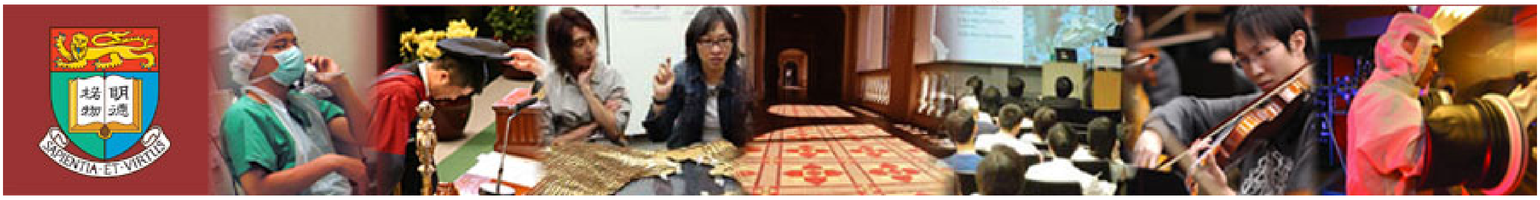

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Finite Element Implementation of the Generalized-Lorenz Gauged $\mathbf{A}$-Φ Formulation for Low-Frequency Circuit Modeling

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***Abstract*—The A-Φ formulation with generalized Lorenz gauge is free of catastrophic breakdown at low frequencies. In the formulation, A and Φ are completely separated and Maxwell's equations are reduced into two independent equations pertinent to A and Φ, respectively. This, however, leads to more complicated equations in contrast to the traditional A-Φ formulation, which makes the numerical representation of the physical quantities challenging, especially for A. By virtue of the differential forms theory and Whitney elements, both A and Φ are appropriately represented. The condition of the resultant matrix system is well-controlled as frequency becomes low, even approaches to 0. The generalized-Lorenz gauged A-Φ formulation is applied to model low-frequency circuits at μm lengthscale.**

I. Introduction

As modern electronic devices advance, their feature sizes are continuously scaled down. To capture their physics completely, full-wave electromagnetic analysis, rather than that based on circuit theory, is highly desired. However, when the size of the device is much smaller than the wavelength, traditional electromagnetic solvers suffer from the so-called low-frequency breakdown problem. It was observed in both integral equation (IE) based methods and partial differential equation (PDE) based methods. To circumvent this problem in the IE methods, one can implement a quasi Helmholtz decomposition by using loop-star or loop-tree basis function [1, 2], or introduce the current continuity equation to balance the vector and scalar potential terms in the augmented electric field integral equation (A-EFIE) [3]. In the PDE community, the tree-cotree splitting technique was applied in the finite element method (FEM) to realize an inexact Helmholtz decomposition of the electric field $\mathbf{E}$ [4, 5], and an eigenvalue based scheme was proposed in [6].

Unfortunately, the tree-cotree splitting technique does not remove the null space of the matrix at low frequency entirely, and the computational cost of the eigenvalue based scheme is high as it begins with finding the eigenvalues. Recently, a novel $\mathbf{A}$-Φ formulation with generalized Lorenz gauge is proved to be free of low-frequency breakdown theoretically, where the electroquasistatic physics and magnetoquasistatic physics are separately captured [7]. Based on this idea, the numerical implementation using FEM is developed in this work. Appropriate expansions of the physical quantities are inspired by the mapping between these quantities and Whitney forms. The condition number of the resultant matrices is well-controlled at extremely low frequencies.

II. Formulation

The vector potential $\mathbf{A}$ and scalar potential Φ are governed by the following equations [7]:

$$\nabla\cdot\varepsilon\nabla\Phi+\chi\omega^2\Phi=-\rho, \tag{1}$$

$$\nabla\times\mu^{-1}\nabla\times\mathbf{A}-\omega^2\varepsilon\mathbf{A}-\varepsilon\nabla\chi^{-1}\nabla\cdot\varepsilon\mathbf{A}=\mathbf{J}, \tag{2}$$

where

$$\chi=\alpha\varepsilon^2\mu, \tag{3}$$

and α can be a function of position. In addition, the generalized Lorenz gauge applied to derive (1) and (2) is

$$\chi^{-1}\nabla\cdot\varepsilon\mathbf{A}=i\omega\Phi. \tag{4}$$

At low frequencies, apparently, (1) degenerates to Poisson equation and thus the resultant matrix system is solvable. In (2), the first and third terms are balanced and thus the null space of the curl operator can be avoided. After solving (1) and (2), $\mathbf{E}$ can be recovered by $\mathbf{E}=i\omega\mathbf{A}-\nabla\Phi$. However, the third term in (2) is so complicated that the weak form of (2) cannot be obtained directly using traditional Ritz method or Galerkin's method. Thus, an intermediate quantity, $\chi^{-1}\nabla\cdot\varepsilon\mathbf{A}$, is introduced to reduce the complexity. In sum, three degrees of freedom (DoFs) are involved and they are approximated by Whitney elements as follows:

$$\Phi=\sum_{n=1}^{N_n}\varphi_n\lambda_n(\mathbf{r}), \tag{5}$$

$$\mathbf{A}=\sum_{n=1}^{N_e}a_n\vec{w}_n(\mathbf{r}), \tag{6}$$

$$\chi^{-1}\nabla\cdot\varepsilon\mathbf{A}=\sum_{n=1}^{N_n}d_n\lambda_n(\mathbf{r}), \tag{7}$$

where $\lambda(\mathbf{r})$ and $\vec{w}(\mathbf{r})$ are Whitney-0 form and Whitney-1 form elements, respectively. N_n and N_e are the numbers of the nodes and edges, respectively. φ_n, a_n, and d_n are the corresponding unknowns.

In order to remove the additional DoF in (7), testing both sides of (7) with $\lambda_m(\mathbf{r})$ yields

$$\sum_{n=1}^{N_e} \left\langle \lambda_m, \chi^{-1} \nabla \cdot \varepsilon \vec{\omega}_n \right\rangle a_n = \sum_{n=1}^{N_n} \left\langle \lambda_m, \lambda_n \right\rangle d_n. \quad (8)$$

Furthermore, the unknown vector $\{d\}$ can be written in the form of the unknown vector $\{a\}$, i.e.,

$$\{d\} = [G_{NN}]^{-1} [G_{NE}] \{a\}, \quad (9)$$

where $[G_{NN}]$ and $[G_{NE}]$ are defined as

$$[G_{NN}]_{mn} = \int_\Omega \lambda_m \lambda_n d\Omega, \quad (10)$$

$$[G_{NE}]_{mn} = \int_\Gamma \frac{1}{\mu\varepsilon} \lambda_m \vec{\omega}_n \cdot \hat{n} d\Gamma - \int_\Omega \frac{1}{\mu\varepsilon} \nabla \lambda_m \cdot \vec{\omega}_n d\Omega. \quad (11)$$

Finally, the matric systems can be obtained by testing both sides of (1) with $\lambda_m(\mathbf{r})$ and those of (2) with $\vec{\omega}_m(\mathbf{r})$.

III. Numerical Verification

To verify our proposed $\mathbf{A}$-Φ formulation, a parallel plate capacitor is firstly considered. Two perfect electric conductor (PEC) plates are of size $35\mu m \times 10\mu m \times 1\mu m$ and the gap between them is $1\mu m$. The sizes are of typical dimensions for on-chip circuits. As shown in Fig. 1 (a), the imaginary part of the impedance Z, obtained by the proposed $\mathbf{A}$-Φ formulation, agrees very well with the analytical reference obtained by the capacitance of parallel plates, while that obtained by the traditional $\mathbf{E}$ formulation diverges suddenly at about 10^7 Hz. The condition numbers of the resultant matrices obtained from (2) and the $\mathbf{E}$ formulation are plotted in Fig. 1 (b) at different frequencies. Obviously, the proposed $\mathbf{A}$-Φ formulation is well conditioned and stable in a wide frequency range, while the low-frequency breakdown emerges in the traditional $\mathbf{E}$ formulation. Similar performance is also achieved for the magnetoquasistatic cases, which is not presented here due to the page limit.

IV. Conclusion

The generalized-Lorenz gauged $\mathbf{A}$-Φ formulation has been verified numerically using FEM. After introducing an intermediate quantity to reduce the complexity of the finite element discretization, accurate and stable matrix systems are established in a wide low-frequency range for both electroquasistatic and magnetoquasistatic problems.

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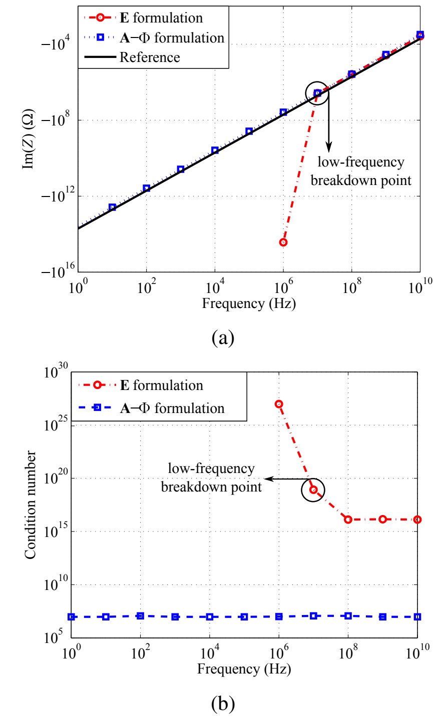

Fig. 1. (a) Imaginary part of Z, and (b) condition number of the matrices.

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