SUMMER PROFESSIONAL DEVELOPMENT

Connecting Algebra to Geometry: A Transition Summer Camp for At-risk Students

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In the summer of 2013, we were tasked with creating and implementing an eight-day algebra camp for students who had just completed eighth grade Algebra I and were entering ninth grade Geometry. The students lived in a district comprised of high percentages of Latino students and students who qualify for free and reduced lunches. Participants for the algebra camp were selected based on poor eighth grade algebra exit exam scores.

The Algebra I exam that the students took at the end of their eighth grade year identified several areas of weakness. Therefore, the content of the camp focused on concepts within these areas that exemplified algebraic thinking, specifically emphasizing algebraic topics that would be utilized in the students' upcoming geometry course. Furthermore, we conveyed content using inquiry and problem-based learning mechanisms as opposed to the traditional lecture method. The purpose of this article is to share how we designed camp curriculum and the impact the algebra camp had on both confidence in mathematics and mathematical ability.

Designing the Curriculum: Connecting Algebra and Geometry

The participating high school wanted to implement a summer camp targeting students who needed to improve upon their algebra skills prior to beginning ninth grade geometry. In the short period of time that we had with the students, it was important to address students' weaknesses on algebraic concepts that would be directly applicable in Geometry. Specifically, we assessed the literature to identify links between algebraic and geometric thinking patterns.

Thinking Algebraically

Before discussing components of algebra that are present in geometry as well as thinking geometrically, it is useful to have an understanding of what it means to think algebraically. Algebraic thinking is recognized as having interconnected components: (1) the use of variables/symbols, (2) the exploration of patterns and relationships, and (3) the use of models and multiple representations (Burrill, 1992; Friel, Rachlin, & Doyle, 2001; Herbert & Brown, 1997; Lee & Freiman, 2006; Usiskin, 1988).

Variables. The word *variable*, though a central concept in algebraic thinking, can be difficult to concisely define, as its purpose can vary depending on context. Usiskin (1988) discusses variables as taking on four different roles:

1. Variables as unknown quantities.
2. Variables as part of "relationships among quantities" (p. 10), such as the relationship described between the area and radius of a circle in the equation $A=\pi r^2$.
3. Variables as part of algebraic structures. For example factorization of $x^2+2xy+y^2$ involves utilizing operations that can be used upon real numbers and polynomials.
4. Variables as "pattern generalizers" (p. 9).

This subtle change of the variable's role naturally leads to confusion in students (Schoenfeld & Arcavi, 1988). Rosnick (1981) found, and we have all certainly seen this, that students struggle to recognize when a variable is used in different contexts such as a parameter versus an unknown quantity. The idea that a variable can take on subtly different roles can be difficult for students to understand.

Patterns. Though pattern finding does not have to be associated with algebra, mathematical patterns are foundational in algebraic thinking (Herbert & Brown, 1997; Lee & Freiman, 2006). By asking the right kinds of questions, pattern recognition can lead to thinking about generalized algebraic concepts and abstract reasoning. Generalizing patterns often requires one to utilize algebraic expressions or equations together with variables. Lee and Freiman argue that through the guidance of "scaffolded questioning, pattern explorations can lead to some very rich algebraic thinking about variables and unknowns, equivalence of algebraic expressions, symbol manipulation, domain and range of expressions and equations, and solving for the unknown" (p. 433). Students often engage in algebra via patterns by first writing about what they "see" and then attempting to formalize their thinking with algebraic notation, which often involves variables.

Multiple representations. When students begin to formalize patterns with algebraic expressions or equations, they can represent the patterns they identify in various ways. Depending on how a student describes a pattern, different students may develop multiple, yet equivalent, representations for the pattern (Lee & Freiman, 2006). When speaking about multiple representations, however, there is no restriction on speaking only about equivalent expressions such as $3n + 1$ and $3(n + 1) - 2$. Multiple representations can refer to different presentations of mathematical information, such as graphical, numerical, symbolic, verbal, etc. The various forms represent the same mathematical information, but in certain situations, one representation is more advantageous to use than the others (Burrill, 1992).

To put it briefly, one explores patterns and analyzes relationships and structures. Variables are used to represent quantities in relationships and structures. In turn, multiple representations of such quantities are used to create models of relationships and structures. These ideas are most certainly present in geometry as well as in algebra.

Connecting Algebra and Geometry

Although there are those who see Algebra and Geometry as two separate courses, algebra and geometry are deeply connected. While the word *geometry* often brings to mind shapes and pictures, there are components of geometry present in algebra. For one, as Banchoff (2008) states, "the geometric demonstration can show why an algebraic argument works" (p.107). Both algebra and geometry allow for multiple representations of concepts, which researchers agree add to the development of conceptual understanding, provide an opportunity to tie the symbolic to the real world, and in turn allow for flexibility in solving mathematical problems (Douglas, 1986; Duval, 2002; Gehrke & Pengelley, 1996; Griffin & Case, 1997; Heinze, Star, & Verschaffel, 2009). The use of multiple representations, a recognized component of algebraic thinking, is clearly tied to geometric thinking, and in fact provides an opportunity to link together concepts from both algebra and geometry.

The use of variables, another component of algebraic thinking, is also prevalent in geometry (Dindyal, 2004, 2007; Schoenfeld & Arcavi, 1988). It is not uncommon for students to have to determine the values of unknown sides or angles of a geometric shape or to set up equations involving angles or sides, thus incorporating the use of variables. Algebra abounds in geometry. As students do in algebra, geometry students also have to explore, understand, and model relationships between variable quantities (Dindyal 2004, 2007). To do this, they explore patterns and generalize results. In fact, one could look at algebra as a tool for exploring within the context of geometry. It is upon these ideas that we built a curriculum for the summer algebra camp.

Algebra Camp Curriculum

When choosing which topics to cover during camp, we first consulted the eighth grade algebra exit exam scores. Looking at student performance on each question, we narrowed our focus on questions on which fewer than 40% of students answered correctly. However, this left a significant amount of material to consider, which was far too much for an eight-day camp. We further pared down content by considering which topics reflected the three key aspects of algebraic thinking, and would be pertinent for students to know entering Geometry. With that in mind, we chose to focus our work on: (1) discovering and generalizing patterns, (2) simplifying expressions and combining like terms, (3) solving equations, (4) expressing word problems with an equation, (5) finding the area and circumference of a circle, (6) representing a line as an equation and a graph, and (7) multiplying binomials.

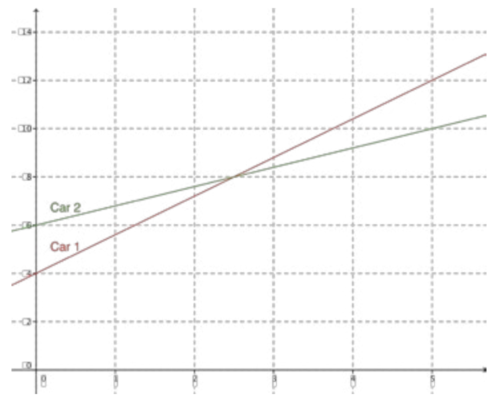
Some of the materials used for the camp were developed, while others came from NCTM publications. In addition, the Common Core State Standards for Mathematical Practice were always considered when designing the curriculum. A few examples of activities will be briefly discussed in the following sections.

Linear Equations and Race cars. We felt it was important to start the camp with a fun, interactive activity. Thus, an activity that tied in technology via the Texas Instruments CBR2™ data collection device was developed. Time was spent at the beginning showing students how to use the technology and set up the race cars. They learned how to interpret the information gathered from the CBR2™, recognizing that the slope of the race cars was always positive because the distance between the car and the CBR2™ grew as time passed, and the y-intercept was the starting location of the race car relative to the CBR2™. Once students understood these concepts and how they connected to the data, they could then explore the impact a weight had when added to a race car, as well as what happened when the cars had different starting points.

Students used the CBR2™ to collect position data from toy race cars, and the activity tied together patterns and multiple representations (physical model/toy race cars, scatter plot/graphical, table/numerical, and linear equation/symbolic). As they worked on questions, students explored and discussed the concepts of slope and the y-intercept of lines:

- Students compared the data between a race car with and without a weight attached to it (slope).
- Students compared the data between race cars that had different starting points. (y-intercept).

A brief example of combining concepts is given. The graph depicts the position of two race cars with respect to time.



After the ideas solidified, questions were posed, such as:

- Describe how the race started. Did either of the cars get a head start? How much of a head start did they have? Explain how you know.
- Was one of the cars faster than the other? Explain how you know.
- At what ordered pair do the two graphs intersect? What are the units of the first coordinate of the ordered pair? What are the units of the second coordinate of the ordered pair? What does this intersection point mean in terms of the race?

The race cars activity not only gave students a simple real-world application of slope and y-intercept, but was also something that they created and could visualize: they set up the race cars, watched them move, and then saw the position data displayed on their calculators. Students connected the multiple representations that position data could have, whether through a physical model or displayed as a discrete point in a scatter plot (or other representation). Connecting through multiple representations provided an opportunity to develop a deeper understanding of the concepts of slope and y-intercept.

Patterns and Polygons. Some of the activities that were created connected to geometry and were done toward the end of camp. One such activity had students explore polygons and the relationship between the number of sides and the sum of the interior angles. Students used sidewalk chalk and drew three different types of triangles on the concrete. For each triangle, they used a protractor to measure the interior angles, recorded data in a table,

and used the data to form a hypothesis about the sum of the interior angles of a triangle. Following this exploration with triangles, students repeated the activity for quadrilaterals and pentagons. Pre-created polygons were then used for a similar exploration.

Students compared findings from the sidewalk-drawn shapes to those that were pre-drawn and encouraged to look for patterns. Patterns that were discovered included that a three-sided figure has a sum of interior angles of 180°, while for four-sided figures the sum is 360°, and five-sided figures the sum is 540°.

The next question to explore was: *What if the shape has n sides?* This pushed students to examine their patterns for structure. When patterns were established, students sometimes had different expressions that depicted those patterns. After comparing answers, students used simplification and like terms to determine whether or not they had equivalent expressions.

In addition to exploring patterns and simplifying algebraic terms, the activity provided an opportunity to discuss possible drawing imperfections and measurement errors, and how these contributed to the accuracy of their approximations for the sum of the measures of the interior angles. Although students had errors, their data was accurate enough to make and test these conjectures and justify their results. Justification and asking students *"How do you know?"* was very common throughout camp.

NCTM Patterns Activities.

"Building with Toothpicks" from NCTM's *Navigating through Algebra in Grades 6-8* (Friel, Rachlin, & Doyle, 2001), and a "growing T" pattern exploration from NCTM's *Mathematics Teaching in the Middle School* (Lee & Freiman, 2006) connected patterns to algebra, explored equivalent symbolic representations, and connected algebra to geometry.

Delivery of Algebra Camp Content

In light of the connections between algebraic and geometric thinking, the content of the summer algebra camp emphasized the three pieces of algebraic thinking identified: the use of variables, the generalization and discovery of patterns, and the use of multiple representations. Problem-Based Learning (PBL) and Inquiry-Based Learning (IBL) mechanisms were implemented to encourage critical thinking.

Students were encouraged to ask themselves questions relating to the three key points of algebraic thinking:

- What I am considering in this problem?
- Are there any patterns that I can identify and use to solve this problem?
- Can I approach this problem differently to find an easier solution or verify my solution?

While some may view PBL and IBL as teaching strategies with minimal guidance, this is not the case. There is an important presence of scaffolding and instructor guidance involved in PBL and IBL (Hmelo-Silver, Duncan, & Chinn, 2006), and appropriately facilitated questions were key in pushing students to explore ideas and concepts.

Accountability and engagement are highly encouraged when participating in PBL and IBL environments. Therefore students become active learners. As active learners, students involved in inquiry develop deeper understandings and problem solving techniques (Kuhn et al., 2000). Thus, students not only leave with a conceptual understanding of the topic they were studying, they are also better prepared to explore other topics which they did not explicitly study.

In the context of the summer camp, PBL and IBL strategies were used to develop skills in algebraic thought and problem solving, avoiding memorization of steps and procedures. Students were able to leave with a conceptual understanding of the algebraic topics covered throughout camp. Furthermore, students also gained the problem solving and inquiry skills necessary to apply these algebraic concepts in a geometry setting.

What Happened?

The intent of the summer camp was to teach concepts of algebraic thinking to students who were lower-performing in algebra, yet proceeding on to take geometry. Therefore, we wanted to know if students (1) had more confidence in doing mathematics after participating in the camp, (2) had increased algebra skills at the end of the camp, and (3) would be successful in geometry in the upcoming year. We had positive results in all three areas.

Confidence, which significantly increased, was measured using the Confidence construct from the Indiana Mathematics Beliefs Scales (Kloosterman & Stage, 1992). A shortened version of the same exit algebra exam was given, and students performed significantly higher on this as well. However, after working on math intensively for eight days, one might expect such results.

Based on midterm grades, 18 of the 19 students who participated in the summer camp continued on to Geometry. Though non-participants performed significantly better at the end of the first semester, there was no statistical difference between non-participants and participants at the end of the second semester. At the end of the academic year, all 18 students that had attended the summer camp were passing with a grade of D or higher, and 12 of the 18 students received a grade of C or better. This is notable, as all of these students prior to entering the summer camp had low scores in algebra, weak mathematical skills, and were lacking confidence in their mathematical ability.

Conclusion

The summer algebra camp was an overall success. Students improved their algebra skills and gained confidence in their ability to solve math problems. In addition, these students succeeded in their ninth grade geometry course, a course many of them might have otherwise failed.

Beyond the data collected, there were anecdotal signs of student improvement. The very first day of the camp, none of the students wanted to attend. However, by the end of the second week, most of the students were disappointed that the camp was over. Furthermore, the chaperone who rode with the students from the participating high school made note that as the camp progressed, the students talked about math subjects more often on the bus ride. During the camp, the improvement in the student's problem solving ability was apparent, as they were able to move through topics quicker as the week progressed. All of this indicates that our task was accomplished, and the students gained skill sets in solving mathematical problems with an unintended side effect of improved confidence and success overall.

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