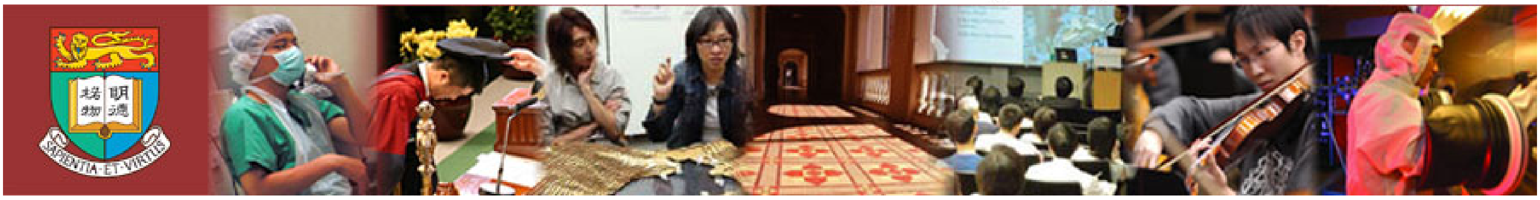

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Convex Ordering for Insurance Preferences*

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Abstract

In this article, we study two broad classes of convex order related optimal insurance decision problems, in which the objective function or the premium valuation is a general functional of the expectation, Value-at-Risk and Average Value-at-Risk of the loss variables. These two classes of problems include many existing and contemporary optimal insurance problems as interesting examples being prevalent in the literature. To solve these problems, we apply the Karlin-Novikoff-Stoyan-Taylor multiple-crossing conditions, which is a useful sufficient criterion in the theory of convex ordering, to replace an arbitrary insurance indemnity by a more favorable one in convex order sense. The convex ordering established provide a unifying approach to solve the special cases of the problem classes. We show that the optimal indemnities for these problems in general take the double layer form.

Key words: Convex ordering; Karlin-Novikoff-Stoyan-Taylor crossing conditions; Value-at-Risk; Average Value-at-Risk; Optimal insurance decision problem

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1 Introduction

Optimal insurance decision problem has long been one of the most popular research topics in the insurance context due to its immediate practical consequence. The optimality of the deductible contracts for policyholders subject to the classical expected value premium principle was first proven by Borch (1960) for the minimization of the variance of the retained loss, and by Arrow (1974) for the maximization of the expected utility (EU) of the terminal wealth. Since then, intense effort has been observed in the literature to solve similar problems under various model settings with different objective functions as well as imposing various constraints that lead to a variety of optimality results. For example, see Asimit et al. (2013a,b), Balbás et al. (2009), Bernard and Tian (2009), Cai et al. (2008), Centeno and Guerra (2010), Cheung et al. (2013, 2014, 2015), Kaluszka and Okolewski (2008), Sung et al. (2011), Tan et al. (2011), and the references therein.

The notion of *stochastic ordering*, in particular *convex ordering*, have been well developed and they are essential for comparing risky alternatives in decision analysis based on different criteria. For example, convex ordering arranges risks by their variations with respect to the value of same means, and consequently allows the decision maker to choose the "least risky" alternative. Convex ordering has been thoroughly applied for solving various problems in economics, finance and actuarial science, which demonstrates its usefulness and importance. For instance, it can be applied to compare the aggregate risk of a portfolio, in which the comonotonicity structure among the risks attains the upper bound of the convex order. For comprehensive studies and other applications in convex ordering, see Denuit et al. (2005), Denuit and Dhaene (2012), Dhaene et al. (2002, 2006, 2012), Kaas et al. (1994, 2008), Müller and Stoyan (2002), Rüschendorf (2013), Shaked and Shanthikumar (2007), and the references therein.

The convex ordering approach to solve the optimal insurance decision problem was first adopted by Ohlin (1969) of minimizing a measure of the dispersion of the retained and ceded losses. The crucial mathematical tool employed by Ohlin (1969) is the 'Karlin-Novikoff once-crossing criterion' by Karlin and Novikoff (1963) for (increasing) convex ordering. Later, Gollier and Schlesinger (1996) used the same approach to extend the result of Arrow (1974) through maximizing an increasing convex order preserving objective functional of the terminal wealth. More recently, this approach was re-exploited to solve various optimal insurance decision problems. For instance, Cai and Wei (2012) solved the multivariate risk minimizing problems in which the risks are positively dependent. Chi and Tan (2013) considered the optimal insurance problems under which the premium principle is a certain convex order preserving functional.

In this paper, we study two broad classes of convex order related optimal insurance decision problems:

(I) maximizing a concave order preserving functional of the terminal wealth of the insured with the premium principle specified by a general function of the expectation, Value-at-Risk (V@R), and Average Value-at-Risk (AV@R) of the indemnity; and,

(II) minimizing another general function of expectation, V@R and AV@R of the terminal loss of the insured with the premium valuated by a general function of the expectation and a convex order preserving functional of the ceded loss.

Both classes include many existing and contemporary optimal insurance problems as interesting examples as we shall show in later sections.

Since the problem settings involve the convex order preserving functionals, it is natural to apply the convex ordering approach on solving for these two problem classes. Instead of using the 'Karlin-Novikoff once-crossing criterion' by Karlin and Novikoff (1963), we adopt the '*Karlin-Novikoff-Stoyan-Taylor crossing conditions*', developed by Stoyan (1983) and Taylor (1983) and named by Hürlimann (1998, 2008a,b), which is a generalization of the once-crossing condition. By exploiting this multiple-crossing criterion, we are able to

(i) rank the insurance indemnities in terms of their convex orders together with a greater flexibility than that through the once-crossing condition; and,

(ii) provide a unifying approach to solve for two classes of optimal insurance decision problems (I) and (II) by using the convex ordering obtained in (i).

The organization of our paper is as follows. In Section 2, two classes of optimal insurance problems with the corresponding optimality criterion and constraint are formulated. The main theorem using the multiple-crossing conditions to establish the convex ordering of the insurance indemnities are presented in Section 3. Resolutions of the special cases of two classes of problems formulated in Section 2 are illustrated as the corollaries in Sections 4 and 5.

2 Preliminaries and Problem Formulation

2.1 Preliminaries

We first recall the definitions and results of several stochastic orderings. For a comprehensive review of the theory and applications, see the references in the first paragraph in Introduction. In this section, Y and Z are random variables with cumulative distribution functions F_Y and F_Z.

Definition 2.1. Y is said to be smaller than Z in the convex (concave, increasing convex, increasing concave, resp.) order if for all convex (concave, increasing convex, increasing concave, resp.) functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[\varphi(Y)] \leq \mathbb{E}[\varphi(Z)]$, provided that the expectations exist. The convex (concave, increasing convex, increasing concave, resp.) ordering is denoted as $Y \leq_{\text{cx}} Z$ ($Y \leq_{\text{cv}} Z$, $Y \leq_{\text{icx}} Z$, $Y \leq_{\text{icv}} Z$, resp.).

Since $Y \leq_{\text{cx}} Z$ is equivalent to $Z \leq_{\text{cv}} Y$, and $Y \leq_{\text{icx}} Z$ is equivalent to $-Y \geq_{\text{icv}} -Z$, and we shall only make use of the results in convex and increasing convex order in this article, we only present the following summary of useful results for the convex and increasing convex order. The counterpart results for the concave and increasing concave order are similar. In what follows, all stated moments are assumed to be finite.

Proposition 2.1.

(i) If $Y \leq_{\text{cx}} Z$, then $\mathbb{E}[Y] = \mathbb{E}[Z]$ and $\text{Var}(Y) \leq \text{Var}(Z)$. Also, if $Y \leq_{\text{icx}} Z$, then $\mathbb{E}[Y] \leq \mathbb{E}[Z]$.

(ii) Define $\pi_Y(t) \triangleq \mathbb{E}[(Y-t)_+]$ as the stop-loss transform of Y. Then, $Y \leq_{\text{icx}} Z$ if, and only if, $\pi_Y(t) \leq \pi_Z(t)$ for any real numbers t. Furthermore, if $\mathbb{E}[Y] = \mathbb{E}[Z]$, then $Y \leq_{\text{cx}} Z$ if, and only if, $\pi_Y(t) \leq \pi_Z(t)$.

Notice that the results of convex order and increasing convex order are analogous to each other; indeed, we have the following equivalence of these two orders provided that the means of Y and Z are equal.

Proposition 2.2. $Y \leq_{\text{cx}} Z$ if, and only if, $Y \leq_{\text{icx}} Z$ and $\mathbb{E}[Y] = \mathbb{E}[Z]$.

To facilitate further use of convex and increasing convex orderings, Karlin and Novikoff (1963) provided sufficient conditions in terms of the cumulative distribution functions, known as 'Karlin-Novikoff once-crossing criterion'.

Definition 2.2. The distribution functions F_Y and F_Z are said to be crossing $r \geq 1$

times if there exist

$$\xi_{0,2} < \xi_{1,1} \leq \xi_{1,2} < \xi_{2,1} \leq \xi_{2,2} < \cdots < \xi_{r,1} \leq \xi_{r,2} < \xi_{r+1,1},$$

where $\xi_{0,2} \triangleq \inf\{x : F_Y(x) \neq F_Z(x)\}$ and $\xi_{r+1,1} \triangleq \sup\{x : F_Y(x) \neq F_Z(x)\}$, such that, for each $i = 1, 2, \ldots, r$,

(i) for any $x \in (\xi_{i-1,2}, \xi_{i,1})$ and $y \in (\xi_{i,2}, \xi_{i+1,1})$,

$$(F_Y(x) - F_Z(x))\,(F_Y(y) - F_Z(y)) < 0; \text{ and}$$

(ii) if $\xi_{i,1} < \xi_{i,2}$, $F_Y(z) = F_Z(z)$ for any $\xi_{i,1} \leq z < \xi_{i,2}$.

Theorem 2.3. Assume that $\mathbb{E}[Y] = \mathbb{E}[Z]$ (resp. $\mathbb{E}[Y] \leq \mathbb{E}[Z]$). If F_Y and F_Z cross once, and $F_Y(x) - F_Z(x) < 0$ for $\xi_{0,2} < x < \xi_{1,1}$, then $Y \leq_{\text{cx}} Z$ (resp. $Y \leq_{\text{icx}} Z$).

In addition, in this paper we shall make use of the following generalization by Stoyan (1983) and Taylor (1983), coined as 'Karlin-Novikoff-Stoyan-Taylor crossing conditions' by Hürlimann (1998, 2008a,b).

Theorem 2.4. Assume that F_Y and F_Z cross $n \geq 1$ times. Then $Y \leq_{\text{icx}} Z$ if, and only if, one of the following two cases is satisfied:

<u>**Case 1**</u>

(i) There is an even number of crossings $n = 2m$ for some $m = 1, 2, \ldots$;

(ii) $F_Y(x) - F_Z(x) > 0$ for $\xi_{0,2} < x < \xi_{1,1}$; and

(iii) for any $j = 1, 2, \ldots, m$, $\pi_Y(\xi_{2j-1,2}) \leq \pi_Z(\xi_{2j-1,2})$.

<u>**Case 2**</u>

(i) $\mathbb{E}[Y] \leq \mathbb{E}[Z]$;

(ii) there is an odd number of crossings $n = 2m - 1$ where $m = 1, 2, \ldots$;

(iii) $F_Y(x) - F_Z(x) < 0$ for $\xi_{0,2} < x < \xi_{1,1}$; and

(iv) if $m \geq 2$, for any $j = 1, 2, \ldots, m-1$, $\pi_Y(\xi_{2j,2}) \leq \pi_Z(\xi_{2j,2})$.

Applying Proposition 2.2 yields an analogous theorem for the convex order by noting that when $\mathbb{E}[Y] = \mathbb{E}[Z]$, Case 1 in Theorem 2.4 cannot be valid as before.

Theorem 2.5. Assume that F_Y and F_Z cross $n \geq 1$ times. Then $Y \leq_{\text{cx}} Z$ if, and only if,

(i) $\mathbb{E}[Y] = \mathbb{E}[Z]$;

(ii) there is an odd number of crossings $n = 2m - 1$ where $m = 1, 2, \ldots$;

(iii) $F_Y(x) - F_Z(x) < 0$ for $\xi_{0,2} < x < \xi_{1,1}$; and

(iv) if $m \geq 2$, for any $j = 1, 2, \ldots, m-1$, $\pi_Y(\xi_{2j,2}) \leq \pi_Z(\xi_{2j,2})$.

We next recall the definitions and results regarding the V@R, AV@R and comonotonicity. For a thorough reference, see Denuit et al. (2005) and Kaas et al. (2008). Define the (left-continuous) quantile function of Y as $F_Y^{-1}(p) \triangleq \inf\{x \in \mathbb{R} | F_Y(x) \geq p\}$ for any $p \in (0, 1]$.

Definition 2.3. Assume that $\alpha \in (0, 1]$.

(i) The V@R of Y at the level α, denoted by $\text{V@R}_\alpha(Y)$, is defined as

$$\text{V@R}_\alpha(Y) \triangleq F_Y^{-1}(\alpha).$$

(ii) The AV@R of Y at the level α, denoted by $\text{AV@R}_\alpha(Y)$, is defined as

$$\text{AV@R}_\alpha(Y) \triangleq \frac{1}{1-\alpha} \int_\alpha^1 \text{V@R}_\lambda(Y) d\lambda = \frac{1}{1-\alpha} \int_\alpha^1 F_Y^{-1}(\lambda) d\lambda$$

for $\alpha < 1$; when $\alpha = 1$, $\text{AV@R}_\alpha(Y)$ is defined as the $\operatorname{ess\,sup} Y$.

(iii) Y and Z are said to be comonotonic if, and only if, there exist a random variable S and two non-decreasing functions f_1 and f_2 such that $Y \stackrel{d}{=} f_1(S)$ and $Z \stackrel{d}{=} f_2(S)$.

By definition, for a non-negative random variable Y, the integral in $\text{AV@R}_\alpha(Y)$ is the area bounded by the horizontal boundaries $y = \alpha$, $y = 1$, the vertical boundary $x = 0$, and the graph F_Y.

Proposition 2.6. Assume that $\alpha \in (0, 1]$.

(i) For any non-decreasing continuous function f, $\text{V@R}_\alpha(f(Y)) = f(\text{V@R}_\alpha(Y))$.

(ii) Both V@R and AV@R are translation invariant and comonotonic additive.

2.2 Problem Formulation

Let X be a non-negative integrable random variable defined on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to model the insurable risk/loss of an agent over a fixed period of time. To avoid unnecessary technical details, we assume that the cumulative distribution function F_X is continuous and strictly increasing on the support of X.[1] Also, denote $S_X = 1 - F_X$ as the survival function. To reduce the risk exposure, the agent seeks for an insurance protection by paying a premium P in advance, while in return the insurer is responsible for a part of the loss (ceded loss), which is denoted by $I(X)$, where the ceded loss function I satisfies certain reasonable assumptions to be specified below. Consequently, the agent retains the remaining part of the loss (retained loss), which is denoted by $R(X) = X - I(X)$; and, the terminal risk/loss exposure and terminal wealth of the agent, with a positive initial capital w, are

$$L_I \triangleq X - I(X) + P = R(X) + P$$

and

$$Z_I \triangleq w - L_I(X) = w - P - X + I(X),$$

respectively.

In this article, any feasible ceded loss function I satisfies $I(0) = 0$ and $0 \leq I(x) \leq x$, which is normalized at zero and depicts that any insurance indemnity cannot be greater than the loss in any ethical manner. Also, to avoid moral hazard and potential swindles from the insured, (i) for any additional loss claim, at least not lesser insurance payment would be requested; and (ii) a unit increment of loss cannot result in more than a unit additional indemnity payment; mathematically, in the sequel, we assume that, for any feasible ceded loss function I,

$$0 \leq I(x_1) - I(x_2) \leq x_1 - x_2, \qquad \text{for } 0 \leq x_2 < x_1 \leq \operatorname{ess\,sup} X.$$

Define the set $\mathcal{I}$ of feasible ceded loss functions as follows:

$$\begin{aligned} \mathcal{I} \triangleq \{ & I : [0, \operatorname{ess\,sup} X] \to [0, \operatorname{ess\,sup} X] | \\ & (I1) : 0 \leq I(x) \leq x, \qquad \text{for any } x \in [0, \operatorname{ess\,sup} X]; \\ & (I2) : 0 \leq I(x_1) - I(x_2) \leq x_1 - x_2, \qquad \text{for any } 0 \leq x_2 < x_1 \leq \operatorname{ess\,sup} X \}. \end{aligned}$$

[1] Our analysis and results include that case that F_X has a jump at 0.

As a result, the corresponding set $\mathcal{R}$ of feasible retained loss functions is given by:

$$\begin{aligned}\mathcal{R} \triangleq \{R : [0, \operatorname{ess\,sup} X] \to [0, \operatorname{ess\,sup} X] | \\ (R1) : 0 \leq R(x) \leq x, \quad \text{for any } x \in [0, \operatorname{ess\,sup} X]; \\ (R2) : 0 \leq R(x_1) - R(x_2) \leq x_1 - x_2, \quad \text{for any } 0 \leq x_2 < x_1 \leq \operatorname{ess\,sup} X\}.\end{aligned}$$

Both $(I2)$ and $(R2)$ are referred as 1-Lipchitz condition. Notice that, for any $I \in \mathcal{I}$, both I and the corresponding R are non-decreasing functions.

Naturally, the agent is interested in purchasing an insurance contract under a certain optimality criterion. In particular, we assume that the agent chooses the insurance contract which would respectively solve the following two general optimization problems.

Problem 2.1. Let g_1 be an arbitrary function from $\mathbb{R}_+^3$ to $\mathbb{R}_+$, and $\alpha \in (0, 1]$ be a threshold risk level. Let P be a fixed positive constant. Let $V_{1,\text{cv}}$ be a functional preserving concave order. Consider the premium principle H_1 defined by:

$$H_1(\cdot) \triangleq g_1(\mathbb{E}[\cdot], \text{V@R}_\alpha(\cdot), \text{AV@R}_\alpha(\cdot)).$$

The agent aims to choose an optimal policy $I \in \mathcal{I}$ that maximizes

$$V_{1,\text{cv}}(w - H_1(I(X)) - X + I(X)),$$

subject to the premium constraint that $H_1(I(X)) = P$.

Problem 2.2. Let g_2 be a function from $\mathbb{R}_+^2$ to $\mathbb{R}_+$, which is non-decreasing in both arguments. Let $V_{2,\text{cx}}$ be a convex order preserving functional. Let G_2 be a real-valued function on $\mathbb{R}_+^3$, which is non-decreasing in all arguments. Let $\alpha \in (0, 1]$ be a threshold risk level. Consider the premium principle H_2 defined by:

$$H_2(\cdot) \triangleq g_2\left(\mathbb{E}[\cdot], V_{2,\text{cx}}(\cdot)\right).$$

The agent would like to choose an optimal policy $I \in \mathcal{I}$ that minimizes

$$G_2(\mathbb{E}\left[X - I(X) + H_2(I(X))\right], \text{V@R}_\alpha(X - I(X) + H_2(I(X))), \text{AV@R}_\alpha(X - I(X) + H_2(I(X)))).$$

Before analyzing and solving these two classes of problems, we emphasize that they are formulated with reasonable, but not maximal, generality and flexibility, and yet with minimal assumptions. These two classes of problems include and generalize many existing models in the literature as shown in the next two sections. In particular, the premium

$H_1(I(X))$ for the first problem includes the well-known actuarial premium principle

$$(1+\theta)\,\mathbb{E}[I(X)],$$

where the risk-loading θ is a constant; on the other hand, it could also take the form of

$$(1+\Theta\left(\mathrm{V@R}_\alpha(I(X))\right))\,\mathbb{E}[I(X)],$$

where $\Theta(\cdot)$ is a nondecreasing function. This premium valuation generalizes the actuarial premium principle by allowing the risk-loading be influenced by the tail risk exposure, which is natural in practice.

In the sequel, for notational simplicity, we denote $a \triangleq \mathrm{V@R}_\alpha(X)$ and $b \triangleq \mathrm{AV@R}_\alpha(X)$.

3 Main Theorem

In this section, we study the convex ordering of the retained losses $R(X)$ among the retained loss functions $R \in \mathcal{R}$. Results for the ceded losses follow the same way by replacing any R by I and $\mathcal{R}$ by $\mathcal{I}$. We first define two sub-classes $\mathcal{R}_1$ and $\mathcal{R}_3$ of $\mathcal{R}$ by

$$\mathcal{R}_1 \triangleq \{R \in \mathcal{R} \mid R(x) = x - (x-d)_+ \text{ for some } 0 \le d \le \operatorname{ess\,sup} X\},$$

and

$$\mathcal{R}_3 \triangleq \{R \in \mathcal{R} \mid R(x) = x - (x-d_1)_+ + (x-d_2)_+ - (x-d_3)_+ \\ \text{for some } 0 \le d_1 \le d_2 \le d_3 \le \operatorname{ess\,sup} X\}.$$

Clearly, $\mathcal{R}_1$ is a subset of $\mathcal{R}_3$. If the class $\mathcal{R}_1$ is partitioned into sub-classes

$$\mathcal{R}_{1,\mu} \triangleq \{R \in \mathcal{R}_1 \mid \mathbb{E}\left[R(X)\right] = \mu\},$$

parametrized by the mean μ of the retained loss, each non-empty sub-class $\mathcal{R}_{1,\mu}$ is a singleton, i.e., the parameter d in each sub-class $\mathcal{R}_{1,\mu}$ is unique. The following proposition is well-known in the literature which is immediate by Theorem 2.3.

Proposition 3.1. For any $\mu \in [0, \mathbb{E}\left[X\right]]$ and $R \in \mathcal{R}$ with $\mathbb{E}\left[R(X)\right] = \mu$, there exists a unique $\tilde{R} \in \mathcal{R}_{1,\mu}$ such that $\tilde{R}(X) \le_{\mathrm{cx}} R(X)$.

By this proposition, for the optimal insurance problem which minimizes the convex order of the retained loss with the premium principle being solely a function of the expectation of the retained loss, the optimal solution takes the single layer form.

Similarly, the idea to solve Problem 2.1 and 2.2 is to find the retained loss function(s), for which the corresponding retained loss(es) has the smallest convex order with the same mean, V@R and AV@R. However, each retained loss function in $\mathcal{R}_1$ contains only one parameter which is not flexible enough for these two problems. Therefore, we need to generalize the result in Proposition 3.1 from $\mathcal{R}_1$ to $\mathcal{R}_3$ which allows a greater flexibility in choosing the parameters. We apply the Karlin-Novikoff-Stoyan-Taylor (multiple-)crossing conditions in Theorem 2.5 to prove the following main theorem, which demonstrates that, to minimize the convex order of $R(X)$ with the same values of mean and with the same functional values ν at some point τ, it suffices to consider those functions R in the sub-class $\mathcal{R}_3$.

Theorem 3.2. For any $\mu \in [0, \mathbb{E}[X]]$, $(\tau, \nu) \in [0, \operatorname{ess\,sup} X] \times [0, \tau]$, and $R \in \mathcal{R} \backslash \mathcal{R}_3$ with $\mathbb{E}[R(X)] = \mu$ and $R(\tau) = \nu$, there exists $\tilde{R} \in \mathcal{R}_3$, with parameters d_1, d_2, d_3, such that (i) $d_2 \leq \tau \leq d_3$, (ii) $\mathbb{E}\big[\tilde{R}(X)\big] = \mathbb{E}[R(X)] = \mu$, (iii) $\tilde{R}(\tau) = R(\tau) = \nu$, and (iv) $\tilde{R}(X) \leq_{\mathrm{cx}} R(X)$.

Proof. Fix $\mu \in [0, \mathbb{E}[X]]$, $(\tau, \nu) \in [0, \operatorname{ess\,sup} X] \times [0, \tau]$, and $R \in \mathcal{R} \backslash \mathcal{R}_3$ with $\mathbb{E}[R(X)] = \mu$ and $R(\tau) = \nu$. Let $(d_1, d_2, d_3) \in [0, \tau] \times [d_1, \tau] \times [\tau, \operatorname{ess\,sup} X]$ be real constants which satisfy the following system:

$$\begin{cases} \text{(a)} \quad \mathbb{E}[X - (X - d_1)_+ + (X - d_2)_+ - (X - d_3)_+] = \mathbb{E}[R(X)] = \mu, \\ \text{(b)} \quad \tau - d_2 + d_1 = R(\tau) = \nu, \\ \text{(c)} \quad \mathbb{E}\left[((X - (X - d_1)_+ + (X - d_2)_+ - (X - d_3)_+) - (\tau - d_2 + d_1))_+\right] = \mathbb{E}\left[(R(X) - R(\tau))_+\right]. \end{cases}$$

The existences of d_1, d_2, d_3 are guaranteed by Intermediate Value Theorem. Define $\tilde{R}(x) \triangleq x - (x - d_1)_+ + (x - d_2)_+ - (x - d_3)_+$ which is clearly in $\mathcal{R}_3$. By the definition of $\tilde{R}$, (a) is equivalent to

$$\mathbb{E}\big[\tilde{R}(X)\big] = \mathbb{E}[R(X)] = \mu;$$

(b) is equivalent to

$$\tilde{R}(\tau) = R(\tau) = \nu;$$

and, (c) is equivalent to

$$\mathbb{E}\big[(\tilde{R}(X) - \tilde{R}(\tau))_+\big] = \mathbb{E}\big[(R(X) - R(\tau))_+\big].$$

It is easy to check that $F_{\tilde{R}(X)}$, the distribution function of $\tilde{R}(X)$, is given by:

$$F_{\tilde{R}(X)}(y) = \begin{cases} F_X(y), & \text{for } 0 \leq y < d_1, \\ F_X(y + d_2 - d_1), & \text{for } d_1 \leq y < d_3 - d_2 + d_1, \\ 1, & \text{for } y \geq d_3 - d_2 + d_1. \end{cases}$$

Suppose that $\tilde{R}$ and R do not overlap on any interval. In this case, $F_{\tilde{R}(X)}$ and $F_{R(X)}$ cross each other three times with $\xi_{1,1} = \xi_{1,2} \triangleq d_1$, $\xi_{2,1} = \xi_{2,2} \triangleq \tau - d_2 + d_1$, and $\xi_{3,1} = \xi_{3,2} \triangleq d_3 - d_2 + d_1$. By condition (a), $\mathbb{E}\big[\tilde{R}(X)\big] = \mathbb{E}\left[R(X)\right]$, while, by condition (c), $\pi_{\tilde{R}(X)}(\xi_{2,2}) = \pi_{R(X)}(\xi_{2,2})$. The form of $F_{\tilde{R}(X)}$ guarantees $F_{\tilde{R}(X)}(y) - F_{R(X)}(y) < 0$ for $\xi_{0,2} < y < \xi_{1,1}$. Therefore, by Theorem 2.5, $\tilde{R}(X) \leq_{\text{cx}} R(X)$.

On the other hand, if $\tilde{R}$ and R overlap on some interval, similar arguments hold as the previous case but it is possible that $F_{\tilde{R}(X)}$ and $F_{R(X)}$ cross only once. For instance, if there exists $0 \leq e_1 \leq \tau \leq e_2 \leq \eta$ such that $R(x) = x - (x - e_1)_+ + (x - e_2)_+$ for $x \leq \eta$, let $d \in [\tau, \operatorname{ess\,sup} X]$ such that

$$\mathbb{E}\left[R(X)\mathbb{1}_{\{X \geq \tau\}}\right] = \mathbb{E}\left[\left(R(\tau) + X - (X - d)_+\right)\mathbb{1}_{\{X \geq \tau\}}\right].$$

The existence of d is again guaranteed by Intermediate Value Theorem. Take $d_1 = e_1$, $d_2 = \tau$ and $d_3 = d$, and hence $F_{\tilde{R}(X)}$ and $F_{R(X)}$ cross once. By the construction of d and Theorem 2.5, $\tilde{R}(X) \leq_{\text{cx}} R(X)$. □

As an illustration, we provide a crossing example of $\tilde{R}$ and R when they do not overlap on any interval. Because the crossing conditions are written using the distribution functions, we express conditions $(R1)$ and $(R2)$ in $\mathcal{R}$ in terms of the distribution function of $R(X)$ as follows:

(F_R1) : $F_{R(X)}(0-) = 0$ and $F_X(z) \leq F_{R(X)}(z)$ for any $z \in [0, \operatorname{ess\,sup} X]$;

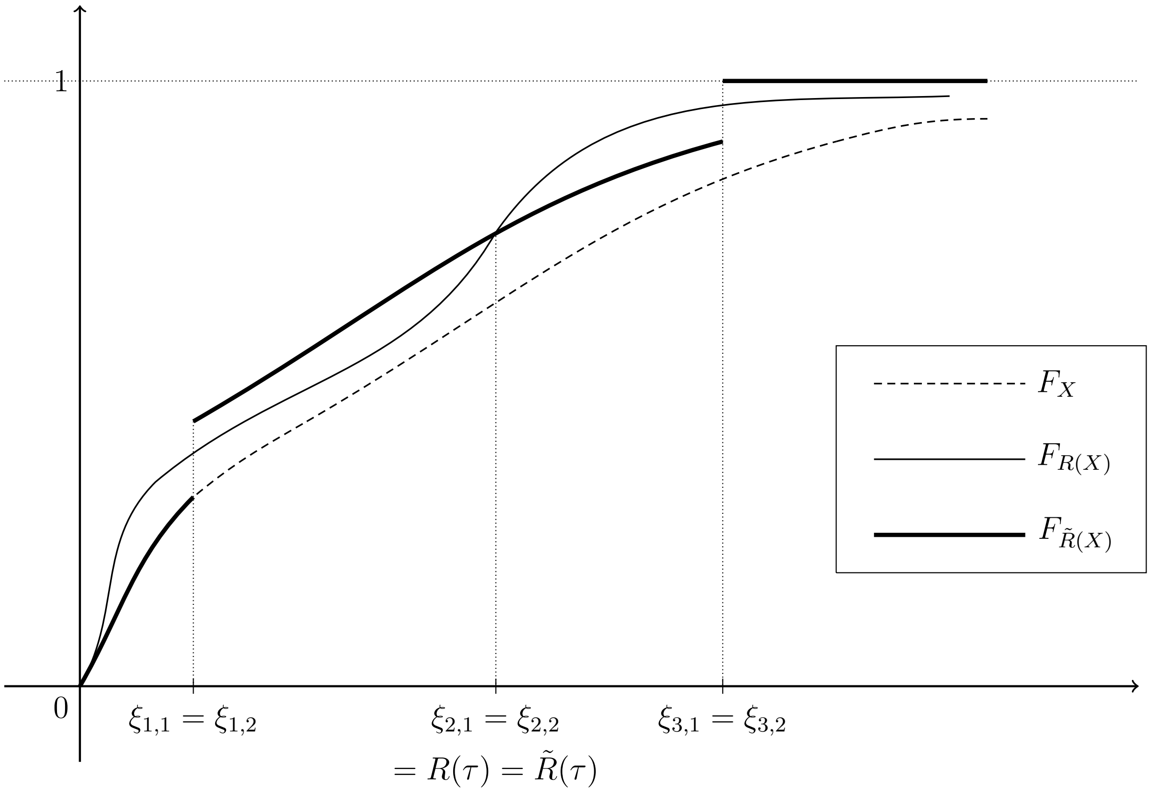
(F_R2) : $0 \leq F^{-1}_{R(X)}(\beta_2) - F^{-1}_{R(X)}(\beta_1) \leq F_X^{-1}(\beta_2) - F_X^{-1}(\beta_1)$ for any $0 \leq \beta_1 < \beta_2 \leq 1$.

Since $\tilde{R}, R \in \mathcal{R}$, their distribution functions $F_{\tilde{R}(X)}$ and $F_{R(X)}$ satisfy conditions (F_R1) and (F_R2), i.e., both $F_{\tilde{R}(X)}$ and $F_{R(X)}$ lie above F_X, and they exhibit a smaller horizontal increment than F_X for the same vertical increment. Figure 1 shows an example of the graphs of F_X (dashed), $F_{R(X)}$ (semi-thick) and $F_{\tilde{R}(X)}$ (ultra-thick) with three crossings.

The construction of $\tilde{R}$ in Theorem 3.2 is actually not necessarily unique. The reason is that the class $\mathcal{R}_3$ can be written as the union of sub-classes

$$\mathcal{R}_{3,\mu,\tau,\nu} \triangleq \{R \in \mathcal{R}_3 \mid \mathbb{E}\left[R(X)\right] = \mu \text{ and } R(\tau) = \nu\},$$

Figure 1: An example of distribution functions in the proof of Theorem 3.2



parametrized by the mean μ of the retained loss and the point (τ, ν), where each non-empty sub-class $\mathcal{R}_{3,\mu,\tau,\nu}$ is not necessarily a singleton since the parameters d_1, d_2, d_3 in each of these sub-classes $\mathcal{R}_{3,\mu,\tau,\nu}$ are not necessarily unique. There could be more than one, and even infinitely many, feasible double layer retained loss functions such that the retained losses have the same mean and the functions pass through the point (τ, ν). Indeed, the condition (c) in the proof can be replaced by the inequality:

$$(\text{c}')\,\mathbb{E}\left[\left((X - (X-d_1)_+ + (X-d_2)_+ - (X-d_3)_+) - (\tau - d_2 + d_1)\right)_+\right] \leq \mathbb{E}\left[(R(X) - R(\tau))_+\right],$$

such that any retained loss function R satisfying conditions (a), (b) and (c′) is also a possible candidate for the existence. However, the $\tilde{R}$ constructed in the proof has the added advantage that, it not only preserves the mean of its retained loss and passes through the same point, but also preserves the mean above and below the point (τ, ν) as those of the arbitrary R. This observation helps us solve some special cases of Problem 2.1 and 2.2 as corollaries by Theorem 3.2.

Nevertheless, Theorem 3.2 only demonstrates that one can exclude those retained loss functions R outside $\mathcal{R}_3$ when seeking for the retained loss with a smaller convex order among the functions $R \in \mathcal{R}$. It does not compare the convex order of the retained losses among the functions $R \in \mathcal{R}_3$. Hence, we have the following theorem which reveals the convex ordering within the sub-class $\mathcal{R}_3$.

Theorem 3.3. Let $\mu \in [0, \mathbb{E}[X]]$ and $(\tau, \nu) \in [0, \operatorname{ess\,sup} X] \times [0, \tau]$. Define $d^* \in [0, \operatorname{ess\,sup} X]$ such that $\mathbb{E}[X - (X - d^*)_+] = \mu$. For any $R \in \mathcal{R}_3$ with parameters $(d_1, d_2, d_3) \in [0, \operatorname{ess\,sup} X] \times [d_1, \operatorname{ess\,sup} X] \times [d_2, \operatorname{ess\,sup} X]$ such that $\mathbb{E}[R(X)] = \mu$, and $R(\tau) = \nu$, let $(d_1', d_2', d_3') \in [0, \operatorname{ess\,sup} X] \times [d_1', \operatorname{ess\,sup} X] \times [d_2', \operatorname{ess\,sup} X]$ be real constants such that

$$\mathbb{E}[X - (X - d_1')_+ + (X - d_2')_+ - (X - d_3')_+] = \mathbb{E}[R(X)],$$

and satisfy one of the following systems of inequalities:

(a)

$$\begin{cases} d_1 \leq d_1' \leq d^*, \\ d_1' \leq d_2' \leq d_1' + d_2 - d_1, \\ d^* \leq d_3' - d_2' + d_1' \leq d_3 - d_2 + d_1, \end{cases}$$

if $0 \leq \tau \leq d_1$,

(b)

$$\begin{cases} d_1' = d_1, \\ \tau \leq d_2' \leq d_2, \\ d^* - d_1 \leq d_3' - d_2' \leq d_3 - d_2, \end{cases}$$

if $d_1 \leq \tau \leq d_2$,

(c)

$$\begin{cases} d_1 \leq d_1' \leq \tau - d_2 + d_1, \\ d_2' = d_1' + d_2 - d_1, \\ d^* + d_2 - d_1 \leq d_3' \leq d_3, \end{cases}$$

if $d_2 \leq \tau \leq d^* + d_2 - d_1$,

(d)

$$\begin{cases} d_1 \leq d_1' \leq d^*, \\ d_2' = d_1' + d_2 - d_1, \\ \tau \leq d_3' \leq d_3, \end{cases}$$

if $d^* + d_2 - d_1 \leq \tau \leq d_3$,

(e)

$$\begin{cases} d_1 \le d_1' \le d^*, \\ d_3' - d_2' + d_1' = d_3 - d_2 + d_1, \\ d_3 \le d_3' \le \tau, \end{cases}$$

if $d_3 \le \tau$.

Then, for $R'(x) \triangleq x - (x - d_1')_+ + (x - d_2')_+ - (x - d_3')_+$, we have (i) $R' \in \mathcal{R}_3$, (ii) $\mathbb{E}\big[R'(X)\big] = \mathbb{E}\,[R(X)] = \mu$, (iii) $R'(\tau) = R(\tau) = \nu$, and (iv) $R'(X) \le_{\text{cx}} R(X)$.

Proof. Fix $\mu \in [0, \mathbb{E}\,[X]]$, $(\tau, \nu) \in [0, \operatorname{ess\,sup} X] \times [0, \tau]$, and $R \in \mathcal{R}_3$ with $\mathbb{E}\,[R(X)] = \mu$, $R(\tau) = \nu$ and parameters d_1, d_2, d_3. We only prove the cases that $d_2 \le \tau \le d^* + d_2 - d_1$ and $d^* + d_2 - d_1 \le \tau \le d_3$ since the remaining cases can be shown similarly. For both of these two cases, let d_1', d_2', d_3', where the existences are guaranteed by Intermediate Value Theorem, and R' be as stated. It is clear that (i)–(iii) are true. For (iv), $F_{R'(X)}$ and $F_{R(X)}$ only cross once, with $\xi_{1,1} = d_1'$ and $\xi_{1,2} = d_3' - d_2 + d_1$. Also, the form of $F_{R'(X)}$ guarantees $F_{R'(X)}(y) - F_{R(X)}(y) < 0$ for $\xi_{0,2} < y < \xi_{1,1}$. Therefore, by Theorem 2.5, $R'(X) \le_{\text{cx}} R(X)$. □

4 Analysis of Problem 2.1

In this section, we consider a general class of optimal insurance decision problems in which the objective is to maximize a concave order preserving functional of the terminal wealth Z_I, and the premium is an arbitrary function of the expectation, V@R, and AV@R of $I(X)$, which is mathematically presented in Problem 2.1 in Section 2. Using Theorem 3.2 and 3.3, we demonstrate, in Corollaries 4.1–4.4, that this problem class includes Gollier and Schlesinger (1996) and Kaluszka and Okolewski (2008) as special cases, and further examples which have not appeared in the literature.

Firstly, using Proposition 2.6, the premium constraint can be expressed in terms of both the ceded loss as:

$$g_1(\mathbb{E}[I(X)], I(a), \text{AV@R}_\alpha(I(X))) = P, \tag{1}$$

or in terms of the retained loss as:

$$g_1(\mathbb{E}[X] - \mathbb{E}[R(X)], a - R(a), b - \text{AV@R}_\alpha(R(X))) = P. \tag{2}$$

Let $\mathcal{P}$ be the feasible set of all positive premium values P such that there is at least one feasible $I \in \mathcal{I}$ (resp. $R \in \mathcal{R}$), satisfying the premium constraint (1) (resp. (2)); and we fix a premium $P \in \mathcal{P}$ throughout this section. We then embed the premium constraint (1) to the feasible class $\mathcal{I}$ and the premium constraint (2) to the feasible class $\mathcal{R}$, by introducing the sets

$$\mathcal{I}_P \triangleq \{I \in \mathcal{I} \mid g_1(\mathbb{E}[I(X)], I(a), \mathrm{AV@R}_\alpha(I(X))) = P\},$$

and

$$\mathcal{R}_P \triangleq \{R \in \mathcal{R} \mid g_1(\mathbb{E}[X] - \mathbb{E}[R(X)], a - R(a), b - \mathrm{AV@R}_\alpha(R(X))) = P\}$$

respectively. By defining $V_{1,\mathrm{cx}}(x) \triangleq -V_{1,\mathrm{cv}}(w - P - x)$, Problem 2.1 is now equivalent to

Problem 4.1.

$$\inf_{R \in \mathcal{R}_P} V_{1,\mathrm{cx}}(R(X)).$$

In words, the objective of Problem 4.1 is to determine an optimal retained loss function R^* such that $R^*(X)$ is the smallest in terms of the convex order subject to the premium constraint (2).

We say that a function $h : \mathbb{R}^n \to \mathbb{R}$ does not depend on the i-th argument if for any $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$, the univariate function

$$x_i \mapsto h(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

is a constant. To indicate explicitly the dependence of g_1 on its three arguments, we introduce the indicators:

$$J_i(g_1) \triangleq \begin{cases} 0 & \text{if } g_1 \text{ does not depend on the } i\text{-th argument,} \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, 3$. In the following corollaries, we demonstrate that Problem 2.1, or equivalently Problem 4.1, includes some existing and contemporary optimal insurance problems as special cases, and we show that their solution forms could be reduced in some cases.

Corollary 4.1. Assume that g_1 depends on all three arguments, such that $J_1(g_1) = J_2(g_1) = J_3(g_1) = 1$. The optimal ceded loss function for Problem 2.1 takes the form of

$$I^*(x) = (x - d_1^*)_+ - (x - d_2^*)_+ + (x - d_3^*)_+,$$

for some $0 \le d_1^* \le d_2^* \le a \le d_3^* \le \operatorname{ess\,sup} X$.

Proof. For any $R \in \mathcal{R}_P$, if $R \in \mathcal{R}_P \backslash \mathcal{R}_{3,P}$, by Theorem 3.2 and taking $\tau = a$, there exists $\tilde{R} \in \mathcal{R}_3$ such that

$$\begin{cases} \text{(a)} \quad \mathbb{E}\big[\tilde{R}(X)\big] = \mathbb{E}\left[R(X)\right], \\ \text{(b)} \quad \tilde{R}(a) = R(a), \\ \text{(c)} \quad \mathbb{E}\big[(\tilde{R}(X) - \tilde{R}(a))_+\big] = \mathbb{E}\big[(R(X) - R(a))_+\big], \\ \text{(d)} \quad \tilde{R}(X) \leq_{\text{cx}} R(X). \end{cases}$$

By (b) and (c),

$$\begin{aligned} (1-\alpha)\,\text{AV@R}_\alpha(\tilde{R}(X)) &= \tilde{R}(a)\,(1-\alpha) + \mathbb{E}\big[(\tilde{R}(X) - \tilde{R}(a))_+\big] \\ &= R(a)\,(1-\alpha) + \mathbb{E}\big[(R(X) - R(a))_+\big] \\ &= \,(1-\alpha)\,\text{AV@R}_\alpha(R(X)). \end{aligned}$$

Therefore, $\tilde{R} \in \mathcal{R}_{3,P}$ and hence, it suffices to consider those $R \in \mathcal{R}_{3,P}$, with parameters d_1, d_2, d_3. Define

$$R'(x) = \begin{cases} x - (x - d^*)_+ & \text{if } 0 \leq a \leq d_1, \\ x - (x - d_1)_+ + (x - a)_+ - (x - d_3')_+ & \text{if } d_1 \leq a < d_2, \\ x - (x - d_1)_+ + (x - d_2)_+ - (x - d_3)_+ & \text{if } d_2 \leq a \leq d_3, \\ x - (x - d_1')_+ + (x - d_2')_+ - (x - a)_+ & \text{if } d_3 < a, \end{cases}$$

where d_1', d_2', d_3' are constructed such that $\mathbb{E}\big[R'(X)\big] = \mathbb{E}\left[R(X)\right]$. Notice that, in any cases, $d_2' \leq a \leq d_3'$. By Theorem 3.3, $R' \in \mathcal{R}_{3,P}$ and $R'(X) \leq_{\text{cx}} R(X)$. □

Corollary 4.2. Assume that g_1 depends only on the first and the third arguments, such that $J_1(g_1) = J_3(g_1) = 1$ but $J_2(g_1) = 0$. The optimal ceded loss function for Problem 2.1 takes the form of

$$I^*(x) = (x - d_1^*)_+ - (x - d_2^*)_+ + (x - d_3^*)_+,$$

for some $0 \leq d_1^* \leq d_2^* \leq d_3^* \leq \operatorname{ess\,sup} X$.

Proof. The proof works essentially the same as that of Corollary 4.1. □

Corollary 4.3. [Extension of Kaluszka and Okolewski (2008)] Assume that g_1 depends only on the first and the second arguments, such that $J_1(g_1) = J_2(g_1) = 1$ but

$J_3(g_1) = 0$. The optimal ceded loss function for Problem 2.1 takes the form of

$$I^*(x) = (x - d_1^*)_+ - (x - a)_+ + (x - d_2^*)_+,$$

for some $0 \leq d_1^* \leq a \leq d_2^* \leq \operatorname{ess\,sup} X$, or

$$I^{**}(x) = (x - d_1^{**})_+ - (x - d_2^{**})_+ + (x - a)_+,$$

for some $0 \leq d_1^{**} \leq d_2^{**} \leq a \leq \operatorname{ess\,sup} X$.

In particular, if $\alpha = 1$, the optimal indemnity is a single layer

$$I^{***}(x) = (x - d_1^{***})_+ - (x - d_2^{***})_+,$$

for some $0 \leq d_1^{***} \leq d_2^{***} \leq \operatorname{ess\,sup} X$. This special case was considered in Kaluszka and Okolewski (2008).

Proof. Similarly, by Theorem 3.2, it suffices to consider any $R \in \mathcal{R}_P \backslash \mathcal{R}_{3,P}$ with parameters d_1, d_2, d_3. Define

$$R'(x) = \begin{cases} x - (x - d^*)_+ & \text{if } 0 \leq a \leq d_1, \\ x - (x - d_1')_+ + (x - a)_+ - (x - d_3') & \text{if } d_1 \leq a \leq d^* + d_2 - d_1, \\ x - (x - d_1')_+ + (x - d_2')_+ - (x - a)_+ & \text{if } d^* + d_2 - d_1 \leq a, \end{cases}$$

where d_1', d_2', d_3' are constructed such that $\mathbb{E}\big[R'(X)\big] = \mathbb{E}\left[R(X)\right]$. By Theorem 3.3 and taking $\tau = a$, we have $R' \in \mathcal{R}_{3,P}$ and $R'(X) \leq_{\text{cx}} \tilde{R}(X)$. Finally, the special case of $\alpha = 1$ corresponds to that $a = \operatorname{ess\,sup} X$. □

Cheung et al. (2015) studied the optimal insurance problems under various disappointment theories. Since the objective functions under the Disappointment Theory by Loomes and Sugden (1986) when the disappointment function is concave and Disappointment Aversion Theory by Gul (1991) preserve the concave order, those results now can be deduced from Corollary 4.3. However, it is not clear that whether the objective function under the generalized Disappointment Theory without prior expectation motivated by Cillo and Delquié (2006) and proposed in Cheung et al. (2015) preserves the concave order and so Corollary 4.3 may not be applicable in general.

Corollary 4.4. [Extension of Gollier and Schlesinger (1996)] Assume that g_1 solely depends on the first argument, such that $J_1(g_1) = 1$ but $J_2(g_1) = J_3(g_1) = 0$. The optimal ceded loss function for Problem 2.1 takes the form of

$$I^*(x) = (x - d^*)_+,$$

for some $0 \le d^* \le \operatorname{ess\,sup} X$.

Proof. It is immediate by Proposition 3.1 or Theorem 3.2 and 3.3 with $\tau = \nu = 0$. □

Corollary 4.4 extends the model studied by Gollier and Schlesinger (1996), who generalized the objective function from the expected utility framework by Arrow (1974) to any increasing convex order (or second order stochastic dominance) preserving functional. Also, this example includes the problems studied by Borch (1960) and Ohlin (1969), in which the objective functions are the variance and a measure of dispersion, respectively.

5 Analysis of Problem 2.2

In this section, we consider another general class of optimal insurance decision problems in which the objective is to minimize a general function of the expectation, V@R and AV@R of the terminal loss L_I with the premium being another general function of the expectation and a convex order preserving functional of $I(X)$, which is mathematically depicted in Problem 2.2 in Section 2. We show, in Corollaries 5.1–5.4, that this class of problems includes Chi (2012b) as special cases, and further examples which have not appeared in the literature. Proofs in this section are similar to those in the previous section but use the main theorems in Section 3 by replacing any R by I and $\mathcal{R}$ by $\mathcal{I}$, and hence are omitted.

As a result of Proposition 2.6 and by substituting the premium principle $H_2(I(X))$ into the objective function, Problem 2.2 is equivalent to

Problem 5.1.

$$\begin{aligned}\inf_{I \in \mathcal{I}} G_2(&\mathbb{E}[X] - \mathbb{E}[I(X)] + g_2(\mathbb{E}[I(X)], V_{2,\mathrm{cx}}(I(X))),\\ &a - I(a) + g_2(\mathbb{E}[I(X)], V_{2,\mathrm{cx}}(I(X))),\\ &b - \mathrm{AV@R}_\alpha(I(X)) + g_2(\mathbb{E}[I(X)], V_{2,\mathrm{cx}}(I(X)))).\end{aligned}$$

Again, to explicitly clarify the dependence of G_2 on its three arguments, define the indicators:

$$J_i\left(G_2\right) \triangleq \begin{cases} 0 & \text{if } G_2 \text{ does not depend on the } i\text{-th argument,} \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, 3$, in which the meaning of argument dependence was defined in Section 4.

Corollary 5.1. Assume that G_2 depends on all three arguments, such that $J_1(G_2) = J_2(G_2) = J_3(G_2) = 1$. The optimal ceded loss function for Problem 2.2 takes the form of

$$I^*(x) = x - (x - d_1^*)_+ + (x - d_2^*)_+ - (x - d_3^*)_+,$$

for some $0 \le d_1^* \le d_2^* \le a \le d_3^* \le \operatorname{ess\,sup} X$.

Corollary 5.2. [Extension of Chi (2012b)] Assume that G_2 depends only on the first and the third arguments, such that $J_1(G_2) = J_3(G_2) = 1$ but $J_2(G_2) = 0$. The optimal ceded loss function for Problem 2.2 takes the form of

$$I^*(x) = x - (x - d_1^*)_+ + (x - d_2^*)_+ - (x - d_3^*)_+, \tag{3}$$

for some $0 \le d_1^* \le d_2^* \le a \le d_3^* \le \operatorname{ess\,sup} X$.

Corollary 5.3. [Extension of Chi (2012b)] Assume that G_2 depends only on the first and the second arguments, such that $J_1(G_2) = J_2(G_2) = 1$ but $J_3(G_2) = 0$. The optimal ceded loss function for Problem 2.2 takes the form of

$$I^*(x) = x - (x - d_1^*)_+ + (x - d_2^*)_+ - (x - a)_+, \tag{4}$$

for some $0 \le d_1^* \le d_2^* \le a \le \operatorname{ess\,sup} X$, or

$$I^{**}(x) = x - (x - d^{**})_+, \tag{5}$$

for some $0 \le d^{**} \le \operatorname{ess\,sup} X$.

Corollaries 5.2 and 5.3 include the work of Chi (2012b); indeed, for $\delta \in [0, 1]$, suppose that

$$G_2\left(\mathbb{E}\left[L_I(X)\right], \mathrm{V@R}\left(L_I(X)\right), \mathrm{AV@R}\left(L_I(X)\right)\right) = (1-\delta)\mathbb{E}\left[L_I(X)\right] + \delta \mathrm{V@R}\left(L_I(X)\right),$$

or

$$G_2\left(\mathbb{E}\left[L_I(X)\right], \mathrm{V@R}\left(L_I(X)\right), \mathrm{AV@R}\left(L_I(X)\right)\right) = (1-\delta)\mathbb{E}\left[L_I(X)\right] + \delta \mathrm{AV@R}\left(L_I(X)\right).$$

Due to the translation invariance of V@R and AV@R, the objective reduces to the risk-adjusted liability studied in Chi (2012b):

$$\mathbb{E}\left[L_I(X)\right]+\delta \mathrm{V@R}\left(L_I(X)-\mathbb{E}\left[L_I(X)\right]\right), \quad \text{or} \quad \mathbb{E}\left[L_I(X)\right]+\delta \mathrm{AV@R}\left(L_I(X)-\mathbb{E}\left[L_I(X)\right]\right),$$

with the same premium valuation. Note that our solution forms (3), (4) and (5) coincide with Chi (2012b).

Corollary 5.4. Assume that G_2 solely depends on the first argument, such that $J_1(G_2) = 1$ but $J_2(G_2) = J_3(G_3) = 0$. The optimal ceded loss function for Problem 2.2 takes the form of

$$I^*(x) = x - (x - d^*)_+,$$

for some $0 \le d^* \le \operatorname{ess\,sup} X$.

We make a final remark before closing this section. Chi (2012a) studied the optimal insurance problems where the objective functions are

$$\mathrm{V@R}_\alpha\left(X - I(X) + \mathbb{E}[I(X)] + g(\mathrm{Var}(I(X)))\right)$$

and

$$\mathrm{AV@R}_\alpha\left(X - I(X) + \mathbb{E}[I(X)] + g(\mathrm{Var}(I(X)))\right),$$

for some increasing function g. This formulation can be regarded as a special case of our Problem 2.2, in which (i) $G_2(x, y, z) = y$ or $G_2(x, y, z) = z$ so that G_2 does not depend on the first argument, (ii) $V_{2,\mathrm{cx}}(Y) = \mathrm{Var}(Y) = \mathbb{E}\left[(Y - \mathbb{E}\left[Y\right])^2\right]$, and (iii) $g_2\left(\mathbb{E}[I(X)], V_{2,\mathrm{cx}}(I(X))\right) = \mathbb{E}[I(X)] + g(\mathrm{Var}(I(X)))$. We also remark that the "flipping argument" used in Chi (2012a) remains applicable to the following general setting: (i') G_2 does not depend on the first argument, (ii') $V_{2,\mathrm{cx}}(Y) = V_{2,\mathrm{cx}}(a - Y)$ for any random variable Y and $a \in \mathbb{R}$, and (iii') g_2 is translational invariant with respect to the first argument. With these extra structures, one can obtain the single layer indemnity as the optimal solution.

6 Concluding Remarks

We studied two broad classes of convex order related optimal insurance decision problems, in which either the objective function or premium valuation depends on the expectation, V@R and AV@R of the losses. We showed that the optimal solution of these problems in general takes the double layer form. To solve these problems, we adopted the convex ordering approach and applied the Karlin-Novikoff-Stoyan-Taylor multiple-crossing conditions. This multiple-crossing conditions allow us to provide a unifying scheme to solve the optimality.

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