Budget Constraints in Prediction Markets

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Abstract

We give a detailed characterization of optimal trades under budget constraints in a prediction market with a cost-function-based automated market maker. We study how the budget constraints of individual traders affect their ability to impact the market price. As a concrete application of our characterization, we give sufficient conditions for a property we call *budget additivity*: two traders with budgets B and B' and the same beliefs would have a combined impact equal to a single trader with budget $B + B'$. That way, even if a single trader cannot move the market much, a crowd of like-minded traders can have the same desired effect. When the set of payoff vectors associated with outcomes, with coordinates corresponding to securities, is affinely independent, we obtain that a generalization of the heavily-used logarithmic market scoring rule is budget additive, but the quadratic market scoring rule is not. Our results may be used both descriptively, to understand if a particular market maker is affected by budget constraints or not, and prescriptively, as a recipe to construct markets.

1 INTRODUCTION

A prediction market is a central clearinghouse for people with differing opinions about the likelihood of an event—say Hillary Clinton to win the 2016 U.S. Presidential election—to trade monetary stakes in the outcome with one another. At equilibrium, the price to buy a contract paying \$1 if Clinton wins reflects a consensus of sorts on the probability of the event. At that price, and given the wagers already placed, no agent is willing to push the price further up or down. Prediction markets have a good track record of forecast accuracy in many domains [11, 19].

The design of *combinatorial* markets spanning multiple logically-related events raises many interesting questions. What information can be elicited—the full probability distribution, or specific properties of the distribution? What securities can the market allow traders to buy and sell? How can the market support and ensure a variety of trades? For example, in addition to the likelihood of Clinton winning the election, we may want to elicit information about the distribution of her electoral votes.[1] If we create one security for each possible outcome between 0 and 538, each paying \$1 iff Clinton gets exactly that many electoral votes, the market is called *complete*, allowing us to elicit a full probability distribution. Alternatively, if we create just two securities, one paying out $\$x$ if Clinton wins x electoral votes, and the other paying out $\$x^2$, we cannot elicit a full distribution, but we can still elicit the mean and variance of the number of electoral votes.

When agents are constrained in how much they can trade only by risk aversion, prediction market prices can be interpreted as a weighted average of traders' beliefs [2, 20], a natural reflection of the "wisdom of the crowd" with a good empirical track record [14] and theoretical support [2]. However, when agents are budget constrained, discontinuities and idiosyncratic results can arise [7, 16] that call into question whether the equilibrium prices can be trusted to reflect any kind of useful aggregation.

We consider prediction markets with an automated market maker [1, 4, 13] that maintains standing offers to trade every security at some price. Unlike a peer-to-peer exchange, all transactions route through the market maker. The common market makers have bounded loss and are (myopically) incentive compatible: the best (immediate) strategy is for a trader to move the market prices of all securities to equal his own belief. The design of such an automated market maker boils down to choosing a convex *cost function* [1]. This amount of design freedom presents an opportunity to seek cost functions that satisfy additional desiderata such as computational tractability [1, 6].

[1] A U.S. Presidential candidate receives a number of electoral votes between 0 and 538. The candidate who receives a plurality of electoral votes wins the election.

Most of the literature assumes either risk-neutral or risk-averse traders with unbounded budgets. In this paper, we consider how agents with budget constraints trade in such markets, a practical reality in almost all prediction markets denominated in both real and virtual currencies. Our results help with a systematic study of the market's *liquidity parameter*, or the parameter controlling the sensitivity of prices to trading volume. Setting the liquidity is a nearly universal practical concern and, at present, is more (black) art than science. We adopt the notion of the "natural budget constraint" introduced by Fortnow and Sami [8]: the agent is allowed only those trades for which the maximum loss for any possible outcome does not exceed the budget.

The main contribution of this paper is a rich, geometric characterization of the impact of budget constraints. Price vectors, outcomes and trader beliefs are embedded in the space of the same dimension as the number of securities. Outcome vectors enumerate security payoffs; belief vectors enumerate the traders' expectations of payoffs. We consider, for a fixed belief, the locus of the resulting price vectors of an optimal trade as a function of the budget. We show that the price vector moves in the convex hull of the belief and the set of tight outcomes, in a direction that is perpendicular to the set of tight outcomes. We also introduce the concept of *budget additivity*: two agents with budgets B and B' and the same beliefs have the same power to move the prices as a single agent with the same belief and budget $B+B'$. An absence of budget additivity points to an inefficiency in incorporating information from the traders. We show that budget additivity is a non-trivial property by giving examples of market makers that do not satisfy budget additivity. We give a set of sufficient conditions on the market maker and the set of securities offered which guarantee budget additivity. Further, for two of the most commonly used market makers (the quadratic and logarithmic market scoring rules), we show sufficient conditions on the set of securities that guarantee budget additivity.

Of greatest practical interest is the application of our results to markets consisting of several independent questions, with each question priced according to a separate logarithmic market scoring rule. This setup constitutes a de facto industry standard, and the companies that use (or used) it include Inkling Markets,[2] Consensus Point,[3] Microsoft and Yahoo! [17]. Our Theorems 5.6 and 5.8 show that these markets are budget additive.

Previously, Fortnow and Sami [8] considered a different question: do budget-constrained bidders always move the market prices in the direction of their beliefs? They showed that the answer to this is no: there always exist market prices, beliefs and budgets such that the direction of price movement is not towards the belief. We give a richer characterization of how the market prices move in the presence of budget constraints, by charting the path the prices take with increasing budgets. The impossibility result of Fortnow and Sami [8] can be easily derived from our characterization (see Appendix D).[4]

A designer of a prediction market has a lot of freedom but little guidance, and our results can be used both descriptively and prescriptively. As a descriptive tool, our results enable us to analyze commonly used market makers and understand if budget constraints hamper information aggregation in these markets. As a prescriptive tool, our results can be used to construct markets that are budget additive. In particular, we speculate that budget additivity simplifies the choice of the liquidity parameter in the markets, because it allows considering trader budgets in aggregate.

Proof overview and techniques. Our analysis borrows heavily from techniques in convex analysis and builds on the notion of Bregman divergence. We use the special case of Euclidean distance (corresponding to a quadratic market scoring rule) to form our geometric intuition which we then extend to arbitrary Bregman divergences. For the sake of an example, consider a complete market over a finite set of outcomes, where the market prices lie in a simplex, exactly coinciding with the set of probability distributions over outcomes. Every possible outcome imposes a constraint on the set of prices to which a trader can move the market, because the trader is not allowed to exceed the budget if that outcome occurs. The prices satisfying this constraint form a ball with the outcome at its center. The set of feasible prices to which the trader can move the market is therefore the intersection of these balls (see Figure 1).

The key structural result we obtain is the chart of the price movement. Suppose that there is an infinite sequence of agents with infinitesimally small budgets all with the same belief. What is the path along which the prices move from some initial values? This is determined by the agents' belief and the set of budget constraints that are tight at any point, corresponding to the highest risk outcomes (outcomes with the highest potential loss). We show that the price vector can always be written as a convex combination of these highest risk outcomes and the agents' belief. Further, the market prices move in a direction that is perpendicular to the affine space of these outcomes.

The agents' belief partitions the simplex interior into regions, where each region is the interior of the convex hull of the agent belief and a particular subset of outcomes. For a region that is full-dimensional, every interior point can be uniquely written as a convex combination of the agent belief and all except one outcome. Assume that the current price vector lies in this region. In the anticipation of the further development, we call this outcome *profitable* and others *risky*. Motivated by the characterization above, we

[2] inklingmarkets.com

[3] www.consensuspoint.com

[4] The full version of this paper on arXiv includes the appendix.

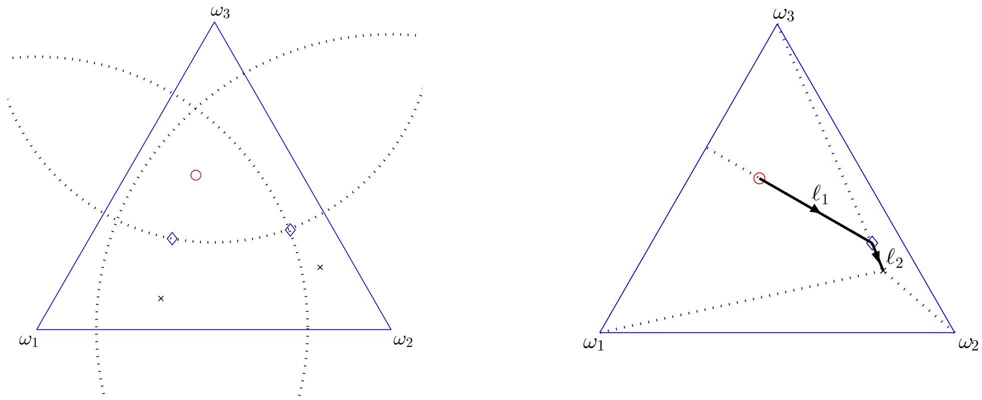

Figure 1: *Left:* $\circ$ —current state, $\times$ —belief, $\diamond$ —optimal action for a given belief and budget. Three circles bound the allowed final states for budget 0.1. We plot optimal actions for two different beliefs. *Right:* A path from the initial state to the belief, consisting of optimal actions for increasing budgets.

move perpendicular to the risky outcomes in the direction towards the agents' belief. As a result, we increase the risk of risky outcomes (equally for all outcomes), while getting closer to the one profitable outcome (and hence increasing its profit). The characterization then guarantees that the prices along this path are indeed those chosen by traders at increasing budgets, because the risky outcomes yield tight constraints.

We would like the same to be true for the lower dimensional regions as well; that is, for the set of tight constraints to be exactly the corresponding set of outcomes defining the convex hull. In fact, this property is sufficient to guarantee budget additivity. The markets for which the tight constraints are exactly the minimal set of outcomes that define the region the price lies in are budget additive. (We conjecture that the converse holds as well.) The entire path is then as follows: w.l.o.g. you start at a full-dimensional region, move along the perpendicular until you hit the boundary of the region and you are in a lower-dimensional region, move along the perpendicular in this lower-dimensional region, and so on until you reach the belief (see Figure 1). The set of tight constraints is monotonically decreasing. We show that such markets are characterized by a certain *acute angles assumption* on the set of possible outcomes. Loosely speaking, this assumption guarantees that outcomes outside the minimal set behave as the profitable outcome in the above example.

Other related work. There is a rich literature on scoring rules and prediction markets. Two of the most studied scoring rules are the quadratic scoring rule [3] and the logarithmic market scoring rule [13]. We consider cost-function-based prediction markets [4, 12], a fully general class under reasonable assumptions [1, 5]. Their equivalence with proper scoring rules has been implicitly noted by Gneiting and Raftery [10]. Several authors have studied relationships between utility functions and price dynamics in prediction markets, drawing a parallel to online learning [2, 5, 9]. Our analysis touches on the problem of setting the market maker's liquidity parameter [15, 17], which determines how (in)sensitive prices are to trading volume. With budget additivity, the market designer can optimize liquidity according to aggregate budgets, without worrying about how budgets are partitioned among traders.

2 PRELIMINARIES

Securities and payoffs. Consider a probability space with a finite set of outcomes $\Omega \subseteq \mathbb{R}^n$. A *security* is a financial instrument whose payoff depends on the realization of an outcome in Ω. In other words, the payoff of a security is a random variable of the probability space. We consider trading with n securities corresponding to n coordinates of the outcomes $\omega \in \Omega$. A security can be traded before the realization is observed with the intention that the price of a security serves as a prediction for the expected payoff, i.e., the expected value of the corresponding coordinate.

Cost function, prices and utilities. An *automated market maker* always offers to trade securities, for the right price. In fact the price vector is the current prediction of the market maker for the expectation of ω. A cost function based market maker is based on a differentiable convex *cost function*, $C : \mathbb{R}^n \to \mathbb{R}$. It is a scalar function of an n-dimensional vector $q \in \mathbb{R}^n$ representing the *number of outstanding shares*[5] for our n securities. We also refer to q as the *state* of the market.

The vector of *instantaneous prices* of the securities is simply the gradient of C at q, denoted by $p(q) := \nabla C(q)$. The prices of securities change continuously as the securities are traded, so it is useful to consider the cost of trading a given quantity of securities. The cost of buying $\delta \in \mathbb{R}^n$ units of securities (where a negative value corresponds to selling) is determined by the path integral $\int_\pi p(\bar{q}) \cdot d\bar{q} = C(q + \delta) - C(q)$, where π is any smooth

[5] We allow trading fractions of a security. Negative values correspond to short-selling.

curve from q to $q + \delta$.

When the outcome ω is realized, the vector of δ units of securities pays off an amount of $\delta \cdot \omega$. Thus, the realized utility of a trader whose trade δ moved the market state from q to $q' = q + \delta$ is

$$U(q', \omega; q) := (q' - q) \cdot \omega - C(q') + C(q) \ .$$

We make a standard assumption that the maximum achievable utility, which is also the maximum loss of the market maker, is bounded by a finite constant (in Section 4, we introduce a standard approach to check this easily). Let $\mathcal{M}$ be the convex hull of the payoff vectors, $\mathcal{M} := \text{conv}(\Omega)$. It is easy to see that $\mathcal{M}$ contains exactly the vectors $\mu \in \mathbb{R}^n$ which can be realized as expected payoffs $\mathbb{E}[\omega]$ for some probability distribution over Ω. For a trader who believes that $\mathbb{E}[\omega] = \mu$, the expected utility takes form

$$U(q', \mu; q) := \mathbb{E}\left[U(q', \omega; q)\right] = (q'-q) \cdot \mu - C(q') + C(q) \ .$$

Throughout, we consider a single *myopic* trader who trades as if he were the last to trade. A key property satisfied by expected utility is *path independence*: for any $q, \bar{q}, q' \in \mathbb{R}^n$, $U(q', \mu; \bar{q}) + U(\bar{q}, \mu; q) = U(q', \mu; q)$, that is, risk-neutral traders have no incentive to split their trades. For a risk-neutral trader, $q' \in \mathbb{R}^n$ is an optimal action if and only if $\mu = \nabla C(q') = p(q')$ (this follows from the first-order optimality conditions). In other words, the trader is incentivized to move the market to the prices corresponding to his belief as long as such prices exist. In general, there may be multiple states yielding the same prices, so the inverse map $p^{-1}(\mu)$ returns a *set*, which can be empty if no state yields the price vector μ.

Commonly-used cost functions include the quadratic cost, logarithmic market-scoring rule (LMSR) and the log-partition function. They are described in detail in Appendix A. The quadratic cost is defined by $C(q) = \frac{1}{2}\|q\|_2^2$ and $p(q) = q$. Log-partition function is defined as $C(q) = \ln(\sum_{\omega \in \Omega} e^{q \cdot \omega})$. It subsumes LMSR as a special case for the *complete market* with the outcomes corresponding to vertices of the simplex. The prices under log-partition cost correspond to the expected value of ω under the distribution $P_q(\omega) = e^{q \cdot \omega - C(q)}$ over Ω, i.e., $p(q) = \mathbb{E}_{P_q}[\omega]$.

Budget constraints. Trading in prediction markets needs an investment of capital. It is possible that an agent loses money on the trade, in particular $U(q', \omega; q)$ could be negative for some ω. One restriction on how an agent trades could be that he is unable to sustain a big loss, due to a budget constraint. We consider the notion of *natural budget constraint* defined by Fortnow and Sami [8] which states that the loss of the agent is at most his budget, for all $\omega \in \Omega$. Given a starting market state q_0 and a budget of $B \geq 0$, a trader with the belief $\mu \in \mathcal{M}$ then solves the problem:

$$\begin{aligned} &\max_{q \in \mathbb{R}^n} \ U(q, \mu; q_0) \\ &\text{s.t. } U(q, \omega; q_0) \geq -B \quad \forall \omega \in \Omega \ . \end{aligned} \tag{2.1}$$

For quadratic costs, each constraint corresponds to a sphere with one of the outcomes at its center, so the feasible region is an intersection of these spheres. We will later see that this generalizes to an intersection of balls w.r.t. a Bregman divergence for general costs.

In general, there may be multiple q optimizing this objective. In the following definition we introduce notation for various solution sets we will be analyzing. The belief μ is fixed throughout most of the discussion, so we suppress the dependence on μ.

Definition 2.1 (Solution sets)**.** Let $\hat{Q}(B; q_0)$ denote the set of solutions of Convex Program (2.1) for a fixed initial state and budget. Let $\hat{Q}(q_0) = \bigcup_{B \geq 0} \hat{Q}(B; q_0)$ denote the set of solutions of (2.1) for a fixed initial state across all budgets. Let $\hat{Q}(\nu; q_0) = p^{-1}(\nu) \cap \hat{Q}(q_0)$ denote the set of states q that optimize (2.1) for some budget B and yield the market price vector ν.

The next theorem shows that solutions for a fixed initial state and budget always yield the same price vector. It is proved in Appendix B.

Theorem 2.2. *If $q, q' \in \hat{Q}(B; q_0)$, then $p(q) = p(q')$.*

Geometry of linear spaces. We finish this section by reviewing a few standard geometric definitions we use in next sections. Let $X \subseteq \mathbb{R}^n$. Then $\text{aff}(X)$ denotes the *affine hull* of the set X (i.e., the smallest affine space including X). We write $X^\perp$ to denote the *orthogonal complement* of X: $X^\perp := \{u \in \mathbb{R}^n : \ u \cdot (x' - x) = 0 \text{ for all } x, x' \in X\}$. We use the convention $\emptyset^\perp = \mathbb{R}^n$. A set $\mathcal{K} \in \mathbb{R}^n$ is called a *cone* if it is closed under multiplication by positive scalars. If a cone is convex, it is also closed under addition. Since Ω is finite, the realizable set $\mathcal{M} = \text{conv}(\Omega)$ is a polytope. Its boundary can be decomposed into *faces*. More precisely, $X \subseteq \Omega$, $X \neq \emptyset$, forms a *face* of $\mathcal{M}$ if X is the set of maximizers over Ω of some linear function.[6] We also view $X = \emptyset$ as a face of $\mathcal{M}$. With this definition, for any two faces X, X', also their intersection $X \cap X'$ is a face.

3 CHARACTERIZING SOLUTION SETS

We start with the optimality (KKT) conditions for the Convex Program (2.1), as characterized by the next lemma. One of the key conditions is that the solution prices must be in the convex hull of the belief μ and all the ω's for which the budget constraints are *tight*. The set of tight constraints is always a face of the polytope $\mathcal{M}$. We allow an empty set as a face, which corresponds to the case when none of the constraints are tight and the solution prices coincide with μ. The proof follows by analyzing KKT conditions (see Appendix C of the full version for details).

[6] Strictly speaking, this is the definition of an *exposed face*, but all faces of a polytope are exposed, so the distinction does not matter here. The exposed face is typically defined to be $\text{conv}(X)$, but in this paper, it is more convenient to work with X directly.

Lemma 3.1 (KKT lemma)**.** *Let* $q_0 \in \mathbb{R}^n$*. Then* $q \in \hat{Q}(B; q_0)$ *if and only if there exists a face* $X \subseteq \Omega$ *such that the following conditions hold:*

(a) $U(q, x; q_0) = U(q, x'; q_0)$*, or equivalently* $(q - q_0) \cdot (x' - x) = 0$*, for all* $x, x' \in X$
(b) $U(q, \omega; q_0) \geq U(q, x; q_0)$*, or equivalently* $(q - q_0) \cdot (\omega - x) \geq 0$*, for all* $x \in X$*,* $\omega \in \Omega \backslash X$
(c) $p(q) \in \mathrm{conv}(X \cup \{\mu\})$
(d) $B = -U(q, x; q_0)$ *for all* $x \in X$ *if* $X \neq \emptyset$*, or* $B \geq \max_{\omega \in \Omega}[-U(q, \omega; q_0)]$ *if* $X = \emptyset$

where conditions (a) and (b) hold vacuously for $X = \emptyset$*.*

The condition (a) requires that $q - q_0$ be orthogonal to the active set X. The set of points satisfying conditions (a) and (c) will be called the Bregman perpendicular and will be defined in the next section. The condition (b) is a statement about acuteness of the angle between $q - q_0$ (the perpendicular) and the outcomes. It will be the basis of our *acute angles* assumption. The condition (d) just states how the budget is related to the active set X.

Witness cones and minimal faces. We now introduce some notation to help us state reinterpretations of the conditions in Lemma 3.1. First of all, given a face X, what is the set of q's that satisfy conditions (a) and (b)? This is captured by what we call the *witness cone*.

Definition 3.2. The *witness cone* for a face $X \subseteq \Omega$ is defined as $\mathcal{K}(X) := \{u \in \mathbb{R}^n : u \cdot (\omega - x) \geq 0 \text{ for all } x \in X, \omega \in \Omega\}$ if $X \neq \emptyset$, and $\mathcal{K}(X) := \mathbb{R}^n$ if $X = \emptyset$.

The following two properties of witness cones are immediate from the definition:

- *Anti-monotonicity:* if $X \subseteq X'$, then $\mathcal{K}(X) \supseteq \mathcal{K}(X')$.
- *Orthogonality:* $\mathcal{K}(X) \subseteq X^{\perp}$.

A state q satisfies conditions (a) and (b) for a given face X if and only if $q - q_0 \in \mathcal{K}(X)$. Now given a state q, consider the set of faces that could satisfy condition (c). This set has a useful structure, namely that there is a unique minimal face (proved in Appendix C of the full version).

Definition 3.3. Given a price vector $\nu \in \mathcal{M}$, the *minimal face* for ν is the minimal face X (under inclusion) s.t. $\nu \in \mathrm{conv}(X \cup \{\mu\})$. The minimal face for ν is denoted as X_ν.

With the existence of a minimal face and the anti-monotonicity of the witness sets, it follows that if q and X satisfy conditions (a), (b) and (c), then so do q and $X_{p(q)}$. Thus we obtain the following version of Lemma 3.1 (proved in Appendix C of the full version).

Theorem 3.4 (Characterization of Solution Sets)**.** $q \in \hat{Q}(q_0)$ *if and only if* $q \in [q_0 + \mathcal{K}(X_{p(q)})]$*.*

Using Theorem 3.4, we immediately obtain a characterization of when a price vector ν could be the price vector of an optimal solution to (2.1).

Corollary 3.5. $\hat{Q}(\nu; q_0) = p^{-1}(\nu) \cap [q_0 + \mathcal{K}(X_\nu)]$*. In particular,* ν *is the price vector of an optimal solution to (2.1) if and only if* $p^{-1}(\nu) \cap [q_0 + \mathcal{K}(X_\nu)] \neq \emptyset$*.*

We now study an example using the above characterization. More examples can be found in Appendix E of the full version.

Example 3.6 (Quadratic cost on an obtuse triangle; see Example E.2 in the full version for details)**.** Consider the following outcome space, belief, and the sequence of market states (depicted in Figure 2):

$$\begin{aligned} \omega_1 &= (0.0,\ 0.0) & q_0 &= \nu_0 = \tfrac{11}{14}\omega_2 + \tfrac{3}{14}\omega_3 \\ \omega_2 &= (1.8,\ 0.0) & q_1 &= \nu_1 = \tfrac{1}{3}\omega_2 + \tfrac{2}{3}\mu \\ \omega_3 &= (6.0,\ 4.2) & q_2 &= \nu_2 = \tfrac{1}{9}\omega_1 + \tfrac{8}{9}\mu \\ \mu = q_\mu &= (2.7,\ 1.8) & q_3 &= \nu_3 \approx \tfrac{1}{19}\omega_1 + \tfrac{18}{19}\mu \end{aligned}$$

Using the KKT lemma, we can show for $j = 1, 2, 3$, that $q_j = \nu_j$ is an optimal action at $q_{j-1} = \nu_{j-1}$ under belief μ, with the corresponding budgets as:

	ω_1	ω_2	ω_3	
$U(q_1, \cdot; q_0)$	0.45	-0.09	-0.09	$B_{01} = 0.09$
$U(q_2, \cdot; q_1)$	-0.56	-0.56	1.12	$B_{12} = 0.56$
$U(q_3, \cdot; q_2)$	-0.565	$-0.28\ldots$	$0.82\ldots$	$B_{23} = 0.565$
$U(q_\mu, \cdot; q_0)$	-1.215	-1.215	2.565	$B_{0\mu} = 1.215$

The above table also shows that the budget $B_{0\mu} = 1.215$ suffices to move directly from q_0 to q_μ. However, note that the sum $B_{01} + B_{12} + B_{23} = 1.215 = B_{0\mu}$, but $\nu_3 \neq \mu$, i.e., after the sequence of optimal actions with budgets B_{01}, B_{12}, and B_{23}, the market is still not at the belief shared by all agents, even though with the budget $B_{0\mu}$, it would have reached it.

Budget additivity. The above example suggests that multiple traders with the same belief may have less power in moving the market state towards their belief compared to a single trader with the same belief and the combined budget. Since prediction markets aim to efficiently aggregate information from agents, it is natural to ask under what conditions multiple traders with the same beliefs do have a combined impact equal to a single trader with the combined budget.

Next, we formally define this property as *budget additivity*. We then define the Euclidean version of the acute angles condition that we show is sufficient for budget additivity.

Definition 3.7 (Budget additivity)**.** We say that a prediction market is budget additive on $\mathcal{M}' \subseteq \mathcal{M}$ if for all beliefs $\mu \in \mathcal{M}'$ and all initial states $q_0 \in p^{-1}(\mathcal{M}')$ the following holds: For any budgets $B, B' \geq 0$ and any sequence of solutions $q \in \hat{Q}(B; q_0)$ and $q' \in \hat{Q}(B'; q)$, we have $p(q), p(q') \in \mathcal{M}'$ and $q' \in \hat{Q}(B + B'; q_0)$.

In other words, the market is budget additive if the sequence of optimal actions of two agents with the same be-

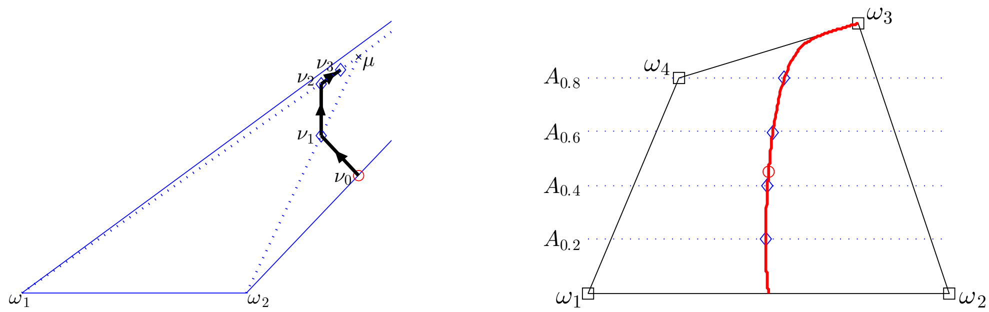

Figure 2: *Left:* An example of non-additive budgets when payoffs form obtuse angles (see Example 3.6 and its extended version Example E.2 in the full version). *Right:* An examples of a non-linear perpendicular for the log-partition cost.

lief and budgets B and B' is also an optimal action of a single agent with the same belief and a larger budget $B + B'$. Thanks to Theorem 2.2 we then also obtain that the price vector following the sequence of optimal actions by the two agents is the same as the price vector after the optimal action by an agent with the combined budget (all with the same beliefs).

We now state the acute angles assumption for the Euclidean case, to give an intuition. Our acute angles assumption (Definition 5.1) is a generalization of this. We later show that the acute angles property is sufficient for budget additivity (Theorem 5.2).

Definition 3.8. We say that the *Euclidean acute angles* hold for a face X, if the angle between any point $\bar{\nu} \in \mathcal{M}$, its projection on the affine hull of X and any payoff $\omega \in \Omega$ is non-obtuse (the angle is measured at the projection).

Based on the above example, one may hypothesize that the obtuse angles are to blame for the lack of budget additivity. In the following sections we will show that this is indeed the case, but that the notion of obtuse/acute angles depends on the Bregman divergence. In particular, the above example would have been budget-additive if we used the log-partition cost instead of the quadratic cost.

4 BREGMAN DIVERGENCE AND PERPENDICULARS

We will see next that the utility function U can be written as the difference of two terms measuring the distance between the belief and the market state before and after the trade. This distance measure is the mixed *Bregman divergence*.[7] To define the Bregman divergence, first let $C^* : \mathbb{R}^n \to (-\infty, \infty]$ be the convex conjugate of C defined as $C^*(\nu) := \sup_{q' \in \mathbb{R}^n} [q' \cdot \nu - C(q')]$. Since C^* is a supremum of linear functions, it is convex lower semi-continuous. Up to a constant, it characterizes the maximum achievable utility on an outcome ω for a fixed initial state q as $\sup_{q' \in \mathbb{R}^n} U(q', \omega; q) = C^*(\omega) + \big[C(q) - q \cdot \omega\big]$. The term in the brackets is always finite, but C^* might be positive infinite. We make a standard assumption that $C^*(\omega) < \infty$ for all $\omega \in \Omega$, i.e., that the maximum achievable utility, which is also the maximum loss of the market maker, is bounded by a finite constant. By convexity, this implies that $C^*(\mu) < \infty$ for all $\mu \in \mathcal{M}$. The *Bregman divergence* derived from C is a function $D : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, \infty]$ measuring the maximum expected utility under belief μ at a state q

$$D(q, \mu) := C(q) + C^*(\mu) - q \cdot \mu = \sup_{q' \in \mathbb{R}^n} U(q', \mu; q) \ .$$

From the convexity of C and C^* and the definition of C^*, it is clear that: (i) D is convex and lower semi-continuous in each argument separately; (ii) D is non-negative; and (iii) D is zero iff $p(q) = \nabla C(q) = \mu$. By the bounded loss assumption, Bregman divergence is finite on $\mu \in \mathcal{M}$. For $\mu \in \mathcal{M}$, we can write

$$U(q', \mu; q) = D(q, \mu) - D(q', \mu) \ . \tag{4.1}$$

Thus, maximizing the expected utility is the same as minimizing the Bregman divergence between the state q' and the belief μ. From Eq. (4.1) it is also clear that each constraint in (2.1) is equivalent to $D(q, \omega) \leq D(q_0, \omega) + B$, and the geometric interpretation is that the agent seeks to find the state closest to his belief, within the intersection of Bregman balls

For the quadratic cost, we have $C^*(\nu) = \frac{1}{2}\|\nu\|^2$ and $D(q, \nu) = \frac{1}{2}\|q - \nu\|^2$, i.e., the Bregman divergence coincides with the Euclidean distance squared. For log-partition cost, we have $C^*(\nu) = \sum_{\omega \in \Omega} P_\nu(\omega) \ln P_\nu(\omega)$ where P_ν is the distribution maximizing entropy among P satisfying $\mathbb{E}_P[\omega] = \nu$. The Bregman divergence is the KL-divergence between P_q and P_ν: $D(q, \nu) = \mathrm{KL}(P_\nu \| P_q)$.

Convex analysis. We overview a few standard definitions and results from convex analysis. For $X \subseteq \mathbb{R}^n$, we write $\mathrm{ri}\, X$ for the *relative interior* of X (i.e., the interior relative to the affine hull). For a convex function $F : \mathbb{R}^n \to (-\infty, \infty]$, we define its *effective domain*

[7] Our notion of Bregman divergence is more general than typically assumed in the literature.

as $\operatorname{dom} F := \{u \in \mathbb{R}^n : F(u) < \infty\}$ (i.e., the set of points where it is finite). The *subdifferential* of F at a point u is the set $\partial F(u) := \{v \in \mathbb{R}^n : F(u') \geq F(u) + (u' - u) \cdot v$ for all $u' \in \mathbb{R}^n\}$. We say that F is *subdifferentiable* at u if $\partial F(u) \neq \emptyset$. A standard result of convex analysis states that F is always subdifferentiable on a superset of $\operatorname{ri}\operatorname{dom} F$. If F is not only convex, but also lower semi-continuous, then ∂F and ∂F^* are inverses in the sense that $v \in \partial F(u)$ iff $u \in \partial F^*(v)$. If F is *differentiable* everywhere on $\mathbb{R}^n$, then F^* is strictly convex on $\operatorname{ri}\operatorname{dom} F^*$.

Let $\operatorname{im} p := \{p(q) : q \in \mathbb{R}^n\}$ denote the set of prices that can be expressed by market states. The implications for our setting are that: (i) C^* is subdifferentiable on $\operatorname{im} p$; (ii) $p^{-1}(\nu) = \partial C^*(\nu)$ for all $\nu \in \mathbb{R}^n$; (iii) all beliefs in $\operatorname{ri}\operatorname{dom} C^*$ can be expressed by some state q; (iv) C^* is strictly convex on $\operatorname{ri}\operatorname{dom} C^*$, and similarly $D(q,\nu)$ is strictly convex on $\operatorname{ri}\operatorname{dom} C^*$ as a function of the second argument.

Assumptions on the cost function.

- *Convexity and differentiability on* $\mathbb{R}^n$. C is convex and differentiable on $\mathbb{R}^n$.
- *Finite loss.* $\mathcal{M} \subseteq \operatorname{dom} C^*$, i.e., C^* is finite on $\mathcal{M}$.
- *Inclusion of the relative interior.* $\operatorname{ri}\mathcal{M} \subseteq \operatorname{ri}\operatorname{dom} C^*$.

The first two assumptions are standard. The third assumption is a regularity condition that we require in our results. Here we briefly discuss how it compares with the finite loss assumption. While the two assumptions look similar, neither of them implies the other. For example, if $\operatorname{dom} C^*$ is an n-dimensional simplex and $\mathcal{M}$ is one of its lower dimensional faces, which are lower dimensional simplices, then the finite loss assumption holds, but the inclusion assumption does not. Similarly, for $n = 1$ and $\mathcal{M} = [0, 1]$, the inclusion assumption is satisfied by the conjugate $C^*(\nu) = 1/\nu + 1/(1-\nu)$ on $\nu \in (0, 1)$ and $C^*(\nu) = \infty$ on $\nu \notin (0, 1)$, but this conjugate does not satisfy the finite loss assumption.

We do not view the inclusion assumption as very restrictive, since it is satisfied by many common cost functions. For instance, it always holds when C is constructed as in [1], because their construction guarantees $\operatorname{dom} C^* = \mathcal{M}$. However, the inclusion assumption might not hold for cost functions that allow arbitrage (e.g., [6]).

Our main result relies on strict convexity of C^* on $\operatorname{ri}\operatorname{dom} C^*$, so some of our statements will require that the market prices and beliefs lie in that set. The inclusion assumption above guarantees that at the minimum $\operatorname{ri}\mathcal{M} \subseteq \operatorname{ri}\operatorname{dom} C^*$, but the boundary of $\mathcal{M}$ is not necessarily included. To allow some generality beyond $\operatorname{ri}\mathcal{M}$, we define the set

$$\tilde{\mathcal{M}} := \begin{cases} \mathcal{M} & \text{if } \mathcal{M} \subseteq \operatorname{ri}\operatorname{dom} C^* \\ \operatorname{ri}\mathcal{M} & \text{otherwise.} \end{cases}$$

In either case we obtain that $\tilde{\mathcal{M}} \subseteq \operatorname{ri}\operatorname{dom} C^* \subseteq \operatorname{im} p$, i.e., beliefs in $\tilde{\mathcal{M}}$ can be expressed by some state q. For the quadratic cost, $\tilde{\mathcal{M}} = \mathcal{M}$. For the log-partition cost, $\tilde{\mathcal{M}} = \operatorname{ri}\mathcal{M}$.

Perpendiculars. We now define the notion of a Bregman perpendicular to an affine space. This is a *constructive* definition. It plays a central role in the definition of the acute angles assumption, and also in the proof of the main result (Theorem 5.2). We will see that the set of optimal price vectors for different budgets is a sequence of Bregman perpendiculars. Naturally, perpendiculars are closely related to the conditions in Lemma 3.1; in particular to the set of q's that satisfy conditions (a) and (c) for a given face X.

For quadratic costs, Bregman perpendiculars coincide with the usual Euclidean perpendiculars. Consider an affine space and a point not in it. A projection of the point onto the space is the point in the space that is closest in Euclidean distance to the given point. Now consider moving this affine space towards the projected point. The locus of the projection as we move the space is the perpendicular to the space through the given point. We extend this definition to arbitrary Bregman divergences by defining the projection using the corresponding Bregman divergence.

A Bregman perpendicular is determined by three geometric objects within the affine hull $\operatorname{aff}(\operatorname{dom} C^*)$. The first of these is an affine space, say $A_0 \subseteq \operatorname{aff}(\operatorname{dom} C^*)$. The second is a point $a_1 \in \operatorname{aff}(\operatorname{dom} C^*) \backslash A_0$. The affine space $A = \operatorname{aff}(A_0 \cup \{a_1\}) \subseteq \operatorname{aff}(\operatorname{dom} C^*)$ will be the ambient space that will contain the perpendicular. Define parallel spaces to A_0 in A, for an arbitrary point $a_0 \in A_0$, as $A_\lambda := A_0 + \lambda(a_1 - a_0)$ for $\lambda \in \mathbb{R}$. Note that the definition of A_λ is independent of the choice of a_0. The third geometric object is a market state $q \in \mathbb{R}^n$ such that $p(q) \in A$. For technical reasons, we will define a perpendicular at q rather than a more natural notion, which would be at $p(q)$. Our reason for switching into q-space is that inner products, defining optimality of the Bregman projection, are between elements of q-space and ν-space (the two spaces coincide for Euclidean distance). For all $\lambda \in \mathbb{R}$ define a Bregman projection of q onto A_λ as

$$\nu_\lambda := \operatorname*{argmin}_{\nu \in A_\lambda} D(q, \nu) \ .$$

Since $D(q,\nu)$ is bounded from below and lower semi-continuous, the minimum is always attained (but it may be equal to ∞). If it is attained at more than one point, we choose an arbitrary minimizer. Whenever we can choose $\nu_\lambda \in \operatorname{ri}\operatorname{dom} C^*$, this ν_λ must be the unique minimizer by strict convexity of $D(q,\cdot)$ on $\operatorname{ri}\operatorname{dom} C^*$, and the minimum is finite. We use these ν_λ's to define the perpendicular:

Definition 4.1. Given A_0, a_1 and q as above, the a_1-*perpendicular* to A_0 at q is a map $\gamma : \lambda \mapsto \nu_\lambda$ defined over $\lambda \in \Lambda := \{\lambda \in \mathbb{R} : \nu_\lambda \in \operatorname{ri}\operatorname{dom} C^*\}$. We call Λ the domain of the perpendicular. We define a total order on

$\nu_\lambda, \nu_{\lambda'} \in \operatorname{im}\gamma$ as $\nu_\lambda \preceq \nu_{\lambda'}$ iff $\lambda \le \lambda'$.

In Appendix F.2 of the full version, we show that perpendiculars are continuous maps. The name perpendicular is justified by the following proposition which matches our Euclidean intuition that the perpendiculars can be obtained by intersecting the ambient space A with the affine space which passes through q and is orthogonal to A_0. It also shows that the perpendicular corresponds to the set of prices that satisfy conditions (a) and (c) with the convex hull relaxed to the affine hull (when A_0 is the affine hull of face X, point a_1 coincides with μ and q is the initial state). Recall that for an arbitrary set $X \subseteq \mathbb{R}^n$, its orthogonal complement is defined as $X^\perp := \{u :\ u \cdot (x' - x) = 0 \text{ for all } x, x' \in X\}$.

Proposition 4.2. *Let γ be the a_1-perpendicular to A_0 at q, and let $A = \operatorname{aff}(A_0 \cup \{a_1\})$. The following two statements are equivalent for any $\nu' \in \mathbb{R}^n$:*

(i) $\nu' \in \operatorname{im}\gamma$
(ii) $\nu' \in A \cap (\operatorname{ri}\operatorname{dom} C^*)$, $p^{-1}(\nu') \cap (q + A_0^\perp) \neq \emptyset$

Proposition 4.2 is proved in Appendix F of the full version. The perpendiculars have the following closure property which is useful for showing budget additivity (also proved in Appendix F of the full version):

Proposition 4.3. *Under the assumptions of Proposition 4.2, γ is also the a_1-perpendicular to A_0 at any $q' \in p^{-1}(\operatorname{im}\gamma) \cap (q + A_0^\perp)$.*

5 BUDGET ADDITIVITY

We now state the *acute angles* property which links the Bregman perpendicular and Corollary 3.5, and is sufficient for budget additivity.

Definition 5.1. We say that the *acute angles* hold for a face X, if for every μ-perpendicular γ to X at q, such that $\mu \in \tilde{\mathcal{M}}$ and $q \in p^{-1}(\tilde{\mathcal{M}})$, the following holds: If $\nu' \in \operatorname{im}\gamma$ and $\nu' \succeq p(q)$, then $p^{-1}(\nu') \cap [q + \mathcal{K}(X)] \neq \emptyset$.

The motivation for the name "acute angles" comes from the Euclidean distance case, where this assumption is equivalent to Definition 3.8 (see Proposition G.1 in the full version). The acute angles property is non-trivial and we have seen that without this property, budget additivity need not hold; we conjecture that it is also a necessary condition. After stating the main theorem, we analyze in more detail when the acute angles are satisfied by the quadratic and log-partition costs.

We now state the main result, that the acute angles are sufficient for budget additivity:

Theorem 5.2 (Sufficient conditions for budget additivity)**.** *If acute angles hold for every face $X \subseteq \Omega$, then the prediction market is budget additive on $\tilde{\mathcal{M}}$.*

Sufficient conditions for acute angles. We next give the sufficient conditions when the acute angles hold for the quadratic and log-partition cost functions. We also show that the acute angles hold for all one-dimensional outcome spaces, and that they are preserved by taking *direct sums* of markets. Recall that a set $\mathcal{K} \in \mathbb{R}^n$ is called a cone if it is closed under multiplication by positive scalars. A cone is called *acute*, if $x \cdot y \ge 0$ for all $x, y \in \mathcal{K}$. An *affine cone* with the vertex a_0 is a set $\mathcal{K}'$ of the form $a_0 + \mathcal{K}$ where $\mathcal{K}$ is a cone.

Theorem 5.3 (Sufficient condition for quadratic cost)**.** *Let X be a face and A' be the affine space $a_0 + X^\perp$ for an arbitrary $a_0 \in \operatorname{aff}(X)$. Acute angles hold for the face X and the quadratic cost if and only if the projection of Ω (or, equivalently, $\mathcal{M}$) on A' is contained in an affine acute cone with the vertex a_0.*

Corollary 5.4. *Acute angles hold for the quadratic cost and a hypercube $\Omega = \{0, 1\}^n$.*

Corollary 5.5. *Acute angles hold for the quadratic cost and simplex $\Omega = \{e_i :\ i \in [n]\}$ where $[n] = \{1, 2, \ldots, n\}$ and e_i is the i-th vector of the standard basis in $\mathbb{R}^n$.*

Theorem 5.6 (Log-partition over affinely independent outcomes)**.** *If the set Ω is affinely independent then acute angles assumption is satisfied for the log-partition cost.*

Theorem 5.7 (One-dimensional outcome spaces)**.** *Acute angles hold for any cost function if $\mathcal{M}$ is a line segment.*

Let $\Omega_1 \subseteq \mathbb{R}^{n_1}$ and $\Omega_2 \subseteq \mathbb{R}^{n_2}$ be outcome spaces with costs C_1 and C_2. We define the *direct sum* of Ω_1 and Ω_2 to be the outcome space $\Omega = \Omega_1 \times \Omega_2$ with the cost $C : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ defined as $C(q_1, q_2) = C_1(q_1) + C_2(q_2)$.

Theorem 5.8 (Acute angles for direct sums)**.** *If acute angles hold for Ω_1 with cost C_1, and Ω_2 with cost C_2, then they also hold for their direct sum.*

As a direct consequence of this theorem, we obtain that the log-partition cost function satisfies the acute angles assumption on a hypercube. More generally, any direct sum of costs on line segments satisfies the acute angles. This means that all cost-based prediction markets consisting of independent binary questions are budget additive, regardless of costs used to price individual questions.

As mentioned in the introduction, a vast number of deployed cost-based prediction markets consists of independent questions (not necessarily binary), each priced according to an LMSR (i.e., a log-partition cost on a simplex). Theorems 5.6 and 5.8 imply that this industry standard is budget additive.

5.1 Proof of Theorem 5.2

In this section we sketch the proof of Theorem 5.2 (for a complete proof see Appendix H of the full version). We proceed in several steps. Let $\nu_0 = p(q_0)$. Assuming acute angles, we begin by constructing an oriented curve L joining ν_0 with μ, by sequentially choosing portions of perpendiculars for monotonically decreasing active sets. We then show that budget additivity holds for any solutions with prices in L, and finally show that the curve L is the locus of the optimal prices of solutions $\hat{Q}(q_0)$, as well as optimal prices of solutions $\hat{Q}(q)$ for any $q \in \hat{Q}(q_0)$.

Part 1: Construction of the solution path L. In this part, we construct:

- a sequence of prices $\nu_0, \nu_1, \ldots, \nu_k$ with $\nu_0 = p(q_0)$ and $\nu_k = \mu$
- a sequence of oriented curves $\ell_0, \ldots, \ell_{k-1}$ where each ℓ_i goes from ν_i to ν_{i+1}
- a monotone sequence of sets $\Omega \supseteq X_0 \supset X_1 \supset \cdots \supset X_k = \emptyset$, such that the following *minimality* property holds: X_i is the minimal face for all $\nu \in (\operatorname{im} \ell_i) \backslash \{\nu_{i+1}\}$ for $i \leq k-1$, and X_k is the minimal face for ν_k.
- a sequence of states $q_1, \ldots, q_{k-1}$ such that $q_i \in p^{-1}(\nu_i) \cap [q_{i-1} + \mathcal{K}(X_{i-1})]$

The curves ℓ_i will be referred to as *segments*. The curve obtained by concatenating the segments ℓ_0 through ℓ_{k-1} will be called the *solution path* and denoted L. In the special case that $\nu_0 = \mu$, we have $k = 0$, $X_0 = \emptyset$ and L is a degenerate curve with $\operatorname{im} L = \{\mu\}$.

If $\nu_0 \neq \mu$, we construct the sequence of segments iteratively. Let $X_0 \neq \emptyset$ be the minimal face such that $\nu_0 \in \operatorname{conv}(X_0 \cup \{\mu\})$. By the minimality, $\mu \notin \operatorname{aff}(X_0)$. Let γ be the μ-perpendicular to $\operatorname{aff}(X_0)$ at q_0. The curve γ passes through ν_0 and eventually reaches the boundary of $\operatorname{conv}(X_0 \cup \{\mu\})$ at some ν_1 by continuity of γ (see Theorem F.3). Let segment ℓ_0 be the portion of γ going from ν_0 to ν_1.

This construction gives us the first segment ℓ_0. There are two possibilities:

1. $\nu_1 = \mu$; in this case we are done;
2. ν_1 lies on a lower-dimensional face of $\operatorname{conv}(X_0 \cup \{\mu\})$; in this case, we pick some $q_1 \in p^{-1}(\nu_1) \cap [q_0 + \mathcal{K}(X_0)]$, which can be done by the acute angles assumption, and use the above construction again, starting with q_1, and obtaining a new set $X_1 \subset X_0$ and a new segment ℓ_1; and iterate.

The above process eventually ends, because with each iteration, the size of the active set decreases. This construction yields monotonicity of X_i and the minimality property.

The above construction yields a specific sequence of $q_i \in p^{-1}(\nu_i) \cap [q_{i-1} + \mathcal{K}(X_{i-1})]$. We show in Appendix H of the full version that actually $q_i \in p^{-1}(\nu_i) \cap (q_0 + X_{i-1}^{\perp})$ and that the construction of L is independent of the choice of $q_1, q_2, \ldots, q_{k-1}$.

Part 2: Budget additivity for points on L. Let $\nu, \nu' \in \operatorname{im} L$ such that $\nu \preceq \nu'$. Let $q \in \hat{Q}(\nu; q_0)$ and $q' \in \hat{Q}(\nu'; q)$ such that $q \in \hat{Q}(B; q_0)$ and $q' \in \hat{Q}(B'; q)$. In this part we show that $q' \in \hat{Q}(B + B'; q_0)$.

First, consider the case that $\nu' = \mu$. To see that $q' \in \hat{Q}(B + B'; q_0)$, first note that the constraints of Convex Program (2.1) hold, because $U(q', \omega; q_0) = U(q', \omega; q) + U(q, \omega; q_0) \geq -B' - B$ for all ω by path independence of the utility function. As noted in the introduction, in the absence of constraints, the utility $U(\bar{q}, \mu; q_0)$ is maximized at any $\bar{q}$ with $p(\bar{q}) = \mu$. Thus, q' is a global maximizer of the utility and satisfies the constraints, so $q' \in \hat{Q}(B + B'; q_0)$. If $\nu = \mu$, we must also have $\nu' = \mu$ and the statement holds by previous reasoning.

In the remainder, we only analyze the case $\nu \preceq \nu' \prec \mu$. This means that $\nu \in (\operatorname{im} \ell_i) \backslash \{\nu_{i+1}\}$ and $\nu' \in (\operatorname{im} \ell_j) \backslash \{\nu_{j+1}\}$ for $i \leq j$. By Theorem 3.4, we therefore must have $q \in [q_0 + \mathcal{K}(X_i)]$ and $q' \in [q + \mathcal{K}(X_j)]$. By anti-monotonicity of witness cones, $\mathcal{K}(X_j) \supseteq \mathcal{K}(X_i)$ and hence, $q' \in [q_0 + \mathcal{K}(X_j)]$, yielding $q' \in \hat{Q}(\nu'; q_0)$.

We now argue that the budgets add up. Let $x \in X_j \subseteq X_i$. By Lemma 3.1, we obtain that $q \in \hat{Q}(B; q_0)$ for $B = -U(q, x; q_0)$, and $q' \in \hat{Q}(B'; q)$ for $B' = -U(q', x; q)$, and finally $q' \in \hat{Q}(\bar{B}; q_0)$ for $\bar{B} = -U(q', x; q_0)$. However, by path independence of the utility function

$$\bar{B} = -U(q', x; q_0) = -U(q', x; q) - U(q, x; q_0) = B' + B.$$

Part 3: L as the locus of all solutions. See Appendix H of the full version for the proof that

$$\hat{Q}(q_0) = \bigcup_{\nu \in \operatorname{im} L} \hat{Q}(\nu; q_0) \ .$$

Part 3': L as the locus of solutions starting at a midpoint. Let $\nu \in \operatorname{im} L$ and $q \in \hat{Q}(\nu; q_0)$. Since $\hat{Q}(\nu; q_0) \subseteq p^{-1}(\nu) \cap (q_0 + X_\nu^{\perp})$, Part 1' (Appendix H of the full version) yields that the solution path L' for q coincides with the portion of L starting at ν. Applying the proof of Part 3 to L', we obtain

$$\hat{Q}(q) = \bigcup_{\nu' \in \operatorname{im} L : \nu' \succeq \nu} \hat{Q}(\nu'; q) \ .$$

Part 4: Proof of the theorem. Let $B, B' \geq 0$ and $q \in \hat{Q}(B; q_0)$ and $q' \in \hat{Q}(B'; q)$. From Parts 3 and 3', we know that $q \in \hat{Q}(\nu; q_0)$ and $q' \in \hat{Q}(\nu'; q)$ for some $\nu, \nu' \in \operatorname{im} L$ such that $\nu \preceq \nu'$. By Part 2, we therefore obtain that $q' \in \hat{Q}(B + B'; q_0)$, proving the theorem.

References

[1] Jacob Abernethy, Yiling Chen, and Jennifer Wortman Vaughan. An optimization-based framework for automated market-making. In *ACM Conference on Electronic Commerce*, pages 297–306, 2011.

[2] Alina Beygelzimer, John Langford, and David M. Pennock. Learning performance of prediction markets with Kelly bettors. In *International Conference on Autonomous Agents and Multiagent Systems*, pages 1317–1318, 2012.

[3] GW Brier. Verication of forecasts expressed in terms of probability. *Monthly Weather Review*, 78:13, 1950.

[4] Yiling Chen and David M. Pennock. A utility framework for bounded-loss market makers. In *Conference on Uncertainty in Artificial Intelligence*, pages 49–56, 2007.

[5] Yiling Chen and Jennifer Wortman Vaughan. A new understanding of prediction markets via no-regret learning. In *ACM Conference on Electronic Commerce*, pages 189–198, 2010.

[6] Miroslav Dudík, Sébastien Lahaie, and David M. Pennock. A tractable combinatorial market maker using constraint generation. In *ACM Conference on Electronic Commerce*, 2012.

[7] Edmund Eisenberg and David Gale. Consensus of subjective probabilities: The pari-mutuel method. *Annals of Mathematical Statistics*, 30:165–168, 1959.

[8] Lance Fortnow and Rahul Sami. Multi-outcome and multidimensional market scoring rules. *CoRR*, abs/1202.1712, 2012.

[9] Rafael Frongillo, Nicolas Della Penna, and Mark Reid. Interpreting prediction markets: a stochastic approach. In *Advances in Neural Information Processing Systems 25*, pages 3275–3283. 2012.

[10] Tilmann Gneiting and Adrian E Raftery. Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102(477): 359–378, 2007.

[11] Sharad Goel, Daniel M. Reeves, Duncan J. Watts, and David M. Pennock. Prediction without markets. In *ACM Conference on Electronic Commerce*, pages 357–366, New York, NY, USA, 2010. ACM.

[12] Robin Hanson. Combinatorial information market design. *Information Systems Frontiers*, 5(1):107–119, 2003.

[13] Robin Hanson. Logarithmic market scoring rules for modular combinatorial information aggregation. *Journal of Prediction Markets*, 1(1):3–15, February 2007.

[14] Robert A. Jacobs. Methods for combining experts' probability assessments. *Neural Computation*, 7(5): 867–888, September 1995.

[15] Xiaolong Li and Jennifer Wortman Vaughan. An axiomatic characterization of adaptive-liquidity market makers. In *ACM Conference on Electronic Commerce*, 2013.

[16] Charles F. Manski. Interpreting the predictions of prediction markets. *Economics Letters*, 91(3):425–429, June 2006.

[17] Abraham Othman, Tuomas Sandholm, David M. Pennock, and Daniel M. Reeves. A practical liquidity-sensitive automated market maker. In *ACM Conference on Electronic Commerce*, 2010.

[18] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1970.

[19] Justin Wolfers and Eric Zitzewitz. Prediction markets. *Journal of Economic Perspectives*, Winter, 2004.

[20] Justin Wolfers and Eric Zitzewitz. Interpreting prediction market prices as probabilities. IZA Discussion Papers 2092, Institute for the Study of Labor (IZA), April 2006.