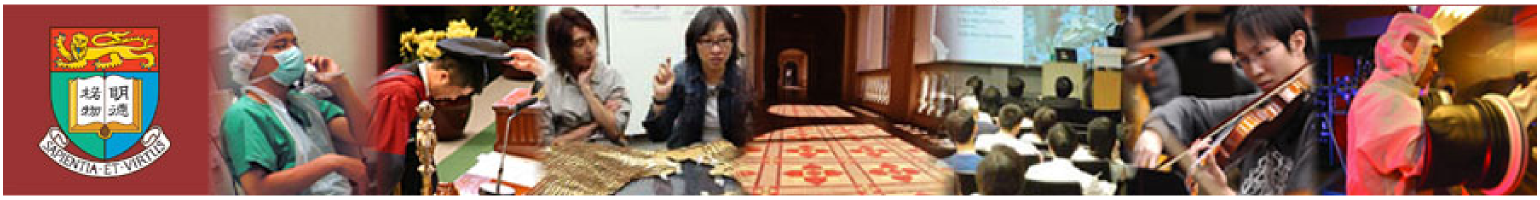

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Minimal Realizations of Three-Port Resistive Networks

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***Abstract*—This paper is concerned with the minimal realization problem of a third-order real symmetric matrix as the admittance of three-port resistive networks. First, a necessary and sufficient condition is derived for a real symmetric matrix to be realizable as the admittance of three-port resistive networks with four terminals and at most k elements, where $k \in \{1, 2, \ldots, 5\}$. Since it is well-known that the matrix must be paramount, necessary and sufficient conditions are obtained for any paramount matrix to be realizable as the admittance of three-port resistive networks with at most k elements, where $k \in \{1, 2, 3, 4\}$. Moreover, a necessary and sufficient condition is derived for a paramount matrix that cannot be realized with less than five elements to be realizable as the admittance of three-port resistive networks with five elements. Finally, some numerical examples are presented to illustrate the results. The results of this paper can contribute to solving minimal realization problems of one-port and multi-port transformerless networks with more than one kind of elements.**

***Index Terms*—Paramountcy, passive network synthesis, three-port resistive networks.**

I. Introduction

PASSIVE network synthesis is an important subject in systems theory, which has experienced its "golden era" from the 1930s to the 1970s [1], [7], [31], [34]. As stated in [1], [31], a linear, lumped, time-invariant, reciprocal n-port network is passive if and only if its impedance (resp. admittance) $H(s)$ is a real-rational symmetric positive-real matrix (symmetric PR) whenever $H(s)$ exists, which can always be physically constructed with a finite number of resistors, inductors, capacitors, and transformers. When $n = 1$, Bott and Duffin [4] established a transformerless realization procedure that can realize any positive-real function as the impedance (resp. admittance) of a one-port passive network consisting of only resistors, inductors, and capacitors, which, however, generates a large number of redundant elements. Although a series of further investigations were made [29], [32], minimal transformerless realization problems for passive one-port networks are far from being solved to date. Moreover, unlike the one-port case, realizability of passive multi-port transformerless networks $(n > 1)$ is unsolved.

Recently, a new two-terminal passive mechanical element named *inerter* [36] was invented with the property that the force applied at the two terminals is proportional to the relative acceleration between them, that is, $F = b(\dot{v}_1 - \dot{v}_2)$, where the constant b is called the *inertance*. Based on the force-current analogy, the inerter is analogous to the capacitor. Therefore, the analogy between passive mechanical and electrical circuit elements has been completed, where dampers, springs, inerters, and levers are analogous to resistors, inductors, capacitors, and transformers, respectively. As a result, the theory of passive network synthesis can be directly transplanted into passive mechanical design (see [14]), which makes the design become more systematic. Based on circuit synthesis theory, any passive mechanical system can be physically constructed with a finite number of dampers, springs, inerters, and levers (levers are not necessary for the one-port case). So far, mechanisms containing the inerter have been successfully applied to a series of mechanical control systems [20], [21], [24]. As a consequence, interest in passive network synthesis has revived [10]–[13], [15]–[17], [25], [26], [40], [41]. Particularly, Kalman made an independent call for renewed effort [27]. In addition to the application in mechanical engineering, the passive network still has its application value in the area of electrical engineering such as circuit-antenna design [28], [30], and external circuits of the mechatronic system utilized in the suspension system control are passive electrical networks [39].

The n-port resistive network is an important class of passive reciprocal n-port networks containing only resistors, which was widely investigated from the 1950s to the 1970s [2], [5], [6], [8], [9], [22], [35], [37]. Through the extraction of elements, realizability results of n-port resistive networks can be applied to solve realization problems of one-port networks containing a limited number of certain two kinds of elements [12], which is also stated in [25]. Moreover, some methods and results of investigating n-port resistive networks can provide guidance on synthesis of general transformerless n-port networks. Because of passivity and reciprocity, the admittance (resp. impedance) of n-port resistive networks must be non-negative definite. Since no transformer is present, there should be further constraints. Cederbaum [8] proved that the admittance (resp. impedance) of n-port resistive networks must be paramount. When $n \leq 3$, Tellegen [37] showed that paramountcy is a necessary and sufficient condition for the realizability. Since Tellegen's work is in Dutch, [10, App. A] presented a completed and better structured reworking in English. When $n > 3$, although the realizability of admittances as n-port resistive networks containing $(n+1)$ terminals has been solved [2], [5], [6], [22], how to obtain a testable realizability condition for the realizability of general n-port resistive networks is still an open problem (see [18]). Recently, Chen *et al.* [16] obtained some new results on the realizability of n-port resistive networks containing $2n$ terminals. Moreover, unlike the synthesis of one-port RLC networks, minimal realizations of n-port resistive networks in terms of the total number of elements have seldom been discussed, which, however, has important practical implications. See [19] for a survey on synthesis of n-port resistive networks.

This paper is concerned with minimal realizations of any third-order real symmetric matrix as the admittance of three-

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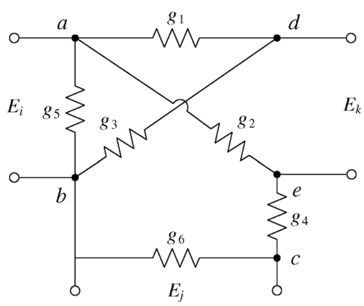


Fig. 1. The canonical configuration that can realize any third-order paramount admittance, where the conductances of elements are positive, 0, or $+\infty$ [35].

port resistive networks. Paramountcy is a necessary and sufficient condition for any third-order real symmetric matrix to be realizable as the admittance of a three-port resistive network [35], [37]. Moreover, it is known that the canonical configuration [35, Fig. 11(B)] (see Fig. 1) that can realize any paramount matrix as the admittance generally contains six elements. Since elements are preferred to be as few as possible, it is essential to investigate realization problems of three-port resistive networks with the least number of elements.

First, a necessary and sufficient condition is derived for a third-order real symmetric matrix to be realizable as the admittance of a three-port four-terminal resistive network with at most k elements, where $k \in \{1, 2, \ldots, 5\}$. Then, together with discussion in [17], one directly obtains a necessary and sufficient condition for a third-order paramount matrix to be realizable as the admittance of a three-port resistive network containing at most k elements for $k \in \{1, 2, 3\}$, where the case of $k = 3$ has been presented in [17]. Together with topological constraints, a four-element configuration is presented whose admittance is not always realizable as a network with four terminals and at most four elements. By deriving the realizability condition of the configuration, one obtains a necessary and sufficient condition for a third-order paramount matrix to be realizable with at most four elements. Similarly, by finding all the five-element configurations whose admittances are not always realizable as networks with four terminals and at most five elements and investigating their realizability conditions, a necessary and sufficient condition is derived for a paramount matrix that cannot be realized with less than five elements to be realizable as the admittance of three-port resistive networks with five elements. Since the augmented graph of three-port resistive networks with no more than five elements must be planar, the results of this paper can be directly transformed to those of the impedance synthesis based on the principle of duality. The investigation of this paper can provide guidance on minimal realization problems of more general n-port resistive networks and can be a critical step toward solving the realizability of n-port RLC transformerless networks. Moreover, the results can be utilized to solve minimal realization problems of one-port or two-port passive mechanical or electrical networks by elements extraction [12], [17], results of which can contribute to passive mechanical control with the inerter [14], [36].

II. Problem Formulation

Consider a real symmetric third-order matrix Y as

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & y_{23} \\ y_{13} & y_{23} & y_{33} \end{bmatrix}. \quad (1)$$

If Y is realizable as the admittance of three-port resistive networks, then it is necessarily paramount [35]. This paper addresses the following question: What are necessary and sufficient conditions for a given paramount matrix Y in the form of (1) to be realizable as the admittance of three-port resistive networks with at most k elements, where $k \in \{1, 2, \ldots, 5\}$, and what about the covering configurations? It is assumed that there is no transformer and values of elements are positive and finite. The classical graph theory is utilized (see [34]).

Notations: $\mathbb{S}^n$ denotes the set of real symmetric $n \times n$ matrices. $Y_n \in \mathbb{S}^n$ denotes an nth-order real symmetric matrix; specially, $Y \in \mathbb{S}^3$ denotes a third-order real symmetric matrix in the form of (1). M_{ij} denotes a minor of Y, where $M_{11} = y_{22}y_{33} - y_{23}^2$, $M_{22} = y_{11}y_{33} - y_{13}^2$, $M_{33} = y_{11}y_{22} - y_{12}^2$, $M_{12} = y_{12}y_{33} - y_{23}y_{13}$, $M_{13} = y_{12}y_{23} - y_{22}y_{13}$, and $M_{23} = y_{11}y_{23} - y_{12}y_{13}$.

III. Preliminaries

Definition 1: [3] For any n-port resistive network with n_e elements and n_v nodes, a graph $\mathcal{G}$ called *augmented graph* is formulated by letting each port or each element correspond to an edge and each node of the network correspond to a vertex. The subgraph $\mathcal{G}_p$ that consists of all the edges corresponding to the ports is called a *port graph*. The subgraph $\mathcal{G}_e$ that consists of all the edges corresponding to the elements is called a *network graph*. Furthermore, the edge belonging to a port graph $\mathcal{G}_p$ is called a *port edge*; the edge belonging to a network graph $\mathcal{G}_e$ is called a *network edge*.

From Definition 1, it is obvious that $\mathcal{G} = \mathcal{G}_p \cup \mathcal{G}_e$. The concept of *isomorphism* for graphs has been defined in [34, p. 13]. Since the augmented graph contains two types of edges: port edges and network edges, it is needed to define the *isomorphism for augmented graphs* as follows.

Definition 2: Two augmented graphs $\mathcal{G}$ and $\mathcal{G}_1$ are isomorphic if there simultaneously exist the one-to-one correspondence between vertices of $\mathcal{G}$ and $\mathcal{G}_1$, the one-to-one correspondence between port edges of $\mathcal{G}$ and $\mathcal{G}_1$, and the one-to-one correspondence between network edges of $\mathcal{G}$ and $\mathcal{G}_1$, which preserves the incidence relationships.

It is obvious that two isomorphic augmented graphs can correspond to the same n-port resistive network. Furthermore, the following assumption is made throughout the paper.

Assumption 1: The augmented graph $\mathcal{G}$ of n-port resistive networks is connected.

If a given n-port resistive network does not satisfy Assumption 1, then it is obvious that there are at least two connected subnetworks. By short-circuiting any two nodes of each two connected subnetworks, the given n-port resistive network with an unconnected augmented graph can always be equivalent to another network satisfying Assumption 1 and containing the same number of elements. Therefore, Assumption 1 does not affect the final realizability results. Since a connected graph always contains a tree [34, p. 25], the following lemma is valid.

Lemma 1: [17] An n-port resistive network has a well-defined admittance if and only if its port graph $\mathcal{G}_p$ is made part of a tree of its augmented graph $\mathcal{G}$.

Definition 3: A node that is not the terminal of a network is called an *internal node*.

For an n-port resistive network, any internal node can be eliminated by the *generalized star-mesh transformation* (see [38]) without increasing the number of elements. Therefore, without loss of generality, one also makes the following assumption in this paper.

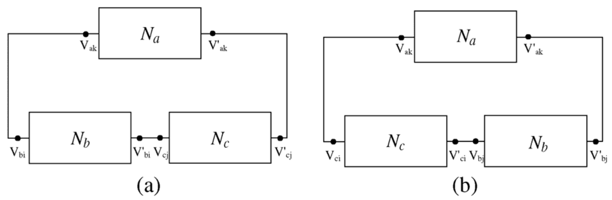


Fig. 2. The structures of two n-port networks stated in Lemma 2, where (a) is for N and (b) is for N_1.

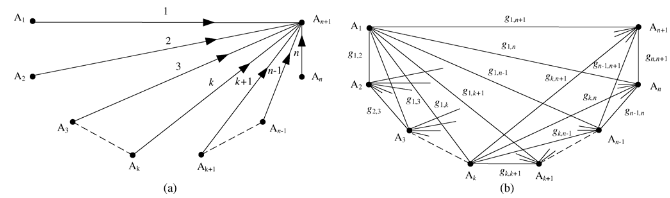


Fig. 3. (a) The directed port graph $\mathcal{G}_p$ that is a Lagrangian tree with all the port edges oriented towards the common vertex A_{n+1}; (b) the graph connecting each two vertices of $\mathcal{G}_p$, where $g_{i,j} \geq 0$ for $i, j \in \{1, 2, \ldots, n+1\}$ with $i < j$.

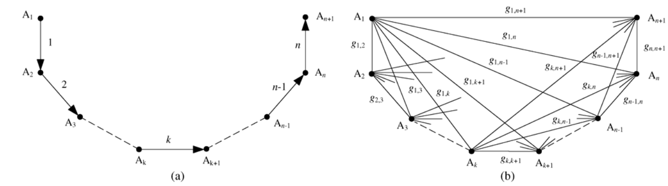


Fig. 4. (a) The directed port graph $\mathcal{G}_p$ that is a path tree with all the port edges ordered and oriented to the same direction; (b) the graph connecting each two vertices of $\mathcal{G}_p$, where $g_{i,j} \geq 0$ for $i, j \in \{1, 2, \ldots, n+1\}$ with $i < j$.

Assumption 2: For an n-port resistive network, there is no internal node, that is to say, all the nodes must be terminals.

Consider an n-port resistive network N consisting of three parts: N_a, N_b and N_c, any one of which has only one common node with another [see Fig. 2(a)]. Exchanging the positions of N_b and N_c without any other alterations yields another n-port resistive network N_1 as in Fig. 2(b).

Lemma 2: Two n-port resistive networks N and N_1, whose structures are as shown in Fig. 2, have the same admittance.

Proof: As derived in [3], the admittance Y of any n-port resistive network is expressed as

$$Y = B_{f2}^T \left(B_{f1} D B_{f1}^T\right)^{-1} B_{f2}, \tag{2}$$

where D is a diagonal matrix, each diagonal entry of which is the impedance of each element, and $B_f = [B_{f1}, B_{f2}]$ is the *fundamental circuit matrix* (see [34, pg. 91]) of the augmented graph $\mathcal{G}$, where the columns of $B_{f1} = [I_{n_e+n-n_v+1}, B_{f12}]$ correspond to edges of the network graph $\mathcal{G}_e$. It is obvious that N and N_1 have the same B_f and D. Hence, the admittance matrices of N and N_1 are the same by (2). ∎

Definition 4: [5] A tree with all the branches incident at a common vertex is called a *Lagrangian tree*. A tree whose branches form a path is called a *path tree*.

Definition 5: [6] A *cross-sign change* of a matrix is to change the sign of each non-zero entry in the ith row and ith column except the diagonal entry.

A cross-sign change corresponds to switching the polarity of a port. Moreover, interchanging two rows and corresponding two columns corresponds to swapping two ports.

Definition 6: [9] A *dominant matrix* is a real symmetric matrix such that each of its main diagonal entries is not less than the sum of absolute values of all other entries in the same row. A *uniformly tapered matrix* is an nth-order real symmetric matrix $Y_n \in \mathbb{S}^n$ whose entries satisfy $y_{i,j-1} - y_{i,j} \geq y_{i-1,j-1} - y_{i-1,j}$ for $i, j \in \{1, 2, \ldots, n+1\}$ with $i < j$, where $y_{0,k} = y_{k-1,n+1} = 0$ for $k \in \{1, 2, \ldots, n+1\}$.

Based on Lemma 1, if the admittance of an n-port resistive network exists, then the least number of terminals is $(n+1)$, where $\mathcal{G}_p$ constitutes a tree of $\mathcal{G}$.

Lemma 3: [5], [22] A matrix $Y_n \in \mathbb{S}^n$ is realizable as the admittance of an n-port resistive network with $(n+1)$ terminals whose port graph is a Lagrangian tree, if and only if after a finite number of cross-sign changes Y_n is a dominant matrix with all the off-diagonal entries non-positive. Moreover, if Y_n is a dominant matrix with all the off-diagonal entries non-positive, then all the port edges are oriented towards (or from) the common vertex [see Fig. 3(a)], and the values of the conductances connecting any two terminals are uniquely determined as $g_{i,j} = -y_{ij}$ for $i, j \in \{1, 2, \ldots, n\}$ with $i < j$ and $g_{i,n+1} = \sum_{k=1}^{n} y_{ik}$ for all $i \in \{1, 2, \ldots, n\}$ [see Fig. 3(b)].

Lemma 4: [5], [22] A matrix $Y_n \in \mathbb{S}^n$ is realizable as the admittance of an n-port resistive network with $(n+1)$ terminals whose port graph is a path tree, if and only if after a finite number of cross-sign changes and a proper rearrangement of rows and columns Y_n is a uniformly tapered matrix. Moreover, if Y_n is a uniformly tapered matrix, then port edges are ordered and oriented to the same direction [see Fig. 4(a)], and the values of the conductances connecting any two terminals are uniquely determined as $g_{i,j} = (y_{i,j-1} - y_{ij}) - (y_{i-1,j-1} - y_{i-1,j})$ for $i, j \in \{1, 2, \ldots, n+1\}$ with $i < j$, where $y_{0,k} = y_{k-1,n+1} = 0$ for $k \in \{1, 2, \ldots, n+1\}$ [see Fig. 4(b)].

IV. Main Results

If the admittance of a three-port resistive network exists, then the number of terminals must be four, five, or six. Lemma 5 presents a necessary and sufficient condition for any third-order real symmetric matrix $Y \in \mathbb{S}^3$ to be realizable as the admittance of three-port resistive networks containing four terminals and at most k elements where $k \in \{1, 2, \ldots, 5\}$. Furthermore, necessary and sufficient conditions are obtained for a paramount matrix $Y \in \mathbb{S}^3$ to be realizable as the admittances of three-port resistive networks containing at most k elements where $k \in \{1, 2, \ldots, 5\}$ in Theorems 1, 3, and 9.

A. Minimal Realizability With Four Terminals

Lemma 5: A matrix $Y \in \mathbb{S}^3$ is realizable as the admittance of three-port resistive networks containing four terminals and at most k elements where $k \in \{1, 2, \ldots, 5\}$, if and only if

1) when $y_{12}y_{13}y_{23} \leq 0$, the following inequalities hold simultaneously with at least $(6-k)$ of the six inequality signs being equality: $|y_{12}| \geq 0$, $|y_{13}| \geq 0$, $|y_{23}| \geq 0$, $y_{11} - |y_{12}| - |y_{13}| \geq 0$, $y_{22} - |y_{12}| - |y_{23}| \geq 0$, and $y_{33} - |y_{13}| - |y_{23}| \geq 0$;
2) when $y_{12}y_{13}y_{23} \geq 0$, at least one of the following three conditions holds with at least $(6-k)$ of the six inequality signs being equality: i) $-|y_{13}| \leq 0$, $|y_{13}| \leq |y_{12}| \leq y_{11}$, $|y_{13}| \leq |y_{23}| \leq y_{33}$, and $|y_{12}| + |y_{23}| - |y_{13}| \leq y_{22}$; ii) $-|y_{12}| \leq 0$, $|y_{12}| \leq |y_{13}| \leq y_{11}$, $|y_{12}| \leq |y_{23}| \leq y_{22}$, and $|y_{13}| + |y_{23}| - |y_{12}| \leq y_{33}$; iii) $-|y_{23}| \leq 0$, $|y_{23}| \leq |y_{12}| \leq y_{22}$, $|y_{23}| \leq |y_{13}| \leq y_{33}$, and $|y_{12}| + |y_{13}| - |y_{23}| \leq y_{11}$.

Proof: When the number of terminals is four, the port graph is either a Lagrangian tree or path tree. Hence, this lemma can follow from Lemmas 3 and 4, where Condition 1 corresponds to the Lagrangian-tree case and Condition 2 corresponds to the path-tree case. ∎

B. Realizability With at Most Four Elements

First, a necessary and sufficient condition is presented for the realizability with at most k elements for $k \in \{1, 2, 3\}$, following discussion in [17].

Theorem 1: A paramount matrix $Y \in \mathbb{S}^3$ is realizable as the admittance of three-port resistive networks with at most k elements where $k \in \{1, 2, 3\}$, if and only if Y satisfies the condition of Lemma 5.

Proof: The case of $k = 3$ follows directly from [17]. It is easy to prove cases of $k = 1$ and $k = 2$ by the method in [17]. ■

In order to obtain a necessary and sufficient condition for any $Y \in \mathbb{S}^3$ to be realizable as the admittance of a three-port resistive network with at most four elements, it suffices to find the configurations that cannot always be equivalent to a three-port four-terminal configuration with at most four elements and derive their realizability conditions, since the condition of Lemma 5 for $k = 4$ is a necessary and sufficient condition for the realizability of three-port resistive networks with four terminals and at most four elements. Prior to the discussion, a series of sufficient conditions are established for the realizability of three-port resistive networks containing four terminals and at most four elements.

Lemma 6: For a paramount matrix $Y \in \mathbb{S}^3$, if at least two of y_{12}, y_{13}, and y_{23} are zero, then Y is realizable as the admittance of a three-port resistive network containing four terminals and at most four elements.

Proof: After a proper rearrangement of rows and corresponding columns, $Y \in \mathbb{S}^3$ satisfies $y_{13} = y_{23} = 0$. The paramountcy of Y implies that $y_{11} \geq |y_{12}| \geq 0$, $y_{22} \geq |y_{12}| \geq 0$, and $y_{33} \geq 0$. Hence, it follows that $y_{12}y_{13}y_{23} = 0$, $|y_{12}| \geq 0$, $|y_{13}| = 0$, $|y_{23}| = 0$, $y_{11} - |y_{12}| - |y_{13}| = y_{11} - |y_{12}| \geq 0$, $y_{22} - |y_{12}| - |y_{23}| = y_{22} - |y_{12}| \geq 0$, and $y_{33} - |y_{13}| - |y_{23}| = y_{33} \geq 0$. Then, the condition of Lemma 5 holds for $k = 4$. Therefore, the given Y is realizable as the admittance of a three-port resistive network containing four terminals and at most four elements. ■

Lemma 6 shows that if a three-port resistive network whose admittance exists cannot be equivalent to the one containing four terminals and at most four elements, then its augmented graph $\mathcal{G}$ must be *nonseparable* (see [34, p. 35]).

Lemma 7: If a paramount matrix $Y \in \mathbb{S}^3$ contains two equal rows or two rows for which one row is the negative of the other, then Y is realizable as the admittance of the three-port resistive network containing four terminals and at most four elements.

Proof: First, consider the case when $Y \in \mathbb{S}^3$ contains two equal rows. Assume them to be the first and second rows. Together with the symmetry, Y is in the form of

$$Y = \begin{bmatrix} y_{11} & y_{11} & y_{13} \\ y_{11} & y_{11} & y_{13} \\ y_{13} & y_{13} & y_{33} \end{bmatrix}.$$

Together with paramountcy of Y, one obtains that $y_{12}y_{13}y_{23} = y_{11}y_{13}^2 \geq 0$, $-|y_{13}| \leq 0$, $y_{11} - |y_{12}| = 0$, $|y_{12}| - |y_{13}| = y_{11} - |y_{13}| \geq 0$, $y_{33} - |y_{23}| = y_{33} - |y_{13}| \geq 0$, $|y_{23}| - |y_{13}| = 0$, and $y_{22} - (|y_{12}| + |y_{23}| - |y_{13}|) = y_{11} - (y_{11} + |y_{13}| - |y_{13}|) = 0$. Hence, Condition 2 of Lemma 5 must hold for $k = 4$. When there is another pair of two equal rows, a similar argument can be applied. Moreover, the other case when Y contains two rows where one row is the negative of the other can be proved using the similar method. ■

Lemma 7 shows that if a three-port resistive network whose admittance exists cannot be equivalent to the one containing

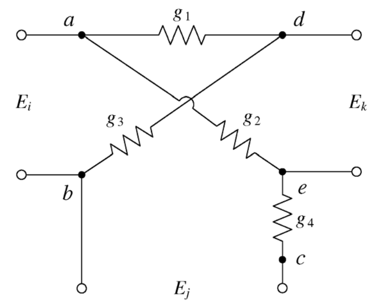
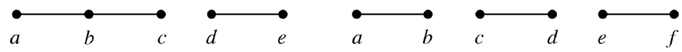


Fig. 5. The configuration discussed in Lemma 8, where $g_1, g_2, g_3, g_4 > 0$.



(a) (b)

Fig. 6. (a) The port graph of three-port resistive networks containing five terminals; (b) the port graph of three-port resistive networks containing six terminals.

four terminals and at most four elements, its augmented graph $\mathcal{G} = \mathcal{G}_p \cup \mathcal{G}_e$ cannot contain two port edges directly in series.

Lemma 8: A matrix $Y \in \mathbb{S}^3$ is realizable as the admittance of a three-port resistive network with at most four elements if and only if Y satisfies the condition of Lemma 5 for $k = 4$ or Y is realizable as the configuration in Fig. 5.

Proof: Sufficiency. The sufficiency is obvious by Lemma 5.

Necessity. As discussed above, the number of possible terminals can only be four, five, or six. If the number of terminals is four, the condition of Lemma 5 holds for $k = 4$.

If the number of terminals is five, then the port graph $\mathcal{G}_p$ must be the one in Fig. 6(a), which consists of two *maximal connected subgraphs* (see [34, p. 16]). If the number of elements is at most three, then the condition of Lemma 5 holds for $k = 3$ by Theorem 1, which further implies that the condition of Lemma 5 holds for $k = 4$. Therefore, Let us assume that the number of elements is four and try to find out the possible configurations whose admittances does not always satisfy the condition of Lemma 5 for $k = 4$. Denote the cut-set that separates $\mathcal{G}$ into two parts respectively containing two maximal connected subgraphs of $\mathcal{G}_p$ as $\mathcal{C}_{ep}$. It is obvious that all the edges belonging to $\mathcal{C}_{ep}$ must be network edges. By Lemma 6, both vertices d and e must be incident by at least one network edge belong to $\mathcal{C}_{ep}$. Otherwise, $\mathcal{G}$ must be separable, implying that the admittance must always satisfy the condition of Lemma 5 for $k = 4$. Moreover, if the number of network edges belonging to $\mathcal{C}_{ep}$ that are incident at vertex d or e is only one, then the network can always be equivalent to another one containing four terminals and at most four elements by Lemma 2, implying that the condition of Lemma 5 holds for $k = 4$. Therefore, each of vertices d and e must be incident by two network edges belonging to $\mathcal{C}_{ep}$. By Lemmas 6 and 7, each of vertices $a - c$ must be incident by at least one network edge belonging to $\mathcal{C}_{ep}$. If one regards two augmented graphs that are isomorphic with each other as the same graph, it is not difficult to see that only the configurations whose graphs $\mathcal{G}$ are as in Fig. 7 are possible, where the bold line segments denote port edges and the light ones denote network edges. Furthermore, utilizing Lemma 2 one can eliminate Fig. 7(b). The configuration whose graph is Fig. 7(a) is as shown in Fig. 5.

If the number of terminals is six, then the port graph $\mathcal{G}_p$ must be the one in Fig. 6(b), which consists of three maximal connected subgraphs. It suffices to show that there is no such class of networks that cannot always be equivalent to another one containing less than six terminals and at most four elements.

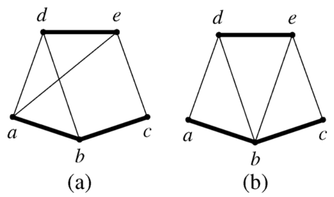


Fig. 7. Augmented graphs discussed in the proof of Theorem 8, where edges in bold line segments are port edges and those in light line segments are network edges.

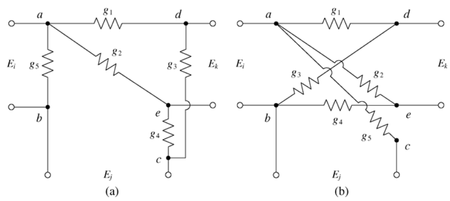


Fig. 8. Three-port resistive configurations containing five elements mentioned in Lemma 9, where $g_1, g_2, g_3, g_4, g_5 > 0$.

Assume that such a network exists. Using a similar argument as above, one can imply that at least six elements are needed by Lemmas 2, 6, and 7, which is impossible. ■

Theorem 2: Consider a paramount matrix $Y \in \mathbb{S}^3$ that cannot be realized as the admittance of three-port resistive networks containing less than four elements. Then, Y is realizable as the admittance of the configuration in Fig. 5 if and only if at least one of the following conditions holds: i) $M_{11} - |M_{12}| = 0$ and $M_{33} - |M_{23}| = 0$; ii) $M_{22} - |M_{23}| = 0$ and $M_{11} - |M_{13}| = 0$; iii) $M_{33} - |M_{13}| = 0$ and $M_{22} - |M_{12}| = 0$.

Proof: See Appendix A for the detail. ■

Theorem 3: A paramount matrix $Y \in \mathbb{S}^3$ is realizable as the admittance of three-port resistive networks with at most four elements if and only if Y satisfies the condition of Lemma 5 for $k = 4$ or the condition of Theorem 2.

Proof: This theorem follows directly from Lemma 8 and Theorems 1 and 2. ■

C. Realizability With Five Elements

Now, let us consider the paramount matrix $Y \in \mathbb{S}^3$ that cannot be realized as the admittance of three-port resistive networks containing less than five elements, that is, does not satisfy the condition of Theorem 3. Then, a necessary and sufficient condition will be established for such Y to be realizable as the admittance of three-port resistive networks containing five elements.

Lemma 9: If $Y \in \mathbb{S}^3$ is the admittance of at least one configuration in Fig. 8, then Y is always realizable as the admittance of a three-port resistive network containing four terminals and at most four elements.

Proof: It suffices to prove that the admittances of two configurations in Fig. 8 always satisfy the condition of Lemma 5 for $k = 4$. The detail is omitted for brevity. ■

Lemma 10: A matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of three-port resistive networks containing five elements if and only if Y satisfies the condition of Lemma 5 for $k = 5$ or Y is realizable as at least one configuration in Figs. 9 and 10.

Proof: Sufficiency. The sufficiency part can be directly proved together with Lemma 5.

Necessity: The number of possible terminals must be four, five, or six. If the number of terminals is four, then the condition of Lemma 5 holds for $k = 5$.

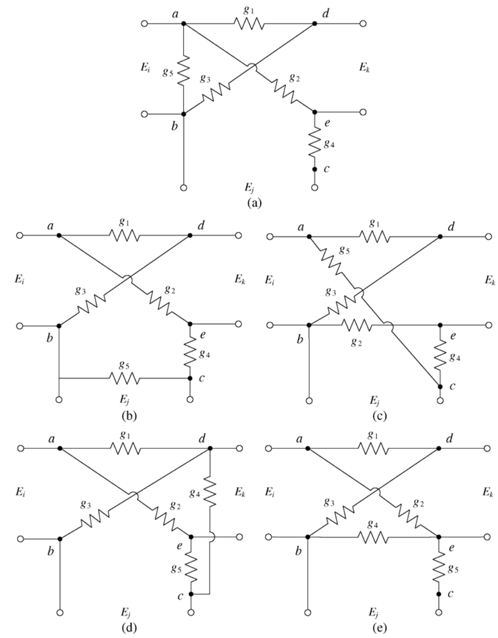


Fig. 9. Three-port resistive configurations containing five elements mentioned in Lemma 10, where $g_1, g_2, g_3, g_4, g_5 > 0$.

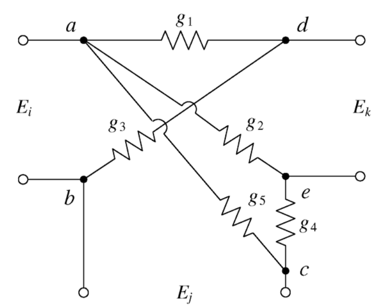


Fig. 10. A three-port resistive configuration containing five elements mentioned in Lemma 10, where $g_1, g_2, g_3, g_4, g_5 > 0$.

If the number of terminals is five, then the port graph $\mathcal{G}_p$ is as shown in Fig. 6(a), consisting of two maximal connected subgraphs. Since the condition of Theorem 3 does not hold, the number of elements cannot be less than five. Then, we will find all possible configurations that cannot always be equivalent to the one with four terminals and at most five elements. Note that the augmented graph $\mathcal{G}$ must be nonseparable by Lemma 6 and the augmented graph $\mathcal{G}$ cannot contain two port edges directly in series by Lemma 7. Denote the cut-set that separates $\mathcal{G}$ into two parts respectively containing two components of $\mathcal{G}_p$ as $\mathcal{C}_{ep}$. Then, together with Lemma 2, each of vertices d and e must be incident by at least two network edges belonging to $\mathcal{C}_{ep}$, and vertices $a - c$ must be incident by at least one network edge belonging to $\mathcal{C}_{ep}$. Hence, the number of edges belonging to $\mathcal{C}_{ep}$ must be either four or five, which are all network edges. If $\mathcal{C}_{ep}$ contains four edges, then only augmented graphs shown in Figs. 11(a)–11(f) are possible utilizing the previous constraints provided that one regards two augmented graphs that are isomorphic with each other as the same graph. When $\mathcal{C}_{ep}$ contains five edges, only augmented graphs shown in Figs. 11(g) and 11(h) are possible utilizing the previous constraints provided that one regards two augmented graphs that are isomorphic with each other as the same graph. Furthermore, Figs. 11(c) and 11(f) are eliminated by Lemma 9. Then, the possible configurations are shown in Figs. 9 and 10.

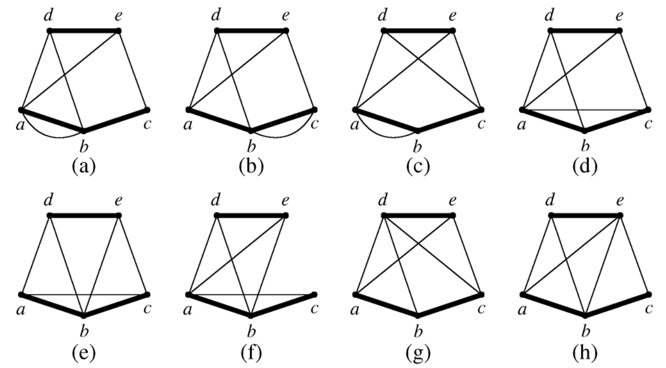


Fig. 11. Augmented graphs discussed in the proof of Lemma 10, where edges in bold line segments are port edges and those in light line segments are network edges.

If the number of terminals is six, then the port graph $\mathcal{G}_p$ must be the one in Fig. 6(b), consisting of three components. By Lemmas 2 and 6, each of six vertices of $\mathcal{G}_p$ must be incident by at least two network edges, provided that a network cannot be equivalent to the one containing less terminals and at most five elements. This means that at least six network edges are needed. Therefore, the admittance of any three-port five-element network with six terminals is always realizable as the one with fewer terminals and at most five elements. ■

Theorem 4: A paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of the configuration in Fig. 9(a) if and only if

1) when $y_{12}y_{13}y_{23} \leq 0$, at least one of the following conditions holds: i) $y_{33} - |y_{13}| - |y_{23}| > 0$ and $(M_{11} - |M_{12}|)(M_{22} - |M_{12}|) = 0$; ii) $y_{22} - |y_{12}| - |y_{23}| > 0$ and $(M_{11} - |M_{13}|)(M_{33} - |M_{13}|) = 0$; iii) $y_{11} - |y_{12}| - |y_{13}| > 0$ and $(M_{22} - |M_{23}|)(M_{33} - |M_{23}|) = 0$;
2) when $y_{12}y_{13}y_{23} > 0$, at least one of the following conditions holds: i) $|y_{13}| < \min\{|y_{12}|, |y_{23}|\}$ and $(M_{11} - |M_{12}|)(M_{33} - |M_{23}|) = 0$; ii) $|y_{23}| < \min\{|y_{12}|, |y_{13}|\}$ and $(M_{22} - |M_{12}|)(M_{33} - |M_{13}|) = 0$; iii) $|y_{12}| < \min\{|y_{13}|, |y_{23}|\}$ and $(M_{11} - |M_{13}|)(M_{22} - |M_{23}|) = 0$.

Proof: See Appendix B for the detail. ■

Theorem 5: A paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of the configuration in Fig. 9(b) if and only if

1) when $y_{12}y_{13}y_{23} \leq 0$, at least one of the following conditions holds: i) $y_{33}(M_{33} - |M_{23}|) - y_{11}(M_{11} - |M_{12}|) = 0$; ii) $y_{22}(M_{22} - |M_{12}|) - y_{33}(M_{33} - |M_{13}|) = 0$; iii) $y_{11}(M_{11} - |M_{13}|) - y_{22}(M_{22} - |M_{23}|) = 0$;
2) when $y_{12}y_{13}y_{23} > 0$, at least one of the following conditions holds: i) $|y_{13}| < \min\{|y_{12}|, |y_{23}|\}$ and $y_{33}(M_{33} - |M_{23}|) - y_{11}(M_{11} - |M_{12}|) = 0$; ii) $|y_{23}| < \min\{|y_{12}|, |y_{13}|\}$ and $y_{22}(M_{22} - |M_{12}|) - y_{33}(M_{33} - |M_{13}|) = 0$; iii) $|y_{12}| < \min\{|y_{13}|, |y_{23}|\}$ and $y_{11}(M_{11} - |M_{13}|) - y_{22}(M_{22} - |M_{23}|) = 0$.

Proof: The method is similar to that of Theorem 4. ■

Theorem 6: A paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of the configuration in Fig. 9(c) if and only if $y_{12}y_{13}y_{23} > 0$, and at least one of the following conditions holds: i) $y_{11}y_{22}y_{33} - y_{12}y_{13}y_{23} - y_{33}(|y_{13}|(y_{22} - |y_{12}|) + |y_{23}|(y_{11} - |y_{12}|)) = (y_{11} + y_{22} - |y_{12}|)(y_{33}|y_{12}| - |y_{13}||y_{23}|) > 0$; ii) $y_{11}y_{22}y_{33} - y_{12}y_{13}y_{23} - y_{11}(|y_{13}|(y_{22} - |y_{13}|) + |y_{12}|(y_{33} - |y_{23}|)) = (y_{33} + y_{22} - |y_{23}|)(y_{11}|y_{23}| - |y_{13}||y_{12}|) > 0$; iii) $y_{11}y_{22}y_{33} - y_{12}y_{13}y_{23} - y_{22}(|y_{12}|(y_{33} - |y_{12}|) + |y_{23}|(y_{11} - |y_{13}|)) = (y_{11} + y_{33} - |y_{13}|)(y_{22}|y_{13}| - |y_{12}||y_{23}|) > 0$.

Proof: The method is similar to that of Theorem 4. ■

Theorem 7: A paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of the configuration in Fig. 9(d) if and only if

1) when $y_{12}y_{13}y_{23} \leq 0$, at least one of the following conditions holds: i) $|y_{12}|(|y_{23}| + |y_{13}|)^2 - (|y_{23}| + |y_{13}|)M_{33} + y_{33}(y_{22} - |y_{12}|)(y_{11} - |y_{12}|) = 0$; ii) $|y_{23}|(|y_{12}| + |y_{13}|)^2 - (|y_{12}| + |y_{13}|)M_{11} + y_{11}(y_{22} - |y_{23}|)(y_{33} - |y_{23}|) = 0$; iii) $|y_{13}|(|y_{23}| + |y_{12}|)^2 - (|y_{23}| + |y_{12}|)M_{22} + y_{22}(y_{33} - |y_{13}|)(y_{11} - |y_{13}|) = 0$;
2) when $y_{12}y_{13}y_{23} > 0$, at least one of the following conditions holds: i) $(|y_{12}||y_{23}| - y_{22}|y_{13}|)(y_{11}|y_{23}| - |y_{12}||y_{13}|) > 0$ and $|(|y_{23}| - |y_{13}|)M_{33}| = |y_{12}|(|y_{23}| - |y_{13}|)^2 + y_{33}(y_{22} - |y_{12}|)(y_{11} - |y_{12}|)$; ii) $(|y_{12}||y_{23}| - y_{22}|y_{13}|)(y_{33}|y_{12}| - |y_{23}||y_{13}|) > 0$ and $|(|y_{12}| - |y_{13}|)M_{11}| = |y_{23}|(|y_{12}| - |y_{13}|)^2 + y_{11}(y_{22} - |y_{23}|)(y_{33} - |y_{23}|)$; iii) $(|y_{23}||y_{13}| - y_{33}|y_{12}|)(y_{11}|y_{23}| - |y_{12}||y_{13}|) > 0$ and $|(|y_{23}| - |y_{12}|)M_{22}| = |y_{13}|(|y_{23}| - |y_{12}|)^2 + y_{22}(y_{33} - |y_{13}|)(y_{11} - |y_{13}|)$.

Proof: The method is similar to that of Theorem 4. ■

Theorem 8: A paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of the configuration in Fig. 9(e) if and only if

1) when $y_{12}y_{13}y_{23} \leq 0$, at least one of the following conditions holds: i) $|y_{12}|(y_{33} - |y_{23}|) - |y_{13}|(y_{22} - |y_{23}|) > 0$ or $|y_{23}|(y_{11} - |y_{12}|) - |y_{13}|(y_{22} - |y_{12}|) > 0$ holds and $y_{11}|y_{23}|(y_{22} - |y_{23}|) - y_{33}|y_{12}|(y_{22} - |y_{12}|) - |y_{12}||y_{23}|(|y_{12}| - |y_{23}|) = 0$; ii) $|y_{12}|(y_{33} - |y_{13}|) - |y_{23}|(y_{11} - |y_{13}|) > 0$ or $|y_{13}|(y_{22} - |y_{12}|) - |y_{23}|(y_{11} - |y_{12}|) > 0$ holds and $y_{22}|y_{13}|(y_{11} - |y_{13}|) - y_{33}|y_{12}|(y_{11} - |y_{12}|) - |y_{12}||y_{13}|(|y_{12}| - |y_{13}|) = 0$; iii) $|y_{13}|(y_{22} - |y_{23}|) - |y_{12}|(y_{33} - |y_{23}|) > 0$ or $|y_{23}|(y_{11} - |y_{13}|) - |y_{12}|(y_{33} - |y_{13}|) > 0$ holds and $y_{11}|y_{23}|(y_{33} - |y_{23}|) - y_{22}|y_{13}|(y_{33} - |y_{13}|) - |y_{13}||y_{23}|(|y_{13}| - |y_{23}|) = 0$;
2) when $y_{12}y_{13}y_{23} > 0$, at least one of the following conditions holds: i) $M_{11} + (|y_{12}||y_{23}| - y_{22}|y_{13}|) > y_{33}|y_{12}| - |y_{23}||y_{13}| > 0$ or $M_{33} + (|y_{12}||y_{23}| - y_{22}|y_{13}|) > y_{11}|y_{23}| - |y_{12}||y_{13}| > 0$ holds and $y_{11}|y_{23}|(y_{22} - |y_{23}|) - y_{33}|y_{12}|(y_{22} - |y_{12}|) - |y_{12}||y_{23}|(|y_{12}| - |y_{23}|) = 0$; ii) $M_{22} + (|y_{12}||y_{13}| - y_{11}|y_{23}|) > y_{33}|y_{12}| - |y_{23}||y_{13}| > 0$ or $M_{33} + (|y_{13}||y_{12}| - y_{11}|y_{23}|) > y_{22}|y_{13}| - |y_{12}||y_{23}| > 0$ holds and $y_{22}|y_{13}|(y_{11} - |y_{13}|) - y_{33}|y_{12}|(y_{11} - |y_{12}|) - |y_{12}||y_{13}|(|y_{12}| - |y_{13}|) = 0$; iii) $M_{11} + (|y_{13}||y_{23}| - y_{33}|y_{12}|) > y_{22}|y_{13}| - |y_{23}||y_{12}| > 0$ or $M_{22} + (|y_{23}||y_{13}| - y_{33}|y_{12}|) > y_{11}|y_{23}| - |y_{13}||y_{12}| > 0$ holds and $y_{11}|y_{23}|(y_{33} - |y_{23}|) - y_{22}|y_{13}|(y_{33} - |y_{13}|) - |y_{13}||y_{23}|(|y_{13}| - |y_{23}|) = 0$.

Proof: The method is similar to that of Theorem 4. ■

Lemma 11: If a paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of the configuration in Fig. 10, then Y is also realizable as the admittance of the configuration in Fig. 9(a).

Proof: It suffices to prove that the admittance of the configuration in Fig. 10 always satisfies the condition of Theorem 4. The detail is omitted for brevity. ■

As a summary, the final result is obtained as follows.

Theorem 9: A paramount matrix $Y \in \mathbb{S}^3$ that does not satisfy the condition of Theorem 3 is realizable as the admittance of a three-port resistive network with five elements, if and only if Y

satisfies the condition of Lemma 5 for $k = 5$ or Y satisfies at least one of the conditions in Theorems 4–8.

Proof: This theorem follows directly from Lemmas 10 and 11 and Theorems 4–8. ■

Remark 1: By Kuratowski's Theorem in [23, p. 109], it can be implied that any graph containing less than nine edges must be planar. Therefore, the augmented graphs of three-port resistive networks containing no more than five elements must be planar, which means that the dual graphs always exist (see [34, Theorem 3.15]). Based on the principle of duality, the results in Theorems 1–9 can be directly transformed to those of the impedance case.

Remark 2: From results of this paper, it is clear that the least number of elements to realize the entire class of third-order paramount matrices is six.

Remark 3: One should note that the theorem of Reichert [33], which states that "any one-port network containing two reactive elements and an arbitrary number of resistors is equivalent to the one with two reactive elements and three resistors," assumes that values of reactive elements are arbitrary. The reduction in the number of resistors is partially due to the variation of reactive element values. In order to investigate synthesis of one-port networks containing two reactive elements with restricted values, which has important practical implications, the results of this paper can be utilized.

V. Numerical Examples

Example 1: Consider a paramount matrix $Y \in \mathbb{S}^3$ with $y_{11} = 2$, $y_{12} = 1$, $y_{13} = -1/3$, $y_{22} = 2$, $y_{23} = 4/3$, $y_{33} = 20/9$, which does not satisfy the condition of Lemma 5 for $k = 4$. Furthermore, it is checked that the condition of Theorem 2 holds with $y_{12}y_{13}y_{23} < 0$ and Condition i) being satisfied. Therefore, Y is realizable as the admittance of the configuration in Fig. 5, where Terminals a and b constitute Port 1 with the current from a to b, Terminals b and c constitute Port 2 with the current from b to c, and Terminals d and e constitute Port 3 with the current from e to d. The values of the elements satisfy $g_1 = 1$, $g_2 = 2$, $g_3 = 3$, and $g_4 = 3$.

Example 2: Consider a paramount matrix $Y \in \mathbb{S}^3$ with $y_{11} = 3$, $y_{12} = 1$, $y_{13} = -2$, $y_{22} = 5$, $y_{23} = 3$, and $y_{33} = 6$, which does not satisfy the condition of Theorem 3. Furthermore, it is checked that the condition of Lemma 5 holds for $k = 5$. Therefore, Y is realizable as the admittance of a network whose network graph containing four terminals and five elements, whose port graph is a Lagrangian tree.

Example 3: Consider a paramount matrix $Y \in \mathbb{S}^3$ with $y_{11} = 7$, $y_{12} = 5/3$, $y_{13} = 1$, $y_{22} = 20/9$, $y_{23} = 1/3$, and $y_{33} = 2$, which neither satisfies the condition of Theorem 3 nor satisfies the condition of Lemma 5 for $k = 5$. Furthermore, it is checked that the condition of Theorem 5 holds with $y_{12}y_{13}y_{23} > 0$ and Condition 2-ii) being satisfied. Therefore, Y is realizable as the admittance of the configuration in Fig. 9(b), where Terminals b and c constitute Port 1 with the current from c to b, Terminals a and b constitute Port 2 with the current from a to b, and Terminals d and e constitute Port 3 with the current from e to d. The values of the elements satisfy $g_1 = 2$, $g_2 = 3$, $g_3 = 1$, $g_4 = 3$, and $g_5 = 5$.

VI. Conclusion

This paper has been concerned with the minimal realization problem of three-port resistive networks. A necessary and sufficient condition was derived for any real symmetric matrix to be realizable as the admittance of four-terminal three-port resistive networks with at most k elements where $k \in \{1, 2, \ldots, 5\}$. Furthermore, necessary and sufficient conditions were obtained for any paramount matrix to be realizable as the admittance of three-port resistive networks with at most k elements where $k \in \{1, 2, 3, 4\}$ in Theorem 1 ($k \in \{1, 2, 3\}$) and Theorem 3 ($k = 4$). Moreover, a necessary and sufficient condition was derived for a paramount matrix that cannot be realized with less than five elements to be realizable as the admittance of three-port resistive networks with five elements in Theorem 9. Finally, some numerical examples were presented for illustration. In practice, the realizability conditions are always tested in the order from $k = 1$ to $k = 5$ to guarantee the minimality of realizations.

Appendix A
Proof of Theorem 2

Necessity: For Fig. 5, assume that Terminals a and b constitute Port 1 with the current from a to b; Terminals b and c constitute Port 2 with the current from b to c; Terminals d and e constitute Port 3 with the current from e to d. The entries of Y can be expressed as follows:

$$\begin{aligned} y_{11} &= (g_1+g_2)(g_3+g_4)/G,\ y_{12} = g_4(g_1+g_2)/G, \\ y_{13} &= (g_1g_4 - g_2g_3)/G,\ y_{22} = g_4(g_1+g_2+g_3)/G, \\ y_{23} &= g_4(g_1+g_3)/G,\ y_{33} = (g_1+g_3)(g_2+g_4)/G, \end{aligned} \tag{3}$$

where $G = g_1 + g_2 + g_3 + g_4$. Since $g_1, g_2, g_3, g_4 > 0$, it follows that $y_{11}, y_{12}, y_{22}, y_{23}, y_{33} > 0$. Furthermore, one can check that $M_{11} - |M_{12}| = 0$ and $M_{33} - |M_{23}| = 0$.

Moreover, other cases can be obtained by properly swapping some of two ports and switching the polarity of some ports, which correspond to a proper rearrangement of rows and corresponding columns and a finite number of cross-sign changes. Condition 1 or 2 also holds for other cases.

Sufficiency: Let the values of elements as in Fig. 5 satisfy

$$g_1 = \frac{(y_{33} - y_{23} + y_{13})M_{12}}{M_{12} - y_{23}(y_{33} - y_{23})}, \tag{4}$$

$$g_2 = \frac{M_{12}}{y_{23}}, \tag{5}$$

$$g_3 = \frac{(y_{23} - y_{13})M_{12}}{M_{12} - y_{23}(y_{33} - y_{23})}, \tag{6}$$

$$g_4 = \frac{M_{12}}{y_{33} - y_{23}}. \tag{7}$$

Since it is assumed that Y is not realizable with less than four elements, the condition of Lemma 5 does not hold for $k = 3$. Together with paramountcy, one implies that $y_{11}, y_{22}, y_{33} > 0$.

Assume that $y_{12}y_{13}y_{23} \leq 0$ and Condition i) holds. Assuming that $y_{12} = 0$ (resp. $y_{23} = 0$), it follows from $M_{33} - |M_{23}| = 0$ (resp. $M_{11} - |M_{12}| = 0$) that $y_{22} - |y_{23}| = 0$ (resp. $y_{22} - |y_{12}| = 0$). Then, together with $M_{11} - |M_{12}| = 0$ (resp. $M_{33} - |M_{23}| = 0$), one derives that $y_{33} - |y_{23}| - |y_{13}| = 0$ (resp. $y_{11} - |y_{12}| - |y_{13}| = 0$). This contradicts the assumption that the condition of Lemma 5 does not hold for $k = 3$. Therefore, $y_{12} \neq 0$ and $y_{23} \neq 0$. After some cross-sign changes, one obtains $y_{12}, y_{23} > 0$ and $y_{13} \leq 0$. Then, it follows that $M_{12} > 0$ and $M_{23} > 0$, implying that $M_{11} - M_{12} = M_{11} - |M_{12}| = 0$ and $M_{33} - M_{23} = M_{33} - |M_{23}| = 0$. From $M_{11} - M_{12} = 0$ and $M_{33} - M_{23} = 0$, one obtains that

$$y_{11}\left(M_{12} - y_{23}(y_{33} - y_{23})\right) - y_{33}y_{12}(y_{12} - y_{13}) = 0, \tag{8}$$

$$y_{33}\left(M_{23} - y_{12}(y_{11} - y_{12})\right) - y_{11}y_{23}(y_{23} - y_{13}) = 0. \tag{9}$$

Then, it is implied that $M_{12} - y_{23}(y_{33} - y_{23}) > 0$ and $M_{23} - y_{12}(y_{11} - y_{12}) > 0$. Moreover, $M_{11} - M_{12} = 0$ and $M_{33} - M_{23} = 0$ can also imply $y_{11} - y_{12} + y_{13} \neq 0$ and $y_{33} - y_{23} + y_{13} \neq 0$. Otherwise, Y would satisfy the condition of Lemma 5 for $k = 3$. Based on paramountcy of Y, one can prove that $y_{11} - y_{12} + y_{13}$ and $y_{33} - y_{23} + y_{13}$ cannot both be negative. Therefore, if $y_{33} - y_{23} + y_{13} < 0$, then exchanging the first and third rows and the columns can yield $y_{33} - y_{23} + y_{13} > 0$ without altering all the previous conditions. Hence, $y_{33} - y_{23} > 0$. As a consequence, the values of elements as expressed in (4)–(7) must be positive and finite. By $M_{11} - M_{12} = 0$ and (8), one can verify that (3) must hold. Therefore, the given Y is realizable as the required network.

Assume that $y_{12}y_{13}y_{23} > 0$ and Condition i) holds. After some cross-sign changes, one obtains $y_{12} > 0$, $y_{13} > 0$, and $y_{23} > 0$. Since M_{22} cannot be negative, it follows that at least one of M_{12} and M_{23} is non-negative. If $M_{12} \geq 0$ (resp. $M_{23} \geq 0$), then $M_{11} - M_{12} = M_{11} - |M_{12}| = 0$ (resp. $M_{33} - M_{23} = M_{33} - |M_{23}| = 0$), which is equivalent to $y_{33}(y_{22} - y_{12}) = y_{23}(y_{23} - y_{13})$ (resp. $y_{11}(y_{22} - y_{23}) = y_{12}(y_{12} - y_{13})$). Consequently, $y_{23} - y_{13} \geq 0$ (resp. $y_{12} - y_{13} \geq 0$), indicating that $M_{23} \geq 0$ (resp. $M_{12} \geq 0$). Therefore, together with the assumption that the condition of Lemma 5 does not hold for $k = 3$, we assert that $y_{12} > y_{13} > 0$, $y_{23} > y_{13} > 0$, $M_{12} > 0$, and $M_{23} > 0$. Therefore, $M_{11} - M_{12} = M_{11} - |M_{12}| = 0$ and $M_{33} - M_{23} = M_{33} - |M_{23}| = 0$, which further implies (8). Then, $M_{12} - y_{23}(y_{33} - y_{23}) > 0$. Supposing that $y_{33} - y_{23} = 0$, together with $M_{11} - M_{12} = 0$ and $M_{33} - M_{23} = 0$ one implies that $y_{11} = y_{12}$, $y_{33} = y_{23}$, and $(y_{22} - y_{23}) - (y_{12} - y_{13}) = 0$, contradicting the condition that the condition of Lemma 5 does not hold for $k = 3$. Hence, $y_{33} - y_{23} > 0$, indicating that $y_{33} - y_{23} + y_{13} > 0$. As a consequence, the values of g_1, g_2, g_3, and g_4 as expressed in (4)–(7) must be positive and finite. By $M_{11} - M_{12} = 0$ and (8), one can verify that (3) must hold. Therefore, the given Y is realizable as the required network.

Besides, if Y satisfies other cases of Condition 1 or 2, then properly arranging the rows and columns of Y can always yield one of the above two cases. Now, the theorem is proved.

Appendix B
Proof of Theorem 4

Necessity: For Fig. 9(a), assume that Terminals a and b constitute Port 1 with the current from a to b; Terminals b and c constitute Port 2 with the current from b to c; Terminals d and e constitute Port 3 with the current from e to d. Then,

$$\begin{aligned} y_{11} &= ((g_1 + g_2)(g_3 + g_4 + g_5) + g_5(g_3 + g_4))/G, \\ y_{12} &= g_4(g_1 + g_2)/G, \; y_{13} = (g_1 g_4 - g_2 g_3)/G, \\ y_{22} &= g_4(g_1 + g_2 + g_3)/G, \; y_{23} = g_4(g_1 + g_3)/G, \\ y_{33} &= (g_1 + g_3)(g_2 + g_4)/G, \end{aligned} \tag{10}$$

where $G = g_1 + g_2 + g_3 + g_4$. Since $g_1, g_2, g_3, g_4, g_5 > 0$, it follows that $y_{11}, y_{12}, y_{22}, y_{23}, y_{33} > 0$. If $y_{13} \leq 0$, then $y_{12}y_{13}y_{23} \leq 0$, $y_{33} - |y_{13}| - |y_{23}| = y_{33} + y_{13} - y_{23} = g_1(g_2 + g_4)/(g_1 + g_2 + g_3 + g_4) > 0$, $M_{12} = y_{12}y_{33} - y_{13}y_{23} > 0$, and $M_{11} - |M_{12}| = M_{11} - M_{12} = 0$, implying that Condition 1 must hold. If $y_{13} > 0$, then $y_{12}y_{13}y_{23} > 0$, $|y_{12}| - |y_{13}| = y_{12} - y_{13} = g_2(g_3 + g_4)/(g_1 + g_2 + g_3 + g_4) > 0$, $|y_{23}| - |y_{13}| = y_{23} - y_{13} = g_3(g_2 + g_4)/(g_1 + g_2 + g_3 + g_4) > 0$, $M_{12} = g_2 g_4(g_1 + g_3)/(g_1 + g_2 + g_3 + g_4) > 0$, and $M_{11} - |M_{12}| = M_{11} - M_{12} = 0$, implying that Condition 2 must hold.

Other cases can be obtained by properly swapping some of two ports and switching the polarity of some ports, which correspond to a proper rearrangement of rows and corresponding columns and a finite number of cross-sign changes. It is noted that Condition 1 or 2 also holds for other cases.

Sufficiency: Let the values of elements as in Fig. 9(a) satisfy

$$g_1 = \frac{(y_{33} - y_{23} + y_{13})M_{12}}{M_{12} - y_{23}(y_{33} - y_{23})}, \tag{11}$$

$$g_2 = \frac{M_{12}}{y_{23}}, \tag{12}$$

$$g_3 = \frac{(y_{23} - y_{13})M_{12}}{M_{12} - y_{23}(y_{33} - y_{23})}, \tag{13}$$

$$g_4 = \frac{M_{12}}{y_{33} - y_{23}}, \tag{14}$$

$$g_5 = \frac{y_{11}\left(M_{12} - y_{23}(y_{33} - y_{23})\right) - y_{33}y_{12}(y_{12} - y_{13})}{M_{12} - y_{23}(y_{33} - y_{23})}. \tag{15}$$

Since it is assumed that the condition of Theorem 3 does not hold, neither the condition of Lemma 5 for $k = 4$ nor that of Theorem 2 holds. Together with the condition of paramountcy, one implies that $y_{11}, y_{22}, y_{33} > 0$. By Lemma 6, it is implied that at most one of y_{12}, y_{13}, and y_{23} is zero.

Assume that $y_{12}y_{13}y_{23} \leq 0$ and Condition 1-i) holds. One can always guarantee $M_{11} - |M_{12}| = 0$ by properly interchanging the first and second rows and columns of Y if necessary. After some cross-sign changes one obtains $y_{12}, y_{23} \geq 0$, and $y_{13} \leq 0$. Since at most one of y_{12}, y_{13}, and y_{23} is zero, it follows that $y_{12} - y_{13} > 0$, $y_{23} - y_{13} > 0$, $M_{12} > 0$, $M_{23} > 0$, and $M_{11} - M_{12} = M_{11} - |M_{12}| = 0$. Moreover, $y_{33} - y_{23} + y_{13} = y_{33} - |y_{13}| - |y_{23}| > 0$, indicating that $y_{33} - y_{23} > 0$. From $M_{11} - M_{12} = 0$, it can be implied that $y_{23} > 0$, since the condition of Lemma 5 does not hold for $k = 4$. Moreover, since $M_{33} - |M_{23}| = 0$ would contradict the assumption that the condition of Theorem 2 does not hold, it follows that $M_{33} - M_{23} = M_{33} - |M_{23}| > 0$ together with the condition of paramountcy. Therefore, $M_{11} - M_{12} = 0$ yields

$$\begin{aligned} &y_{11}\left(M_{12} - y_{23}(y_{33} - y_{23})\right) \\ &\quad - y_{33}y_{12}(y_{12} - y_{13}) = y_{33}(M_{33} - M_{23}) > 0, \end{aligned} \tag{16}$$

further implying that $M_{12} - y_{23}(y_{33} - y_{23}) > 0$. As a consequence, the values of elements as expressed in (11)–(15) must be positive and finite. Moreover, (11)–(15) and $M_{11} - |M_{12}| = 0$ imply (10). Therefore, the given Y is realizable as the required network.

Assume that $y_{12}y_{13}y_{23} > 0$ and Condition 2-i) holds. After some cross-sign changes, one obtains $y_{12} > y_{13} > 0$ and $y_{23} > y_{13} > 0$. Moreover, one makes $M_{11} - |M_{12}| = 0$ by properly interchanging the first and third rows and columns of Y if necessary, which does not alter the assumption. Together with the condition of paramountcy, it follows that $M_{12} > y_{13}(y_{33} - y_{23}) \geq 0$ and $M_{23} > y_{13}(y_{11} - y_{12}) \geq 0$, which implies that $M_{11} - M_{12} = M_{11} - |M_{12}| = 0$. Since the condition of Lemma 5 does not hold for $k = 4$, it follows that $y_{33} - y_{23} > 0$, implying that $y_{33} - y_{23} + y_{13} > 0$. Considering that $M_{33} - |M_{23}| = 0$ would contradict the assumption that the condition of Theorem 2 does not hold, it follows that $M_{33} - M_{23} = M_{33} - |M_{23}| > 0$ together with the condition of paramountcy. Since $M_{11} - |M_{12}| = 0$ yields (16), it follows that $M_{12} - y_{23}(y_{33} - y_{23}) > 0$. As a consequence, the values of elements as expressed in (11)–(15) must be positive and finite. Moreover, (11)–(15) and $M_{11} - |M_{12}| = 0$ imply (10). Therefore, the given Y is realizable as the required network.

Besides, if Y satisfies other cases of Condition 1 or 2, then properly arranging the rows and corresponding columns of Y can always yield one of the above two cases.

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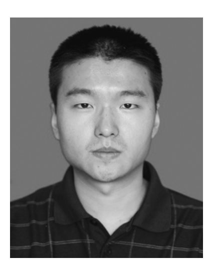

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