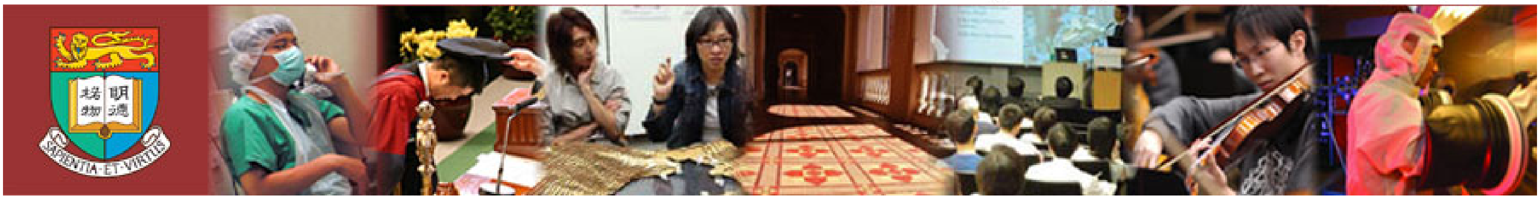

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Performance evaluation for inerter-based dynamic vibration absorbers

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Abstract

This paper is concerned with the H_∞ and H_2 optimization problem for inerter-based dynamic vibration absorbers (IDVAs). The proposed IDVAs are obtained by replacing the damper in the traditional dynamic vibration absorber (TDVA) with some inerter-based mechanical networks. It is demonstrated in this paper that adding one inerter alone to the TDVA provides no benefits for the H_∞ performance and negligible improvement (less than 0.32% improvement over the TDVA when the mass ratio less than 1) for the H_2 performance. This implies the necessity of introducing another degree of freedom (element) together with inerter to the TDVA. Therefore, four different IDVAs are proposed by adding an inerter together with a spring to the TDVA, and significant improvement for both the H_∞ and H_2 performances is obtained. Numerical simulations in dimensionless form show that more than 20% and 10% improvement can be obtained for the H_∞ and H_2 performances, respectively. Besides, for the H_∞ performance, the effective frequency band can be further widened by using inerter.

Keywords: Inerter, dynamic vibration absorber, H_∞ optimization, H_2 optimization.

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1. Introduction

Dynamic vibration absorber (DVA) is an auxiliary mass system attached to a vibrating primary system to reduce undesired vibration, which is widely used in the fields of civil and mechanical engineering for its simple design and high reliability [1]. In the first DVA proposed by Frahm in 1909 [2], only a spring was employed, and it was useful only in a narrow band of frequency. In 1928, the damping mechanism was introduced by Ormondroyd and Den Hartog [3], which is a parallel arrangement of a spring and a damper, and as a result, the effective frequency band was significantly widened. It was also pointed out in [3] that for the spring-damper DVA (in this paper, it is called the traditional DVA or TDVA) and undamped primary system, there were two frequencies called fixed points, where the magnitudes were independent of the damping, and the optimal setting of the spring stiffness was the one equalizing the magnitudes at the fixed points, and the optimal damping was the one making the curves of the frequency response horizontally pass through the fixed points. Such a tuning method is still in use today and currently known as the fixed-point method [1], which has been demonstrated to be a suboptimal H_∞ optimization method [4]. The exact

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solutions were analytically derived in [4] and it was also shown that the fixed-point method actually yielded an approximate but highly precise solution (with less than 0.5% deviation when the mass ratio less than 1). Another common performance measure of tuning DVA is the H_2 performance measure, which is desirable when the primary system subjected to random excitations. The objective of H_2 optimization is to optimize the total vibration energy of the system over all frequencies [5]. For the TDVA with undamped primary systems, the optimal tuning frequency and damping ratio were investigated in [5], and then the analytical solutions were derived in [6]. For damped primary systems, various design methods and tuning criteria have been proposed, such as those in [7, 9, 8, 10], and the applications of the TDVA in nonlinear and distributed primary systems have been investigated [11, 12, 13]. The active DVAs utilizing feedback control actions have also been proposed [14, 15, 16].

Inerter is a two-terminal mechanical device proposed by Smith in 2002 [17], which has the property that the applied force at its two terminals is proportional to the relative acceleration between them and the constant of proportionality is called inertance with a unit of kilogram. Inerter has been successfully applied in Formula One racing car suspension systems [18], and now, applications of inerter in various mechanical systems, such as vehicle suspensions [19, 20, 21, 22] and vibration suppression systems [23, 24, 25, 26], have been investigated. Recently, the reduction of vibration systems' natural frequencies by using inerter has been theoretically demonstrated [27], and the interest in passive network synthesis has also been rekindled [28, 29, 30, 31, 32, 33, 34, 35, 38].

Vibration absorption is one of the potential applications of inerter [17]. In [17], the problem of designing inerter-based networks to absorb vibration at a specific frequency was studied. Thereafter, the suppression of vibration over a broadband frequency by using inerter has been proposed. In [23], an inerter-based configuration ($C4$ in this paper) was employed between adjacent storeys to suppress the vibration of a multi-storey building. In [24], optimal solutions for several inerter-based isolators (including all the configurations except $C5$ in this paper) were algebraically derived based on a "uni-axial" vibration isolation system. In [25], a new configuration incorporating an inerter was proposed and applied to a mechanical cascaded (chain-like) systems. In [26], the dynamics of a tuned mass absorber with an additional viscous damper and an inerter attached to the pendulum was investigated.

In this paper, a novel structure for inerter-based DVAs (IDVAs) is proposed by replacing the damper in the TDVA with some inerter-based mechanical networks, and both the H_∞ and H_2 performances of the proposed IDVAs are investigated. It is demonstrated in this paper that adding an inerter alone to the TDVA, no matter it is in parallel connection or in series connection, provides no benefits for the H_∞ performance and negligible benefits (less than 0.32% improvement over the TDVA when the mass ratio less than 1) for the H_2 performance. In contrast, by adding an inerter together with a spring to the TDVA (e.g. $C3$, $C4$, $C5$, and $C6$ in this paper), both H_∞ and H_2 performances can be significantly improved. Over 20% improvement compared with the TDVA can be obtained for the H_∞ performance, and the effective frequency band can also be further widened by using inerter. For the H_2 performance, it is analytically demonstrated that the IDVAs proposed in this paper perform surely better than the TDVA and over 10% improvement is obtained in numerical simulation. Moreover, a minmax framework directly using the resonance frequencies is proposed for the H_∞ optimization, and an algebraic method to analytically calculate the H_2 norm is employed for the H_2 optimization. All these constitute the main contributions of this paper.

The organization of this paper is as follows. In Section 2, the IDVAs in this paper are introduced and the dimensionless representations of displacement transfer functions are derived. In Section 3 and Section 4, the H_∞ and H_2 optimization problems are solved for the IDVAs and the comparison between the IDVAs and the TDVA is conducted. Conclusions are drawn in Section 5.

2. Inerter-based dynamic vibration absorbers

Fig. 1 shows the comparison between the IDVAs proposed in this paper and the TDVA, where the IDVA is obtained by replacing the damper in the TDVA with some inerter-based mechanical networks. The entire networks employed in this paper are shown in Fig. 2. The equations of motion for the whole system in the Laplace domain are

$$Ms^2x = F + F_d - Kx, \tag{1}$$

$$ms^2x_a = -F_d, \tag{2}$$

$$F_d = (k + sY(s))(x_a - x), \tag{3}$$

where $Y(s)$ is the admittance of the inerter-based passive mechanical networks and F_d is the force of the DVA imposed on the primary mass M.

From (2) and (3), one obtains,

$$F_d = -R(s)x,$$

where

$$R(s) = \frac{(k + sY(s))ms^2}{k + ms^2 + sY(s)}.$$

Then, one obtains the displacement transfer function as

$$H(s) = \frac{x}{x_s} = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{1}{K}R(s) + 1}, \tag{4}$$

where $x_s = F/K$ and $\omega_n = \sqrt{\frac{K}{M}}$ are the static displacement and natural frequency of the primary system, respectively.

The admittance of each network in Fig. 2 is shown in Table 1, where $Y_i(s)$, $i = 1, \ldots, 6$ corresponds to C_i, $i = 1, \ldots, 6$ in Fig. 2, respectively. Substituting each $Y_i(s)$ into (4), one can obtain the detailed transfer function for each configuration. To obtain the dimensionless representation of each configuration, the following dimensionless parameters are defined as

$$\left.\begin{array}{c} \mu = \frac{m}{M} : \text{ mass ratio} \\ \delta = \frac{b}{m} : \text{ inertance-to-mass ratio} \\ \zeta = \frac{c}{2\sqrt{mk}} : \text{ damping ratio} \\ \eta = \frac{\omega_b}{\omega_m} : \text{ corner frequency ratio} \\ \gamma = \frac{\omega_m}{\omega_n} : \text{ natural frequency ratio} \\ \lambda = \frac{\omega}{\omega_n} : \text{ forced frequency ratio} \end{array}\right\} \tag{5}$$

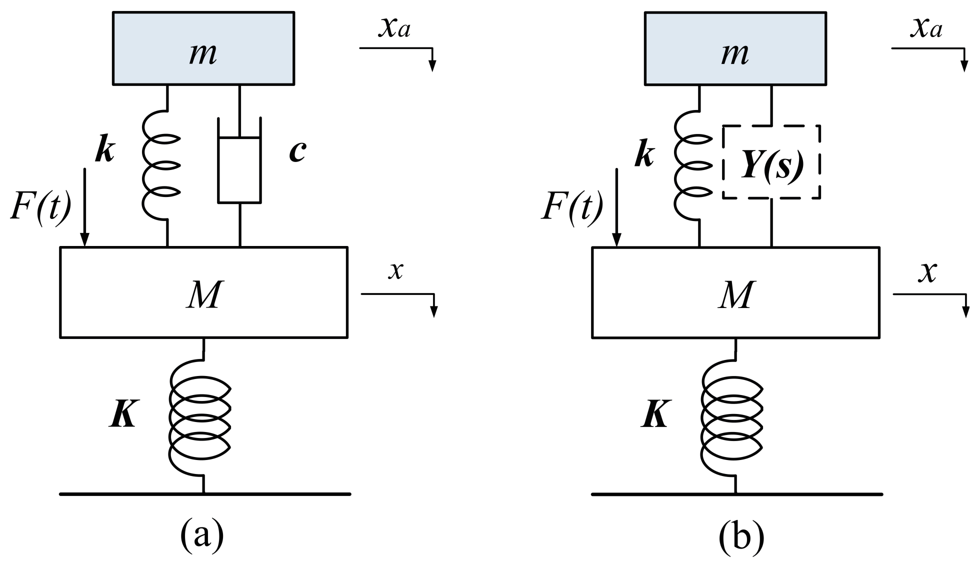


Figure 1: Dynamic vibration absorbers (DVA): (a) traditional dynamic vibration absorber (TDVA); (b) inerter-based dynamic vibration absorber (IDVA).

where

$$\left.\begin{aligned}\omega_m &= \sqrt{\tfrac{k}{m}}: \text{ natural frequency of the DVA}\\ \omega_b &= \sqrt{\tfrac{k_1}{b}}: \text{ corner frequency of the DVA}\\ \omega_n &= \sqrt{\tfrac{K}{M}}: \text{ natural frequency of the primary system}\end{aligned}\right\} \tag{6}$$

Remark 1. *In this paper, the force-current analogy between mechanical and electrical networks is employed, and admittance is defined to be the ratio of force to velocity, which agrees with the usual electrical terminology (see [17] for details). Such a definition is consistent with some books [36, p. 328], but not others which use the force-voltage analogy [37].*

Remark 2. *Since the natural frequencies would be perturbed by using inerter as demonstrated in [27], ω_m and ω_n are not the real natural frequencies of the whole system. Neither is ω_b the real corner frequency. Here, these notations are employed just for dimensionless representations.*

Replacing s with $j\omega$ in (4), the frequency response functions in a dimensionless form can be obtained as

$$H_i(j\lambda) = \frac{R_{ni} + jI_{ni}}{R_{mi} + jI_{mi}},\ i = 1, \ldots, 6, \tag{7}$$

where R_{ni}, I_{ni}, R_{mi}, and I_{mi}, $i = 1, \ldots, 6$ are functions with respect to λ, γ, δ, and ζ. The detailed representations are given in Appendix A.

3. H_∞ optimization for the IDVAs

3.1. Minmax optimization problem formulation

The objective of the H_∞ optimization is to minimize the maximum magnitude of the frequency response $|H_i(j\lambda)|$, $i = 1, \ldots, 6$, which is known as the H_∞ norm of $H_i(s)$ with

Table 1: Admittance $Y(s)$ for each configuration in Fig. 2.

$Y_1(s) = bs + c$	$Y_2(s) = \frac{1}{\frac{1}{bs}+\frac{1}{c}}$	$Y_3(s) = \frac{1}{\frac{s}{k_1}+\frac{1}{c}+\frac{1}{bs}}$
$Y_4(s) = \frac{1}{\frac{1}{\frac{k_1}{s}+c}+\frac{1}{bs}}$	$Y_5(s) = \frac{1}{\frac{1}{\frac{k_1}{s}+bs}+\frac{1}{c}}$	$Y_6(s) = \frac{1}{\frac{1}{bs+c}+\frac{s}{k_1}}$

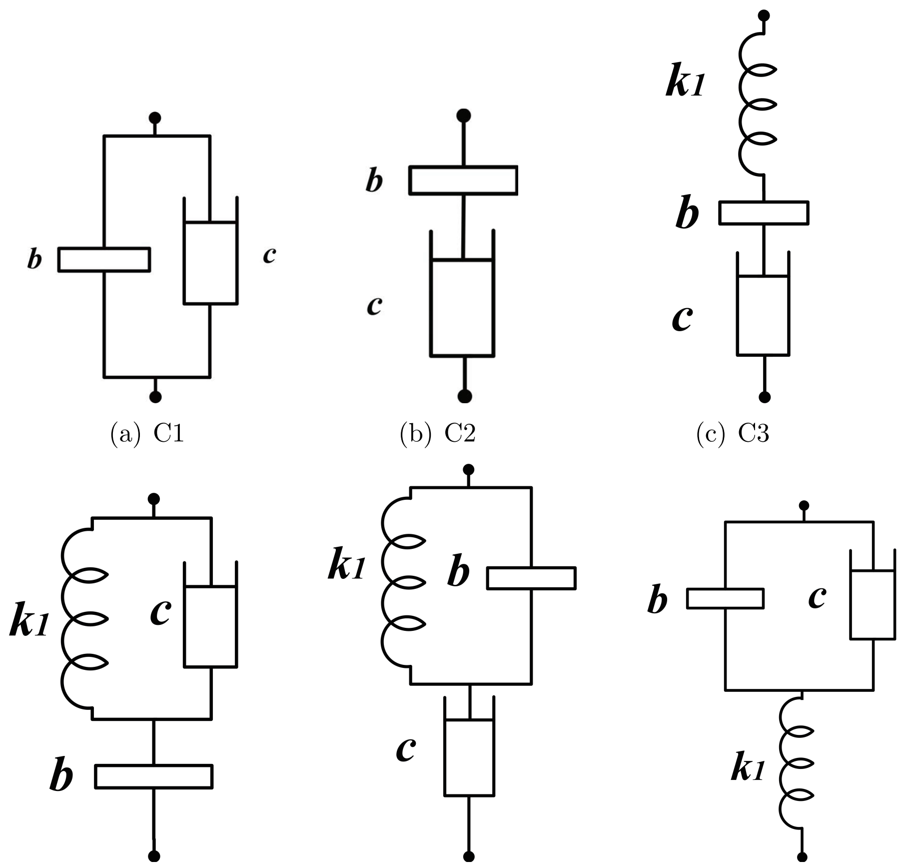


Figure 2: The employed inerter-based networks as $Y(s)$ in Fig. 1.

$s = j\lambda$. For the TDVA, the fixed-point method [1] is commonly used to analytically obtain the optimal parameters [1, Chapter 3.3]. Since there always exist more than two fixed points with respect to the damping ratio for IDVAs, it is difficult to obtain simple and analytical representations for optimal parameters. Given this fact, in this paper, a minmax optimization problem is formulated as follows to directly minimize the magnitude at resonance frequencies.

For a given mass ratio μ, solving the follow minmax problem

$$\min_{\delta,\gamma,\eta,\zeta}\left(\max_{\lambda_l}\left(|H_i(j\lambda_l)|\right)\right), i = 1, \ldots, 6 \tag{8}$$

subject to $\delta \geq 0$, $\gamma \geq 0$, $\eta \geq 0$, $\zeta \geq 0$, and λ_l, $l = 1, \ldots, N$, are the real and positive solutions of the following equation

$$\frac{\partial |H_i(j\lambda)|^2}{\partial \lambda^2} = 0, \tag{9}$$

where $i = 1, \ldots, 6$ corresponds to the six IDVAs in Fig. 2, respectively.

The underlying idea of the minmax problem (8) and (9) is, instead of using the fixed points to approximately minimize the H_∞ norm as done in the fixed-point method [1], here the resonance frequencies are directly used to exactly minimize the H_∞ norm. This is inspired by the method in [4], where the two resonance frequencies were employed to derive the exact solutions for the TDVA. Note that the solution set of (9), that is λ_l, $l = 1, \ldots, N$, contains the resonance frequencies, anti-resonance frequencies, and other frequencies where the curves horizontally pass through. Since the largest magnitude of the frequency response, representing the H_∞ norm of the transfer function, only occurs at resonance frequencies, it is sufficient to minimize $\max_{\lambda_l}\left(|H_i(j\lambda_l)|\right)$, $l = 1, \ldots, N$, to obtain the optimal H_∞ norm of the transfer function $H_i(s)$.

Equation (9) can be transformed into a polynomial function with respect to λ^2 as follows. From (7), $|H_i(j\lambda)|^2$ can be written as

$$|H_i(j\lambda)|^2 = \frac{n}{m},$$

where $n = R_{ni}^2 + I_{ni}^2$, $m = R_{mi}^2 + I_{mi}^2$. Since

$$\frac{\partial |H_i(j\lambda)|^2}{\partial \lambda^2} = \frac{n'm - m'n}{m^2},$$

where $n' = \frac{\partial n}{\partial \lambda^2}$ and $m' = \frac{\partial m}{\partial \lambda^2}$, (9) is equivalent to

$$n'm - m'n = 0, \tag{10}$$

which is an equation of λ^2 with different orders for different configurations.

Problem (8) and (10) is a constrained optimization problem, and the equality constraint (10) can be transformed into the objective function by employing $\lambda_l = f(\delta, \gamma, \eta, \zeta)$. In this paper, a direct search method is employed to solve the constrained optimization problem (8) and (10) by using the Matlab solver *patternsearch* with multiple starting points.

3.2. Comparison between the TDVA and IDVAs

For the TDVA, the optimal parameters can be analytically obtained as [1]:

$$\gamma_{opt} = \sqrt{\frac{1}{1+\mu}},\ \zeta_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)}},$$

and the optimal height at the two fixed points are $\sqrt{\frac{2+\mu}{\mu}}$.

3.2.1. Performance limitation of $C1$ and $C2$

In this subsection, it will be demonstrated that configurations $C1$ and $C2$ provide no improvement for the H_∞ performance compared with the TDVA.

For configuration $C1$, by directly using the fixed-point method in [1], the optimal parameters for $C1$ can be analytically obtained as

$$\gamma_{opt} = \frac{\sqrt{1+(1+\mu)\delta}}{1+\mu},\ \zeta_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)}},$$

and the optimal height at the two fixed points are $\sqrt{\frac{2+\mu+2\delta(1+\mu)}{\mu}}$. It is obvious that the optimal δ is 0, which means that the parallel inerter in configuration $C1$ provides no improvement in the H_∞ optimization. Such an observation is shown in Fig. 3 with $\mu = 0.1$.

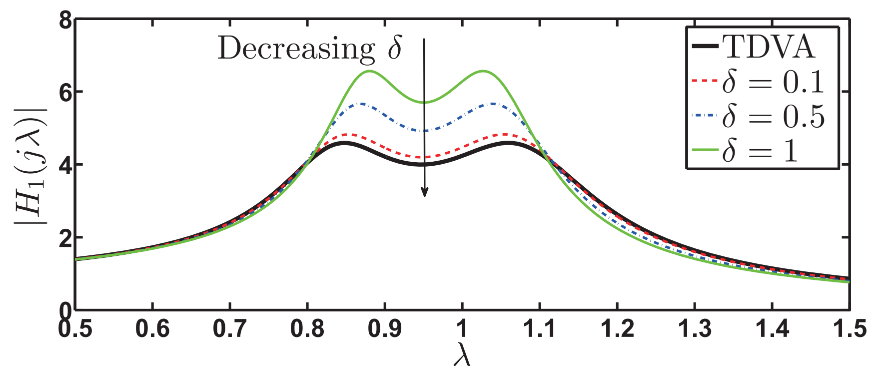


Figure 3: Comparison between the TDVA and $C1$ when $\mu = 0.1$ with different δ.

The minmax optimization method proposed in this paper is also applicable for $C1$ and a comparison between the method in this paper and the fixed-point method is shown in Fig. 4. As shown in Fig. 4, the results by these two methods highly coincide with each other and the results are consistent with the analytical solutions in [4, Table 2], which demonstrates the effectiveness of the method in this paper.

In what follows, it will be shown that for configuration $C2$, the series-connected inerter provides no improvement for the H_∞ performance as well. To show the influence of δ, the problem (8) is slightly modified as: for a given μ and δ,

$$\min_{\gamma,\zeta}\left(\max_{\lambda_l}\left(|H_2(j\lambda_l)|\right)\right),$$

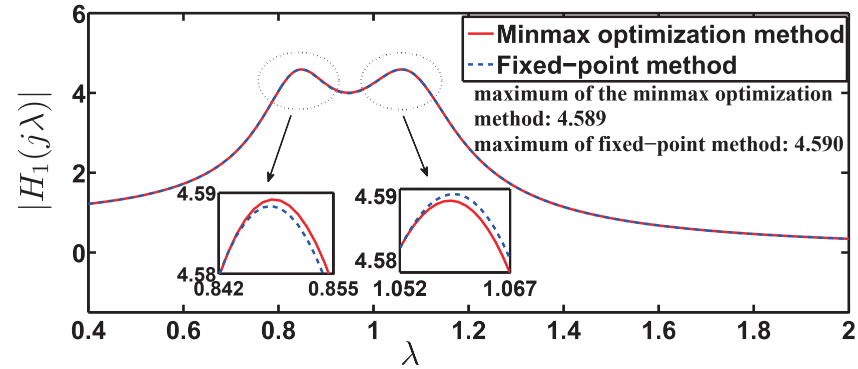


Figure 4: Comparison between the minmax optimization method in this paper and the fixed-point method when $\mu = 0.1$.

subject to $\gamma \geq 0$, $\eta \geq 0$, $\zeta \geq 0$, and λ_l, $l = 1, \ldots, N$, are the real and positive solutions of (10). Fig. 5 shows the comparison between $C2$ with different δ and the TDVA when $\mu = 0.1$, where it is clearly shown that the maximum of $|H_2(j\lambda)|$ is decreased by increasing δ and if δ is sufficiently large, the frequency response of $C2$ coincides with that of the TDVA. Such an observation is also confirmed by other choices of μ, as shown in Fig. 6. Therefore, it is sufficient to conclude that for a single series arrangement of an inerter and a damper, the series inerter provides no improvement for the H_∞ performance of the isolation system.

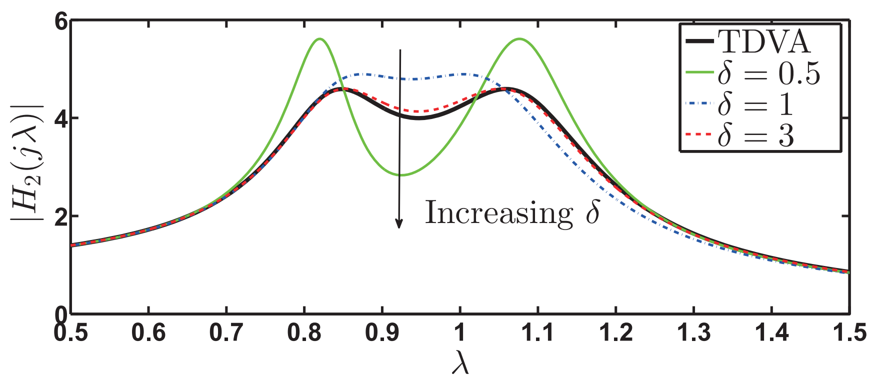


Figure 5: Comparison between the TDVA and $C2$ when $\mu = 0.1$ with different δ.

The IDVAs $C1$ and $C2$ represent the two ways of adding an inerter to the TDVA, that is, the parallel connection ($C1$) and the series connection ($C2$). Now, it has been demonstrated that adding a single inerter alone to the TDVA, no matter it is in parallel connection or in series connection, provides no improvement for the H_∞ performance. Therefore, other degrees of freedom should be introduced, which is the motivation of introducing IDVAs $C3$, $C4$, $C5$, and $C6$ by adding an inerter together with a spring to the TDVA.

3.2.2. Performance benefits of $C3$, $C4$, $C5$, and $C6$

In this subsection, it will be shown that after adding another degree of freedom, that is the spring k_1, the H_∞ performance will be significantly improved compared with the TDVA.

The optimization problem (8) with the constraint (10) is solved for configurations $C3$, $C4$, $C5$ and $C6$, separately, where a 9th-order polynomial of equation (10) with respect to

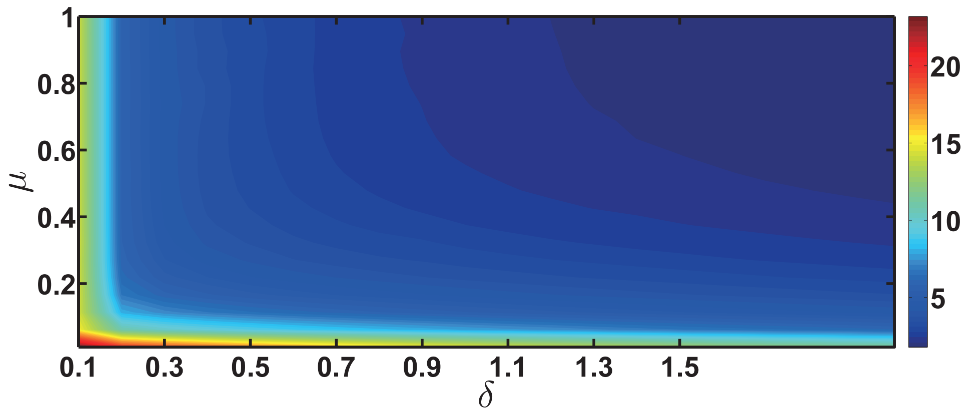


Figure 6: $\max(|H_2(j\lambda)|)$ with different μ and δ.

λ^2 is obtained. The exact solutions of the TDVA in [4] are employed for comparison and the detailed parameter values are shown in Tables 2, 3, and 4. Table 2 shows that all the IDVAs $C3$, $C4$, $C5$ and $C6$ can improve the H_∞ performance compared with the TDVA, where $C3$ performs the best and the order of the performance is $C3 > C6 > C4 > C5 > TDVA$ (">" means performing better) with an exception for $\mu >= 1$. However, since the mass ratio is normally quite small and practically less than 0.25 [42, 43], it is sufficient to conclude that $C3 > C6 > C4 > C5 > TDVA$. Such an conclusion is also confirmed by Fig. 7, where the comparison of the IDVAs over the TDVA in the range of $0 < \mu \leq 0.25$ is shown. As shown in the right figure of Fig. 7, 8% to 26% improvement can be obtained for the IDVAs. The other parameters in the range of $0 < \mu \leq 0.25$ are depicted in Fig. 8. It should be noted that although the optimal γ and ζ for $C3$ are almost identical to the TDVA, as shown in Table 3 and Fig. 8, over 22% improvement can be provided by $C3$ compared with the TDVA. Moreover, the spring k_1 is better to be in series connection for the H_∞ performance, given the fact that $C3$ and $C6$ are superior to $C4$ and $C5$.

The frequency responses of the IDVAs and the TDVA when $\mu = 0.1$ are shown in Fig. 9, where one sees that the magnitudes of the IDVAs around 1 are much flatter than those of the TDVA, and the effective frequency band is much larger than that of the TDVA.

Table 2: maximum magnitude $\max|H(j\lambda)|$ in the H_∞ optimization.

μ	TDVA [4]	$C3$	$C4$	$C5$	$C6$
0.01	14.1796	11.0330	11.0860	12.9216	11.0351
0.02	10.0530	7.8340	7.9064	9.1498	7.8352
0.05	6.4080	5.0159	5.1194	5.8051	5.0210
0.1	4.5892	3.6175	3.7448	4.1379	3.6208
0.2	3.3254	2.6552	2.7986	2.9877	2.6616
0.5	2.2480	1.8513	1.9941	2.0198	1.8521
1	1.7457	1.4893	1.6127	1.5809	1.4893
2	1.4279	1.2697	1.3629	1.3157	1.2697
5	1.1942	1.1166	1.1702	1.1766	1.1166
10	1.1033	1.0602	1.0918	1.0934	1.0603

Table 3: Optimal natural frequency ratio γ and damping ratio ζ in the H_∞ optimization.

(a) Optimal natural frequency ratio γ

μ	TDVA [4]	$C3$	$C4$	$C5$	$C6$
0.01	0.9902	0.9900	0.9957	0.9712	0.9842
0.02	0.9802	0.9802	0.9911	0.9493	0.9684
0.05	0.9520	0.9520	0.9766	0.9090	0.9242
0.1	0.9083	0.9083	0.9499	0.8501	0.8642
0.2	0.8319	0.8319	0.8931	0.7538	0.7693
0.5	0.6642	0.6643	0.7514	0.5681	0.5604
1	0.4973	0.4971	0.5882	0.4041	0.3979
2	0.3307	0.3302	0.4100	0.2547	0.2526
5	0.1646	0.1641	0.2145	0.2004	0.1197
10	0.0889	0.0893	0.1198	0.1118	0.0652

(b) Optimal damping ratio ζ

μ	TDVA [4]	$C3$	$C4$	$C5$	$C6$
0.01	0.0603	0.0547	0.0025	0.0655	0.0025
0.02	0.0841	0.0769	0.0065	0.0973	0.0073
0.05	0.1276	0.1199	0.0224	0.1477	0.0270
0.1	0.1686	0.1657	0.0505	0.2086	0.0593
0.2	0.2101	0.2244	0.0981	0.2919	0.1180
0.5	0.2402	0.3175	0.2012	0.4294	0.3047
1	0.2235	0.3894	0.2905	0.5359	0.4354
2	0.1749	0.4505	0.3779	0.6325	0.5498
5	0.1002	0.5057	0.4525	0.5163	0.6593
10	0.0581	0.5288	0.4804	0.5313	0.6841

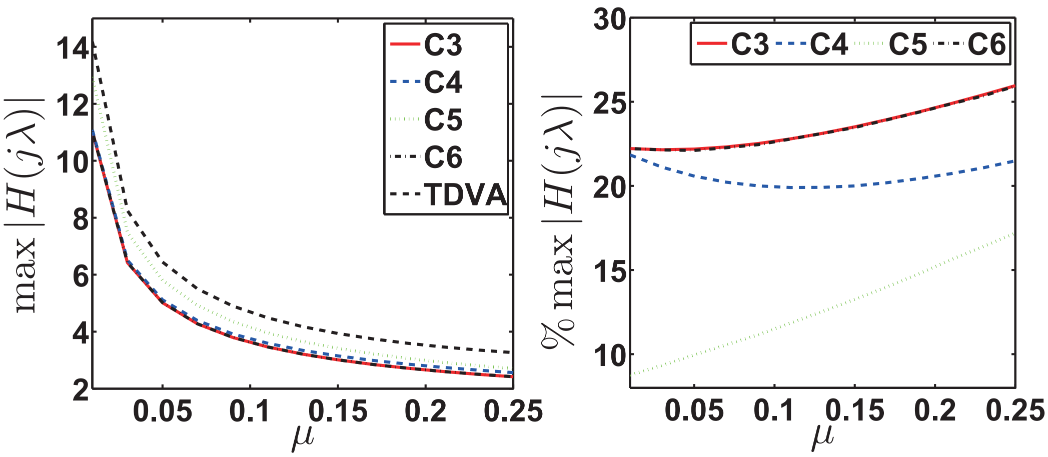


Figure 7: Maximum magnitude comparison between the IDVAs and the TDVA (left figure) and percentage improvement of the IDVAs with respect to the TDVA (right figure).

Table 4: Optimal inertance-to-mass ratio δ and corner frequency ratio η in the H_∞ optimization.

(a) Optimal inertance-to-mass ratio δ

μ	$C3$	$C4$	$C5$	$C6$
0.01	0.0238	0.0234	2.2791	0.0228
0.02	0.0473	0.0453	1.8105	0.0448
0.05	0.1156	0.1069	1.6782	0.0989
0.1	0.2208	0.1930	1.5320	0.1538
0.2	0.4082	0.3212	1.1521	0.2126
0.5	0.8256	0.5719	0.6919	0.2426
1	1.2552	0.7785	0.3130	0.2009
2	1.7228	0.9703	0.1423	0.1364
5	2.2540	1.1307	3.9018	0.0627
10	2.4989	1.2089	3.6257	0.0339

(b) Optimal corner frequency ratio η

μ	$C3$	$C4$	$C5$	$C6$
0.01	1.0051	0.9864	1.1242	1.0248
0.02	1.0098	0.9745	1.1982	1.0492
0.05	1.0248	0.9420	1.3341	1.1288
0.1	1.0485	0.9013	1.5181	1.2454
0.2	1.0940	0.8563	1.8754	1.4560
0.5	1.2219	0.7713	2.8856	2.2775
1	1.4061	0.7163	4.9686	3.5386
2	1.7178	0.6629	9.6074	6.0835
5	2.4169	0.6141	0.5009	14.5775
10	3.2632	0.5780	0.4739	27.6261

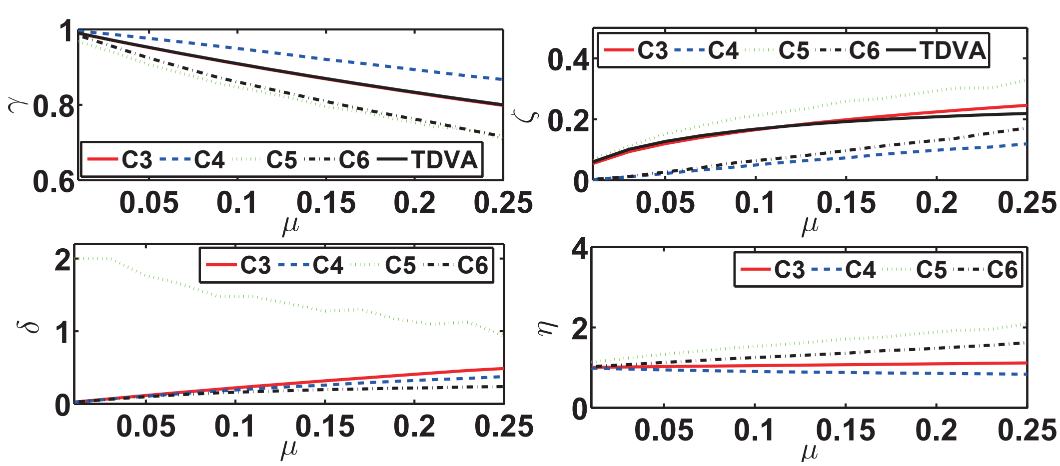


Figure 8: Optimal parameters in the H_∞ optimization: natural frequency ratio γ (up left); damping ratio ζ (up right); inertance-to-mass ratio δ (bottom left); corner frequency ratio η (bottom right).

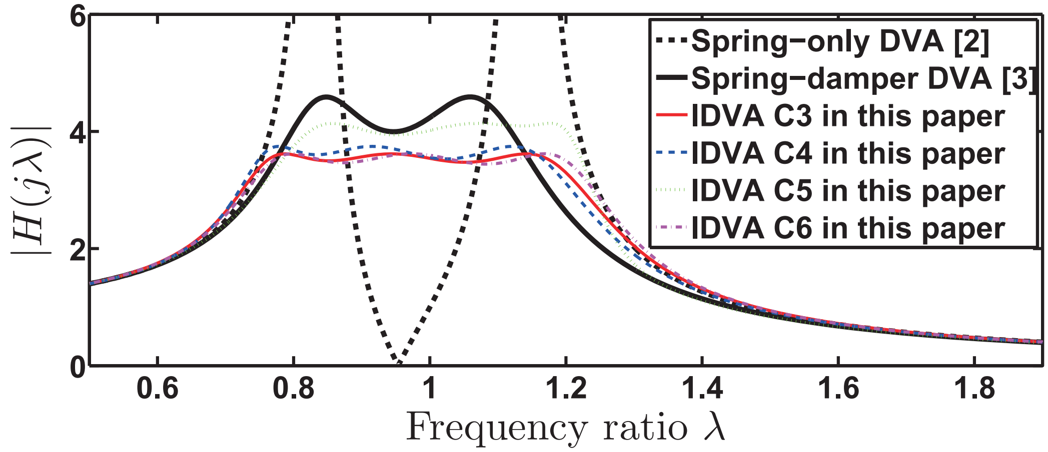


Figure 9: Comparison between the IDVAs and TDVA when $\mu = 0.1$. The spring-only DVA is the first DVA proposed by Frahm in 1909 [2]. The spring-damper DVA is the TDVA proposed by Ormondroyd and Den Hartog in 1928 [3].

4. H_2 optimization for the IDVAs

4.1. H_2 performance measure and its analytical solution

If the system is subjected to random excitation instead of sinusoidal excitation, the H_2 optimization would be more desirable than the H_∞ optimization [9, 39, 40]. The performance measure in the H_2 optimization is defined as [9, 39, 40]

$$I = \frac{E\left[x^2\right]}{2\pi S_0 \omega_n}, \tag{11}$$

where S_0 is the uniform power spectrum density function. The mean square value of x of the object mass m can be calculated as

$$E\left[x^2\right] = S_0 \int_{-\infty}^{\infty} |H(j\lambda)|^2 \, d_\omega = S_0 \omega_n \int_{-\infty}^{\infty} |H(j\lambda)|^2 \, d_\lambda, \tag{12}$$

where $H(j\lambda)$ is given in (7). Substituting (12) into (11), one obtains

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\lambda)|^2 \, d_\lambda, \tag{13}$$

which is exactly the definition of the H_2 norm of the transfer function $\hat{H}(s)$ by replacing $j\lambda$ in $H(j\lambda)$ with the Laplace variable s.

Therefore, the H_2 performance measure is rewritten as

$$I = \left\|\hat{H}(s)\right\|_2^2. \tag{14}$$

The analytical approach provided in [41, Chapter 2.6] will be employed to derive analytical solutions for IDVAs in the H_2 optimization, which is briefly presented as follows.

For a stable transfer function $\hat{H}(s)$, its H_2 norm can be calculated as [41, Chapter 2.6]

$$\|\hat{H}(s)\|_2^2 = \|C(sI - A)^{-1}B\|_2^2 = CLC^T,$$

where A, B, C are the minimal state-space realization $\hat{H}(s) = C(sI - A)^{-1}B$ and L is the unique solution of the Lyapunov equation

$$AL + LA^T + BB^T = 0. \tag{15}$$

We can write $\hat{H}(s)$

$$\hat{H}(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

in its controllable canonical form below

$$\dot{x} \;=\; Ax + Bu,\ y = Cx,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C = [b_0,\ b_1,\ b_2\ \ldots\ b_{n-1}].$$

4.2. Comparison between the TDVA and IDVAs

For the TDVA, the H_2 performance measure can be obtained as

$$I_{TDVA} = \frac{\gamma(1+\mu)\zeta}{\mu} + \frac{1 - (\mu+2)\gamma^2 + (1+\mu)^2\gamma^4}{4\mu\gamma\zeta}, \tag{16}$$

and the optimal γ and ζ are

$$\gamma_{TDVA,opt} \;=\; \sqrt{\frac{\mu+2}{2(1+\mu)^2}}, \tag{17}$$

$$\zeta_{TDVA,opt} \;=\; \sqrt{\frac{(3\mu+4)\mu}{8(\mu+1)(\mu+2)}}. \tag{18}$$

Substituting $\gamma_{TDVA,opt}$ and $\zeta_{TDVA,opt}$ into (16), one obtains the optimal $I_{TDVA,opt}$ as

$$I_{TDVA,opt} = \sqrt{\frac{3\mu+4}{4(\mu+1)\mu}}. \tag{19}$$

4.2.1. Performance limitation of $C1$ and $C2$

The H_2 performance measures for $C1$ and $C2$ can be obtained as

$$I_{C1} \;=\; \frac{\gamma(1+\mu)\zeta}{\mu} + \frac{1}{4\mu\gamma\zeta}\left(\delta^2 - 2((1+\mu)\gamma^2 - 1)\delta + 1 - (\mu+2)\gamma^2 + (1+\mu)^2\gamma^4\right) \tag{20}$$

$$\;=\; I_{TDVA} + \frac{1}{4\mu\gamma\zeta}\left(\delta^2 + a_{C1,1}\delta\right), \tag{21}$$

$$I_{C2} \;=\; \left(a_{C2,2}\delta^{-2} + a_{C2,1}\delta^{-1} + a_{C2,0}\right)\zeta + \frac{1 - (\mu+2)\gamma^2 + (1+\mu)^2\gamma^4}{4\mu\gamma\zeta} \tag{22}$$

$$\;=\; I_{TDVA} + \left(a_{C2,2}\delta^{-2} + a_{C2,1}\delta^{-1}\right)\zeta, \tag{23}$$

where

$$\begin{aligned} a_{C1,1} &= -2((1+\mu)\gamma^2 - 1), \\ a_{C2,2} &= \frac{\gamma}{\mu}\left((1+\mu)^3\gamma^4 - 2(1+\mu)\gamma^2 + 1\right), \\ a_{C2,1} &= \frac{\gamma}{\mu}\left(2+\mu-2(1+\mu)^2\gamma^2\right), \\ a_{C2,0} &= \frac{\gamma(1+\mu)}{\mu}. \end{aligned}$$

The following proposition can be obtained.

Proposition 1. *For the H_2 performance, $C1$ performs no better than the TDVA.*

Proof. See Appendix B. □

Proposition 2. *For the H_2 performance, $C2$ performs slightly better than the TDVA, but only at most 0.32% improvement can be achieved when $\mu \leq 1$.*

Proof. See Appendix C. □

Now, we have demonstrated that for the H_2 performance, $C1$ performs no better than the TDVA and $C2$ provides negligible improvement over the TDVA. This means that adding an inerter alone to the TDVA provides limited improvement for the H_2 performance, and therefore, another four IDVAs $C3$, $C4$, $C5$, and $C6$ are proposed by adding an inerter together with a spring to the TDVA. It will be shown in the following sections that in this way, the H_2 performance can be significantly improved.

4.2.2. Performance benefits of $C3$, $C4$, $C5$, and $C6$

In this subsection, it will be analytically demonstrated that for the H_2 performance, IDVAs $C3$, $C4$, $C5$, and $C6$ perform surely better than the TDVA, and an optimization problem will be formulated to find the optimal parameters.

By using the method shown in Subsection 4.1, the analytical representations of the H_2 performance measures for $C3$, $C4$, $C5$, and $C6$ are calculated and the detailed equations are shown in Appendix D. Denote the optimal H_2 performances of $C3$, $C4$, $C5$, and $C6$ as $I_{C3,opt}$, $I_{C4,opt}$, $I_{C5,opt}$, $I_{C6,opt}$, respectively. The following proposition can be obtained.

Proposition 3. *For the H_2 performance, IDVAs $C3$ and $C5$ always perform better than the TDVA, that is, the following inequalities hold:*

$$I_{C3,opt} < I_{TDVA,opt}, \tag{24}$$

$$I_{C5,opt} < I_{TDVA,opt}, \tag{25}$$

and if $\mu \leq 1$, IDVAs $C4$ and $C6$ always perform better than the TDVA, that is, the following inequalities hold:

$$I_{C4,opt} < I_{TDVA,opt}, \tag{26}$$

$$I_{C6,opt} < I_{TDVA,opt}, \tag{27}$$

where $I_{TDVA,opt}$ is the optimal H_2 performance for the TDVA given by (19).

Proof. See Appendix E. □

Remark 3. *The condition $\mu \leq 1$ for $C4$ and $C6$ in Proposition 3 is only a sufficient condition, which means that for the case $\mu > 1$, it is also possible that the inequalities (26) and (27) hold. However, such a condition introduces no conservativeness for DVA applications, as the mass ratio μ is normally less than 1 in practice (typically less than 0.25) [42, 43].*

Since the IDVAs $C3$, $C4$, $C5$, $C6$ can always reduce to the TDVA by setting the spring stiffness k_1 (or η) and inertance b (or δ) to 0 or ∞, the conclusions $I_{Ci,opt} \leq I_{TDVA,opt}$, $i = 3, 4, 5, 6$ always hold. However, Proposition 3 demonstrates the existence of finite η and δ such that the IDVAs $C3$, $C4$, $C5$, and $C6$ are surely better than the TDVA.

To determine the optimal values of δ, γ, η, and ζ, the following optimization problem should be solved.

$$\min_{\delta,\gamma,\eta,\zeta} I_{Ci}, i = 3, 4, 5, 6, \tag{28}$$

subject to $\delta > 0$, $\gamma > 0$, $\eta > 0$, and $\zeta > 0$.

Analytical solutions of $C3$: Problem (28) can be analytically solved for $C3$, where the optimal parameters for $C3$ are obtained as follows

$$\gamma_{C3,opt} = \sqrt{\frac{\sqrt{17\mu^2 + 32\mu + 16} - \mu}{4(1+\mu)^2}}, \tag{29}$$

$$\eta_{C3,opt} = \sqrt{\frac{1 - 2(1+\mu)\gamma_{C3,opt}^2 + (1+\mu)\gamma_{C3,opt}^4}{(1 - (2+3\mu)\gamma_{C3,opt}^2 + (1+\mu)^2\gamma_{C3,opt}64)\gamma_{C3,opt}^2}}, \tag{30}$$

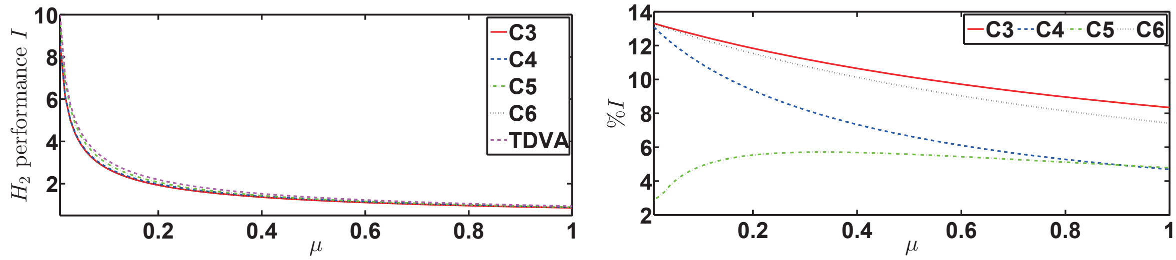
$$\delta_{C3,opt} = -\frac{2\hat{a}_{C3,2}}{\hat{a}_{C3,1}}, \tag{31}$$

$$\zeta_{C3,opt} = \sqrt{\frac{1 - (\mu+2)\gamma_{C3,opt}^2 + (1+\mu)^2\gamma_{C3,opt}^4}{4\mu\gamma_{C3,opt}(\hat{a}_{C3,2}\delta_{C3,opt}^{-2} + \hat{a}_{C3,1}\delta_{C3,opt}^{-1} + \hat{a}_{C3,0})}}, \tag{32}$$

where $\hat{a}_{C3,2}$, $\hat{a}_{C3,1}$, and $\hat{a}_{C3,0}$ are obtained by setting $\gamma = \gamma_{C3,opt}$ and $\eta = \eta_{C3,opt}$ for $a_{C3,2}$, $a_{C3,1}$, and $a_{C3,0}$, respectively. For the representations of $a_{C3,2}$, $a_{C3,1}$, and $a_{C3,0}$, see Appendix D.

The analytical solutions δ, γ, and η are derived by successively setting the first derivatives of I_{C3} with respect to δ, η, and γ as 0, and then checking the sign of the second derivatives at stationary points. The optimal $\zeta_{C3,opt}$ is derived due to the fact that both parts on the right hand side of (D.1) of I_{C3} are positive.

Solutions of $C4$, $C5$, and $C6$: The analytical solutions of $C4$, $C5$, and $C6$ cannot be obtained due to the high order equations (more than 4th-order) involved in the derivation. However, the optimal solutions of η and ζ can be analytically represented with respect to δ



(a) (b)

Figure 10: Comparison between IDVAs and the TDVA. (a) the H_2 performance; (b) Percentage improvement of IDVAs with respect to the TDVA.

and γ as follows:

$$\eta_{C4,opt} = \frac{\sqrt{-(g_{C4,1}\delta + f_{C4,1})(2f_{C4,2} + 2g_{C4,2}\delta + 2l_{C4,2}\delta^2)}}{2(f_{C4,2} + g_{C4,2}\delta + l_{C4,2}\delta^2)}, \tag{33}$$

$$\zeta_{C4,opt} = \sqrt{\frac{l_{C4,2}\eta^4\delta^2 + l_{C4,1}\delta + l_{C4,0}}{a_{C4,2}\delta^{-2} + a_{C4,1}\delta^{-1} + a_{C4,0}}}, \tag{34}$$

$$\delta_{C5,opt} = -\frac{2a_{C5,2}}{a_{C5,1}}, \tag{35}$$

$$\zeta_{C5,opt} = \sqrt{\frac{1 - (\mu + 2)\gamma^2 + (1 + \mu)^2\gamma^4}{4\mu\gamma(a_{C5,2}\delta_{C5,opt}^{-2} + a_{C5,1}\delta_{C5,opt}^{-1} + a_{C5,0})}}, \tag{36}$$

$$\zeta_{C6,opt} = \sqrt{\frac{l_{C6,2}\eta^4\delta^2 + l_{C6,1}\delta + l_{C6,0}}{a_{C6,2}\delta^{-2} + a_{C6,1}\delta^{-1} + a_{C6,0}}}. \tag{37}$$

Correspondingly substituting the optimal representations above into I_{Ci}, $i = 4, 5, 6$, the problem (28) for Ci, $i = 4, 5, 6$ reduces to a nonlinear programming problem with two unknown variables δ and γ for $C4$ and $C5$, and with three unknown variables δ, γ and η for $C6$, which can be efficiently solved by using the Matlab solver *fmincon* and *GlobalSearch* in Global Optimization Toolbox.

Fig. 10 and Fig. 11 depict the comparison between IDVAs $C3$, $C4$, $C5$, $C6$ and the TDVA when $0 \leq \mu \leq 1$. As shown in Fig. 10(b), $C3$ performs the best, and more than 10% improvement with respect to the TDVA can be obtained by $C3$, $C4$ and $C6$. Similar to the H_∞ performance, the spring k_1 is better to be in series connection for the H_2 performance, given the fact that $C3$ and $C6$ are superior to $C4$ and $C5$.

5. Conclusions

In this paper, the performance of inerter-based dynamic vibration absorbers (IDVAs) has been investigated, where the proposed IDVAs were a parallel arrangement of a spring and an inerter-based mechanical network. Both H_∞ and H_2 performances were considered. The H_∞ performance optimization problem was formulated in a minmax framework and solved by using a direct search optimization method; while in the H_2 optimization, an analytical method

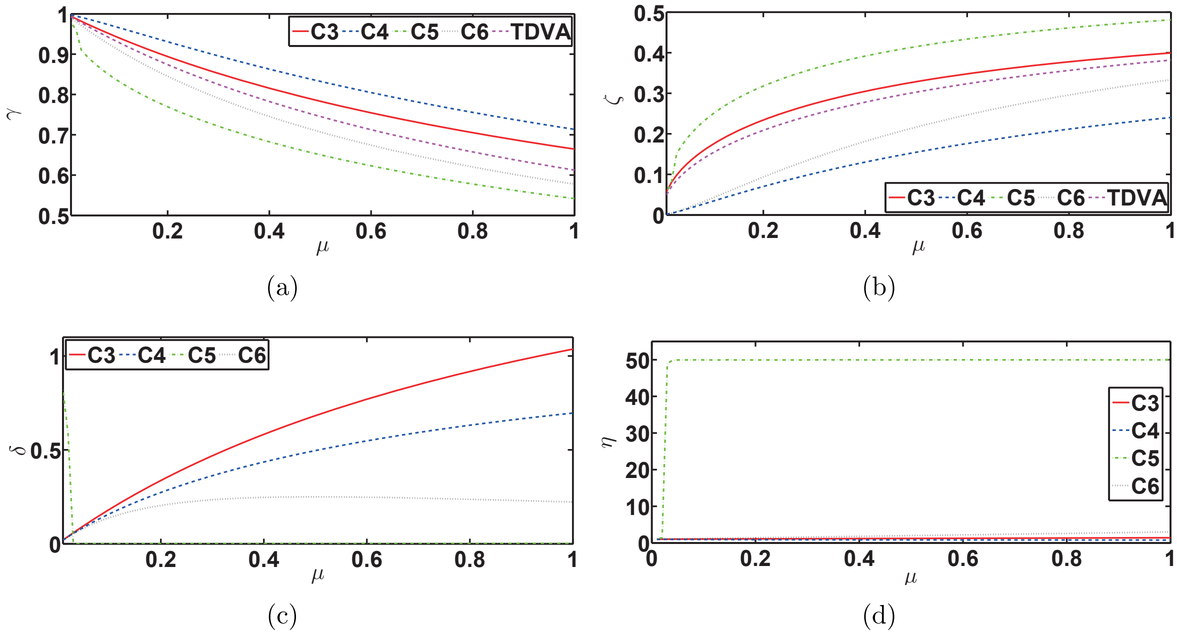


Figure 11: Optimal parameters: (a) optimal γ; (b) optimal ζ; (c) optimal δ; (d) optimal η.

was employed to calculate the H_2 performance measures. Comparisons between the proposed IDVAs and the traditional dynamic vibration absorber (TDVA) were conducted. The results showed that adding one inerter alone to the TDVA, no matter it is in parallel connection ($C1$) or in series connection ($C2$), provided no improvement for the H_∞ performance, and negligible improvement (less than 0.32% improvement over the TDVA when the mass ratio less than 1) for the H_2 performance. This demonstrated the necessity of introducing another degree of freedom together with the inerter to the TDVA, and then the IDVAs $C3$, $C4$, $C5$, and $C6$ were proposed by adding an inerter together with a spring to the TDVA. Significant improvement was obtained by IDVAs $C3$, $C4$, $C5$, and $C6$. For the H_∞ performance, numerical simulations showed that over 20% improvement was achieved compared with the TDVA and the effective frequency band can be enlarged by using inerter; while for the H_2 performance, it was analytically demonstrated that IDVAs $C3$, $C4$, $C5$, and $C6$ were surely better than the TDVA by carefully choosing the parameters, and over 10% improvement was obtained in the numerical simulation.

Appendix A. Detailed representations of R_{ni}, I_{ni}, R_{mi}, and I_{mi}, $i = 1, \ldots, 6$.

$$
\begin{aligned}
R_{n1} &= \lambda^2 - \gamma^2 + \delta\lambda^2, \\
I_{n1} &= -2\lambda\gamma\zeta, \\
R_{m1} &= (-\mu\delta - \delta - 1)\lambda^4 + (\gamma^2 + \mu\gamma^2 + 1 + \delta)\lambda^2 - \gamma^2, \\
I_{m1} &= 2\lambda\gamma\zeta(\lambda^2 - 1 + \mu\lambda^2), \\
R_{n2} &= \delta\lambda(\gamma^2 - \lambda^2), \\
I_{n2} &= -2\gamma\zeta(\gamma^2 - (1+\delta)\lambda^2), \\
R_{m2} &= \delta\lambda(\lambda^4 - (\gamma^2 + \mu\gamma^2 + 1)\lambda^2 + \gamma^2), \\
I_{m2} &= -2\gamma\zeta((1 + \delta + \mu\delta)\lambda^4 - (\gamma^2 + \mu\gamma^2 + 1 + \delta)\lambda^2 + \gamma^2), \\
R_{n3} &= \delta\eta^2\gamma\lambda(\gamma^2 - \lambda^2), \\
I_{n3} &= -2\zeta(\gamma^4\eta^2 - (1 + \delta\eta^2 + \eta^2)\lambda^2\gamma^2 + \lambda^4), \\
R_{m3} &= \delta\eta^2\gamma\lambda(\lambda^4 - (1 + \gamma^2 + \mu\gamma^2)\lambda^2 + \gamma^2), \\
I_{m3} &= 2\zeta(\lambda^6 - (1 + \mu + \eta^2 + \delta\eta^2 + \mu\delta\eta^2)\lambda^4 + ((\mu+1)\eta^2\gamma^2 + 1 + \eta^2 + \delta\eta^2)\gamma^2\lambda^2 - \gamma^4\eta^2, \\
R_{n4} &= -\delta(\lambda^4 - (1 + \eta^2 + \delta\eta^2)\gamma^2\lambda^2 + \gamma^4\eta^2), \\
I_{n4} &= -2\gamma\lambda\zeta(\gamma^2 - \lambda^2 - \delta\lambda^2), \\
R_{m4} &= \delta(\lambda^6 - (1 + (1 + \mu + \eta^2 + \delta\eta^2 + \delta\mu\eta^2)\gamma^2)\lambda^4 + ((\mu+1)\eta^2\gamma^2 + (1 + \eta^2 + \delta\eta^2))\gamma^2\eta^2 - \gamma^4\eta^2), \\
I_{m4} &= -2\gamma\lambda\zeta((1 + \delta + \mu\delta)\lambda^4 - (1 + \delta + \gamma^2 + \mu\gamma^2)\lambda^2 + \gamma^2), \\
R_{n5} &= \delta(\gamma^2 - \lambda^2)(\lambda^2 - \eta^2\gamma^2), \\
I_{n5} &= -2\gamma\lambda\zeta((1 + \delta\eta^2)\gamma^2 - (1+\delta)\lambda^2), \\
R_{m5} &= \delta(\lambda^2 - \eta^2\gamma^2)(\lambda^4 - (1 + \gamma^2 + \mu\gamma^2)\lambda^2 + \gamma^2), \\
I_{m5} &= -2\gamma\lambda\zeta((1 + \delta + \mu\delta)\lambda^4 - ((1 + \mu + \delta\eta^2 + \mu\delta\eta^2)\gamma^2 + 1 + \delta)\lambda^2 + (1 + \delta\eta^2)\gamma^2), \\
R_{n6} &= -\delta(\lambda^4 - (1 + \eta^2 + \delta\eta^2)\gamma^2\lambda^2 + \gamma^4\eta^2), \\
I_{n6} &= 2\lambda\gamma\zeta(\lambda^2 - (1 + \delta\eta^2)\gamma^2), \\
R_{m6} &= \delta(\lambda^6 - (1 + (1 + \mu + \eta^2 + \delta\eta^2 + \mu\delta\eta^2))\lambda^4 + ((\mu+1)\eta^2\gamma^2 + (1 + \eta^2 + \delta\eta^2))\gamma^2\lambda^2 - \gamma^4\eta^2), \\
I_{m6} &= -2\gamma\lambda\zeta(\lambda^4 - (1 + (1 + \mu + \delta\eta^2 + \mu\delta\eta^2)\gamma^2)\lambda^2 + (1 + \delta\eta^2)\gamma^2).
\end{aligned}
$$

Appendix B. Proof of Proposition 1

From (21), if $C1$ performs better than the TDVA, that is $I_{C1} < I_{TDVA}$, the second term of (21) must be less than 0, which means

$$\delta^2 + a_{C1,1}\delta < 0.$$

Since $\delta \geq 0$, if $\gamma^2 < \frac{1}{1+\mu}$, the optimal δ denoted as δ_{opt} is 0. If $\gamma^2 \geq \frac{1}{1+\mu}$, the optimal $\delta_{opt} = (1+\mu)\gamma^2 - 1$, and it can be checked that the optimal γ is $\frac{1}{1+\mu}$ by substituting δ_{opt} into (21), which means that the optimal δ is also 0.

Appendix C. Proof of Proposition 2

First, we prove that $C2$ performs better than the TDVA, that is $I_{C2,opt} < I_{TDVA,opt}$, where $I_{C2,opt}$ denotes the optimal I_{C2}. From (23), if $C2$ performs better than the TDVA, the following inequality must hold

$$a_{C2,2}\delta^{-2} + a_{C2,1}\delta^{-1} < 0,$$

which requires that

$$a_{C2,1} < 0 \text{ or } \gamma^2 > \frac{2+\mu}{2(1+\mu)^2},$$

as $a_{C2,2} \geq 0$ for any $\gamma \geq 0$. If $\gamma^2 > \frac{2+\mu}{2(1+\mu)^2}$, the optimal δ^{-1} is

$$\delta_{opt}^{-1} = -\frac{a_{C2,1}}{2a_{C2,2}},$$

and I_{C2} can be represented as

$$I_{C2} = \sqrt{\frac{(1-(2+\mu)\gamma^2 + (1+\mu)^2\gamma^4)(4(1+\mu)^2\gamma^2 - \mu)}{4\mu(1-2(1+\mu)\gamma^2 + (1+\mu)^3\gamma^4)}}. \tag{C.1}$$

Using $I_{TDVA,opt}$ given in (16), one obtains

$$I_{C2}^2 - I_{TDVA,opt}^2 = \frac{((\mu+1)\gamma^2 - 1)(2(\mu+1)^2\gamma^2 - 2 - \mu)^2}{4\mu(1-2(\mu+1)\gamma^2 + (\mu+1)^3\gamma^4)(\mu+1)},$$

Clearly, if $\gamma^2 < \frac{1}{1+\mu}$, then $I_{C2} < I_{TDVA,opt}$. Since $\frac{1}{1+\mu} > \frac{2+\mu}{2(1+\mu)^2}$, one can always find a γ such that $I_{C2} < I_{TDVA,opt}$. Since $I_{C2,opt} \leq I_{C2}$, one obtains $I_{C2,opt} < I_{TDVA,opt}$.

Second, we graphically prove that only at most 0.32% improvement can be obtained by $C2$ when $\mu \leq 1$. The optimal γ can be obtained by solving $\frac{\partial I_{C2}^2}{\partial \gamma^2} = 0$, which is equivalent to

$$(2\alpha^2\gamma^2 - 1 - \alpha)(2\alpha^5\gamma^6 + (\alpha^4 - 7\alpha^3)\gamma^4 + (8\alpha^2 - 2\alpha^3)\gamma^2 - 3\alpha + 1) = 0, \tag{C.2}$$

where $\alpha = \mu + 1$. It is easy to check that (C.2) has two real positive solutions denoted as γ_1 and γ_2, $\gamma_1 < \gamma_2$, where

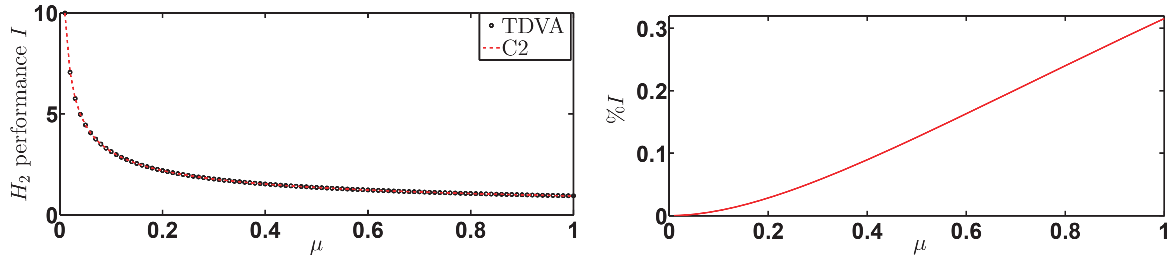
$$\gamma_1 = \sqrt{\frac{1+\alpha}{2\alpha^2}},$$

and $\gamma_1 < \gamma_2 < \sqrt{2}\gamma_1$. Also, γ_2^2 is the unique real solution of equation

$$2\alpha^5\gamma^6 + (\alpha^4 - 7\alpha^3)\gamma^4 + (8\alpha^2 - 2\alpha^3)\gamma^2 - 3\alpha + 1 = 0,$$

and the optimal γ is γ_2.

For $0 \leq \mu \leq 1$, a graphical comparison with the TDVA is shown in Fig. C.12, where it is clearly shown that at most 0.32% improvement is obtained for $C2$.



(a) (b)

Figure C.12: Comparison between $C2$ and TDVA when $0 \leq \mu \leq 1$. (a) the H_2 performance; (b) Percentage improvement of $C2$ with respect to TDVA.

Appendix D. Analytical representations of the H_2 performance measures for $C3$, $C4$, $C5$, and $C6$

Denote I_{C3}, I_{C4}, I_{C5}, and I_{C6} as the H_2 performance measures for $C3$, $C4$, $C5$, and $C6$, respectively. The detailed representations are obtained as follows:

$$\begin{aligned}
I_{C3} &= \left(a_{C3,2}\delta^{-2} + a_{C3,1}\delta^{-1} + a_{C3,0}\right)\zeta + \frac{1 - (\mu+2)\gamma^2 + (1+\mu)^2\gamma^4}{4\gamma\mu\zeta} \\
&= I_{TDVA} + \left(a_{C3,2}\delta^{-2} + a_{C3,1}\delta^{-1}\right)\zeta, \qquad \text{(D.1)} \\
I_{C4} &= \left(a_{C4,2}\delta^{-2} + a_{C4,1}\delta^{-1} + a_{C4,0}\right)\zeta + \left(l_{C4,2}\eta^4\delta^2 + l_{C4,1}\delta + l_{C4,0}\right)\frac{1}{\zeta} \\
&= I_{TDVA} + \left(a_{C4,2}\delta^{-2} + a_{C4,1}\delta^{-1}\right)\zeta + \left(l_{C4,2}\eta^4\delta^2 + l_{C4,1}\delta + f_{C4,2}\eta^4 + f_{C4,1}\eta^2\right)\frac{1}{\zeta}, \\
I_{C5} &= \left(a_{C5,2}\delta^{-2} + a_{C5,1}\delta^{-1} + a_{C5,0}\right)\zeta + \frac{1}{4\gamma\mu\zeta}\left(1 - (\mu+2)\gamma^2 + (1+\mu)^2\gamma^4\right) \\
&= I_{TDVA} + \left(a_{C5,2}\delta^{-2} + a_{C5,1}\delta^{-1}\right)\zeta, \qquad \text{(D.2)} \\
I_{C6} &= \left(a_{C6,2}\delta^{-2}\eta^{-4} + a_{C6,1}\delta^{-1}\eta^{-2} + a_{C6,0}\right)\zeta + \left(l_{C6,2}\delta^2 + l_{C6,1}\delta + l_{C6,0}\right)\frac{1}{\zeta} \\
&= I_{TDVA} + \left(a_{C6,2}\delta^{-2}\eta^{-4} + a_{C6,1}\delta^{-1}\eta^{-2}\right)\zeta + \left(l_{C6,2}\delta^2 + l_{C6,1}\delta + f_{C6,2}\eta^{-4} + f_{C6,1}\eta^{-2}\right)\frac{1}{\zeta},
\end{aligned}$$

315 where

$$
\begin{aligned}
a_{C3,2} &= d_{C3,2}\eta^{-4} + d_{C3,1}\eta^{-2} + d_{C3,0},\ a_{C3,1} = g_{C3,1}\eta^{-2} + g_{C3,0},\ a_{C3,0} = \frac{\gamma(1+\mu)}{\mu},\\
d_{C3,2} &= \frac{1}{\gamma^3\mu}\left(1 - 2\gamma^2 + (1+\mu)\gamma^4\right),\ d_{C3,1} = -\frac{2}{\gamma\mu}\left(1 - (2+\mu)\gamma^2 + (1+\mu)^2\gamma^4\right),\\
d_{C3,0} &= \frac{\gamma}{\mu}\left(1 - 2(1+\mu)\gamma^2 + (1+\mu)^3\gamma^4\right),\ g_{C3,1} = -\frac{2}{\mu\gamma}\left(1 - (1+\mu)\gamma^2\right),\\
g_{C3,0} &= -\frac{\gamma}{\mu}\left(2(1+\mu)^2\gamma^2 - 2 - \mu\right),\\
a_{C4,2} &= \frac{\gamma}{\mu}\left(1 - (2+\mu)\gamma^2 + (1+\mu)^3\gamma^4\right),\ a_{C4,1} = \frac{\gamma}{\mu}\left(2 + \mu - 2(1+\mu)^2\gamma^2\right),\ a_{C4,0} = \frac{\gamma(1+\mu)}{\mu},\\
l_{C4,2} &= \frac{\gamma^3(1+\mu)^2}{4\mu},\ l_{C4,1} = g_{C4,2}\eta^4 + g_{C4,1}\eta^2,\ l_{C4,0} = f_{C4,2}\eta^4 + f_{C4,1}\eta^2 + f_{C4,0},\\
g_{C4,2} &= \frac{\gamma^3}{2\mu}\left(1 + \mu - (1+\mu)^3\gamma^2\right),\ g_{C4,1} = \frac{\gamma}{4\mu}\left(2(1+\mu)^2\gamma^2 - \mu - 2\right),\\
f_{C4,2} &= \frac{\gamma^3}{4\mu}\left((1+\mu)^4\gamma^4 + (\mu-2)(\mu+1)^2\gamma^2 + 1\right),\ f_{C4,1} = -\frac{\gamma}{2\mu}\left((1+\mu)^3\gamma^4 - 2(1+\mu)\gamma^2 + 1\right),\\
f_{C4,0} &= \frac{1}{4\mu\gamma}\left(1 - (\mu+2)\gamma^2 + (1+\mu)^2\gamma^4\right),\\
a_{C5,2} &= \frac{g_{C5,2}\eta^4 + g_{C5,1}\eta^2 + g_{C5,0}}{\mu(1 + f_{C5,1}\eta^2 + f_{C5,2}\eta^4)^2},\ a_{C5,1} = \frac{l_{C5,3}\eta^6 + l_{C5,2}\eta^4 + l_{C5,1}\eta^2 + l_{C5,0}}{\mu(1 + f_{C5,1}\eta^2 + f_{C5,2}\eta^4)^2},\ a_{C5,0} = \frac{\gamma(1+\mu)}{\mu},\\
g_{C5,2} &= \gamma\left((1+\mu)\gamma^4 - 2\gamma^2 + 1\right),\ g_{C5,1} = -2\gamma\left((1+\mu)^2\gamma^4 - (\mu+2)\gamma^2 + 1\right),\\
g_{C5,0} &= \gamma\left((1+\mu)^3\gamma^4 - 2(1+\mu)\gamma^2 + 1\right),\ f_{C5,1} = -(1 + \gamma^2(1+\mu)),\ f_{C5,2} = \gamma^2,\\
l_{C5,3} &= 2\gamma^3((1+\mu)^3 - 1),\ l_{C5,2} = -\gamma\left(4(1+\mu)^2\gamma^4 - 2\gamma^2 - \mu - 2\right),\\
l_{C5,1} &= 2\gamma\left((1+\mu)^3\gamma^4 + (1+\mu)^2\gamma^2 - \mu - 2\right),\ l_{C5,0} = \gamma\left(\mu + 2 - 2(1+\mu)^2\gamma^2\right),\\
a_{C6,2} &= \frac{1 - 2\gamma^2 + (1+\mu)\gamma^4}{\gamma^3\mu},\ a_{C6,1} = \frac{2((1+\mu)\gamma^2 - 1)}{\gamma\mu},\ a_{C6,0} = \frac{\gamma(1+\mu)}{\mu},\\
l_{C6,2} &= \frac{1}{4\gamma\mu},\ l_{C6,1} = g_{C6,1}\eta^{-2} + g_{C6,0},\ l_{C6,0} = f_{C6,2}\eta^{-4} + f_{C6,1}\eta^{-2} + f_{C6,0},\\
g_{C6,1} &= \frac{\mu - 2 + 2\gamma^2}{4\gamma^3\mu},\ g_{C6,0} = \frac{1 - (1+\mu)\gamma^2}{2\gamma\mu},\\
f_{C6,2} &= \frac{1 + (\mu-2)\gamma^2 + \gamma^4}{4\mu\gamma^5},\ f_{C6,1} = -\frac{1 - 2\gamma^2 + (1+\mu)\gamma^4}{2\mu\gamma^3},\ f_{C6,0} = \frac{1 - (2+\mu)\gamma^2 + (1+\mu)^2\gamma^4}{4\gamma\mu}.
\end{aligned}
$$

316 **Appendix E. Proof of Proposition 3**

317 For $C3$, substituting $\gamma_{TDVA,opt}$ and $\zeta_{TDVA,opt}$ into (D.1), one obtains

$$I'_{C3} = I_{TDVA,opt} + \left(a'_{C3,2}\delta^{-2} + a'_{C3,1}\delta^{-1}\right)\zeta_{TDVA,opt},$$

where $a'_{C3,2}$ and $a'_{C3,1}$ are obtained by setting $\gamma = \gamma_{TDVA,opt}$ for $a_{C3,2}$ and $a_{C3,1}$, respectively. It can be checked that $a'_{C3,2} > 0$ and

$$a'_{C3,1} = -\sqrt{\frac{2}{2+\mu}}\eta^{-2} < 0,$$

which means that there exist finite δ and η such that $I'_{C3} < I_{TDVA,opt}$. Since $I_{C3,opt} \leq I'_{C3}$, then one obtains $I_{C3,opt} < I_{TDVA,opt}$.

For $C4$, denote

$$I'_{C4} = 2\sqrt{\left(a'_{C4,2}\delta^{-2} + a'_{C4,1}\delta^{-1} + a'_{C4,0}\right)\left(l'_{C4,2}\eta^4\delta^2 + l'_{C4,1}\delta + l'_{C4,0}\right)},$$

where $a'_{C4,2}$, $a'_{C4,1}$, $a'_{C4,0}$, $l'_{C4,2}$, $l'_{C4,2}$, and $l'_{C4,0}$ are obtained by setting $\gamma = \gamma_{TDVA,opt}$.

Expanding I'_{C4}, one obtains

$$I'_{C4} = 2\sqrt{a'_{C4,0}f'_{C4,0} + f_{C4,\eta}}, \tag{E.1}$$

where

$$f_{C4,\eta} = \left(l'_{C4,2}\delta^2 + g'_{C4,2}\delta + f'_{C4,2}\right)(a'_{C4,2}\delta^{-2} + a'_{C4,0})\eta^4 + f'_{C4,1}(a'_{C4,2}\delta^{-2} + a'_{C4,0})\eta^2 + f'_{C4,0}a'_{C4,2}\delta^{-2}.$$

Note that

$$I_{TDVA,opt} = 2\sqrt{a'_{C4,0}f'_{C4,0}}.$$

Then, we will prove that there exist finite δ and η so that $f_{C4,\eta} < 0$. It can be checked that $l'_{C4,2}\delta^2 + g'_{C4,2}\delta + f'_{C4,2} > 0$, $a'_{C4,2}\delta^{-2} + a'_{C4,0} > 0$, and $f'_{C4,1}(a'_{C4,2}\delta^{-2} + a'_{C4,0}) < 0$. The discriminant of $f_{C4,\eta} = 0$ is

$$\Delta = (a'_{C4,2}\delta^2 + a'_{C4,0})\Big(({f'_{C4,1}}^2 - 4f'_{C4,2}f'_{C4,0})a'_{C4,2}\delta^{-2} - 4g'_{C4,2}f'_{C4,0}a'_{C4,2}\delta^{-1} + {f'_{C4,1}}^2 a'_{C4,0} - 4l'_{C4,2}f'_{C4,0}a'_{C4,2}\Big).$$

It can be checked that if $\mu < \frac{8\sqrt{2}-4}{7} \approx 1.045$, there exists a finite δ such that the second term of Δ is positive, which means that if $\mu < 1.045$, there exists a finite η such that $f_{C4,\eta} < 0$. For example, if choosing

$$\delta^{-1} = \frac{2g'_{C4,2}f'_{C4,0}}{{f'_{C4,1}}^2 - 4f'_{C4,2}f'_{C4,0}} = \frac{(3\mu+4)(1+\mu)}{4\mu(\mu+2)}, \tag{E.2}$$

and

$$\eta = \sqrt{\frac{-f'_{C4,1}}{l'_{C4,2}\delta^2 + g'_{C4,2}\delta + f'_{C4,2}}} = \sqrt{\frac{2(3\mu+4)^2(1+\mu)(4+\mu)}{(\mu+2)(43\mu^3 + 204\mu^2 + 272\mu + 64)}}, \tag{E.3}$$

one obtains

$$f_{C4,\eta} = \frac{1}{128}\frac{(7\mu^2 + 8\mu - 16)(\mu+4)(3\mu+4)^2}{\mu(43\mu^3 + 204\mu^2 + 272\mu + 64)(1+\mu)(\mu+2)} < 0.$$

From (E.1) and for the δ and η given by (E.2) and (E.3), one obtains that if $\mu < 1.045$,

$$I'_{C4} < I_{TDVA,opt}.$$

Since $I_{C4,opt} \leq I'_{C4}$, one obtains that if $\mu < 1.045$, $I_{C4,opt} < I_{TDVA,opt}$.

For $C5$, setting $\gamma = \gamma_{TDVA,opt}$ and $\zeta = \zeta_{TDVA,opt}$ in (D.2), one obtains

$$I'_{C5} = I_{TDVA,opt} + \left(a'_{C5,2}\delta^{-2} + a'_{C5,1}\delta^{-1}\right)\zeta_{TDVA,opt}. \tag{E.4}$$

Then, we will show that there exist finite δ and η such that $a'_{C5,2}\delta^{-2} + a'_{C5,1}\delta^{-1} < 0$. It can be checked that $a'_{C5,2} > 0$. Therefore, we only need to prove that there exists a finite η such that $a'_{C5,1} < 0$. Since

$$a'_{C5,1} = \frac{l'_{C5,3}\eta^6 + l'_{C5,2}\eta^4 + l'_{C5,1}\eta^2}{\mu(1 + f'_{C5,1}\eta^2 + f'_{C5,2}\eta^4)^2},$$

it is easy to check that $a'_{C5,1} < 0$ if $\eta^2 > (\mu+1)\left(\mu + 1 + \sqrt{\mu^2 + 2\mu}\right)$ or $\eta^2 < (\mu + 1)\left(\mu + 1 - \sqrt{\mu^2 + 2\mu}\right)$. For example, if choosing

$$\eta = \sqrt{2(1+\mu)^2}, \tag{E.5}$$

$$\delta^{-1} = \frac{2(2+\mu)(\mu+1)^2}{(1+8\mu+4\mu^2)(4+9\mu+4\mu^2)}, \tag{E.6}$$

one obtains

$$f_\delta = -\frac{\sqrt{2}(2+\mu)^{5/2}(\mu+1)^2}{(1+8\mu+4\mu^2)(4+9\mu+4\mu^2)(1+3\mu+5\mu^2+2\mu^3)^2} < 0,$$

which means that for the η and δ given by (E.5) and (E.6), $I'_{C5} < I_{TDVA,opt}$. Since $I_{C5,opt} \leq I'_{C5}$, one obtains $I_{C5,opt} < I_{TDVA,opt}$.

For $C6$, setting $\gamma = \gamma_{TDVA,opt}$ and $\zeta = \zeta_{TDVA,opt}$, one obtains

$$I'_{C6} = I_{TDVA,opt} + f_{C6,\eta},$$

where $f_{C6,\eta} = d_2\eta^{-4} + d_1\eta^{-2} + d_0$, with

$$\begin{aligned} d_2 &= a'_{C6,2}\zeta_{TDVA,opt}\delta^{-2} + f'_{C6,2}/\zeta_{TDVA,opt}, \\ d_1 &= a'_{C6,1}\zeta_{TDVA,opt}\delta^{-1} + (g'_{C6,1}\delta + f'_{C6,1})/\zeta_{TDVA,opt}, \\ d_0 &= (l'_{C6,2}\delta^2 + g'_{C6,0}\delta)/\zeta_{TDVA,opt}. \end{aligned}$$

It can be checked that $d_2 > 0$ for any δ and if $\mu < \sqrt{2}$, $d_1 < 0$. Thus, it remains to prove that there exists a finite $\eta > 0$ such that $f_{C6,\eta} < 0$. This can be done by checking the discriminant of $f_{C6,\eta}$, which is

$$\begin{aligned} \Delta &= d_1^2 - 4d_2d_0 \\ &= 16(\mu-4)(\mu+1)^8\delta^4 - 16\mu(4\mu^3 + 11\mu^2 + 5\mu - 4)(\mu+1)^4\delta^3 + \\ &\quad 8\mu^2(5\mu^2 + 21\mu + 20)(\mu+1)^3\delta^2 + \mu^3(3\mu+4)^2. \end{aligned}$$

It is easy to see that there always exists a finite δ such that $\Delta > 0$. For example, if choosing

$$\delta = \frac{\mu(4\mu^3 + 11\mu^2 + 5\mu - 4 - \sqrt{6\mu^6 + 56\mu^5 + 253\mu^4 + 606\mu^3 + 799\mu^2 + 568\mu + 176})}{2(\mu - 4)(\mu + 1)^4},$$

which is larger than 0 if $\mu < 4$, one obtains

$$\Delta = \mu^3(3\mu + 4)^2 > 0.$$

Therefore, we can always find a η^{-2} between the two real positive solutions of $f_{C6,\eta} = 0$ such that $f_{C6,\eta} < 0$. A possible choice is $\eta^{-2} = -\frac{d_1}{2d_2}$. This means that if carefully choosing δ and η, the inequality $I'_{C6} < I_{IDVA,opt}$ holds. Since $I_{C6,opt} \leq I'_{C6}$, one obtains $I_{C6,opt} < I_{TDVA,opt}$.

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