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Subspace constraints for joint measurability

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Abstract. The structure of quantum effects, positive operators of norm at most one, played a central role in the work of Paul Busch on uncertainty, complementarity, and joint measurability in quantum measurement theory. Here we focus on one aspect of this structure, called “strength of an effect along a ray” [Lett. Math. Phys. **47** 329 (1999)], presenting a few observations not explicit in the existing literature. In fact, the strength function turns out to be useful for characterising positivity and complementarity of quantum effects of a suitable block matrix form, and for studying extensions of joint measurements defined on a subspace of codimension one.

1. Introduction

Uncertainty, complementarity, and joint measurability have been central to quantum mechanics since the beginning [1–3], and have acquired renewed interest in recent years in the context of quantum information and correlations. In particular, the existence of *incompatible* observables, i.e., those which cannot be measured using a single device, is a fundamental non-classical feature not only interesting in its own right [4, 5], but also essential for protocols such quantum steering [6–10] and state discrimination [11, 12].

While joint measurability was originally formulated as commutativity of Hermitian operators, the concept only acquired its current richness when applied to generalised observables, POVMs [13–15], which are necessary for practically relevant situations involving noise and open systems. Joint measurability of a set of observables does not require their commutativity; the general definition (e.g. [16]) is stated in terms of classical post-processing from a common observable, which immediately relates the concept to hidden state models for bipartite quantum states [17]. While the general definition is valid for any set of observables, it simplifies if the number of observables is finite - in particular, two observables are jointly measurable if they are marginals of a single common observable, see e.g. [15].

Restriction to two observables is motivated by the study of pairs of maximally incompatible, *complementary*, observables, especially position and momentum, which can be approximated by jointly measurable pairs obtained by introducing noise. In fact, this form of joint measurability appeared most often in the work of Paul Busch; for review see [18]. Similar ideas have appeared in the context of *Mutually Unbiased Bases* (MUB) which represent complementary observables in finite-dimensional systems, see e.g. [19]. Complementarity in the sense of a pair of MUBs formalises the idea that a state with a definite outcome for one of the observables has a completely uniform outcome distribution for the other one. While this works well in the finite-dimensional case, complementarity of continuous variables is more subtle. A considerable part of the work



of Paul Busch was aimed at forming and studying an exact definition of complementarity based on these intuitive ideas [18].

Central to Paul’s definition of complementarity was the order structure of quantum effects; as described in [18], one aspect of this structure is the notion of “strength of an effect along a ray” in Hilbert space [20]. The purpose of the present paper is to focus on this concept, which turns out to have various uses in the context of complementarity and joint measurability. We first briefly review the definition, making a connection to Schur complements of block matrices, and then provide a characterisation of complementary effects which goes somewhat beyond the existing literature (including the review [18]). Finally, we show how this concept naturally features in the study of joint measurability of pairs of observables having a suitable block structure relative to a subspace of co-dimension one.

2. Strength of an effect along a ray

Let \mathcal{H} be a separable (possibly infinite-dimensional) Hilbert space, and $\mathcal{B}(\mathcal{H})$ the set of bounded operators on \mathcal{H} . For each vector $\psi \in \mathcal{H}$ we let $|\psi\rangle\langle\psi| \in \mathcal{B}(\mathcal{H})$ denote the rank-1 operator $\varphi \mapsto \langle\psi|\varphi\rangle\psi$.

An operator $E \in \mathcal{B}(\mathcal{H})$ is called an *effect* if $0 \leq E \leq \mathbb{I}$ where \mathbb{I} is the identity operator. We recall that effects can be operationally associated to detection events, say, of a measurement yielding to a specific outcome. Given a quantum state ρ (a positive operator of unit trace), the probability of the event is given by $\text{tr}[\rho E]$. Each projection P is an effect; in the terminology of Paul Busch, an effect is called *unsharp* [21] if it is not a projection.

Each vector $\psi \in \mathcal{H}$ with $\|\psi\| \leq 1$ determines the rank one effect $|\psi\rangle\langle\psi|$. These are the elementary units from which more complex effects are composed of, hence they are sometimes called *weak atoms* [20]. Accordingly, given an effect E and a fixed vector $\psi \in \mathcal{H}$, one may ask whether E can be decomposed as $E = \lambda|\psi\rangle\langle\psi| + E'$ where $\lambda \geq 0$ and $E' \geq 0$. Operationally, this corresponds to a *fine-graining*: we consider the event associated to E as being split into two mutually exclusive events, assigned to the weak atom $\lambda|\psi\rangle\langle\psi|$, and the remainder E' , respectively—indeed, $\text{tr}[\rho(\lambda|\psi\rangle\langle\psi|)] + \text{tr}[\rho E'] = \text{tr}[\rho E]$ for any state ρ . The maximum λ for which such a decomposition holds is called *the strength of E along ψ* ; explicitly,

$$\lambda(E, \psi) := \sup\{\lambda \geq 0 \mid \lambda|\psi\rangle\langle\psi| \leq E\}. \tag{1}$$

Note that $\lambda(E, \psi) \in [0, \|\psi\|^{-2}]$, and $\lambda(E, \psi) = \infty$ iff $\psi = 0$. The optimisation problem (1) can be solved explicitly by using the connection between effect order and range inclusion [22]. In fact, there is a $\lambda > 0$ with $\lambda|\psi\rangle\langle\psi| \leq E$, if and only if $\psi \in \text{ran } E^{\frac{1}{2}}$. Noting that the restriction of $E^{\frac{1}{2}}$ to $(\ker E)^{\perp}$ is injective, we may define $E^{-\frac{1}{2}}$ as the unbounded operator on \mathcal{H} with domain $\text{ran } E^{\frac{1}{2}}$ and range $(\ker E)^{\perp}$. Denoting $\|E^{-\frac{1}{2}}\varphi\| = \infty$ whenever φ is not in the domain, the following result was obtained in [20]:

$$\lambda(E, \psi)^{-1} = \|E^{-\frac{1}{2}}\psi\|^2 \tag{2}$$

for any effect E and vector $\psi \in \mathcal{H}$. We then note that $\text{ran } E^{\frac{1}{2}} = \{\psi \in (\ker E)^{\perp} \mid \int_{[0,1]} x^{-1}\langle\psi|\mathbf{E}(dx)\psi\rangle < \infty\}$, where \mathbf{E} is the spectral measure of E , and, furthermore,

$$\lambda(E, \psi)^{-1} = \int_{[0,1]} x^{-1}\langle\psi|\mathbf{E}(dx)\psi\rangle, \tag{3}$$

which holds for all $\psi \in \mathcal{H}$ (the left hand side being finite exactly when the right hand side is). In particular, we obtain the following equality from functional calculus.

Lemma 1.

$$\lambda(E, \psi)^{-1} + \lambda(\mathbb{I} - E, \psi)^{-1} = \lambda(E(\mathbb{I} - E), \psi)^{-1}, \tag{4}$$

for each $\psi \in \mathcal{H}$, and each effect $E \in \mathcal{B}(\mathcal{H})$.

3. Positive extensions of positive subspace operators

We now consider the above Hilbert space \mathcal{H} as a subspace of codimension one in the “minimally” enlarged Hilbert space $\tilde{\mathcal{H}} = \mathbb{C} \oplus \mathcal{H}$. As an example, one could think of $\tilde{\mathcal{H}}$ as the Fock space truncated to the first two sectors, the vacuum state $|\text{vac}\rangle$ and the single system space. We write the associated vectors in $\tilde{\mathcal{H}}$, linear functionals $\tilde{\mathcal{H}} \rightarrow \mathbb{C}$, and operators in $\mathcal{B}(\tilde{\mathcal{H}})$ in the convenient (slightly nonstandard) matrix notation

$$c|\text{vac}\rangle \oplus \varphi = \begin{pmatrix} c \\ |\varphi\rangle \end{pmatrix} \in \tilde{\mathcal{H}}, \quad \bar{c}\langle\text{vac}| \oplus \langle\varphi| = (\bar{c} \quad \langle\varphi|) : \tilde{\mathcal{H}} \rightarrow \mathbb{C}, \quad \tilde{E} = \begin{pmatrix} p & \langle\psi'| \\ |\psi\rangle & E \end{pmatrix} \in \mathcal{B}(\tilde{\mathcal{H}}),$$

where $p, c \in \mathbb{C}$, $\psi, \psi', \varphi \in \mathcal{H}$, and $E \in \mathcal{B}(\mathcal{H})$, and the action of the operator and the functional on a vector is computed by usual matrix multiplication rule and “bra-ket” notation. In the Fock space picture, vectors of the above form represent coherent superpositions of a single system and vacuum. In what follows we sometimes use P_0 to denote the projection on $\tilde{\mathcal{H}}$ with range \mathcal{H} .

We then define the *Schur complements* associated to \tilde{E} by

$$\tilde{E}/p := E - p^{-1}|\psi\rangle\langle\psi'|, \quad \tilde{E}/E := p - \langle E^{-\frac{1}{2}}\psi'|E^{-\frac{1}{2}}\psi\rangle,$$

where the first one is defined when $p \neq 0$, and the second one when $E \geq 0$ and $\psi, \psi' \in \text{ran } E^{\frac{1}{2}}$. In the case where $\dim H < \infty$ these coincide with the usual Schur complements for block matrices [23]. In the general case, they are closely related to the strength function $\lambda(E, \psi)$, which characterises the positivity of \tilde{E} as the following proposition shows.

Proposition 1. *Let*

$$\tilde{E} = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & E \end{pmatrix}$$

where $E \geq 0$, $p > 0$, and $\psi \in \mathcal{H}$. Then the following are equivalent:

- (i) $\tilde{E} \geq 0$;
- (ii) $\tilde{E}/p \geq 0$;
- (iii) $p \lambda(E, \psi) \geq 1$;
- (iv) \tilde{E}/E is defined and $\tilde{E}/E \geq 0$.

Proof. From equations (1) and (2), it is clear that $\tilde{E}/p \geq 0$ iff $p^{-1} \leq \lambda(E, \psi)$ iff $p - \|E^{-\frac{1}{2}}\psi\|^2 \geq 0$ iff $\tilde{E}/E \geq 0$. The equivalence between (i) and (ii) can be found in [24, p.27]. \square

We note that this result is well-known in the finite-dimensional case [23]. In general, the following corollary is immediate.

Corollary 1. *In the notation of the preceding proposition, if $\tilde{E} \geq 0$, then $\|\psi\|^2 \leq p\|E\|$.*

Proof.

$$\|\psi\|^2 = \|E^{\frac{1}{2}}E^{-\frac{1}{2}}\psi\|^2 \leq \|E^{\frac{1}{2}}\|^2 \lambda(E, \psi)^{-1} \leq \|E^{\frac{1}{2}}\|^2 p = \|E\|p.$$

\square

4. Coherent extensions of effects and connection to complementarity

Any effect $E \in \mathcal{B}(\mathcal{H})$ trivially defines an effect on $\tilde{\mathcal{H}}$ as \mathcal{H} is a subspace. However, if we require that the resulting effect be used to *detect (vacuum) coherence* in states, that is, distinguish coherent superposition states between the vacuum and \mathcal{H} from classical mixtures, the extension is not entirely straightforward. We first define, for each $p > 0$, $\psi \in \mathcal{H}$, a map $I_{p,\psi} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\tilde{\mathcal{H}})$ via

$$I_{p,\psi}(E) = \begin{pmatrix} p & \langle \psi | \\ | \psi \rangle & E \end{pmatrix}.$$

This extension map is called *coherent* if $\psi \neq 0$, and we call ψ the associated *coherence*. Clearly, a coherent extension is needed for detecting coherence in states, as $\text{tr}[\rho I_{p,0}(E)] = \text{tr}[\rho' I_{p,0}(E)]$ for the density operators of the form

$$\rho = \begin{pmatrix} q & \langle \xi | \\ | \xi \rangle & \rho_0 \end{pmatrix}, \quad \rho' = \begin{pmatrix} q & 0 \\ 0 & \rho_0 \end{pmatrix}.$$

By Proposition 1, $I_{p,\psi}(E) \geq 0$ if and only if $p\lambda(E, \psi) \geq 1$. We now easily obtain the following.

Proposition 2. *Let $E \in \mathcal{B}(\mathcal{H})$ be an effect, and $\psi \in \mathcal{H} \setminus \{0\}$. The following are equivalent:*

- (i) $I_{p,\psi}(E)$ is an effect for some $p \in [0, 1]$;
- (ii) $\lambda(E(\mathbb{I} - E), \psi) \geq 1$.

In this case, $I_{p,\psi}(E)$ is an effect if and only if $\lambda(E, \psi)^{-1} \leq p \leq 1 - \lambda(\mathbb{I} - E, \psi)^{-1}$.

Proof. Denote $a := \lambda(E, \psi)^{-1}$ and $b := 1 - \lambda(\mathbb{I} - E, \psi)^{-1}$, and note that $a > 0$ and $b < 1$ because $\neq 0$. Now, for any $p \in [0, 1]$ we have $\mathbb{I} - I_{p,\psi}(E) = I_{1-p,-\psi}(\mathbb{I} - E)$. Hence $I_{p,\psi}(E)$ is an effect iff $p \in (0, 1)$ and $p^{-1} \leq \lambda(E, \psi)$ and $(1 - p)^{-1} \leq \lambda(\mathbb{I} - E, \psi)$. But this happens exactly when $a \leq p \leq b$. Hence, (i) is equivalent to $a \leq b$, which by Lemma 1, is equivalent to (ii). In this case $I_{p,\psi}(E)$ is an effect exactly when $p \in [a, b]$. □

Corollary 2. *If P is a projection, it has no coherent effect extension $I_{p,\psi}(P)$. In other words, only unsharp effects can be coherently extended into effects.*

Proof. Now $P(\mathbb{I} - P) = 0$ so $\lambda(E(\mathbb{I} - E), \psi) \geq 1$ only when $\psi = 0$, and hence the claim follows from Proposition 2. Alternatively (and more explicitly), we note that $I_{p,\psi}(P) \geq 0$ and $\mathbb{I} - I_{p,\psi}(P) = I_{1-p,-\psi}(\mathbb{I} - P) \geq 0$ implies $\psi \in \text{ran } P \cap \text{ran } (\mathbb{I} - P) = \{0\}$. □

Complementarity of quantum effects was one of the key notions in the work of Paul Busch. Here we only quote the basics from the review [18], before proceeding to show how complementarity is related to the extension of subspace effects.

Two effects $E, F \in \mathcal{B}(\mathcal{H})$ are called *complementary*, if there is no nonzero effect $A \in \mathcal{B}(\mathcal{H})$ such that $A \leq E$ and $A \leq F$. The following characterisation can be found in [18] (see references therein for the original papers).

Proposition 3. *For two effects $E, F \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

- (i) E and F are complementary.
- (ii) $\text{ran } E^{\frac{1}{2}} \cap \text{ran } F^{\frac{1}{2}} = \{0\}$;
- (iii) *There is no $\psi \in \mathcal{H} \setminus \{0\}$ such that $|\psi\rangle\langle\psi| \leq E$ and $|\psi\rangle\langle\psi| \leq F$;*

We can now state the following result, which reveals another aspect of complementarity.

Proposition 4. *For two effects $E, F \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

- (i) E and F are complementary;

- (ii) $\lambda(E, \psi) = 0$ or $\lambda(F, \psi) = 0$ for each $\psi \in \mathcal{H} \setminus \{0\}$;
 (iii) The coherent extensions $I_{p,\psi}(E)$ and $I_{p,\psi}(F)$ are not both positive for any choice of $\psi \in \mathcal{H} \setminus \{0\}$, $p \in [0, 1]$.

Proof. Proposition 3 shows that (i) is equivalent to (ii), and the equivalence of (ii) and (iii) is clear from Proposition 1. \square

Proposition 5. For two effects $E, F \in \mathcal{B}(\mathcal{H})$, the following are equivalent:

- (i) The effects $E(\mathbb{I} - E)$ and $F(\mathbb{I} - F)$ are complementary;
 (ii) The coherent extensions $I_{p,\psi}(E)$ and $I_{p',\psi}(F)$ are not both effects for any choice of $\psi \in \mathcal{H} \setminus \{0\}$, $p, p' \in [0, 1]$.

Proof. Follows immediately from Proposition 2. \square

5. Coherent extensions of operations, observables, and instruments

In this section we consider constraints for extending the standard quantum objects related to measurement processes. We also give an operational interpretation to such extensions within the above mentioned Fock space context, which allows us to describe the idea that a measurement may leave the system in a coherent superposition with the vacuum after the measurement.

5.1. Operations

The basic definitions can be found in [15]. An *operation* is a conditional transformation between the state spaces of two quantum systems. Denoting by \mathcal{H} and \mathcal{H}' the corresponding Hilbert spaces, an operation is any completely positive normal linear map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$ with $\Phi(\mathbb{I}) \leq \mathbb{I}$. Here the effect $\Phi(\mathbb{I})$ (called the effect of Φ) has the following operational meaning: for any state ρ on \mathcal{H}' , the number $\text{tr}[\rho\Phi(\mathbb{I})]$ is the probability of the event which triggers the transformation (typically an occurrence of a specific outcome in a measurement). Every operation can be decomposed in terms of its *Kraus operators* $K_k : \mathcal{H} \rightarrow \mathcal{H}'$ as $\Phi(A) = \sum_k K_k^* A K_k$, where the sum may be (countably) infinite, converging in the weak operator topology. An operation Φ is called *channel* if $\Phi(\mathbb{I}) = \mathbb{I}$. Clearly, if E is the effect of an operation Φ , it is also the effect of $\Phi \circ \Lambda$ where Λ is any channel.

An operation is called *pure* if it can be decomposed using only one Kraus operator K . In this case the transformation preserves pure states, acting as $\psi \mapsto K\psi$ in the Hilbert space level. We note that *any* effect $E \in \mathcal{B}(\mathcal{H})$ can be decomposed in various ways as $E = K^*K$ with $K : \mathcal{H} \rightarrow \mathcal{H}'$ where \mathcal{H}' can be any Hilbert space of dimension at least the rank of E . Operationally, these correspond to different pure operations whose triggering event is described by E . The case where $E = |\psi\rangle\langle\psi|$ for some $\psi \in \mathcal{H}$ is special in that the final normalised state after the operation is just the weak atom $\|\psi\|^{-2}|\psi\rangle\langle\psi|$, i.e., all information on the initial state has been lost in the transformation.

Consider now a fine-graining of E discussed in section 2, that is, $E = |\eta\rangle\langle\eta| + E'$, where E' is an effect. Hence we must have $\lambda(E, \eta) \geq 1$, and in particular, $\eta \in \text{ran } E^{\frac{1}{2}}$. Take any Kraus operator $K : \mathcal{H} \rightarrow \mathcal{H}'$ such that $E = K^*K$. It follows (see [18, Lemma 1]) that there is a unit vector $\varphi \in \mathcal{H}'$ such that

$$K^*\varphi = \sqrt{\lambda(E, \eta)}\eta.$$

Denoting $\mu = \lambda(E, \eta)^{-1}$ we define $K_1 := \sqrt{\mu}|\varphi\rangle\langle\varphi|$ and $K_0 := \sqrt{1-\mu}|\varphi\rangle\langle\varphi| + \mathbb{I} - |\varphi\rangle\langle\varphi|$; then $K_1^*K_1 = \mu|\varphi\rangle\langle\varphi|$, $K_0^*K_0 = \mathbb{I} - \mu|\varphi\rangle\langle\varphi|$, so we may define a channel $\Lambda(A) = K_0^*AK_0 + K_1^*AK_1$ with two Kraus operators, corresponding to the above decomposition: $E' = K^*K_0^*K_0K$ and $|\eta\rangle\langle\eta| = K^*K_1^*K_1K$. Hence the effect E can be associated to a sequential detection where exactly one of the two pure operations corresponding to K_0 and K_1 are performed after K ,

with respective probabilities μ and $1 - \mu$. In particular, the choice between the two is entirely classical, i.e., *incoherent*, as described by the operation $\Phi(A) = K^* \Lambda(A) K$.

The detection can be modified into a coherent one using the Fock space extension considered above: we first look at the pure operation acting on a vector $\phi \in \mathcal{H}$ in two stages as follows:

$$\begin{aligned} \phi &\mapsto K\phi = \langle \varphi | K\phi \rangle \varphi + (\mathbb{I} - |\varphi\rangle\langle \varphi|) K\phi \\ &\mapsto \sqrt{\mu} \langle \varphi | K\phi \rangle |\text{vac}\rangle \oplus (\sqrt{1 - \mu} \langle \varphi | K\phi \rangle \varphi + (\mathbb{I} - |\varphi\rangle\langle \varphi|) K\phi) = \langle \eta | \phi \rangle |\text{vac}\rangle \oplus K_0 K\phi. \end{aligned}$$

Here, following the application of K , the component of $K\phi$ along φ transforms into a superposition of the vacuum and φ (with amplitudes squaring to the probabilities μ and $1 - \mu$ appearing above), while the rest remains unaffected. The case $\mu = 1$, (i.e., $\lambda(E, \eta) = 1$) corresponds to the situation where the entire component along φ gets absorbed. Assuming also $|\text{vac}\rangle \mapsto 0$, this operator reads

$$\tilde{K} = \begin{pmatrix} 0 & \langle \eta | \\ 0 & K_0 K \end{pmatrix},$$

and one can readily check that $\tilde{K}^* \tilde{K} = 0 \oplus E = E$, that is, \tilde{K} defines a pure operation acting on the extended Hilbert space but whose effect is again E . Hence, instead of the incoherent operation Φ with two Kraus operators acting on the single system subspace, this operation combines the two processes coherently.

The detection corresponding to the new operation can *create* coherence, but cannot *detect* existing coherence, as its *effect* E is still incoherent. This is because the operation discards any existing vacuum contribution as $\tilde{K}|\text{vac}\rangle = 0$. In order to have a coherent effect, we only need to replace this by $\tilde{K}|\text{vac}\rangle = u|\text{vac}\rangle$ where $u \in \mathbb{C}$ is nonzero. This gives

$$\tilde{K} = \begin{pmatrix} u & \langle \eta | \\ 0 & K_0 K \end{pmatrix}, \quad (5)$$

leading to

$$\tilde{K}^* \tilde{K} = \begin{pmatrix} \bar{u} & 0 \\ |\eta\rangle & K^* K_0^* \end{pmatrix} \begin{pmatrix} u & \langle \eta | \\ 0 & K_0 K \end{pmatrix} = \begin{pmatrix} |u|^2 & \bar{u} \langle \eta | \\ u |\eta\rangle & |\eta\rangle \langle \eta | + E' \end{pmatrix} = \begin{pmatrix} |u|^2 & \bar{u} \langle \eta | \\ u |\eta\rangle & E \end{pmatrix} = I_{|u|^2, u\eta}(E),$$

and hence showing that we have indeed obtained a coherent positive extension of E . (While positivity is clear by construction, we also see it from the constraint on the strength function, as $\lambda(E, \eta) = |u|^2 \lambda(E, u\eta) \geq 1$.) However, an extra constraint now emerges, as the coherent extension $I_{|u|^2, u\eta}(E)$ is not necessarily an effect (see Proposition 2); we also need $(1 - |u|^2) \lambda(\mathbb{I} - E, u\eta) \geq 1$, that is

$$\frac{|u|^2}{1 - |u|^2} \leq \lambda(\mathbb{I} - E, \eta).$$

When this condition holds, the operator (5) describes a pure operation whose effect is $I_{|u|^2, u\eta}(E)$, a coherent effect extension of E , and the complementary effect $I_{1 - |u|^2, -u\eta}(\mathbb{I} - E)$ can be associated with a pure operation of the same kind.

In conclusion, we have shown that any coherent effect extension can be associated to a detection event where the system, when detected, proceeds into a coherent superposition of vacuum state and “no-absorption” relative to a single vector state. We also remark that *all* coherent effect extensions $I_{p, \psi}(E)$ of E can be associated in this way to coherent pure operations of the form (5), with $\psi = u\eta$ and $p = |u|^2$.

5.2. Observables

In the preceding section we considered coherent extensions of single effects and operations. In general, *quantum measurements* are described by collections of these, so we may use the above procedure to extend them as well. Beginning at the general level, we fix Ω be the set of outcomes for the measurements, with \mathcal{A} a σ -algebra of subsets of Ω .

An *observable* on the Hilbert space \mathcal{H} is a normalised positive operator valued measure (POVM), that is, a weakly (or, equivalently, strongly) σ -additive map $E : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, such that $E(X) \geq 0$ for each $X \in \mathcal{A}$, and $E(\Omega) = \mathbb{I}$. The set Ω represents possible outcomes of the observable, in the sense that the event of detecting an outcome in a set $X \in \mathcal{A}$ is associated to the effect $E(X)$.

Similarly, we can of course define observables on the extended Hilbert space $\tilde{\mathcal{H}}$. Given such an observable \tilde{E} , we can clearly compress it to an observable E on the subspace \mathcal{H} by setting $E(X) = P_0 \tilde{E}(X) P_0$. We call this the *subspace observable* of \tilde{E} .

Conversely, given an observable in \mathcal{H} , it is natural to ask whether it can be extended to an observable on the Hilbert space $\tilde{\mathcal{H}}$ by applying the procedure discussed for effects above. Clearly, we need a probability measure $p : \mathcal{A} \rightarrow [0, 1]$, and a (norm) σ -additive vector measure $\Psi : \mathcal{A} \rightarrow \mathcal{H}$ such that $\Psi(\Omega) = 0$ and $p(X) \lambda(E(X), \Psi(X)) \geq 1$ (see Proposition 1) for each $X \in \mathcal{A}$, as then

$$\tilde{E}(X) := \begin{pmatrix} p(X) & \langle \Psi(X) | \\ | \Psi(X) \rangle & E(X) \end{pmatrix}$$

defines an observable $\tilde{E} : \mathcal{A} \rightarrow \mathcal{B}(\tilde{\mathcal{H}})$. Note that now a single positivity constraint for each $\tilde{E}(X)$ is sufficient, as the overall normalisation condition $\Psi(\Omega) = 0$ (together with $E(\Omega) = \mathbb{I}$) then forces it to be an effect. From Corollary 1 we get a necessary condition

$$\|\Psi(X)\|^2 \leq \|E(X)\| p(X) \leq p(X),$$

which shows that the vector measure Ψ is p -continuous and of bounded variation. Since Hilbert spaces have the Radon-Nikodym property, it follows [25] that Ψ has a density with respect to p . However, E does not need to have an operator valued density relative to p , and the extension problem appears somewhat intractable in full generality. The problem simplifies if both E and p have a densities relative to fixed (not necessarily finite) positive scalar measure μ ; suppose therefore that $p(X) = \int_X p_x d\mu(x)$, $E(X) = \int_X E_x d\mu(x)$ (weakly), where $x \mapsto p_x$ is a positive measurable function, and $x \mapsto E_x$ is a (weakly) measurable $\mathcal{B}(\mathcal{H})$ -valued function such that $E_x \geq 0$ for each x . Since Ψ has a density relative to p , it follows that $\Psi(X) = \int_X \psi_x d\mu(x)$ for some measurable \mathcal{H} -valued function $x \mapsto \psi_x$. The above extension then takes the form

$$\tilde{E}(X) := \int_X \begin{pmatrix} p_x & \langle \psi_x | \\ | \psi_x \rangle & E_x \end{pmatrix} d\mu(x),$$

where the integrand is a weakly measurable $\mathcal{B}(\tilde{\mathcal{H}})$ -valued function, the integral is understood in the weak sense, and the constraints for normalisation and positivity take the form

$$\int_{\Omega} \psi_x d\mu(x) = 0, \quad p_x \lambda(E_x, \psi_x) \geq 1 \quad \text{for all } x \in \Omega. \quad (6)$$

One could proceed further with these constraints alone. However, we continue with quantum instruments, which provide a more concrete view to the extension problem.

5.3. Instruments

An *instrument* describes the measurement process, including the state change conditional to the outcome being observed. Mathematically, an instrument \mathcal{I} associates to each set $X \in \mathcal{A}$ an

operation $\mathcal{I}(X)$ on $\mathcal{B}(\mathcal{H})$, such that $X \mapsto \mathcal{I}(X)$ is weakly σ -additive, and $\mathcal{I}(\Omega)$ is a channel. This channel is interpreted as the transformation made on the system as a result of the measurement, without conditioning on a particular outcome. For an instrument \mathcal{I} , the map $X \mapsto \mathcal{I}(X)(\mathbb{I})$ is an observable, interpreted as the observable measured by the instrument.

An observable of the form $E(X) = \int_X E_x d\mu(x)$ considered above naturally arises from the Lüders instrument

$$\mathcal{I}(X)(A) = \int_X K_x^* A K_x d\mu(x), \quad (7)$$

where $x \mapsto K_x$ is measurable, and satisfies $E_x = K_x^* K_x$ for each $x \in \Omega$ ¹. We can now apply the construction of the preceding section pointwise for each x , so as to extend this into a Lüders instrument on $\mathcal{B}(\tilde{\mathcal{H}})$:

$$\tilde{\mathcal{I}}(X)(\tilde{A}) := \int_X \begin{pmatrix} \bar{u}_x & 0 \\ |\eta_x\rangle & K_x^* K_{0,x}^* \end{pmatrix} \tilde{A} \begin{pmatrix} u_x & \langle \eta_x | \\ 0 & K_{0,x} K_x \end{pmatrix} d\mu(x). \quad (8)$$

Here we require that $x \mapsto u_x$ is measurable, with $\int |u_x|^2 d\mu(x) = 1$, $\int u_x \eta_x d\mu(x) = 0$, and $\lambda(\eta_x, E_x) \geq 1$ for each x . The map $K_{0,x}$ is defined as in the preceding section, pointwise using K_x and η_x , and we assume this is done in a measurable way. The interpretation is that conditional on the outcome of the measurement being x , the system has the possibility of subsequently absorbed against the vector state φ_x defined by $K_x^* \varphi_x = \sqrt{\lambda(E_x, \psi_x)} \eta_x$ as described above.

6. Joint measurability and subspace compatibility

A pair of observables E and F on \mathcal{H} with outcome sets Ω_1 and Ω_2 , respectively, are said to be *jointly measurable* or *compatible*, if there exists an observable G with outcome set $\Omega_1 \times \Omega_2$, such that $G(X \times \Omega_2) = E(X)$ for all X , and $G(\Omega_1 \times Y) = F(Y)$ for all Y . The definition can be modified to include arbitrary collections of observables, but pairs are sufficient for our purpose. A concrete way of constructing joint measurements is through sequential combination: if \mathcal{I} is an instrument with observable E , and F is an observable, then (under fairly general conditions, see [15]) the sequential measurement defines a joint observable $G(X \times Y) := \mathcal{I}(X)(F(Y))$ with $G(X \times \Omega_2) = E(X)$. There other marginal $G(\Omega_1 \times Y)$ is by construction jointly measurable with E . Of course, the latter does not usually equal F , due to the disturbance caused by the first measurement.

Obviously, the same considerations apply to observables on the extended Hilbert space $\tilde{\mathcal{H}}$. Now, we introduce the following concept: two observables on the extended Hilbert space $\tilde{\mathcal{H}}$ are called *subspace compatible* if their subspace observables (see the preceding subsection) are compatible. The following observation is immediate:

Proposition 6. *If \tilde{E} and \tilde{F} are compatible observables on $\tilde{\mathcal{H}}$, then they are also subspace compatible.*

Proof. If \tilde{G} is a joint observable for \tilde{E} and \tilde{F} , then the compression $G(\cdot) = P_0 \tilde{G}(\cdot) P_0$ is a joint observable for the subspace observables $X \mapsto P_0 \tilde{E}(X) P_0$ and $Y \mapsto P_0 \tilde{F}(Y) P_0$. \square

The converse problem—whether given subspace compatible observables are also compatible—is interesting, and clearly nontrivial in general. Here we only show how observables jointly measurable with a given coherently extended subspace observable may arise, and then proceed with an example in the next section.

Fix an observable E in \mathcal{H} , and let \tilde{E} be one of its coherent extensions as described in the preceding section. A very general way of constructing observables jointly measurable with \tilde{E} is

¹ We do not consider here the conditions which ensure measurability in case E_x is given and K_x is obtained by pointwise decomposition; in most relevant cases, K_x is given explicitly.

by dilation (see [26]). In our case, this can be achieved by using the decomposition appearing in (8). We define

$$\tilde{\mathbf{G}}(X \times Y) = \int_X \begin{pmatrix} \bar{u}_x & 0 \\ |\eta_x\rangle & K_x^* K_{0,x}^* \end{pmatrix} \tilde{\mathbf{H}}_x(Y) \begin{pmatrix} u_x & \langle \eta_x | \\ 0 & K_{0,x} K_x \end{pmatrix} d\mu(x),$$

where $\tilde{\mathbf{H}}_x(Y)$ is an observable with a fixed outcome set Ω' , for each $x \in \Omega$ (and we again assume that the dependence on x is sufficiently measurable). Then $\tilde{\mathbf{G}}(X \times \Omega') = \tilde{\mathbf{E}}(X)$ for each X by construction, and hence by computing the other marginal we find an observable $\tilde{\mathbf{F}}(Y) := \tilde{\mathbf{G}}(\Omega \times Y)$ jointly measurable with $\tilde{\mathbf{E}}$. In fact, under certain conditions *all* observables jointly measurable with $\tilde{\mathbf{E}}$ can be constructed in this way [26].

A particular but practically relevant special case is obtained by choosing $\tilde{\mathbf{H}}_x(Y)$ independent of x , in which case $\tilde{\mathbf{G}}(X \times Y) = \tilde{\mathbf{I}}(X)(\tilde{\mathbf{F}}(Y))$, that is, the joint observable can be implemented as a sequential measurement consisting of the above absorption implementation of $\tilde{\mathbf{E}}$, followed by a measurement of $\tilde{\mathbf{F}}$. Explicitly, let

$$\tilde{\mathbf{H}}(Y) = \begin{pmatrix} q(Y) & \langle \Phi(Y) | \\ |\Phi(Y)\rangle & \mathbf{H}(Y) \end{pmatrix};$$

then the joint measurement is

$$\begin{aligned} \tilde{\mathbf{G}}(X \times Y) &= \int_X \begin{pmatrix} \bar{u}_x & 0 \\ |\eta_x\rangle & K_x^* K_{0,x}^* \end{pmatrix} \begin{pmatrix} q(Y) & \langle \Phi(Y) | \\ |\Phi(Y)\rangle & \mathbf{H}(Y) \end{pmatrix} \begin{pmatrix} u_x & \langle \eta_x | \\ 0 & K_{0,x} K_x \end{pmatrix} d\mu(x) \\ &= \int_X \begin{pmatrix} \bar{u}_x & 0 \\ |\eta_x\rangle & K_x^* K_{0,x}^* \end{pmatrix} \begin{pmatrix} u_x q(Y) & q(Y)\langle \eta_x | + \langle \Phi(Y) | K_{0,x} K_x \\ |u_x \Phi(Y)\rangle & |\Phi(Y)\rangle \langle \eta_x | + \mathbf{H}(Y) K_{0,x} K_x \end{pmatrix} d\mu(x) \\ &= \int_X \begin{pmatrix} |u_x|^2 q(Y) & \langle q(Y)\psi_x + u_x K_x^* K_{0,x}^* \Phi(Y) | \\ |q(Y)\psi_x + u_x K_x^* K_{0,x}^* \Phi(Y)\rangle & M_x(Y) \end{pmatrix} d\mu(x) \\ &= \begin{pmatrix} p(X)q(Y) & \langle q(Y)\Psi(X) + K(X)\Phi(Y) | \\ |q(Y)\Psi(X) + K(X)\Phi(Y)\rangle & \mathbf{G}(X \times Y) \end{pmatrix}, \end{aligned}$$

where $K(X) := \int_X u_x K_x^* K_{0,x}^* d\mu(x)$, and the subspace observable is given by

$$\mathbf{G}(X \times Y) := \int_X M_x(Y) d\mu(x)$$

with

$$\begin{aligned} M_x(Y) &= q(Y)|\eta_x\rangle \langle \eta_x | + I_x(Y) + K_x^* K_{0,x}^* H(Y) K_{0,x} K_x \\ &= q(Y)\mu_x K_x^* |\varphi_x\rangle \langle \varphi_x | K_x + K_x^* K_{0,x}^* H(Y) K_{0,x} K_x + I_x(Y) \\ &= K_x^* (q(Y)\mu_x |\varphi_x\rangle \langle \varphi_x | + K_{0,x}^* H(Y) K_{0,x} + I_x(Y)) K_x \end{aligned}$$

and $I_x(Y) = 2\text{Re}(|K_{0,x}^* \Phi(Y)\rangle \langle \sqrt{\mu} \varphi_x |)$. From this we can understand the structure of the joint observable, the central feature of which is that the particle is only subjected to the subspace part of the second measurement if it does not get absorbed into the vacuum by the first measurement.

First of all, the coherence $q(Y)\Psi(X) + K(X)\Phi(Y)$ is a superposition of the coherence of $\Psi(X)$ the first measurement, and the coherence $\Phi(Y)$ of the second measurement transformed by $K(X)$ conditional to the first measurement taking place without absorption. Note that $K(X)$ is indeed the coherent combination of all the conditional transformations associated to the no-absorption.

Secondly, the subspace observable $\mathbf{G}(X \times Y)$ has the following structure: the term $q(Y)\mu_x |\varphi_x\rangle \langle \varphi_x |$ corresponds to the event of the system being absorbed into the vacuum in

the first measurement, with the subsequent measurement $q(Y)$ of the vacuum component in the second measurement. The second term $K_{0,x}^* H(Y) K_{0,x}$ corresponds to the event of no-absorption, consisting of the associated conditional transformation $K_{0,x}$ followed by the subspace component $H(Y)$ of the second measurement. Finally, $I_x(Y)$ is the interference term for the coherences arising from the two choices, the coherence of the second measurement modified by the no-absorption transform of the first measurement, and absorption in the first measurement. In particular, the interference term only appears if the second measurement is coherent.

As a by-product, we now obtain a new family of observables in \mathcal{H} jointly measurable with the original observable E we started with. Indeed, by Proposition 6, the subspace observable of \tilde{F} must be jointly measurable with E since the latter is the subspace observable of the first marginal \tilde{E} . Explicitly, the observable

$$F'(Y) := \int_{\Omega} K_x^* (q(Y)\mu_x|\varphi_x\rangle\langle\varphi_x| + K_{0,x}^* H(Y) K_{0,x} + I_x(Y)) K_x d\mu(x)$$

is jointly measurable with E , and the joint measurement is the G constructed above. This describes the adjustment in the usual sequential Lüders measurement due to the absorption process as just described.

7. Example: unsharp position and momentum

In this final section we illustrate the use of the above absorption-supplemented measurement model, in the case of a standard model position measurement followed by the sharp momentum measurement.

7.1. Supplementing the standard model of position measurement with rank one absorption

Let $\mathcal{H} = L^2(\mathbb{R})$, and Q the usual position operator, with spectral measure E^Q . Since Q is projection valued, it cannot be extended coherently into $\tilde{\mathcal{H}}$ (see Corollary 2), that is, the only way possible extensions are of the form

$$\tilde{E}^Q(X) = \begin{pmatrix} p(X) & 0 \\ 0 & E^Q(X) \end{pmatrix},$$

where p is some probability measure. However, realistic position measurements are unsharp, i.e., not projection valued, which makes it possible to extend them using the above procedure.

Fix a function $f \in L^2(\mathbb{R})$, and assume that f is (essentially) bounded. We consider a position measurement of the form (7) where μ is the Lebesgue measure on the real line $\Omega = \mathbb{R}$, and $K_x = f(Q - x)$ is the operator of multiplication by the bounded function $y \mapsto f(y - x)$, that is,

$$\mathcal{I}(X)(A) = \int_X f(Q - x)^* A f(Q - x) dx. \tag{9}$$

Such an instrument is obtained, for instance, from the *standard model* of position measurement (again we refer to [15] and the references therein). This measurement approximately localizes the particle in the sense that the post-measurement state conditional on the outcome x has the wave function proportional to $y \mapsto f(y - x)\psi(y)$ (which is in $L^2(\mathbb{R})$ for almost all x). In particular, if f peaks at zero, the conditional state after the measurement yielding value x is concentrated around x in the position space, indicating that the outcome is likely to occur there. The measured observable

$$E(X) = \int_X f(Q - x)^* f(Q - x) dx \tag{10}$$

is called an unsharp position observable.

We can now supplement this measurement with rank one absorption. We only consider the simple case in which the normalisation is ensured by taking the coherence of the approximating position observable to be proportional to a single state χ . This requires the condition $\chi \in \text{ran } \sqrt{K_x^* K_x}$ for all $x \in \mathbb{R}$, which reads as

$$\lambda(E_x, \chi)^{-1} = \|E_x^{-\frac{1}{2}} \chi\|^2 = \int_{-\infty}^{\infty} \frac{|\chi(y)|^2}{|f(y-x)|^2} dy < \infty.$$

In order to illustrate this, we consider the Gaussian case, with $\chi(x) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2/2}$, and $f(x) = \left(\frac{\lambda}{\pi}\right)^{\frac{1}{4}} e^{-\lambda x^2/2}$, where $a > 0$ and $\lambda > 0$ can be adjusted to change the spreads of χ and the localisation operators. We recall that in the standard model of position measurement, λ can be associated to the scaling of the pointer function of the measurement device [15]. In our case the supports are full, and the possibility of absorption is determined by the relative spread

$$\lambda(E_x, \chi)^{-1} = \sqrt{\frac{a}{\lambda}} \int_{-\infty}^{\infty} e^{-ay^2 + \lambda(y-x)^2} dy = e^{\lambda x^2} \sqrt{\frac{a}{\lambda}} \int_{-\infty}^{\infty} e^{-(a-\lambda)y^2 - 2\lambda xy} dy = \frac{\sqrt{\pi r}}{\lambda} e^{rx^2},$$

where $r := \lambda a(a - \lambda)^{-1}$. Clearly, $\lambda(E_x, \chi)^{-1} < \infty$ exactly when $a > \lambda$. Hence, the spread of χ must be strictly smaller than the spread of the localisation function f of the measurement, to ensure that χ is in the range of each $f(Q - x)$. Next, we need to modulate χ with an x -dependent factor to fulfil the condition for positivity: we define $\eta_x = g(x)\chi$, and choose the coherence function $g(x)$ so that $\lambda(E_x, \eta_x) \geq 1$, that is, $|g(x)|^2 \leq \lambda(E_x, \chi)$, which reads

$$|g(x)|^2 \leq \frac{\lambda}{\sqrt{\pi r}} e^{-rx^2}, \quad \text{for all } x \in \mathbb{R}.$$

In order to achieve maximal coherence (and also to simplify the result), we choose $g(x) = \lambda^{\frac{1}{2}}(\pi r)^{-\frac{1}{4}} e^{-\frac{1}{2}rx^2}$, so that $\lambda(E_x, \eta_x) \geq 1$ for each x . Finally, we need to ensure normalisation, choosing u_x so that $\int \psi_x dx = 0$; this condition now reads $\int u_x g(x) dx = 0$. Since g is even, any odd function $x \mapsto u_x$ with $\int |u_x|^2 dx = 1$ works here. Perhaps the simplest choice which combines easily with g is $u_x = (\pi/r)^{-\frac{1}{4}} e^{-rx^2/2} \text{sign}(x)$, leading to

$$\psi_x = u_x \eta_x = \sqrt{\lambda/\pi} e^{-rx^2} \text{sign}(x) \chi.$$

Note that due to maximal coherence, $K_{0,x} = \mathbb{I} - |\varphi_x\rangle\langle\varphi_x|$ is a projection. We can now find the absorption ‘‘sink’’ unit vector φ_x given by the condition $K_x^* \varphi_x = \eta_x$; explicitly,

$$\varphi_x(y) = (a\lambda/(r\pi))^{\frac{1}{4}} e^{-\frac{1}{2}\frac{\lambda a}{r}(y + \frac{r}{a}x)^2} = \varphi_0\left(y + \frac{r}{a}x\right),$$

that is, $\varphi_x = U_x \varphi_0$ where U_x is the scaled translation semigroup. Hence, the sink associated to outcome x is obtained from the sink at $x = 0$ by translation. Next, we define $B_x := U_x^* f(Q - x)$, which is a combination of localisation around x and translation by x . This allows us to write

$$g(x)\chi = \eta_x = B_x^* \varphi_0, \quad \text{for all } x. \quad (11)$$

Note also that $B_x^* B_x = f(Q - x)^2$. Next, letting $R := \mathbb{I} - |\varphi_0\rangle\langle\varphi_0|$ we have $K_{0,x} K_x = U_x R B_x$, and we can write the instrument as

$$\tilde{\mathcal{I}}(X)(\tilde{A}) = \int_X \begin{pmatrix} u_x & 0 \\ |g(x)\chi\rangle & B_x^* R U_x^* \end{pmatrix} \tilde{A} \begin{pmatrix} u_x & \langle g(x)\chi | \\ 0 & U_x R B_x \end{pmatrix} dx, \quad (12)$$

and the associated coherent extension of the position observable (10) is given explicitly by

$$\tilde{E}(X) = \int_X \begin{pmatrix} |u_x|^2 & u_x g(x) \langle \chi | \\ u_x g(x) | \chi \rangle & f(Q-x)^2 \end{pmatrix} dx = \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}} \int_X \begin{pmatrix} (r/\lambda)^{\frac{1}{2}} & \text{sign}(x) \langle \chi | \\ \text{sign}(x) | \chi \rangle & e^{-\lambda(Q-x)^2} e^{rx^2} \end{pmatrix} e^{-rx^2} dx. \tag{13}$$

One can easily check directly that this is indeed an observable whose compression to \mathcal{H} (i.e., the lower right block of the matrix) is (10).

7.2. *Approximate joint measurements of extended position and momentum*

A straightforward way of implementing approximate joint measurements of position and momentum is their sequential combination. In particular, measuring first an unsharp position using the instrument (9), and then subsequently a sharp momentum observable $E^P(Y)$, yields the joint observable

$$G(X \times Y) = \int_X f(Q-x) E^P(Y) f(Q-x) dx$$

whose marginal observables (10) and

$$F(Y) := \int f(Q-x) E^P(Y) f(Q-x) dx \tag{14}$$

are unsharp versions of position and momentum [15]. From the point of view of the extended Hilbert space, any extensions of these observables are therefore (by definition) subspace compatible. We now proceed to find a “distorted” momentum observable, whose incoherent extension is not only subspace compatible with (10), but also compatible with the above constructed coherent extension (13).

A natural way to do this is to apply the above instrument to an extension of the sharp momentum observable E^P . Due to being sharp, the latter can only be extended incoherently, and if we wish to keep the extension sharp, there is only one option:

$$\tilde{E}^P(Y) = \begin{pmatrix} 1 & 0 \\ 0 & E^P(Y) \end{pmatrix}.$$

Now, measuring first \tilde{E} using the above constructed instrument (12), and then measuring the sharp observable \tilde{E}^P gives the joint observable $\tilde{G}(X \times Y) = \tilde{I}(X)(\tilde{E}^P(Y))$, whose first marginal is \tilde{E} . It is interesting to find the second marginal \tilde{F} , which is then by construction jointly measurable with \tilde{E} . First, we compute the joint observable

$$\begin{aligned} \tilde{G}(X \times Y) &:= \int_X \begin{pmatrix} u_x & 0 \\ |g(x)\chi\rangle & B_x^* R U_x^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E^P(Y) \end{pmatrix} \begin{pmatrix} u_x & \langle g(x)\chi | \\ 0 & U_x R B_x \end{pmatrix} dx \\ &= \int_X \begin{pmatrix} u_x & 0 \\ |g(x)\chi\rangle & B_x^* R U_x^* \end{pmatrix} \begin{pmatrix} u_x & \langle g(x)\chi | \\ 0 & E^P(Y) U_x R B_x \end{pmatrix} dx \\ &= \int_X \begin{pmatrix} u_x^2 & \langle u_x g(x)\chi | \\ |u_x g(x)\chi\rangle & g(x)^2 | \chi \rangle \langle \chi | + B_x^* R U_x^* E^P(Y) U_x R B_x \end{pmatrix} dx \\ &= \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}} \int_X \begin{pmatrix} (r/\lambda)^{\frac{1}{2}} & \text{sign}(x) \langle \chi | \\ \text{sign}(x) | \chi \rangle & M_x(Y) \end{pmatrix} e^{-rx^2} dx, \end{aligned}$$

where

$$M_x(Y) = \sqrt{\frac{\lambda}{r}} | \chi \rangle \langle \chi | + e^{rx^2} \sqrt{\frac{\pi}{\lambda}} B_x^* R E^P(Y) R B_x,$$

and we have used the fact that U_x commutes with $E^P(Y)$. In particular, the joint observable is coherent, with the same coherences as \tilde{E} , and its subspace observable is

$$G(X \times Y) := \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}} \int_X M_x(Y) e^{-rx^2} dx = \int_X B_x^*(|\varphi_0\rangle\langle\varphi_0| + RE^P(Y)R) B_x dx.$$

Now $G(X \times \mathbb{R}) = E(X)$, the standard position observable, but the other marginal is different from (14), and given instead by

$$F'(Y) = \int_{\mathbb{R}} B_x^*(|\varphi_0\rangle\langle\varphi_0| + RE^P(Y)R) B_x dx.$$

We now observe how the structure of the joint observable reflects the detection scheme—the momentum measurement is in fact performed only conditional on the no-absorption event (represented by the projection R), in accordance with the obvious intuition: if the particle gets absorbed during the position measurement, it cannot experience the following momentum measurement. Finally, we note that the marginal of the extended joint observable is

$$\tilde{F}(Y) := \tilde{G}(\mathbb{R} \times Y) = \begin{pmatrix} 1 & 0 \\ 0 & F'(Y) \end{pmatrix},$$

that is, the momentum observable is distorted, but remains incoherent.

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