m\in\mathbb{N}}$ and $(\widehat{\mathfrak{m}}_{\lambda,m})_{m\in\mathbb{N}}$.

Lemma 7.2.12 *Suppose that condition (a) of Assumption 7.2.1 holds for some $\lambda > 2$. Then the following three assertions hold.*

(i) $\widehat{\mathfrak{m}}_m - \overline{\mathfrak{m}}_m \to 0$ $\mathbb{P}$*-a.s. Moreover* $\widehat{\mathfrak{m}}_{\widetilde{\lambda},m} - \overline{\mathfrak{m}}_{\widetilde{\lambda},m} \to 0$ $\mathbb{P}$*-a.s. for every* $\widetilde{\lambda} \in [1, \lambda]$.

(ii) $\widehat{\mathfrak{s}}_m - \overline{\mathfrak{s}}_m \to 0$ $\mathbb{P}$*-a.s.*

(iii) *If in addition* $\lim_{m\to\infty} \overline{\mathfrak{s}}_m = \mathfrak{s}$ *for some* $\mathfrak{s} \in \mathbb{R}_{>0}$*, then* $\widehat{\mathfrak{s}}_m \to \mathfrak{s}$ $\mathbb{P}$*-a.s.*

Proof At first, the claims in (i) are an immediate consequence of Theorem 6.7 in [71] along with condition (a) of Assumption 7.2.1.

To prove (ii), we observe at first for every $m \in \mathbb{N}$ and $\omega \in \Omega$

$$\begin{aligned}
\left|\widehat{\mathfrak{s}}_m(\omega)^2 - (\overline{\mathfrak{s}}_m)^2\right| &= \left|\left(\widehat{m}_{2,m}(\omega) - \widehat{\mathfrak{m}}_m(\omega)^2\right) - \left(\overline{\mathfrak{m}}_{2,m} - (\overline{\mathfrak{m}}_m)^2\right)\right| \\
&\leq \left|\widehat{m}_{2,m}(\omega) - \overline{\mathfrak{m}}_{2,m}\right| + \left|\widehat{\mathfrak{m}}_m(\omega)^2 - (\overline{\mathfrak{m}}_m)^2\right| \\
&= \left|\widehat{m}_{2,m}(\omega) - \overline{\mathfrak{m}}_{2,m}\right| + \left|\widehat{\mathfrak{m}}_m(\omega) - \overline{\mathfrak{m}}_m\right| \cdot \left|\widehat{\mathfrak{m}}_m(\omega) + \overline{\mathfrak{m}}_m\right| \\
&= \left|\widehat{m}_{2,m}(\omega) - \overline{\mathfrak{m}}_{2,m}\right| + \left|\widehat{\mathfrak{m}}_m(\omega) - \overline{\mathfrak{m}}_m\right| \cdot \left|\left(\widehat{\mathfrak{m}}_m(\omega) - \overline{\mathfrak{m}}_m\right) + 2\overline{\mathfrak{m}}_n\right| \\
&\leq \left|\widehat{m}_{2,m}(\omega) - \overline{\mathfrak{m}}_{2,m}\right| + \left|\widehat{\mathfrak{m}}_m(\omega) - \overline{\mathfrak{m}}_m\right|^2 + 2\overline{\mathfrak{m}}_{1,m}\left|\widehat{\mathfrak{m}}_m(\omega) - \overline{\mathfrak{m}}_m\right| \\
&=: S_1(m,\omega) + S_2(m,\omega) + S_3(m,\omega).
\end{aligned}$$

Then $\lim_{m\to\infty} S_1(m,\omega) + S_2(m,\omega) = 0$ for $\mathbb{P}$-a.e. ω by part (i). Finally, in view of condition (a) of Assumption 7.2.1, there exists a finite constant $C > 0$ such that the summand $S_3(m,\omega)$ is bounded above by $2C|\widehat{\mathfrak{m}}_m(\omega) - \overline{\mathfrak{m}}_m|$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$. Using part (i) again, this implies $\lim_{m\to\infty} S_3(m,\omega) = 0$ for $\mathbb{P}$-a.e. ω. Thus we have shown

$$\left|\widehat{\mathfrak{s}}_m(\omega)^2 - (\overline{\mathfrak{s}}_m)^2\right| \to 0 \quad \mathbb{P}\text{-a.e. } \omega. \tag{7.15}$$

Since for any $\omega \in \Omega$ and $m \in \mathbb{N}$

$$\left|\widehat{\mathfrak{s}}_m(\omega) - \overline{\mathfrak{s}}_m\right| \leq \left|\widehat{\mathfrak{s}}_m(\omega)^2 - (\overline{\mathfrak{s}}_m)^2\right|^{1/2},$$

the assertion in (ii) follows from (7.15).

For (iii), note that

$$|\widehat{\mathfrak{s}}_m(\omega) - \mathfrak{s}| \leq |\widehat{\mathfrak{s}}_m(\omega) - \overline{\mathfrak{s}}_m| + |\overline{\mathfrak{s}}_m - \mathfrak{s}|$$

for every $\omega \in \Omega$ and $m \in \mathbb{N}$. In view of $\lim_{m\to\infty} \overline{\mathfrak{s}}_m = \mathfrak{s}$ (by assumption) and part (ii), we end up with $\lim_{m\to\infty} \widehat{\mathfrak{s}}_m(\omega) = \mathfrak{s}$ for $\mathbb{P}$-a.e. ω. This shows (iii). □

We are now in the position to proof Theorems 7.2.5 and 7.2.6.

Proof of Theorem 7.2.5: We will adapt arguments of the proof of Theorem 2.2 in [61].

(i): By condition (c) of Assumption 7.2.1 and the representation (6.1), for any $m \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) &= \sqrt{m}\widehat{\mathfrak{s}}_m \mathcal{R}_\rho(\mathrm{N}_{0,1}) + m\widehat{\mathfrak{m}}_m, \\
\mathcal{R}_\rho(\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2}) &= \sqrt{m}\overline{\mathfrak{s}}_m \mathcal{R}_\rho(\mathrm{N}_{0,1}) + m\overline{\mathfrak{m}}_m.
\end{aligned}$$

Thus, we obtain for any $m \in \mathbb{N}$

$$\frac{1}{m}\left(\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \mathcal{R}_\rho(\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2})\right) = m^{-1/2}(\widehat{\mathfrak{s}}_m - \overline{\mathfrak{s}}_m)\mathcal{R}_\rho(\mathrm{N}_{0,1}) + (\widehat{\mathfrak{m}}_m - \overline{\mathfrak{m}}_m).$$

Hence, the claim follows from part (ii) of Lemma 7.2.12.

(ii): For every $m \in \mathbb{N}$, let S_m and N_m be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are distributed according to $\ast_{i=1}^m \mu_i$ and $\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2}$, respectively. Moreover, for any $m \in \mathbb{N}$ we set

$$Z_m := \frac{S_m - m\overline{\mathfrak{m}}_m}{\sqrt{m}\overline{\mathfrak{s}}_m} \quad \text{and} \quad \widetilde{Z}_m := \frac{N_m - m\overline{\mathfrak{m}}_m}{\sqrt{m}\overline{\mathfrak{s}}_m}.$$

Note that $\sqrt{m}\bar{\mathfrak{s}}_m Z_m + m\bar{\mathfrak{m}}_m$ and $\widetilde{Z}_m$ has law $\ast_{i=1}^m \mu_i$ and $\mathrm{N}_{0,1}$, respectively. Then, in view of condition (c) of Assumption 7.2.1, we observe

$$\begin{aligned}\mathcal{R}_\rho(\mathrm{N}_{m\bar{\mathfrak{m}}_m, m\bar{\mathfrak{s}}_m^2}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i) &= \rho(\sqrt{m}\bar{\mathfrak{s}}_m \widetilde{Z}_m + m\bar{\mathfrak{m}}_m) - \rho(\sqrt{m}\bar{\mathfrak{s}}_m Z_m + m\bar{\mathfrak{m}}_m)\\ &= \sqrt{m}\bar{\mathfrak{s}}_m\big(\rho(\widetilde{Z}_m) - \rho(Z_m)\big)\\ &= \sqrt{m}\bar{\mathfrak{s}}_m\big(\mathcal{R}_\rho(\mathrm{N}_{0,1}) - \mathcal{R}_\rho(\mathfrak{z}_m)\big) \qquad (7.16)\end{aligned}$$

for any $m \in \mathbb{N}$, where $\mathfrak{z}_m$ denotes the law of Z_m. By Theorem 7.2.10, we find some finite constant $C_\lambda > 0$ such that $d_{\mathrm{Wass},\lambda}(\mathrm{N}_{0,1}, \mathfrak{z}_m) \le C_\lambda\, m^{1/\lambda - 1/2}$ for all $m \in \mathbb{N}$. In virtue of (7.16) as well as conditions (b) and (d) of Assumption 7.2.1 this implies

$$\frac{1}{m}\big|\mathcal{R}_\rho(\mathrm{N}_{m\bar{\mathfrak{m}}_m, m\bar{\mathfrak{s}}_m^2}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big| \;=\; m^{-1/2}\bar{\mathfrak{s}}_m\big|\mathcal{R}_\rho(\mathrm{N}_{0,1}) - \mathcal{R}_\rho(\mathfrak{z}_m)\big| \;=\; o(m^{-1/2}).$$

This shows the assertion in (ii).

(iii): The claim follows immediately from parts (i)–(ii) of Theorem 7.2.5.

(iv): In view of condition (a) of Assumption 7.2.1, Theorem 6.7 in [71] entails that $m^r(\widehat{\mathfrak{m}}_m - \bar{\mathfrak{m}}_m)$ converges $\mathbb{P}$-a.s. to 0 for any $r < \frac{1}{2}$. Hence the assertion arises from part (iii) of Theorem 7.2.5.

(v): Under conditions (a)–(b) of Assumption 7.2.1, part (ii) of Corollary 7.2.11 implies that the law of $\sqrt{m}(\widehat{\mathfrak{m}}_m - \bar{\mathfrak{m}}_m)$ converges weakly to the normal distribution $\mathrm{N}_{0,\mathfrak{s}^2}$, where $\mathfrak{s} \in \mathbb{R}_{>0}$ is as in condition (b) of Assumption 7.2.1. Thus the assertion follows from part (iii) of Theorem 7.2.5 and Slutzky's lemma. This completes the proof of Theorem 7.2.5. □

Proof of Theorem 7.2.6: We will adapt arguments of the proof of Theorem 2.3 in [61].

(i): Analogously to (7.16), for every $\omega \in \Omega$ and $m \in \mathbb{N}$ we have

$$\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m(\omega), m\widehat{\mathfrak{s}}_m^2(\omega)}) - \mathcal{R}_\rho(\widehat{\mu}_m^{*m}(\omega;\bullet)) \;=\; \sqrt{m}\widehat{\mathfrak{s}}_m(\omega)\big(\mathcal{R}_\rho(\mathrm{N}_{0,1}) - \mathcal{R}_\rho(\mathfrak{z}_m(\omega;\bullet))\big) \qquad (7.17)$$

with $\mathfrak{z}_m(\omega;\bullet)$ denoting the law of the random variable

$$\widehat{Z}_m^\omega(\cdot) \;:=\; \frac{\widehat{S}_m^\omega(\cdot) - m\widehat{\mathfrak{m}}_m(\omega)}{\sqrt{m}\widehat{\mathfrak{s}}_m(\omega)},$$

where $\widehat{S}_m^\omega(\cdot)$ is any random variable on some probability space $(\Omega^\omega, \mathcal{F}^\omega, \mathbb{P}^\omega)$ with law $\widehat{\mu}_m^{*m}(\omega;\bullet)$. Note that it is easily seen that $\mathfrak{z}_m(\omega;\bullet) \in \mathcal{M}(L^\lambda)$ for every $\omega \in \Omega$ and $m \in \mathbb{N}$. Also note for (7.17) that $\widehat{\mu}_m(\omega;\bullet)$ has mean $\widehat{\mathfrak{m}}_m(\omega)$ and standard deviation $\widehat{\mathfrak{s}}_m(\omega)$ for every fixed $\omega \in \Omega$ and any $m \in \mathbb{N}$. Then, similarly to (7.12) and (7.14), we obtain for all $m \in \mathbb{N}$

$$\begin{aligned}d_{\mathrm{Wass},\lambda}(\mathrm{N}_{0,1}, \mathfrak{z}_m(\omega;\bullet))^\lambda &\le C_\lambda \frac{\int |x - \widehat{\mathfrak{m}}_m(\omega)|^\lambda\, \widehat{\mu}_m(\omega;dx)}{\big(\int (x - \widehat{\mathfrak{m}}_m(\omega))^2\, \widehat{\mu}_m(\omega;dx)\big)^{\lambda/2}}\, m^{1-\lambda/2}\\ &\le C_\lambda\, 2^\lambda \frac{\widehat{\mathfrak{m}}_{\lambda,m}(\omega)}{\widehat{\mathfrak{s}}_m(\omega)^\lambda}\, m^{1-\lambda/2}, \qquad (7.18)\end{aligned}$$

where $C_\lambda \in \mathbb{R}_{>0}$ is a constant depending only on λ and being independent of m and ω. By condition (a) of Assumption 7.2.1 there exists some finite constant $c_\lambda > 0$ such that $\bar{\mathfrak{m}}_{\lambda,m} \le c_\lambda$ for all $m \in \mathbb{N}$. In view of part (i) of Lemma 7.2.12, this implies

$$\limsup_{m\to\infty} |\widehat{\mathfrak{m}}_{\lambda,m}(\omega)| \;\le\; \limsup_{m\to\infty} |\widehat{\mathfrak{m}}_{\lambda,m}(\omega) - \bar{\mathfrak{m}}_{\lambda,m}| + \limsup_{m\to\infty} \bar{\mathfrak{m}}_{\lambda,m} \;\le\; 0 + c_\lambda \;=\; c_\lambda \;<\; \infty \qquad (7.19)$$

for $\mathbb{P}$-a.e. ω. Hence the numerator of

$$\frac{\widehat{\mathfrak{m}}_{\lambda,m}(\omega)}{\widehat{\mathfrak{s}}_m(\omega)^\lambda} \tag{7.20}$$

is bounded above by a finite constant for $\mathbb{P}$-a.e. ω. Moreover it follows from part (iii) of Lemma 7.2.12 that the denominator of (7.20) converges to $\mathfrak{s}^\lambda$ for $\mathbb{P}$-a.e. ω. In particular, the expression in (7.20) converges to a positive constant for $\mathbb{P}$-a.e. ω. Hence the right-hand side of (7.18) converges to 0 for $\mathbb{P}$-a.e. ω which entails that

$$d_{\mathrm{Wass},\lambda}(\mathrm{N}_{0,1}, \mathfrak{z}_m(\omega;\bullet)) \to 0 \quad \mathbb{P}\text{-a.e. } \omega. \tag{7.21}$$

Thus, in view of the $(d_{\mathrm{Wass},\lambda}, |\cdot|)$-continuity of $\mathcal{R}_\rho$ at $\mathrm{N}_{0,1}$ (by condition (d) of Assumption 7.2.1), Lemma 7.2.12(iii), and (7.17), we arrive at

$$\frac{1}{m}\big|\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m(\omega), m\widehat{\mathfrak{s}}_m^2(\omega)}) - \mathcal{R}_\rho(\widehat{\mu}_m^{*m}(\omega;\bullet))\big| \;=\; m^{-1/2}\widehat{\mathfrak{s}}_m(\omega)\big|\mathcal{R}_\rho(\mathrm{N}_{0,1}) - \mathcal{R}_\rho(\mathfrak{z}_m(\omega;\bullet))\big| \;=\; o(m^{-1/2})$$

for $\mathbb{P}$-a.e. ω. This shows (i).

(ii): The claim follows from part (i) of Theorem 7.2.6 along with part (iii) of Theorem 7.2.5.

(iii)–(iv): Analogously to the proof of parts (iv)–(v) of Theorem 7.2.5, we get the assertions just by replacing part (iii) of Theorem 7.2.5 by part (ii) of Theorem 7.2.6. This completes the proof of Theorem 7.2.6. □

It can be deduced from the proofs of Theorems 7.2.5 and 7.2.6 that the statements in part (ii) of Theorem 7.2.5 and part (i) of Theorem 7.2.6 can be improved if the risk functional $\mathcal{R}_\rho$ is Lipschitz continuous in a certain sense. This will be shown in the following remark.

Remark 7.2.13 The rates of convergence in part (ii) of Theorem 7.2.5 and part (i) of Theorem 7.2.6 can be improved if condition (d) of Assumption 7.2.1 is replaced by the following slightly stronger condition:

(d') For every sequence $(\mathfrak{z}_m)_{m\in\mathbb{N}}$ in $\mathcal{M}(L^\lambda)$ with $d_{\mathrm{Wass},\lambda}(\mathfrak{z}_m, \mathrm{N}_{0,1}) \to 0$, there exists a finite constant $C_\rho > 0$ such that $|\mathcal{R}_\rho(\mathfrak{z}_m) - \mathcal{R}_\rho(\mathrm{N}_{0,1})| \le C_\rho\, d_{\mathrm{Wass},\lambda}(\mathfrak{z}_m, \mathrm{N}_{0,1})$ for all $m \in \mathbb{N}$.

Indeed, we obtain (under Assumption 7.2.1 for some $\lambda > 2$ with (d') in place of (d)) the following two assertions.

(i) $\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m \mu_i) \;=\; \mathcal{O}(m^{1/\lambda-1})$.

(ii) $\frac{1}{m}\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m, m\widehat{\mathfrak{s}}_m^2}) - \frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{*m}) \;=\; \mathcal{O}_{\mathbb{P}\text{-}a.s.}(m^{1/\lambda-1})$.

Note that $\mathcal{O}_{\mathbb{P}\text{-}a.s.}(m^{1/\lambda-1})$ refers to any sequence $(\xi_m)_{m\in\mathbb{N}}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ for which the sequence $(m^{1-1/\lambda}\xi_m)_{m\in\mathbb{N}}$ is bounded $\mathbb{P}$-a.s.

Proof Maintain the notation introduced in the proofs of Theorems 7.2.5 and 7.2.6.

For the claim in (i), note that we obtain by means of (7.16), condition (d'), Theorem 7.2.10, and condition (b) of Assumption 7.2.1

$$\frac{1}{m}\big|\mathcal{R}_\rho(\mathrm{N}_{m\overline{\mathfrak{m}}_m, m\overline{\mathfrak{s}}_m^2}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big| \;=\; m^{-1/2}\overline{\mathfrak{s}}_m\big|\mathcal{R}_\rho(\mathrm{N}_{0,1}) - \mathcal{R}_\rho(\mathfrak{z}_m)\big|$$

$$\leq \; m^{-1/2}\overline{\mathfrak{s}}_m\, C_\rho\, d_{\mathrm{Wass},\lambda}(\mathrm{N}_{0,1},\mathfrak{z}_m) \;\leq\; m^{-1/2}C_{\mathfrak{s}}\, C_\rho C_\lambda\, m^{1/\lambda-1/2} \;=\; C_{\mathfrak{s}}\, C_\rho C_\lambda\, m^{1/\lambda-1}$$

for every $m \in \mathbb{N}$, where $C_{\mathfrak{s}}, C_\rho, C_\lambda > 0$ are finite constants (independent of m). Take into account that condition (d') is applicable because under the imposed assumptions Theorem 7.2.10 implies that $d_{\mathrm{Wass},\lambda}(\mathfrak{z}_m, \mathrm{N}_{0,1}) \to 0$. This shows (i).

To prove (ii), we observe at first in view of condition (d'), (7.17)–(7.18), and (7.19)

$$\begin{aligned}
&\frac{1}{m}\big|\mathcal{R}_\rho(\mathrm{N}_{m\widehat{\mathfrak{m}}_m(\omega),m\widehat{\mathfrak{s}}^2_m(\omega)}) - \mathcal{R}_\rho(\widehat{\mu}^{*m}_m(\omega;\bullet))\big| \;=\; \sqrt{m}\widehat{\mathfrak{s}}_m(\omega)\big|\mathcal{R}_\rho(\mathrm{N}_{0,1}) - \mathcal{R}_\rho(\mathfrak{z}_m(\omega;\bullet))\big| \\
&\quad\leq\; m^{-1/2}\widehat{\mathfrak{s}}_m(\omega)C_\rho\, d_{\mathrm{Wass},\lambda}(\mathrm{N}_{0,1},\mathfrak{z}_m(\omega;\bullet)) \;\leq\; C_\rho C_\lambda m^{1/\lambda-1}\widehat{\mathfrak{m}}_{\lambda,m}(\omega)^{1/\lambda} \;\leq\; C_\rho C_\lambda c_\lambda^{1/\lambda}\, m^{1/\lambda-1}
\end{aligned}$$

for any $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. ω, where $C_\rho, C_\lambda, c_\lambda > 0$ are finite constants being independent of ω and m. Take into account that an application of condition (d') is justified by (7.21). Thus the assertion in (ii) follows. $\diamond$

Note that condition (d') in Remark 7.2.13 is also not very restrictive. Examples 6.4.1–6.4.5 in Section 6.4 reveal that the respective law-invariant risk measures satisfy the latter condition.

7.3 Qualitative robustness of the sequence of empirical convolution estimators

In this section we will show for some suitable class of risk measures that the sequence $(\widehat{\mathcal{R}}^{\mathtt{CE}}_m)_{m\in\mathbb{N}}$ of empirical convolution estimators introduced in (7.5) is qualitatively robust in the sense of Definition 7.3.1 below. Our investigation is motivated by the fact that, in contrast to our approach in Section 7.1, in actuarial practice insurers generally assume a homogeneous individual model (in which the individual risks from an insurance collective are modelled by a sequence of i.i.d. random variables with a common law μ) in order to calculate a future premium for each insurance contract for pragmatic reasons. Then analogously to our elaborations in Section 7.1, an appropriate individual premium (for the next insurance period) in such a homogeneous risk model based on collective size $m \in \mathbb{N}$ is of the form

$$\mathcal{R}_{m;\mu} \;:=\; \frac{1}{m}\mathcal{R}_\rho(\mu^{*m}) \tag{7.22}$$

for some predetermined law-invariant risk measure ρ that describes the insurer's risk position, and the empirical convolution estimator $\widehat{\mathcal{R}}^{\mathtt{CE}}_m$ introduced in (7.5) is in this setting a reasonable estimator for $\mathcal{R}_{m;\mu}$. Since in practice the actually observed distributions $\mu_1, \ldots, \mu_m$ of the single claims in an insurance collective of size m differ, sometimes considerably, from the hypothetically assumed law μ, the quantity $\mathcal{R}_m$ given by (7.1) based on $\mu_1, \ldots, \mu_m$ can also be seen as a candidate for the (exact) individual premium. However, if (under certain topological assumptions) the distance between each law μ_i and μ is 'small' in some 'weak' sense, then by Display (7.29) below, the distance between the individual premiums based on $\mathcal{R}_{m;\mu}$ and $\mathcal{R}_m$ is also 'small' whenever the corresponding risk functional associated with ρ is Lipschitz continuous w.r.t. some Wasserstein metric. Under these conditions, the quantity $\mathcal{R}_{m;\mu}$ in (7.22) may be regarded as an appropriate individual premium in a 'slightly' non-homogeneous insurance collective.

In the following we want to deal with the question whether an analogous statement for the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ can be transferred. To put it another way, under which conditions does the quantity $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ from (7.5) provide a reasonable estimator for the individual premium in a 'slightly' non-homogeneous insurance collective if the insurer assumes a homogeneous risk model and if the distance between all observed claim distributions and the hypothetically assumed law is 'small' in some 'weak' sense? In Definition 7.3.1 below we will introduce a notion of qualitatively robustness of the sequence $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ which formulates our intention mathematically, and Theorem 7.3.4 ahead will provide sufficient conditions under which the sequence $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ is qualitatively robust. The latter result justifies (under certain conditions) to some extent the choice of $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ for estimating the individual premium in 'large' insurance collectives with 'small' nonhomogeneities when the insurer assumes a homogeneous individual model for calculating the future single premium.

To formulate our considerations mathematically more precisely, we consider in the sequel the non-parametric statistical infinite product model

$$(\Omega,\mathcal{F},\{\mathbb{P}^\theta:\theta\in\Theta\}) \;:=\; \big(\mathbb{R}^{\mathbb{N}},\mathcal{B}(\mathbb{R})^{\otimes\mathbb{N}},\{\mathbb{P}^{(\mu_i)}:(\mu_i)\in\mathcal{M}^{\mathbb{N}}\}\big), \tag{7.23}$$

where $\mathcal{M}\subseteq\mathcal{M}_1(\mathbb{R})$ is any set of Borel probability measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and

$$\mathbb{P}^{(\mu_i)} \;:=\; \otimes_{i\in\mathbb{N}}\mu_i$$

is the infinite product measure of $\mu_1,\mu_2,\ldots\in\mathcal{M}$. Hence, the coordinate projections $Y_1,Y_2,\ldots$ on $\Omega=\mathbb{R}^{\mathbb{N}}$ are independent (under $\mathbb{P}^{(\mu_i)}$) and each Y_i has law μ_i under $\mathbb{P}^{(\mu_i)}$ for every $\mu_i\in\mathcal{M}$, $i\in\mathbb{N}$. Note that the random variables $Y_1,\ldots,Y_m$ can be seen as observed historical single claims with laws $\mu_1,\ldots,\mu_m$ (under $\mathbb{P}^{(\mu_i)}$) in an insurance collective of size $m\in\mathbb{N}$.

Fix $p\in\mathbb{R}_{\geq 1}$, and let $\rho: L^p\to\mathbb{R}$ be a law-invariant coherent risk measure with associated risk functional $\mathcal{R}_\rho:\mathcal{M}(L^p)\to\mathbb{R}$ introduced in (6.1). In the sequel, we consider the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}}=\frac{1}{m}\mathcal{R}_\rho(\widehat{\mu}_m^{*m})$ as defined in (7.5), where $\widehat{\mu}_m^{*m}$ is introduced in (7.4). As seen in Theorem 7.2.6, $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ provides (under Assumption 7.2.1) a suitable estimator for the individual premium $\mathcal{R}_m=\frac{1}{m}\mathcal{R}_\rho(\ast_{i=1}^m\mu_i)$ given by (7.1). Also note that it follows from Remark 7.2.7 that the latter estimator is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable for any $m\in\mathbb{N}$.

The following definition is in line with Definition 1.1 in [58]. For a motivation of this definition, see the discussion in the paragraph above of Display (7.23). Recall from Theorem 2.14 in [45] that the so-called *Prohorov metric* d_{Proh} as defined in [45, p. 27]) generates the weak topology on $\mathcal{M}_1(\mathbb{R})$.

Definition 7.3.1 (Qualitative robustness) *Let $M\subseteq\mathcal{M}$ and $\mu\in M$. The sequence of estimators $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ is said to be (asymptotically) robust at μ if for every $\varepsilon>0$ there exist $m_0\in\mathbb{N}$ and an open neighbourhood $U=U(\mu,\varepsilon;M)$ of μ for the relative weak topology $\mathcal{O}_{\mathrm{w}}\cap M$ such that*

$$\mu_i\in U,\ i\in\mathbb{N}\quad\Longrightarrow\quad d_{\mathrm{Proh}}\big(\mathbb{P}^{(\mu)}\circ\{\widehat{\mathcal{R}}_m^{\mathtt{CE}}\}^{-1},\mathbb{P}^{(\mu_i)}\circ\{\widehat{\mathcal{R}}_m^{\mathtt{CE}}\}^{-1}\big)\leq\varepsilon\quad\textit{for all } m\geq m_0.$$

The sequence $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ is said to be robust on M if it is M-robust at every $\mu\in M$.

The notion of qualitative robustness of the sequence of estimators $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ introduced in Definition 7.3.1 can be interpreted as follows. Let μ be some arbitrary law from a set M of probability measures which is assumed to be the single-claim distribution 'used' to determine the (exact) individual premium. If the observed single claim distributions $\mu_1, \mu_2, \ldots$ in the insurance collective all lie in M and are 'close' to μ in a weak sense, then qualitative robustness of the sequence $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ at μ means that the laws of the empirical convolution estimator $\widehat{\mathcal{R}}_m^{\mathtt{CE}}$ w.r.t. $\mathbb{P}^{(\mu)}$ and $\mathbb{P}^{(\mu_i)}$ are 'close' (w.r.t. the Prohorov distance) for 'large' insurance collectives.

Conditions (a)–(b) of Theorem 7.3.4 ahead use the terminologies introduced in the following Definitions 7.3.2–7.3.3. Let ψ_p be the gauge function introduced in (6.6), and set $\overline{\mu}_m := \frac{1}{m}\sum_{i=1}^m \mu_i$ ($\subseteq \mathcal{M}_1(\mathbb{R})$) for the *average measure* of $\mu_1, \ldots, \mu_m \in \mathcal{M}_1(\mathbb{R})$, $m \in \mathbb{N}$. The following definition is motivated by [21, Definition 2.1] as well as [22, Remark 1], and generalizes to some extent Definition 2.2 in [58] and Definition 3.1 in [94].

Definition 7.3.2 *Fix $p \in \mathbb{R}_{\geq 1}$, and let $M \subseteq \mathcal{M}_1(\mathbb{R})$. The set M is said to be locally uniformly p-integrating if for every $\mu \in M$ and $\varepsilon > 0$ there exist $a > 0$, $m_0 \in \mathbb{N}$, and a weakly open neighbourhood U of μ such that*

$$\mu_i \in U \cap M,\ i \in \mathbb{N} \quad \Longrightarrow \quad \int \psi_p \mathbb{1}_{\{\psi_p \geq a\}}\, d\overline{\mu}_m \leq \varepsilon \quad \textit{for all } m \geq m_0. \tag{7.24}$$

Note that a locally uniformly p-integrating set M is a subset of $\mathcal{M}_1^p(\mathbb{R})$. Also note that every locally uniformly p'-integrating set M is also locally uniformly p-integrating whenever $p \leq p'$. For the statement of Theorem 7.3.4 below it is necessary to restrict oneself to those subsets of $\mathcal{M}_1^p(\mathbb{R})$ on which the relative ψ_p-weak topology and the relative weak topology coincide.

Definition 7.3.3 *Fix $p \in \mathbb{R}_{\geq 1}$, and let $M \subseteq \mathcal{M}_1^p(\mathbb{R})$. The set M is said to be a* w-set in $\mathcal{M}_1^p(\mathbb{R})$ *if $\mathcal{O}_{\mathrm{w}}^{\psi_p} \cap M = \mathcal{O}_{\mathrm{w}} \cap M$.*

According to Lemma 3.4 in [94], every locally uniformly p-integrating set $M \subseteq \mathcal{M}_1^p(\mathbb{R})$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$. In our general setting, however, we do not know if the reverse statement is true. If the distribution class considered in the statistical model in (7.23) is given by $\{\mathbb{P}^{(\mu)} : \mu \in \mathcal{M}\}$ for some $\mathcal{M} \subseteq \mathcal{M}_1(\mathbb{R})$, then the coordinate projections $Y_1, Y_2, \ldots$ on $\Omega = \mathbb{R}^{\mathbb{N}}$ are i.i.d. according to μ under $\mathbb{P}^{(\mu)} = \mu^{\otimes\mathbb{N}}$ for every $\mu \in \mathcal{M}$. In this case we may replace in condition (7.24) any $\mu_i \in U \cap M$ by some $\nu \in U \cap M$ and skip the suffix 'for all $m \geq m_0$', and it follows from Theorem 2.3 in [58] that every locally uniformly p-integrating set $M \subseteq \mathcal{M}_1^p(\mathbb{R})$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$. We refer to Examples 7.3.11–7.3.12 below for an illustration of w-sets in $\mathcal{M}_1^p(\mathbb{R})$.

The following theorem shows (under suitable assumptions) the qualitative robustness of the sequence $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ of empirical convolution estimators. Recall from (6.7) the definition of the L^p-Wasserstein metric $d_{\mathrm{Wass},p}$, and that there is a one-to-one correspondence between the sets $\mathcal{M}_1^p(\mathbb{R})$ and $\mathcal{M}(L^p)$, where the latter is defined as in Section 6.1.

Theorem 7.3.4 (Qualitative robustness of $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})$) *Fix $p \in \mathbb{R}_{\geq 1}$, and let $\mathcal{M} \subseteq \mathcal{M}_1^p(\mathbb{R})$ as well as $M \subseteq \mathcal{M}$. Let $\rho : L^p \to \mathbb{R}$ be a law-invariant coherent risk measure. Moreover let $\mathcal{R}_\rho : \mathcal{M}(L^p) \to \mathbb{R}$ be the associated risk functional introduced in (6.1), and assume that the following two conditions hold.*

(a) *M is locally uniformly p-integrating.*

(b) *$\mathcal{R}_\rho$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},p}, |\cdot|)$.*

Then the sequence of estimators $(\widehat{\mathcal{R}}_m^{\mathtt{CE}})_{m\in\mathbb{N}}$ is robust on M.

Note that assumption (b) of Theorem 7.3.4 is similar to condition (d') of Remark 7.2.13 which was already illustrated at the end of the last section. An illustration of condition (a) of Theorem 7.3.4 is in general difficult and is therefore omitted.

The proof of Theorem 7.3.4 avails Lemmas 7.3.5–7.3.8 below. The following lemma involve the metric

$$d_{\mathrm{vag}}(\mu,\nu) \;:=\; \sum_{k=1}^{\infty} 2^{-k}\Big(1\wedge\Big|\int h_k\,d\mu-\int h_k\,d\nu\Big|\Big), \qquad \mu,\nu\in\mathcal{M}_1(\mathbb{R}),$$

where $(h_k)_{k\in\mathbb{N}}$ is a sequence of real-valued continuous functions on $\mathbb{R}$ with compact support. Note that the latter sequence exists because $(\mathbb{R},|\cdot|)$ is a locally compact, separable and complete metric space. It follows from the proof of Theorem 31.5 in [6] that d_{vag} generates the weak topology $\mathcal{O}_{\mathrm{w}}$ on $\mathcal{M}_1(\mathbb{R})$. In particular, we may assume without loss of generality that the metric d_{w} in Display (2.3) is given by d_{vag}, i.e. we have

$$d_{\psi_p}(\mu,\nu) \;=\; d_{\mathrm{vag}}(\mu,\nu)+\Big|\int \psi_p\,d\mu-\int\psi_p\,d\nu\Big| \tag{7.25}$$

for any $\mu,\nu\in\mathcal{M}_1^p(\mathbb{R})$.

Lemma 7.3.5 *For every $\eta>0$ we have*

$$\lim_{m\to\infty}\sup_{(\mu_i)\in\mathcal{M}^{\mathbb{N}}}\mathbb{P}^{(\mu_i)}\big[d_{\mathrm{vag}}(\widehat{\mu}_m,\overline{\mu}_m)\ge\eta\big]=0.$$

Proof Let $\eta>0$, and choose $k_0=k_0(\eta)\in\mathbb{N}$ such that $\sum_{k=k_0+1}^{\infty}2^{-k}<\eta/2$. Then

$$\begin{aligned}
\mathbb{P}^{(\mu_i)}\big[d_{\mathrm{vag}}(\widehat{\mu}_m,\overline{\mu}_m)\ge\eta\big] \;&\le\; \mathbb{P}^{(\mu_i)}\Big[\sum_{k=1}^{k_0}\Big|\int h_k\,d\widehat{\mu}_m-\int h_k\,d\overline{\mu}_m\Big|\ge\frac{\eta}{2}\Big]\\
\le\; \sum_{k=1}^{k_0}\mathbb{P}^{(\mu_i)}\Big[\Big|\int h_k\,d\widehat{\mu}_m-\int h_k\,d\overline{\mu}_m\Big|\ge\frac{\eta}{2k_0}\Big] \;&=\; \sum_{k=1}^{k_0}\mathbb{P}^{(\mu_i)}\Big[\Big|\frac{1}{m}\sum_{i=1}^{m}\big(h_k(Y_i)-\mathbb{E}^{(\mu_i)}[h_k(Y_i)]\big)\Big|\ge\frac{\eta}{2k_0}\Big]\\
\le\; \sum_{k=1}^{k_0}\frac{16\,k_0^2\,\|h_k\|_\infty^2}{\eta^2}\cdot\frac{1}{m} \;\le\; \frac{16\,k_0^3\,\max_{k=1,\dots,k_0}\|h_k\|_\infty^2}{\eta^2}\cdot\frac{1}{m} \;&=:\; C(k_0,\eta)\cdot\frac{1}{m}
\end{aligned} \tag{7.26}$$

for all $m\in\mathbb{N}$ and $\mu_i\in\mathcal{M}$, $i\in\mathbb{N}$, where we used Chebyshev's inequality for the second last "$\le$" in (7.26). The constant $C(k_0,\eta)$ is (independent of m as well as the sequence $(\mu_i)_{i\in\mathbb{N}}$ and) finite because every h_k is clearly bounded. Thus the expression in the last line of (7.26) converges to 0 as $m\to\infty$ uniformly in $(\mu_i)\in\mathcal{M}^{\mathbb{N}}$. Hence the assertion follows. □

Note for the following lemma that the bounded Lipschitz metric d_{BL} on $\mathcal{M}_1(\mathbb{R})$ introduced in Example 2.1.4 (with $E:=\mathbb{R}$) generates the weak topology $\mathcal{O}_{\mathrm{w}}$ on $\mathcal{M}_1(\mathbb{R})$.

Lemma 7.3.6 *Let $\mu,\mu_m\in\mathcal{M}_1(\mathbb{R})$ for every $m\in\mathbb{N}$. Then the following two assertions hold for any $m\in\mathbb{N}$.*

(i) $d_{\mathrm{BL}}(\mu, \overline{\mu}_m) \leq \frac{1}{m}\sum_{i=1}^m d_{\mathrm{BL}}(\mu, \mu_i)$.

If in addition $\mu, \mu_m \in \mathcal{M}_1^p(\mathbb{R})$ *for some fixed* $p \in \mathbb{R}_{\geq 1}$ *and every* $m \in \mathbb{N}$ *then*

(ii) $\widetilde{d}_{\psi_p}(\mu, \overline{\mu}_m) \leq \frac{1}{m}\sum_{i=1}^m \widetilde{d}_{\psi_p}(\mu, \mu_i)$, *where* $\widetilde{d}_{\psi_p}$ *is defined as in (2.3) with* d_{BL} *in place of* d_{w}.

Proof To verify (i), note that in view of (2.5) we have

$$d_{\mathrm{BL}}(\mu, \overline{\mu}_m) = \sup_{h \in \mathbb{M}_{\mathrm{BL}}} \Big| \int h\, d\mu - \int h\, d\overline{\mu}_m \Big| = \frac{1}{m} \cdot \sup_{h \in \mathbb{M}_{\mathrm{BL}}} \Big| \sum_{i=1}^m \Big(\int h\, d\mu - \int h\, d\mu_i \Big) \Big|$$

$$\leq \frac{1}{m} \cdot \sup_{h \in \mathbb{M}_{\mathrm{BL}}} \sum_{i=1}^m \Big| \int h\, d\mu - \int h\, d\mu_i \Big| \leq \frac{1}{m} \sum_{i=1}^m \sup_{h \in \mathbb{M}_{\mathrm{BL}}} \Big| \int h\, d\mu - \int h\, d\mu_i \Big| = \frac{1}{m} \sum_{i=1}^m d_{\mathrm{BL}}(\mu, \mu_i)$$

for any $m \in \mathbb{N}$. This shows (i).

For (ii), it follows from (2.3) along with part (i)

$$\widetilde{d}_{\psi_p}(\mu, \overline{\mu}_m) = d_{\mathrm{BL}}(\mu, \overline{\mu}_m) + \Big| \int \psi_p\, d\mu - \int \psi_p\, d\overline{\mu}_m \Big| = d_{\mathrm{BL}}(\mu, \overline{\mu}_m) + \frac{1}{m} \Big| \sum_{i=1}^m \Big(\int \psi_p\, d\mu - \int \psi_p\, d\mu_i \Big) \Big|$$

$$\leq \frac{1}{m} \sum_{i=1}^m \Big(d_{\mathrm{BL}}(\mu, \mu_i) + \Big| \int \psi_p\, d\mu - \int \psi_p\, d\mu_i \Big| \Big) = \frac{1}{m} \sum_{i=1}^m \widetilde{d}_{\psi_p}(\mu, \mu_i).$$

Thus shows the claim in (ii). □

Lemma 7.3.7 *Fix* $p \in \mathbb{R}_{\geq 1}$, *and let* $M \subseteq \mathcal{M}$ *be a locally uniformly* p*-integrating set. Then for every* $\mu \in M$, $\varepsilon > 0$ *and* $\eta > 0$ *there exist* $\delta > 0$ *and* $m_0 \in \mathbb{N}$ *such that*

$$\mu_i \in M, \quad d_{\psi_p}(\mu, \mu_i) \leq \delta,\, i \in \mathbb{N} \implies \mathbb{P}^{(\mu_i)}\Big[\Big| \int \psi_p\, d\widehat{\mu}_m - \int \psi_p\, d\overline{\mu}_m \Big| \geq \eta \Big] \leq \varepsilon \quad \textit{for all } m \geq m_0.$$

Proof Let $\mu \in M$, $\varepsilon > 0$, and $\eta > 0$ be arbitrary but fixed. Since M is locally uniformly p-integrating, we find in view of Definition 7.3.2 some $\delta > 0$, $a > 0$, and $m_1 \in \mathbb{N}$ such that $\int \psi_p \mathbb{1}_{\{\psi_p \geq a\}}\, d\overline{\mu}_m < \min\{\eta/3, \eta\varepsilon/6\}$ for all $m \geq m_1$ and all $\mu_i \in M$ with $d_{\mathrm{vag}}(\mu, \mu_i) \leq \delta$. Then, using a truncation argument, we obtain for every $m \geq m_1$ and $\mu_i \in M$ with $d_{\mathrm{vag}}(\mu, \mu_i) \leq \delta$ that

$$\begin{aligned}
\mathbb{P}^{(\mu_i)}\Big[\Big| \int \psi_p\, d\widehat{\mu}_m - \int \psi_p\, d\overline{\mu}_m \Big| \geq \eta \Big] \quad &\leq \quad \mathbb{P}^{(\mu_i)}\Big[\Big| \int \psi_p \mathbb{1}_{\{\psi_p < a\}}\, d\widehat{\mu}_m - \int \psi_p \mathbb{1}_{\{\psi_p < a\}}\, d\overline{\mu}_m \Big| \geq \frac{\eta}{3} \Big] \\
&\quad + \mathbb{P}^{(\mu_i)}\Big[\int \psi_p \mathbb{1}_{\{\psi_p \geq a\}}\, d\widehat{\mu}_m \geq \frac{\eta}{3} \Big] \\
&\quad + \mathbb{P}^{(\mu_i)}\Big[\int \psi_p \mathbb{1}_{\{\psi_p \geq a\}}\, d\overline{\mu}_m \geq \frac{\eta}{3} \Big] \\
&=: \quad S_1(m, (\mu_i)) + S_2(m, (\mu_i)) + S_3(m, (\mu_i)),
\end{aligned}$$

where $S_3(m, (\mu_i)) = 0$ and $S_2(m, (\mu_i)) \leq (3/\eta) \int \psi_p \mathbb{1}_{\{\psi_p \geq a\}}\, d\overline{\mu}_m \leq \varepsilon/2$ (by Markov's inequality). Moreover, by Chebyshev's inequality we find some $m_2 \in \mathbb{N}$ such that $S_1(m, (\mu_i)) \leq 9\eta^{-2}a^2m^{-1} \leq \varepsilon/2$ for all $m \geq m_2$ and $\mu_i \in M$, $i \in \mathbb{N}$. Setting $m_0 := \max\{m_1, m_2\}$, this shows the claim with d_{ψ_p} replaced by d_{vag}. Since $d_{\mathrm{vag}} \leq d_{\psi_p}$ (by (7.25)), the proof is now complete. □

Recall for the following lemma that δ_x refers to the Dirac measure at point x.

Lemma 7.3.8 *Under the assumptions of Theorem 7.3.4 and for every $\mu \in M$ and $\varepsilon > 0$ there exist $\delta > 0$ and $m_0 \in \mathbb{N}$ such that*

$$\mu_i \in M, \quad d_{\psi_p}(\mu, \mu_i) \le \delta,\, i \in \mathbb{N} \implies d_{\mathrm{Proh}}\big(\delta_{\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)/m}, \mathbb{P}^{(\mu_i)} \circ \{\widehat{\mathcal{R}}_m^{\mathtt{CE}}\}^{-1}\big) \le \varepsilon \quad \textit{for all } m \ge m_0.$$

Proof Due to Strassen's theorem (see, e.g., [45, Theorem 2.13]) it suffices for the assertion to show that for every $\mu \in M$ and $\varepsilon > 0$ there exist $\delta > 0$ and $m_0 \in \mathbb{N}$ such that

$$\mu_i \in M, \quad d_{\psi_p}(\mu, \mu_i) \le \delta,\, i \in \mathbb{N} \implies \mathbb{P}^{(\mu_i)}\Big[\Big|\frac{1}{m}\big(\mathcal{R}_\rho(\widehat{\mu}_m^{*m}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big)\Big| \ge \varepsilon\Big] \le \varepsilon \quad \text{for all } m \ge m_0. \tag{7.27}$$

Let $\mu \in M$ and $\varepsilon > 0$ be arbitrary but fixed. Since $\mathcal{R}_\rho$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},p}, |\cdot|)$ (by condition (c) of Theorem 7.3.4), we find some finite constant $L_\rho > 0$ such that for all $\nu_1, \nu_2 \in \mathcal{M}_1^p(\mathbb{R})$ we have

$$\big|\mathcal{R}_\rho(\nu_1) - \mathcal{R}_\rho(\nu_2)\big| \;\le\; L_\rho\, d_{\mathrm{Wass},p}(\nu_1, \nu_2). \tag{7.28}$$

Moreover note that in view of [17, Lemma 8.6] the L^p-Wasserstein metric $d_{\mathrm{Wass},p}$ satisfies

$$d_{\mathrm{Wass},p}(\ast_{i=1}^m \nu_i, \ast_{i=1}^m \nu_i') \;\le\; \sum_{i=1}^m d_{\mathrm{Wass},p}(\nu_i, \nu_i') \tag{7.29}$$

for every $m \in \mathbb{N}$ and $\nu_i, \nu_i' \in \mathcal{M}_1^p(\mathbb{R})$, $i = 1, \ldots, m$. As the metrics d_{ψ_p} and $d_{\mathrm{Wass},p}$ are topologically equivalent on $\mathcal{M}_1^p(\mathbb{R})$, we find some $\delta' > 0$ such that in view of (7.28)–(7.29) the right-hand side of (7.27) can be estimated by

$$\begin{aligned}
&\mathbb{P}^{(\mu_i)}\Big[\Big|\frac{1}{m}\big(\mathcal{R}_\rho(\widehat{\mu}_m^{*m}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big)\Big| \ge \varepsilon\Big] \\
&\le \;\; \mathbb{P}^{(\mu_i)}\Big[d_{\mathrm{Wass},p}(\widehat{\mu}_m^{*m}, \ast_{i=1}^m \mu_i) \ge \frac{m\varepsilon}{L_\rho}\Big] \\
&\le \;\; \mathbb{P}^{(\mu_i)}\Big[d_{\mathrm{Wass},p}(\widehat{\mu}_m^{*m}, \mu^{*m}) + d_{\mathrm{Wass},p}(\mu^{*m}, \ast_{i=1}^m \mu_i) \ge \frac{m\varepsilon}{L_\rho}\Big] \\
&\le \;\; \mathbb{P}^{(\mu_i)}\Big[d_{\mathrm{Wass},p}(\widehat{\mu}_m, \mu) \ge \frac{\varepsilon}{2L_\rho}\Big] + \mathbb{P}^{(\mu_i)}\Big[\frac{1}{m}\sum_{i=1}^m d_{\mathrm{Wass},p}(\mu, \mu_i) \ge \frac{\varepsilon}{2L_\rho}\Big] \\
&\le \;\; \mathbb{P}^{(\mu_i)}\big[d_{\psi_p}(\widehat{\mu}_m, \mu) \ge \delta'\big] + \mathbb{P}^{(\mu_i)}\Big[\frac{1}{m}\sum_{i=1}^m d_{\psi_p}(\mu, \mu_i) \ge \delta'\Big] \\
&=: \;\; S_1(m, (\mu_i)) + S_2(m, (\mu_i))
\end{aligned} \tag{7.30}$$

for every $m \in \mathbb{N}$ and $\mu_i \in M$, $i \in \mathbb{N}$. Thus $S_2(m, (\mu_i)) = 0$ for any $m \in \mathbb{N}$ and $\mu_i \in M$ with $d_{\psi_p}(\mu, \mu_i) \le \delta_1 := \delta''/2$. For the first summand in the last line of formula display (7.30), we observe in view of (7.25)

$$\begin{aligned}
&\mathbb{P}^{(\mu_i)}\big[d_{\psi_p}(\widehat{\mu}_m, \mu) \ge \delta'\big] \\
&\le \;\; \mathbb{P}^{(\mu_i)}\big[d_{\psi_p}(\widehat{\mu}_m, \overline{\mu}_m) \ge \delta'/2\big] + \mathbb{P}^{(\mu_i)}\big[d_{\psi_p}(\mu, \overline{\mu}_m) \ge \delta'/2\big] \\
&\le \;\; \mathbb{P}^{(\mu_i)}\Big[d_{\mathrm{vag}}(\widehat{\mu}_m, \overline{\mu}_m) \ge \delta'/4\Big] + \mathbb{P}^{(\mu_i)}\Big[\Big|\int \psi_p\, d\widehat{\mu}_m - \int \psi_p\, d\overline{\mu}_m\Big| \ge \delta'/4\Big] \\
&\qquad + \mathbb{P}^{(\mu_i)}\big[d_{\psi_p}(\mu, \overline{\mu}_m) \ge \delta'/2\big] \\
&=: \;\; S_{1,1}(m, (\mu_i)) + S_{1,2}(m, (\mu_i)) + S_{1,3}(m, (\mu_i))
\end{aligned}$$

for every $m \in \mathbb{N}$ and $\mu_i \in M$, $i \in \mathbb{N}$. Since M ($\subseteq \mathcal{M}_1(\mathbb{R})$) is locally uniformly p-integrating (by condition (b) of Theorem 7.3.4) there exists in view of Lemma 7.3.7 (applied to $\eta := \delta'/4$) some $m_1 \in \mathbb{N}$ and $\delta_2 > 0$ such that $S_{1,2}(m, (\mu_i)) \leq \varepsilon/2$ for all $m \geq m_1$ and all $\mu_i \in M$ with $d_{\psi_p}(\mu, \mu_i) \leq \delta_2$. By Lemma 7.3.5 (applied to $\eta := \delta'/4$), we find some $m_2 \in \mathbb{N}$ such that $S_{1,1}(m, (\mu_i)) \leq \varepsilon/2$ for all $m \geq m_2$ and $\mu_i \in M$ ($\subseteq \mathcal{M}$), $i \in \mathbb{N}$.

Moreover, the metrics d_{vag} and d_{BL} are topologically equivalent. Recall that both d_{vag} and d_{BL} generates the weak topology. In particular, in view of Lemma 2.1.1, this implies that the metrics d_{ψ_p} (given by (7.25)) and $\widetilde{d}_{\psi_p}$ (given by (2.3) with d_{BL} in place of d_{w}) are topologically equivalent, too. Thus we find some $\delta'', \delta''' > 0$ such that in view of Lemma 7.3.6(ii)

$$\begin{aligned}
\mathbb{P}^{(\mu_i)}\big[d_{\psi_p}(\mu, \overline{\mu}_m) \geq \delta'/2\big] \quad &\leq \quad \mathbb{P}^{(\mu_i)}\big[\widetilde{d}_{\psi_p}(\mu, \overline{\mu}_m) \geq \delta''\big] \;\leq\; \mathbb{P}^{(\mu_i)}\Big[\frac{1}{m}\sum_{i=1}^{m} \widetilde{d}_{\psi_p}(\mu, \mu_i) \geq \delta''\Big] \\
&\leq \quad \mathbb{P}^{(\mu_i)}\Big[\frac{1}{m}\sum_{i=1}^{m} d_{\psi_p}(\mu, \mu_i) \geq \delta'''\Big]
\end{aligned}$$

for any $m \in \mathbb{N}$ and $\mu_i \in M$, $i \in \mathbb{N}$. Hence $S_{1,3}(m, (\mu_i)) = 0$ for every $m \in \mathbb{N}$ and all $\mu_i \in M$ with $d_{\psi_p}(\mu, \mu_i) \leq \delta_3 := \delta'''/2$. Consequently, setting $\delta := \min\{\delta_1, \delta_1, \delta_3\}$ and $m_0 := \max\{m_1, m_2\}$, we arrive at (7.27). □

Now, let us turn to the proof of Theorem 7.3.4.

Proof of Theorem 7.3.4: We have to show that for every $\mu \in M$ and $\varepsilon > 0$ there are some $m_0 \in \mathbb{N}$ and an open neighbourhood $U = U(\mu, \varepsilon; M)$ of μ for the relative weak topology $\mathcal{O}_{\text{w}} \cap M$ such that

$$\mu_i \in U,\; i \in \mathbb{N} \quad \Longrightarrow \quad d_{\text{Proh}}\big(\mathbb{P}^{(\mu)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}, \mathbb{P}^{(\mu_i)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}\big) \leq \varepsilon \quad \text{for all } m \geq m_0. \qquad (7.31)$$

Since M is locally uniformly p-integrating and therefore a w-set in $\mathcal{M}_1^p(\mathbb{R})$, it suffices for (7.31) to show that for every $\mu \in M$ and $\varepsilon > 0$ there exist $m_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\mu_i \in M, \quad d_{\psi_p}(\mu, \mu_i) \leq \delta,\; i \in \mathbb{N} \;\Longrightarrow\; d_{\text{Proh}}\big(\mathbb{P}^{(\mu)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}, \mathbb{P}^{(\mu_i)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}\big) \leq \varepsilon \quad \text{for all } m \geq m_0. \qquad (7.32)$$

Take into account that d_{ψ_p} given by (7.25) generates in view of Lemma 2.1.1 the ψ_p-weak topology on $\mathcal{M}_1^p(\mathbb{R})$.

Let $\mu \in M$ and $\varepsilon > 0$ be arbitrary but fixed. In the following we will verify that (7.32) holds for some $m_0 \in \mathbb{N}$ and $\delta > 0$. Note at first, that the right-hand side of (7.32) can be estimated by

$$\begin{aligned}
d_{\text{Proh}}\big(\mathbb{P}^{(\mu)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}, \mathbb{P}^{(\mu_i)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}\big) \quad &\leq \quad d_{\text{Proh}}\big(\mathbb{P}^{(\mu)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}, \delta_{\mathcal{R}_\rho(\mu^{*m})/m}\big) \\
&\qquad + d_{\text{Proh}}\big(\delta_{\mathcal{R}_\rho(\mu^{*m})/m}, \delta_{\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)/m}\big) \\
&\qquad + d_{\text{Proh}}\big(\delta_{\mathcal{R}_\rho(\ast_{i=1}^m \mu_i)/m}, \mathbb{P}^{(\mu_i)} \circ \{\widehat{\mathcal{R}}_m^{\texttt{CE}}\}^{-1}\big) \\
&=: \quad S_1(m) + S_2(m, (\mu_i)) + S_3(m, (\mu_i))
\end{aligned}$$

for any $m \in \mathbb{N}$ and $\mu_i \in M$, $i \in \mathbb{N}$. It follows from Lemma 7.3.8 that there exists $\delta_1 > 0$ and $m_1 \in \mathbb{N}$ such that $S_3(m, (\mu_i)) \leq \varepsilon/3$ for all $m \geq m_1$ and $\mu_i \in M$ with $d_{\psi_p}(\mu, \mu_i) \leq \delta_1$. Since the summand

$S_1(m)$ is for any $m \in \mathbb{N}$ equal to $S_3(m,(\mu_i))$ by choosing $\mu_i := \mu$, $i \in \mathbb{N}$, we get $S_1(m) \le \varepsilon/3$ for every $m \ge m_1$. Moreover since $\mathcal{R}_\rho$ is Lipschitz continuous w.r.t. $(d_{\mathrm{Wass},p}, |\cdot|)$ (by condition (b)) and since $d_{\mathrm{Wass},p}$ and d_{ψ_p} are topologically equivalent on $\mathcal{M}_1^p(\mathbb{R})$ there exist $L_\rho \in \mathbb{R}_{>0}$ and $\delta_2 > 0$ such that in view of (7.28)–(7.29)

$$\begin{aligned} S_2(m,(\mu_i)) \;&=\; \min\Big\{\frac{1}{m}\big|\mathcal{R}_\rho(\mu^{*m}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big|;1\Big\} \;\le\; \frac{1}{m}\big|\mathcal{R}_\rho(\mu^{*m}) - \mathcal{R}_\rho(\ast_{i=1}^m \mu_i)\big| \\ &\le\; \frac{1}{m}L_\rho\, d_{\mathrm{Wass},p}(\mu^{*m}, \ast_{i=1}^m \mu_i) \;\le\; \frac{1}{m}L_\rho \sum_{i=1}^m d_{\mathrm{Wass},p}(\mu,\mu_i) \;\le\; \frac{1}{m}L_\rho m \frac{\varepsilon}{3L_\rho} \;=\; \varepsilon/3 \end{aligned}$$

for every $m \in \mathbb{N}$ and $\mu_i \in M$ with $d_{\psi_p}(\mu,\mu_i) \le \delta_2$. Setting $\delta := \min\{\delta_1,\delta_2\}$ and $m_0 := m_1$, this implies the assertion in (7.32). The proof of Theorem 7.3.4 is now complete. □

In the rest of this section we will illustrate the notion of a w-set in $\mathcal{M}_1^p(\mathbb{R})$. Examples 7.3.11–7.3.12 below involve in each case a parametric class of distributions which gives an example of a w-set in the sense of Definition 7.3.3. The key for the verification of the assertions in these examples will be Lemma 7.3.10 (a variant of Proposition 3.3 in [58]) which is an immediate consequence of the following Theorem 7.3.9 (see also Theorem 2.3 in [58]). Note that in view of Lemma 2.1.1 the equivalence of the statements in (i) and (ii) of this theorem is obvious because the respective topologies are metrizable. Recall that $\xrightarrow{\mathrm{w}}$ refers to weak convergence of probability measures.

Theorem 7.3.9 *Fix $p \in \mathbb{R}_{\ge 1}$, and let $M \subseteq \mathcal{M}_1^p(\mathbb{R})$. Then the following two assertions are equivalent.*

(i) $\mathcal{O}_{\mathrm{w}}^{\psi_p} \cap M = \mathcal{O}_{\mathrm{w}} \cap M$.

(ii) *For every choice of $\nu, \nu_1, \nu_2 \ldots \in M$ for which $\nu_m \xrightarrow{\mathrm{w}} \nu$, the convergence $\int \psi_p \, d\nu_m \to \int \psi_p \, d\nu$ holds.*

Recall for the following lemma that ψ_p is given by (6.6), and let $\|\cdot\|$ be the usual Euclidean norm on $\mathbb{R}^d$ (with $d \in \mathbb{N}$ fixed).

Lemma 7.3.10 *Fix $p \in \mathbb{R}_{\ge 1}$ as well as $\Theta \subseteq \mathbb{R}^d$, and let $\nu_\theta \in \mathcal{M}_1^p(\mathbb{R})$ for every $\theta \in \Theta$. Then the set $\mathcal{M}_\Theta := \{\nu_\theta : \theta \in \Theta\}$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$ if for every choice of $\theta, \theta_1, \theta_2, \ldots \in \Theta$ the following two conditions hold.*

(a) $\nu_{\theta_m} \xrightarrow{\mathrm{w}} \nu_\theta \implies \|\theta_m - \theta\| \to 0$.

(b) $\|\theta_m - \theta\| \to 0 \implies \int \psi_p \, d\nu_{\theta_m} \to \int \psi_p \, d\nu_\theta$.

Using Lemma 7.3.10, Examples 7.3.11 and 7.3.12 give us two illustrations for w-sets in $\mathcal{M}_1^p(\mathbb{R})$.

Example 7.3.11 (Normal distribution) Let $\Theta := \mathbb{R}_{>0}$ $(\subseteq \mathbb{R})$, and let $\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}^2}$ be the normal distribution with known (and therefore fixed) location parameter $\overline{\mathfrak{m}} \in \mathbb{R}$ and unknown (squared) scale parameter $\mathfrak{s}^2 \in \Theta$. Recall that $\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}^2}$ is given by the standard Lebesgue density

$$\varphi^{\mathrm{N}}_{\overline{\mathfrak{m}},\mathfrak{s}^2}(x) \;:=\; (2\pi\mathfrak{s}^2)^{-1/2}\, e^{-(x-\overline{\mathfrak{m}})^2/(2\mathfrak{s}^2)}, \qquad x \in \mathbb{R}.$$

Then the family $\mathcal{N}_\Theta := \{\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}^2} : \mathfrak{s}^2 \in \Theta\}$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$ for any fixed $p \in \mathbb{R}_{\ge 1}$.

Proof It suffices to verify conditions (a)–(b) of Lemma 7.3.10 in order to show that $\mathcal{N}_\Theta$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$. First of all, it is clear that $\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}^2} \in \mathcal{M}_1^p(\mathbb{R})$ for every $\mathfrak{s}^2 \in \Theta$. Now, let $\mathfrak{s}^2 \in \Theta$ and $(\mathfrak{s}_m^2)_{m\in\mathbb{N}}$ be any sequence in Θ.

(a) Suppose that the sequence $(\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}_m^2})_{m\in\mathbb{N}}$ in $\mathcal{N}_\Theta$ converges weakly to some $\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}^2} \in \mathcal{N}_\Theta$. Then the corresponding sequence $(F_{\overline{\mathfrak{m}},\mathfrak{s}_m^2})_{m\in\mathbb{N}}$ of distribution functions satisfies

$$\Phi_{0,1}\Big(\frac{x-\overline{\mathfrak{m}}}{\mathfrak{s}_m}\Big) \;=\; F_{\overline{\mathfrak{m}},\mathfrak{s}_m^2}(x) \;\longrightarrow\; F_{\overline{\mathfrak{m}},\mathfrak{s}^2}(x) \;=\; \Phi_{0,1}\Big(\frac{x-\overline{\mathfrak{m}}}{\mathfrak{s}}\Big)$$

for all $x \in \mathbb{R}$. Recall that $\Phi_{0,1}$ refers to the distribution function of the standard normal distribution. Hence, we have necessarily $\mathfrak{s}_m \to \mathfrak{s}$ and thus $\mathfrak{s}_m^2 \to \mathfrak{s}$.

(b) Suppose that $\mathfrak{s}_m^2 \to \mathfrak{s}^2$. Set $\underline{\mathfrak{s}}^2 := \inf_{m\in\mathbb{N}} \mathfrak{s}_m^2$ and $\bar{\mathfrak{s}}^2 := \sup_{m\in\mathbb{N}} \mathfrak{s}_m^2$, and note that $0 < \underline{\mathfrak{s}}^2 < \bar{\mathfrak{s}}^2 < \infty$. Then

$$\begin{aligned}\lim_{m\to\infty} \psi_p(x)\,\varphi^{\mathrm{N}}_{\overline{\mathfrak{m}},\mathfrak{s}_m^2}(x) &= \lim_{m\to\infty} \psi_p(x)\,(2\pi\mathfrak{s}_m^2)^{-1/2}\,e^{-(x-\overline{\mathfrak{m}})^2/(2\mathfrak{s}_m^2)}\\ &= \psi_p(x)\,(2\pi\mathfrak{s}^2)^{-1/2}\,e^{-(x-\overline{\mathfrak{m}})^2/(2\mathfrak{s}^2)} \;=\; \psi_p(x)\,\varphi^{\mathrm{N}}_{\overline{\mathfrak{m}},\mathfrak{s}^2}(x)\end{aligned}$$

for all $x \in \mathbb{R}$. Since for any $x \in \mathbb{R}$ the mapping $\mathfrak{s}^2 \mapsto -(x-\overline{\mathfrak{m}})^2/(2\mathfrak{s}^2)$ is non-decreasing on Θ, we get

$$\sup_{m\in\mathbb{N}} \psi_p(x)\,\varphi^{\mathrm{N}}_{\overline{\mathfrak{m}},\mathfrak{s}_m^2}(x) \;=\; \sup_{m\in\mathbb{N}} \psi_p(x)\,(2\pi\mathfrak{s}_m^2)^{-1/2}\,e^{-(x-\overline{\mathfrak{m}})^2/(2\mathfrak{s}_m^2)} \;\le\; \psi_p(x)\,(2\pi\underline{\mathfrak{s}}^2)^{-1/2}\,e^{-(x-\overline{\mathfrak{m}})^2/(2\bar{\mathfrak{s}}^2)}$$

for all $x \in \mathbb{R}$. In view of $\mathrm{N}_{\overline{\mathfrak{m}},\bar{\mathfrak{s}}^2} \in \mathcal{M}_1^p(\mathbb{R})$, an application of the dominated convergence theorem yields

$$\int \psi_p(x)\,\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}_m^2}(dx) = \int \psi_p(x)\,\varphi^{\mathrm{N}}_{\overline{\mathfrak{m}},\mathfrak{s}_m^2}(x)\ell(dx) \longrightarrow \int \psi_p(x)\,\varphi^{\mathrm{N}}_{\overline{\mathfrak{m}},\mathfrak{s}^2}(x)\ell(dx) = \int \psi_p(x)\,\mathrm{N}_{\overline{\mathfrak{m}},\mathfrak{s}^2}(dx).$$

Therefore conditions (a)–(b) of Lemma 7.3.10 hold and the latter result implies that $\mathcal{N}_\Theta$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$. ◇

The following example is in line with Example 3.7 in [58].

Example 7.3.12 (Type-1 Gumbel distribution) Let $\Theta := \mathbb{R}_{>0}$ $(\subseteq \mathbb{R})$, and let $\mathrm{G}_{a,\bar{b}}$ be the type-1 Gumbel distribution with unknown scale parameter $a \in \Theta$ and known (and therefore fixed) shape parameter $\bar{b} > 0$. Recall that $\mathrm{G}_{a,\bar{b}}$ is given by the standard Lebesgue density

$$\varphi^{\mathrm{G}}_{a,\bar{b}}(x) \;:=\; a\bar{b}e^{-ax-\bar{b}e^{-ax}}, \qquad x \in \mathbb{R}.$$

Note that $\mathrm{G}_{a,1}$ is nothing but the usual Gumbel distribution. Then the family $\mathcal{G}_\Theta := \{\mathrm{G}_{a,\bar{b}} : a \in \Theta\}$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$ for any fixed $p \in \mathbb{R}_{\ge 1}$.

Proof It suffices to verify conditions (a)–(b) of Lemma 7.3.10 in order to show that $\mathcal{G}_\Theta$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$. At first, it is easily seen that $\mathrm{G}_{a,\bar{b}} \in \mathcal{M}_1^p(\mathbb{R})$ for any $a \in \Theta$. Now, let $a \in \Theta$ and $(a_m)_{m\in\mathbb{N}}$ be any sequence in Θ.

(a) Suppose that the sequence $(\mathrm{G}_{a_m,\bar{b}})_{m\in\mathbb{N}}$ in $\mathcal{G}_\Theta$ converges weakly to some $\mathrm{G}_{a,\bar{b}} \in \mathcal{G}_\Theta$. Then the corresponding sequence $(F_{a_m,\bar{b}})_{m\in\mathbb{N}}$ of distribution functions satisfies

$$e^{-\bar{b}e^{-a_m x}} = F_{a_m,\bar{b}}(x) \longrightarrow F_{a,\bar{b}}(x) = e^{-\bar{b}e^{-ax}}$$

for all $x \in \mathbb{R}$. Hence, we have necessarily $a_m \to a$.

(b) Suppose that $a_m \to a$. Set $\underline{a} := \inf_{m\in\mathbb{N}} a_m$ and $\overline{a} := \sup_{m\in\mathbb{N}} a_m$, and note that $0 < \underline{a} < \overline{a} < \infty$. Then

$$\lim_{m\to\infty} \psi_p(x)\varphi^{\mathrm{G}}_{a_m,\bar{b}}(x) = \lim_{m\to\infty} \psi_p(x)\, a_m\bar{b}e^{-a_m x-\bar{b}e^{-a_m x}} = \psi_p(x)\, a\bar{b}e^{-ax-\bar{b}e^{-ax}} = \psi_p(x)\varphi^{\mathrm{G}}_{a,\bar{b}}(x)$$

for every $x \in \mathbb{R}$. Since for any $x \in \mathbb{R}$ the mapping $a \mapsto -ax - \bar{b}e^{-ax}$ is non-increasing on Θ, we have

$$\sup_{m\in\mathbb{N}} \psi_p(x)\,\varphi^{\mathrm{G}}_{a_m,\bar{b}}(x) \;=\; \sup_{m\in\mathbb{N}} \psi_p(x)\, a_m\bar{b}e^{-a_m x-\bar{b}e^{-a_m x}} \le \psi_p(x)\,\overline{a}\bar{b}e^{-\underline{a}x-\bar{b}e^{-\underline{a}x}}$$

for all $x \in \mathbb{R}$. In view of $\mathrm{G}_{\underline{a},\bar{b}} \in \mathcal{M}_1^p(\mathbb{R})$, an application of the dominated convergence theorem yields

$$\int \psi_p(x)\,\mathrm{G}_{a_m,\bar{b}}(dx) = \int \psi_p(x)\varphi^{\mathrm{G}}_{a_m,\bar{b}}(x)\,\ell(dx) \longrightarrow \int \psi_p(x)\,\varphi^{\mathrm{G}}_{a,\bar{b}}(x)\,\ell(dx) = \int \psi_p(x)\,\mathrm{G}_{a,\bar{b}}(dx).$$

Hence conditions (a)–(b) of Lemma 7.3.10 hold and the latter result implies that $\mathcal{G}_\Theta$ is a w-set in $\mathcal{M}_1^p(\mathbb{R})$. $\diamond$

Appendix A

Quasi-Hadamard differentiability and quasi-Lipschitz continuity

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be vector spaces, and $\boldsymbol{E} \subseteq \boldsymbol{V}$ as well as $\boldsymbol{W}', \boldsymbol{W}'' \subseteq \boldsymbol{W}$ be subspaces. Let $\|\cdot\|_{\boldsymbol{E}}$ and $\|\cdot\|_{\boldsymbol{W}''}$ be norms on $\boldsymbol{E}$ and $\boldsymbol{W}''$, respectively. Moreover let

$$H : \boldsymbol{V}_H \to \boldsymbol{W}'$$

be a map defined on a subset $\boldsymbol{V}_H \subseteq \boldsymbol{V}$.

A.1 Definition of quasi-Hadamard differentiability

In this section we recall in the following Definition A.1 the notion of quasi-Hadamard differentiability introduced in [59]. Note that $\mathbb{R}_{>0} := (0, \infty)$.

Definition A.1 *Let $\boldsymbol{E}_0$ be a subset of $\boldsymbol{E}$, and fix $v \in \boldsymbol{V}_H$.*

(i) The map H is said to be quasi-Hadamard differentiable at v tangentially to $\boldsymbol{E}_0\langle\boldsymbol{E}\rangle$ with trace $\boldsymbol{W}''$ if $H(w) - H(v) \in \boldsymbol{W}''$ for all $w \in \boldsymbol{V}_H$ and there exists a continuous map $\dot{H}_v : \boldsymbol{E}_0 \to \boldsymbol{W}''$ such that

$$\lim_{m\to\infty} \Big\| \frac{H(v + \varepsilon_m v_m) - H(v)}{\varepsilon_m} - \dot{H}_v(v_0) \Big\|_{\boldsymbol{W}''} = 0$$

holds for every triplet $(v_0, (v_m), (\varepsilon_m)) \in \boldsymbol{E}_0 \times \boldsymbol{E}^{\mathbb{N}} \times \mathbb{R}_{>0}^{\mathbb{N}}$ satisfying $\|v_m - v_0\|_{\boldsymbol{E}} \to 0$, $\varepsilon_m \to 0$ as well as $(v + \varepsilon_m v_m) \subseteq \boldsymbol{V}_H$. In this case, the map $\dot{H}_v$ is called quasi-Hadamard derivative of H at v tangentially to $\boldsymbol{E}_0\langle\boldsymbol{E}\rangle$ with trace $\boldsymbol{W}''$.

(ii) If $\boldsymbol{E}_0 = \boldsymbol{E} = \boldsymbol{V}$, then we skip in (i) the phrases "quasi-" as well as "tangentially to $\boldsymbol{E}_0\langle\boldsymbol{E}\rangle$".

(iii) If $\boldsymbol{W}'' = \boldsymbol{W}'$, then we skip in (i) the phrase "with trace $\boldsymbol{W}''$".

Note that Definition A.1 extends the notion of quasi-Hadamard differentiability from [13, 15, 57]. Indeed, this follows from the concept of differentiability in (i) of the latter definition with $\boldsymbol{W}''$ as in (iii).

Remark A.2 Consider the case where $\boldsymbol{W}'' = \boldsymbol{W}'$, $\boldsymbol{E}_0 = \boldsymbol{E}$, and $\|\cdot\|_{\boldsymbol{E}}$ provides a norm on all of $\boldsymbol{V}$. Then the notion of quasi-Hadamard differentiability of H at (fixed) v in part (i) of Definition A.1 coincides with the classical notion of Hadamard differentiability of H at v tangentially to $\boldsymbol{E}$

as introduced in [36, p. 102]. Here we stress the fact that in general the concept of Hadamard differentiability of H at v tangentially to $\boldsymbol{E}$ is not the same as the notion of quasi-Hadamard differentiability of H at v tangentially to $\boldsymbol{E}\langle\boldsymbol{E}\rangle$ because in the latter case the norm $\|\cdot\|_{\boldsymbol{E}}$ may be defined only on $\boldsymbol{E}$ (and not on all of $\boldsymbol{V}$). For an example of the latter situation, see Definition 4.3.7 in Subsection 4.3.2. $\diamond$

A.2 Definition of quasi-Lipschitz continuity, and an auxiliary lemma

In this section we give in Definition A.3 a notion of quasi-Lipschitz continuity which was originally introduced in [57]. The notion of quasi-Lipschitz continuity in the latter reference corresponds to the continuity concept in part (i) of Definition A.3 with $\boldsymbol{W}''$ as in (iii) of this definition. In the sequel, we denote by $0_{\boldsymbol{V}}$ the null in $\boldsymbol{V}$.

Definition A.3 *Let $v \in \boldsymbol{V}_H$.*

(i) The map H is said to be quasi-Lipschitz continuous at v along $\boldsymbol{E}$ with trace $\boldsymbol{W}''$ if $H(w)-H(v) \in \boldsymbol{W}''$ for all $w \in \boldsymbol{V}_H$ and

$$\|H(v+u_m) - H(v)\|_{\boldsymbol{W}''} = \mathcal{O}(\|u_m\|_{\boldsymbol{E}})$$

holds for every sequence $(u_m)_{m\in\mathbb{N}}$ in $\boldsymbol{E}\setminus\{0_{\boldsymbol{V}}\}$ satisfying $\|u_m\|_{\boldsymbol{E}} \to 0$ as well as $(v+u_m) \subseteq \boldsymbol{V}_H$.

(ii) If $\boldsymbol{E} = \boldsymbol{V}$, then we skip in (i) the phrases "quasi-" and "along $\boldsymbol{E}$".

(iii) If $\boldsymbol{W}'' = \boldsymbol{W}'$, then we skip in (i) the phrase "with trace $\boldsymbol{W}''$".

The following lemma is an adapted version of Lemma A.4 in [57]. Its statement can be obtained by following the lines in the proof of Lemma A.4 in [57].

Lemma A.4 *Let $v \in \boldsymbol{V}_H$. Then H is quasi-Lipschitz continuous at v along $\boldsymbol{E}$ with trace $\boldsymbol{W}''$ if and only if $H(w) - H(v) \in \boldsymbol{W}''$ for all $w \in \boldsymbol{V}_H$ and*

$$\|H(v+\varepsilon_m v_m) - H(v)\|_{\boldsymbol{W}''} = o(\varepsilon_m)$$

holds for every doublet $((v_m),(\varepsilon_m)) \in \boldsymbol{E}^{\mathbb{N}} \times \mathbb{R}_{>0}^{\mathbb{N}}$ satisfying $\|v_m\|_{\boldsymbol{E}} \to 0$, $\varepsilon_m \to 0$ as well as $(v + \varepsilon_m v_m) \subseteq \boldsymbol{V}_H$.

Lemma A.5 below provides a tool to obtain quasi-Lipschitz continuity of the map H based on quasi-Hadamard differentiability of H. Note that it follows from Lemma A.4 that quasi-Lipschitz continuity of H at (fixed) v along $\boldsymbol{E}$ with trace $\boldsymbol{W}''$ exactly coincides with quasi-Hadamard differentiability of H at v tangentially to $\{0_{\boldsymbol{V}}\}\langle\boldsymbol{E}\rangle$ with trace $\boldsymbol{W}''$ (in the sense of Definition A.1(i)) and quasi-Hadamard derivative $\dot{H}_v(0_{\boldsymbol{V}}) = 0_{\boldsymbol{W}}$, where $0_{\boldsymbol{W}}$ stands for the null in $\boldsymbol{W}$. Therefore we obtain immediately the following lemma which slightly generalizes Lemma A.5 in [57]. For an application of this lemma, see the proof of Lemma 4.5.5 in Subsection 4.5.2.

Lemma A.5 *Let $v \in \boldsymbol{V}_H$. Moreover let $\boldsymbol{E}_0$ be a subset of $\boldsymbol{E}$ with $0_{\boldsymbol{V}} \in \boldsymbol{E}_0$. If H is quasi-Hadamard differentiable at v tangentially to $\boldsymbol{E}_0\langle\boldsymbol{E}\rangle$ with trace $\boldsymbol{W}''$ and quasi-Hadamard derivative $\dot{H}_v : \boldsymbol{E}_0 \to \boldsymbol{W}''$ satisfying $\dot{H}_v(0_{\boldsymbol{V}}) = 0_{\boldsymbol{W}}$, then H is quasi-Lipschitz continuous at v along $\boldsymbol{E}$ with trace $\boldsymbol{W}''$.*

Appendix B

Lebesgue–Stieltjes integrals and an integration-by-parts formula

B.1 Definition of Lebesgue–Stieltjes integrals, and auxiliary lemmas

In this section we first want to introduce Lebesgue–Stieltjes integrals defined on $\mathbb{R}_{\geq 0}$. Note that we used the notation $\mathbb{R}_{\geq 0} := [0, \infty)$. Denote by $\mathcal{B}(\mathbb{R}_{\geq 0})$ the Borel σ-algebra on $\mathbb{R}_{\geq 0}$, and set $\Delta_a^b v := v(b) - v(a)$ for any map $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and any $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$.

The following proposition can be deduced from an analogue of Theorem 6.5 in [6]. Recall that any locally finite measure on $\mathcal{B}(\mathbb{R}_{\geq 0})$ is finite on every bounded interval in $\mathbb{R}_{\geq 0}$.

Proposition B.1 *Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a map. If v is non-decreasing and right-continuous, then there exists exactly one locally finite measure μ_v on $\mathcal{B}(\mathbb{R}_{\geq 0})$ satisfying*

$$\mu_v[(a, b]] \;=\; \Delta_a^b v \quad \text{for all } a, b \in \mathbb{R}_{\geq 0} \text{ with } a < b. \tag{B.1}$$

In this case, v is called measure-generating function (or Stieltjes measure function) and μ_v is the Stieltjes measure w.r.t. v.

Note that the (unique) locally finite Stieltjes-measure μ_v w.r.t. the measure-generating function v is clearly σ-finite.

Remark B.2 The statement in Proposition B.1 (and thus the following elaborations) can be extended to measure-generating functions v defined on the whole real line. In particular, the corresponding (unique) locally finite Stieltjes measure μ_v w.r.t. v is then defined on $\mathcal{B}(\mathbb{R})$. ◇

For any measure-generating function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with corresponding Stieltjes measure μ_v on $\mathcal{B}(\mathbb{R}_{\geq 0})$ and any Borel measurable map $u : \mathbb{R} \to \mathbb{R}$ for which the Lebesgue-integral $\int_{\mathbb{R}_{\geq 0}} |u| \, d\mu_v$ is finite, we define the *Lebesgue–Stieltjes integral of u w.r.t. v* by

$$\int_{\mathbb{R}_{\geq 0}} u \, dv \;:=\; \int_{\mathbb{R}_{\geq 0}} u \, d\mu_v.$$

The following Lemmas B.3 and B.4 are simple consequences of Proposition B.1 (along with the change-of-variables formula). Their statements will be used in Subsections 5.2.2 and 5.2.3. Let ℓ be the usual Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

Lemma B.3 *Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a measure-generating function (in the sense of Proposition B.1) and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a Borel measurable map with*

$$\int_{(a,b]} u\,d\ell \;=\; \Delta_a^b v \quad \textit{for all } a, b \in \mathbb{R}_{\geq 0} \textit{ with } a < b.$$

Then the Stieltjes measure μ_v w.r.t. v on $\mathcal{B}(\mathbb{R}_{\geq 0})$ has density u w.r.t. ℓ in the sense that

$$\mu_v[B] \;=\; \int_B u\,d\ell \quad \textit{for all } B \in \mathcal{B}(\mathbb{R}_{\geq 0}).$$

In particular, for any Borel measurable map $w : \mathbb{R} \to \mathbb{R}$ satisfying $\int_{\mathbb{R}_{\geq 0}} |w|\,d\mu_v < \infty$ or $\int_{\mathbb{R}_{\geq 0}} |wu|\,d\ell < \infty$ we have the following identity

$$\int_B w\,dv \;=\; \int_B wu\,d\ell \quad \textit{for all } B \in \mathcal{B}(\mathbb{R}_{\geq 0}).$$

Lemma B.4 *Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a measure-generating function (in the sense of Proposition B.1) with corresponding Stieltjes measure μ_v on $\mathcal{B}(\mathbb{R}_{\geq 0})$. Moreover let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a strictly increasing and right-continuous function. Then $v \circ u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a measure-generating function, and the corresponding Stieltjes measure $\mu_{v \circ u}$ on $\mathcal{B}(\mathbb{R}_{\geq 0})$ satisfies*

$$\mu_{v\circ u}[B] \;=\; \mu_v \circ (u^{-1})^{-1}[B] \quad \textit{for all } B \in \mathcal{B}(\mathbb{R}_{\geq 0}).$$

In particular, for any Borel measurable map $w : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}_{\geq 0}} |w| d\mu_{v\circ u} < \infty$ we have the following identity

$$\int_{\mathbb{R}_{\geq 0}} w\,d(v \circ u) \;=\; \int_{\mathbb{R}_{\geq 0}} w \circ u^{-1}\,dv.$$

Next, we want to introduce a Lebesgue–Stieltjes integral for functions $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which are right-continuous and of so-called locally bounded variation. For the latter concept, however, we still need to introduce some additional notation. This extension of the Lebesgue–Stieltjes integral will be needed to formulate the integration-by-parts formula presented in the next section.

In the sequel, we let either $I = \mathbb{R}_{\geq 0}$ or $I = [a, b]$ for some $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$. Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be any map. According to Section 4.4 in [25], the *variation of v on I* is defined by

$$\mathbb{V}_v(I) \;:=\; \sup\Big\{ \sum_i |v(x_i) - v(x_{i-1})| : \{x_i\} \in \mathcal{S} \Big\}, \tag{B.2}$$

where $\mathcal{S}$ consists of all finite sequences $\{x_i\}_{i=0}^n$ such that $x_0, \ldots, x_n \in I$, $x_0 < \ldots < x_n$, and $n \in \mathbb{N}$. In the same way, the positive (resp. negative) variation $\mathbb{V}_v^+(I)$ (resp. $\mathbb{V}_v^-(I)$) of v on I is defined as in (B.2) with $|\cdot|$ replaced by the positive part $(\cdot)^+$ (resp. negative part $(\cdot)^-$). The map v is said to be *of locally bounded variation on* $\mathbb{R}_{\geq 0}$ if $\mathbb{V}_v([a, b]) < \infty$ for every $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$. We denote by $\mathbb{BV}_{\text{loc}}(\mathbb{R}_{\geq 0})$ the linear space of all maps $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that are of locally bounded variation on $\mathbb{R}_{\geq 0}$.

Moreover it can be deduced from Proposition 2.18 in [46] that any $v \in \mathbb{BV}_{\text{loc}}(\mathbb{R}_{\geq 0})$ can be represented by two non-decreasing functions $v_+, v_- : \mathbb{R}_{\geq 0} \to \mathbb{R}$ through $v = v_+ - v_-$. For such v

we have $\mathbb{V}_v([a,b]) \le \Delta_a^b v_+ + \Delta_a^b v_-$ for all $a, b \in \mathbb{R}_{\ge 0}$ with $a < b$ with equality if and only if $\Delta_a^b v_\pm = \mathbb{V}_v^\pm([a,b])$. Therefore any $v \in \mathbb{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0})$ admits the representation

$$v(\cdot) \; = \; v(0) + v_{0,+}(\cdot) - v_{0,-}(\cdot), \tag{B.3}$$

where $v_{0,\pm}$ is defined by $v_{0,\pm}(x) := \mathbb{V}_v^\pm([0,x])$, $x \in \mathbb{R}_{\ge 0}$. Note that (B.3) refers to the so-called *Jordan decomposition of* $v \in \mathbb{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0})$. Also note that $\Delta_a^b[v] = \mathbb{V}_v([a,b])$ for every $a, b \in \mathbb{R}_{\ge 0}$ with $a < b$, where $[v] := v_{0,+} + v_{0,-}$.

In the sequel, we use $\mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\ge 0})$ to denote the linear space of all right-continuous functions in $\mathbb{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0})$, and fix $v \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\ge 0})$. As a consequence of Proposition 2.19 in [46], the non-decreasing components $v_{0,\pm}$ in the Jordan composition (B.3) of $v \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\ge 0})$ are right-continuous. Hence, since for any monotonic function the left-sided limits exist at every point, the functions $v_{0,\pm}$ in (B.3) and thus v and $[v]$ are càdlàg. In particular, the functions $v_{0,+}$ as well as $v_{0,-}$ (and thus $[v]$) are even measure-generating (in the sense of Proposition B.1) with corresponding Stieltjes measures $\mu_{v_{0,+}}$ and $\mu_{v_{0,-}}$, respectively. Thus for any Borel measurable map $u : \mathbb{R} \to \mathbb{R}$, the Lebesgue–Stieltjes integral of u w.r.t. v is defined by

$$\int_{\mathbb{R}_{\ge 0}} u \, dv \; := \; \int_{\mathbb{R}_{\ge 0}} u \, dv_{0,+} - \int_{\mathbb{R}_{\ge 0}} u \, dv_{0,-} \; = \; \int_{\mathbb{R}_{\ge 0}} u \, d\mu_{v_{0,+}} - \int_{\mathbb{R}_{\ge 0}} u \, d\mu_{v_{0,-}}$$

whenever $\int_{\mathbb{R}_{\ge 0}} |u| \, d[v] < \infty$.

B.2 An integration-by-parts formula

Maintain the notation and terminology introduced in Section B.1. In this section we present in Lemma B.5 below an integration-by-parts formula for Lebesgue–Stieltjes integrals defined on $\mathbb{R}_{\ge 0}$. This formula will be needed in Subsection 5.2.2 to show quasi-Hadamard differentiability of the value functional of the terminal wealth problem introduced in Subsection 5.2.1.

For the formulation of Lemma B.5, we denote by v^- the left-sided limit of $v \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\ge 0})$ defined by $v^-(x) := \lim_{y \nearrow x} v(y)$, $x \in \mathbb{R}_{\ge 0}$. Recall that v^- exists whenever $v \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\ge 0})$.

Lemma B.5 *Let* $u, v \in \mathbb{BV}_{\mathrm{loc,r}}(\mathbb{R}_{\ge 0})$ *with* $\lim_{x \searrow 0} u(x)v(x) = \lim_{x \to \infty} u(x)v(x) = 0$*, and assume that* $\int_{\mathbb{R}_{\ge 0}} |v| \, d[u] < \infty$ *as well as* $\int_{\mathbb{R}_{\ge 0}} |u^-| \, d[v] < \infty$*. Then* $\int_{\mathbb{R}_{\ge 0}} v \, du = -\int_{\mathbb{R}_{\ge 0}} u^- \, dv$*.*

The proof of Lemma B.5 can be carried out with similar arguments as in the proof of Lemma B.1 in [14].

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