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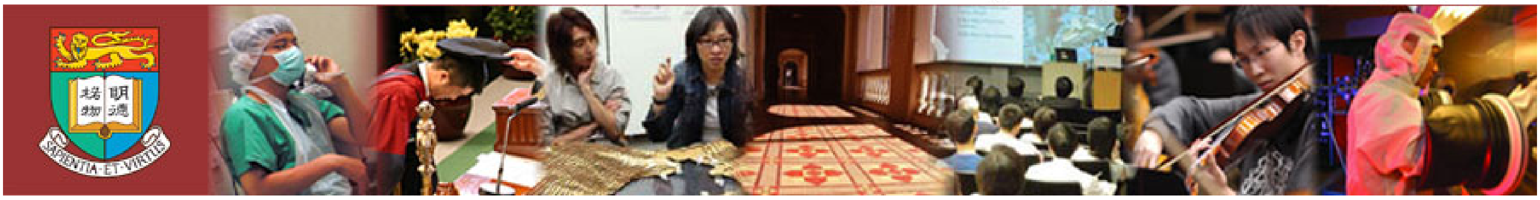

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On a Gerber-Shiu type function and its applications in a dual semi-Markovian risk model

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Abstract

In this paper, we consider a dual risk process which can be used to model the surplus of a business that invests money constantly and earns gains randomly in both time and amount. The occurrences of the gains and their amounts are assumed follow a semi-Markovian structure (e.g. Reinhard (1984)). We analyze a quantity resembling the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) that incorporates random variables defined before and after the time of ruin, such as the minimum surplus level before ruin and the time of the first gain after ruin. General properties of the function are studied, and some exact results are derived upon exponential distributional assumptions on either the inter-arrival times or the gain amounts. Applications in a perpetual insurance and the last inter-arrival time containing the time of ruin are given along with some numerical examples.

Keywords: Dual risk model; Semi-Markovian risk process; Gerber-Shiu function; Generalized penalty function; Perpetual insurance; Last inter-arrival time.

1 Introduction

In a dual risk model, the surplus process $\{U(t)\}_{t\geq 0}$ of a business enterprise is described by

$$U(t) = u - ct + \sum_{n=1}^{N(t)} Y_n, \qquad t \geq 0, \tag{1.1}$$

where $u = U(0) \geq 0$ is the initial surplus, $c > 0$ is the constant rate of expenses per unit time, $\{Y_n\}_{n=1}^{\infty}$ is a sequence of positive random variables with Y_n being the size of the n-th gain (also known as innovation), and $\{N(t)\}_{t\geq 0}$ is a counting process that counts the number of gains. The time of ruin is defined by $\tau_U = \inf\{t \geq 0 : U(t) = 0\}$, with the usual convention that $\tau_U = \infty$ if $U(t) > 0$ for all $t \geq 0$. Note that if the process starts with zero initial surplus, then ruin occurs immediately at time 0. The dual model is appropriate for companies which incur expenses at a fixed rate and earn gains that are random in both time and amount. According to e.g. Avanzi et al. (2007), these include pharmaceutical and petroleum companies where one can view each upward jump Y_n as the net present value of future income arising from an invention or discovery. If $\{N(t)\}_{t\geq 0}$ is assumed to be a Poisson process and $\{Y_n\}_{n=1}^{\infty}$ is an

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independent and identically distributed sequence independent of $\{N(t)\}_{t\geq 0}$, then the model (1.1) reduces to the dual compound Poisson model, for which classical results about ruin probability are available in e.g. Cramér (1955, Section 5.13), Takács (1967, pp. 152-154), Seal (1969, pp. 116-119), and Grandell (1991, p.8). Under such a case, Avanzi et al. (2007, 2013), Cheung and Drekic (2008), Gerber and Smith (2008), and Ng (2009) have recently studied dividend problems under a barrier or threshold dividend strategy; whereas Landriault and Sendova (2011) considered the case in which the expense rate can be reduced if no gain occurs within an Erlang(n) period of time. Further results concerning the time of ruin when $\{N(t)\}_{t\geq 0}$ is more generally a renewal process instead of a Poisson process have also been derived by e.g. Mazza and Rullière (2004) and Albrecher et al. (2008, Appendix). Related risk processes in which the deterministic expense in (1.1) is replaced by a stochastic component have been studied by e.g. Boikov (2002), Temnov (2004), Labbé and Sendova (2009), Albrecher et al. (2010), Labbé et al. (2011).

The afore-mentioned analyses of the dual risk model are performed under the assumption that the inter-arrival times and the resulting gain sizes are all independent. However, in reality this is too good to be true. Motivated by the inadequacy of the independence assumption, a few researchers have considered dual models with dependence. For example, Yang and Zhu (2008) obtained some inequalities for the infinite-time and finite-time ruin probabilities in a dual Markov-modulated risk process; whereas Cheung (2008) looked at a threshold dividend strategy when gains occur according to a Markovian arrival process using a connection to a fluid flow process. In these two contributions, the dependency is modelled via a continuous-time Markov chain (see Remark 1). Cheung (2012) studied another dependency structure where a given gain size has an impact on the next inter-arrival time, which is commonly referred to as a dependent Sparre Andersen model in the terminology of insurance ruin theory (see e.g. Albrecher and Teugels (2006) and Cheung et al. (2010b)).

In this paper, we shall study a different risk model that belongs to the class of semi-Markovian risk processes. In the context of an insurance risk process (which is a reflection of the model (1.1)), semi-Markovian models were defined under fairly general terms and studied in some early papers by e.g. Reinhard (1984, Equation (1.1)) and Janssen and Reinhard (1985, Equation (1.1)). The dual semi-Markovian risk model to be considered here is described as follows. First, we define $\{G_n\}_{n=0}^{\infty}$ to be a time-homogeneous and irreducible discrete-time Markov chain on the state space $\mathcal{E} = \{1, 2, \ldots, m\}$, where G_0 is the environmental state at time 0 and G_n is the environmental state immediately after the n-th gain for $n = 1, 2, \ldots$. The above Markov chain is assumed to have one-period transition probability matrix $\mathbf{P} = [p_{ij}]_{i,j=1}^{m}$. Defining $T_0 = 0$ and denoting the time of the n-th gain by T_n for $n = 1, 2, \ldots$, the gain counting process $\{N(t)\}_{t\geq 0}$ is given by $N(t) = \sup\{n \in \mathbb{N} : T_n \leq t\}$. Moreover, for $n = 1, 2, \ldots$ we define $V_n = T_n - T_{n-1}$ to be the time between the $(n-1)$-th and the n-th gain arrivals. Then, the dependency structure in our model is summarized by, for $n = 1, 2, \ldots$; $i, j \in \mathcal{E}$ and $t, y \geq 0$,

$$\begin{aligned}
& \Pr\{V_n \leq t, Y_n \leq y, G_n = j | G_{n-1} = i, (V_k, Y_k, G_k) \text{ for } k = 0, 1, \ldots, n-1\} \\
= \; & \Pr\{V_1 \leq t, Y_1 \leq y, G_1 = j | G_0 = i\} \\
= \; & K_i(t)\, B_i(y)\, p_{ij}. \qquad (1.2)
\end{aligned}$$

(For notational convenience we define $V_0 = Y_0 = 0$.) The above dynamics imply that, for $n = 1, 2, \ldots$, the inter-arrival time V_n and the resulting gain size Y_n are conditionally independent given the state G_{n-1}. In particular, $V_n | G_{n-1} = i$ has cumulative distribution function (c.d.f.) $K_i(\cdot)$ with corresponding density $k_i(\cdot)$ and mean κ_i; whereas $Y_n | G_{n-1} = i$ has c.d.f. $B_i(\cdot)$ with corresponding density $b_i(\cdot)$ and mean β_i. In other words, if the environmental state immediately after the previous gain is i, then the time until the next gain and the size of the resulting gain have densities $k_i(\cdot)$ and $b_i(\cdot)$ respectively, and the environmental state will become j with probability p_{ij} immediately after the next gain.

Remark 1 Markovian arrival process (MAP) is very well documented in the literature of applied probability (e.g. Neuts (1989), Latouche and Ramaswami (1999), and Asmussen (2003, Chapter XI)), and insurance risk processes where claims occur according to a MAP have also gained popularity in recent years (e.g. Badescu et al. (2005), Ahn and Badescu (2007), and Cheung and Landriault (2010)). In a dual MAP model, $\{N(t)\}_{t\geq 0}$ in (1.1) is replaced by a MAP, and the background process $\{J(t)\}_{t\geq 0}$ is a time-homogeneous irreducible continuous-time Markov chain with finite state space $\{1, 2, \ldots, m\}$ governed by the generators $\mathbf{D}_0 = [D_{0,ij}]_{i,j=1}^m$ and $\mathbf{D}_1 = [D_{1,ij}]_{i,j=1}^m$. While $D_{0,ij} \geq 0$ is the transition rate of $\{J(t)\}_{t\geq 0}$ from state i to state $j \neq i$ without a gain; $D_{1,ij} \geq 0$ represents the transition rate from state i to state j with an accompanying gain. The diagonal elements of $\mathbf{D}_0$ are such that $\mathbf{D}_0 + \mathbf{D}_1$ is zero. If the n-th gain Y_n is a result of a transition of $\{J(t)\}_{t\geq 0}$ from state i to state j, then it is assumed to have density $f_{ij}(\cdot)$. Note that the dual Markov-modulated model in Yang and Zhu (2008) can be retrieved from the above MAP model by letting $\mathbf{D}_1$ be a diagonal matrix and assuming that $f_{ij}(\cdot)$ does not depend on j.

It is instructive to note that the present semi-Markovian risk process defined via (1.2) inherits certain characteristics of the MAP model and the Sparre Andersen (or renewal) model. For example, the dependency structures in the semi-Markovian and the MAP models are similar in the sense that they are both introduced via a Markov chain (though it is a discrete-time Markov chain in the former and a continuous-time one in the latter). However, the inter-arrival times in a MAP model must be phase-type distributed; whereas the semi-Markovian model allows for general inter-arrival time distribution. In this aspect, the semi-Markovian model resembles the Sparre Andersen model. Nonetheless, we also note that if one lets $K_i(t) = 1 - e^{D_{0,ii}t}$ (see Section 3) and $p_{ij} = -D_{1,ij}/D_{0,ii}$ in the semi-Markovian model (1.2), then it corresponds to a MAP model in which $\mathbf{D}_0$ is diagonal and $f_{ij}(\cdot) = b_i(\cdot)$. $\square$

In this paper, we aim at studying (a generalization of) the Gerber-Shiu type function under the afore-mentioned dual semi-Markovian model. Recall that Gerber and Shiu (1998) defined the expected discounted penalty function (now commonly known as the Gerber-Shiu function) in an insurance risk model to be the expectation of the present value of a 'penalty' applied at the time of ruin, with the 'penalty' being a function of the surplus prior to ruin and the deficit at ruin. However, in a dual risk model, both the surplus $U(\tau_U^-)$ prior to ruin and the deficit $|U(\tau_U)|$ at ruin are zero and therefore the Gerber-Shiu function is simply (a constant multiple of) the Laplace transform of the time of ruin. In most situations, even when the surplus of a business line drops below zero, it can usually survive negative surplus for a while by obtaining funds from another line of the same business or by borrowing. Due to the positive security loading condition (see (2.10)), the ruined surplus process will be able to recover eventually, and a quick recovery is always desirable. Therefore, quantities in relation to the survival of the business line *after* ruin are also of critical importance. See also comments in e.g. Egidio dos Reis (1993) and Gerber (1990). These motivated Cheung (2012) to consider the random variables

$$\tau_U^* = \sum_{n=1}^{N(\tau_U)+1} V_n \tag{1.3}$$

and

$$|U(\tau_U^{*-})| = \left| u + \sum_{n=1}^{N(\tau_U)} (Y_n - cV_n) - cV_{N(\tau_U)+1} \right| = \left| u + \sum_{n=1}^{N(\tau_U)} Y_n - c\tau_U^* \right|, \tag{1.4}$$

given an initial surplus of $U(0) = u \geq 0$. Clearly, τ_U^* represents the time of the first gain after the ruin time τ_U; whereas $|U(\tau_U^{*-})|$ is the amount of shortfall just before the first gain after ruin. On the other hand, from a risk management perspective, it can be important to keep track the behaviour of the sample paths leading to ruin. In this regard, various researchers (see e.g. Biffis and Morales (2010), Cheung and Landriault (2010), and Cheung et al. (2010a,b)) have attempted to incorporate various random variables defined *before* ruin into the Gerber-Shiu function to gain additional insights, albeit in the insurance context. In the present dual model, we also consider the surplus level immediately after the last gain before ruin, namely $U(T_{N(\tau_U)})$ which is given by

$$U(T_{N(\tau_U)}) = u + \sum_{n=1}^{N(\tau_U)} (Y_n - cV_n) = u + \sum_{n=1}^{N(\tau_U)} Y_n - cT_{N(\tau_U)}. \tag{1.5}$$

In addition, for $n = 1, 2, \ldots$ we define $R_{U,n} = u - c\sum_{k=1}^{n} V_k + \sum_{k=1}^{n-1} Y_k$ to be the surplus level just before the n-th gain. Although the surplus level just before the 0-th gain is not well-defined, for $n = 0$ we simply set $R_{U,0} = U(0) = u$ (see Remark 2). In the case where there is at least one gain before ruin (i.e. $N(\tau_U) \geq 1$), the surplus level just before the last gain before ruin is $R_{U,N(\tau_U)}$. Moreover, the minimum value of the sequence $\{R_{U,n}\}_{n=0}^{\infty}$ before ruin is given by $\underline{R}_U = \min_{n\in\{0,1,\ldots,N(\tau_U)\}} R_{U,n}$. With the above definitions, we propose to study the Gerber-Shiu type function, for $i, j \in \mathcal{E}$ and $u \geq 0$,

$$\phi_{\delta,ij,U}(u) = E\left[e^{-\delta\tau_U^*} w(U(T_{N(\tau_U)}), |U(\tau_U^{*-})|, \underline{R}_U, R_{U,N(\tau_U)}) 1\{\tau_U < \infty\} 1\{G_{N(\tau_U)} = j\} | G_0 = i, U(0) = u\right], \tag{1.6}$$

where $w(\cdot,\cdot,\cdot,\cdot)$ is the penalty function that satisfies some mild integrability conditions, $1\{A\}$ is the indicator function of the event A, and $\delta \geq 0$ is the force of interest or the Laplace transform argument with respect to τ_U^*. A sample path of the surplus process $\{U(t)\}_{t\geq 0}$ and the associated random variables defined above are shown in Figure 1. Because the ruin time τ_U is related to τ_U^* and $|U(\tau_U^{*-})|$ via the identity $\tau_U = \tau_U^* - |U(\tau_U^{*-})|/c$, its Laplace transform (with argument δ) is a special case of (1.6) by setting $w(x, y, v, r) = e^{(\delta/c)y}$. Similarly, information about the last inter-arrival time $V_{N(\tau_U)+1} = (U(T_{N(\tau_U)}) + |U(\tau_U^{*-})|)/c$ containing the time of ruin (see Section 5.2) as well as the last gain before ruin $Y_{N(\tau_U)} = U(T_{N(\tau_U)}) - R_{U,N(\tau_U)}$ (in the case where $N(\tau_U) \geq 1$) can be retrieved from (1.6) via appropriate choices of the penalty function. We remark that the Gerber-Shiu function, being defined as an expectation, represents an average value. Therefore, it is applicable in e.g. pricing insurance contract and stochastic ordering where expected values are concerned (see Section 5). Moreover, in principle the discounted densities associated with the variables $(U(T_{N(\tau_U)}), |U(\tau_U^{*-})|, \underline{R}_U, R_{U,N(\tau_U)})$ in the penalty function are also obtainable from the Gerber-Shiu function (see Remark 3 in Section 2). However, one of the limitations of the above Gerber-Shiu function is that it only involves a few selected random variables along sample paths that lead to ruin. We refer interested readers to Cai et al. (2009) and Cheung and Feng (2013) for the study of an alternative function that depends on the entire sample path until ruin as well as its connection with the usual Gerber-Shiu function.

INSERT FIGURE 1

Figure 1: Sample path of $\{U(t)\}_{t\geq 0}$ and related random variables

In order to study the Gerber-Shiu function $\phi_{\delta,ij,U}(u)$ defined by (1.6) that contains information both *before* and *after* the time of ruin, in Section 2 we define a useful auxiliary process $\{Z(t)\}_{t\geq 0}$ to aid our analysis. In particular, $\phi_{\delta,ij,U}(u)$ can be expressed in terms of another Gerber-Shiu function pertaining to $\{Z(t)\}_{t\geq 0}$. In general, this latter Gerber-Shiu function is shown to satisfy a Markov renewal equation without any specific distributional assumptions. Sections 3 and 4 are respectively concerned with the derivations of some exact results when either the inter-arrival times or the gain sizes are exponential. Section 5 provides applications of our results in (i) the fair price of a perpetual insurance that keeps the business alive whenever the surplus reaches zero, and (ii) the distribution of the last inter-arrival time containing the time of ruin. These are accompanied by some numerical illustrations as well.

2 General structures

Following Cheung (2012) (who studied a dual dependent Sparre Andersen model), we first start by defining an auxiliary process $\{Z(t)\}_{t\geq 0}$ as follows. For convenience we let $W_n = V_{n+1}$ be the shifted inter-arrival time and $X_n = Y_{n+1}$ be the shifted gain amount for $n = 1, 2, \ldots$. The counting process $\{M(t)\}_{t\geq 0}$ corresponding to the sequence $\{W_n\}_{n=1}^{\infty}$ is then $M(t) = \sup\{n \in \mathbb{N} : \sum_{k=1}^{n} W_k \leq t\}$. Under an initial level of $Z(0^-) = z \geq 0$, the process $\{Z(t)\}_{t\geq 0}$ is defined by

$$Z(t) = z + Y_1 - ct + \sum_{n=1}^{M(t)} X_n, \qquad t \geq 0. \tag{2.1}$$

Note that the initial level here is defined as the level just before time 0, and $\{Z(t)\}_{t\geq 0}$ indeed starts with an upward jump of size Y_1 at time 0 and therefore $Z(0) = z + Y_1$. The ruin time of $\{Z(t)\}_{t\geq 0}$ is $\tau_Z = \inf\{t \geq 0 : Z(t) = 0\}$. In addition, the analogs of (1.3) and (1.4) in the process $\{Z(t)\}_{t\geq 0}$, both defined *after* the time of ruin, are given by

$$\tau_Z^* = \sum_{n=1}^{M(\tau_Z)+1} W_n$$

and

$$|Z(\tau_Z^{*-})| = \left| z + Y_1 + \sum_{n=1}^{M(\tau_Z)} (X_n - cW_n) - cW_{M(\tau_Z)+1} \right| = \left| z + Y_1 + \sum_{n=1}^{M(\tau_Z)} X_n - c\tau_Z^* \right|.$$

Letting $R_{Z,n} = z + \sum_{i=1}^{n}(Y_i - cW_i)$ for $n = 1, 2, \ldots$ with starting value $R_{Z,0} = z$, we are interested in three random variables defined *before* ruin, namely

$$Z(\tau_Z^* - W_{M(\tau_Z)+1}) = z + Y_1 + \sum_{n=1}^{M(\tau_Z)} (X_n - cW_n)$$

(which corresponds to (1.5)), $R_{Z,M(\tau_Z)}$ and $\underline{R}_Z = \min_{n\in\{0,1,\ldots,M(\tau_Z)\}} R_{Z,n}$. See Figure 2. Clearly, the relationships $\tau_Z = \tau_Z^* - |Z(\tau_Z^{*-})|/c$; $W_{M(\tau_Z)+1} = (Z(\tau_Z^* - W_{M(\tau_Z)+1}) + |Z(\tau_Z^{*-})|)/c$ and $X_{M(\tau_U)} = Z(\tau_Z^* - W_{M(\tau_Z)+1}) - R_{Z,M(\tau_Z)}$ hold. Now, the Gerber-Shiu function pertaining to $\{Z(t)\}_{t\geq 0}$ is defined by, for $i, j \in \mathcal{E}$ and $z \geq 0$,

$$\begin{aligned}
&\phi_{\delta,ij,Z}(z) \\
&= E\left[e^{-\delta\tau_Z^*} w(Z(\tau_Z^* - W_{M(\tau_Z)+1}), |Z(\tau_Z^{*-})|, \underline{R}_Z, R_{Z,M(\tau_Z)}) 1\{\tau_Z < \infty\} 1\{G_{M(\tau_Z)+1} = j\} | G_0 = i, Z(0^-) = z\right].
\end{aligned}$$

Here, G_n is still the environmental state immediately after the n-th jump for $n = 1, 2, \ldots$. Since a jump occurs at time 0, G_0 is the environmental state at time 0^-.

INSERT FIGURE 2

Figure 2: Sample path of $\{Z(t)\}_{t\geq 0}$ and related random variables

Comparing the definitions of the processes $\{U(t)\}_{t\geq 0}$ and $\{Z(t)\}_{t\geq 0}$, it can be seen that (apart from a shift of the initial level) $\{Z(t)\}_{t\geq 0}$ behaves like a copy of $\{U(t)\}_{t\geq 0}$ with the first V_1 time units removed. Hence, to study the Gerber-Shiu function $\phi_{\delta,ij,U}(u)$ pertaining to $\{U(t)\}_{t\geq 0}$ with initial surplus $U(0) = u \geq 0$, we distinguish between two cases when conditioning on the time V_1 of the first gain.

1. If the first gain occurs at time $t < u/c$, then $\{U(t)\}_{t\geq 0}$ simply reverts to $\{Z(t)\}_{t\geq 0}$ at time t with the newly established initial level $Z(0^-) = u - ct$.
2. If the first gain occurs at time $t \geq u/c$, then ruin of $\{U(t)\}_{t\geq 0}$ occurs at time $\tau_U = u/c$ with $\tau_U^* = t$; $U(T_{N(\tau_U)}) = \underline{R}_U = R_{U,N(\tau_U)} = u$ and $|U(\tau_U^{*-})| = ct - u$ (see Remark 2).

Further taking into account the states of the underlying Markov chain $\{G_n\}_{n=0}^{\infty}$, we arrive at, for $i, j \in \mathcal{E}$ and $u \geq 0$,

$$\phi_{\delta,ij,U}(u) = \int_0^{\frac{u}{c}} e^{-\delta t} k_i(t)\, \phi_{\delta,ij,Z}(u - ct)\, dt + 1\{i = j\} \int_{\frac{u}{c}}^{\infty} e^{-\delta t} k_i(t)\, w(u, ct - u, u, u)\, dt. \tag{2.2}$$

Since the second term in the above equation is known explicitly, it is clear that the Gerber-Shiu function $\phi_{\delta,ij,U}(\cdot)$ can be characterized by $\phi_{\delta,ij,Z}(\cdot)$. It remains to analyze $\phi_{\delta,ij,Z}(\cdot)$ in general terms.

Remark 2 Because the surplus level just before the 0-th gain is not well-defined, the definition of $R_{U,N(\tau_U)}$ (and hence $\underline{R}_U$) is of slightly different nature depending on whether $V_1 < u/c$ (i.e. $N(\tau_U) \geq 1$) or $V_1 \geq u/c$ (i.e. $N(\tau_U) = 0$). When $N(\tau_U) = 0$, although the definition $R_{U,N(\tau_U)} = R_{U,0} = u$ is artificial, its contribution only appears via the second term in (2.2) which can be readily modified. Since our focus will be to identify $\phi_{\delta,ij,Z}(\cdot)$, this definition would not affect our upcoming analysis. In addition, one can also allow the time V_1 of the first gain to follow a different density by simply replacing $k_i(\cdot)$ in (2.2) with the appropriate density, so that $\{U(t)\}_{t\geq 0}$ resembles a delayed risk process (see e.g. Willmot (2004), and Woo (2010)). $\square$

In order to analyze $\phi_{\delta,ij,Z}(z)$, it is sufficient to focus on the process $\{Z(t)\}_{t\geq 0}$. Because of (1.2), the dynamics of $\{Z(t)\}_{t\geq 0}$ can be described by, for $n = 1, 2, \ldots$; $i, j \in \mathcal{E}$ and $t, y \geq 0$,

$$\begin{aligned} & \Pr\{Y_n \leq y, W_n \leq t, G_n = j | G_{n-1} = i, (Y_k, W_k, G_k) \text{ for } k = 0, 1, \ldots, n-1\} \\ = \; & \Pr\{Y_1 \leq y, W_1 \leq t, G_1 = j | G_0 = i\} \\ = \; & B_i(y)\, p_{ij}\, K_j(t), \end{aligned} \tag{2.3}$$

with the definition $W_0 = 0$. This resembles Albrecher and Boxma (2005, Equation (2)) and Cheung and Landriault (2009, Equation (1.1)). Suppose that we observe $\{Z(t)\}_{t\geq 0}$ at times $\{(\sum_{k=1}^{n} W_k)^-\}_{n=0}^{\infty}$ so that the sequence of increments is $\{Y_n - cW_n\}_{n=1}^{\infty}$. Then (2.3) indicates a semi-Markovian version of a 'random walk' structure, and therefore one can proceed by conditioning on the first 'drop' of $\{Z(t)\}_{t\geq 0}$ below its initial level $Z(0^-)$ (see Figure 2) and the state of $\{G_n\}_{n=0}^{\infty}$ at the time of the 'drop'. To do so, we need to introduce various (discounted) joint densities in relation to $(Z(\tau_Z^* - W_{M(\tau_Z)+1}), |Z(\tau_Z^{*-})|, R_{Z,M(\tau_Z)})$ as follows. First, given $G_0 = i$ and $Z(0^-) = z$, we note that the joint density of the quadruple $(\tau_Z^*, Z(\tau_Z^* - W_{M(\tau_Z)+1}), |Z(\tau_Z^{*-})|, R_{Z,M(\tau_Z)})$ is of different forms depending on whether there is a second jump (including the one at time 0) before ruin of $\{Z(t)\}_{t\geq 0}$ occurs. More specifically, for $M(\tau_Z) = 0$, the joint density of $(Z(\tau_Z^* - W_{M(\tau_Z)+1}), |Z(\tau_Z^{*-})|)$ at (x, y) together with $G_1 = j$ is simply

$$h^*_{1,ij,Z}(x,y|z) = \frac{1}{c}\, b_i(x-z)\, p_{ij}\, k_j \left(\frac{x+y}{c} \right), \qquad x > z; y > 0. \tag{2.4}$$

The above density of sufficient to characterize the distribution of $(\tau^*_Z, Z(\tau^*_Z - W_{M(\tau_Z)+1}), |Z(\tau^{*-}_Z)|, R_{Z,M(\tau_Z)})$, since one has the relationships $\tau^*_Z = (x+y)/c$ and $R_{Z,M(\tau_Z)} = R_{Z,0} = z$. On the other hand, for $M(\tau_Z) \geq 1$ there are no simple relationships among the random variables, and we denote the joint density of $(\tau^*_Z, Z(\tau^*_Z - W_{M(\tau_Z)+1}), |Z(\tau^{*-}_Z)|, R_{Z,M(\tau_Z)})$ at (t,x,y,v) together with $G_{M(\tau_Z)} = j$ by $h^*_{2,ij,Z}(t,x,y,v|z)$ for $x > v > 0; y > 0$ and $t > (\max(x,z)+y)/c$. (Note that $h^*_{2,ij,Z}(t,x,y,v|z) = 0$ for $t \leq (\max(x,z)+y)/c$ because it takes at least $(\max(x,z)+y)/c$ time units for $\{Z(t)\}_{t\geq 0}$ to reach $-y$ from $\max(x,z)$.) By defining the diagonal matrices $\mathbf{k}(t) = \mathrm{diag}\{k_1(t), \ldots, k_m(t)\}$ and $\mathbf{b}(y) = \mathrm{diag}\{b_1(y), \ldots, b_m(y)\}$, the discounted (with respect to τ^*_Z) densities associated to $\mathbf{h}^*_{1,Z}(x,y|z) = [h^*_{1,ij,Z}(x,y|z)]^m_{i,j=1}$ and $\mathbf{h}^*_{2,Z}(t,x,y,v|z) = [h^*_{2,ij,Z}(t,x,y,v|z)]^m_{i,j=1}$ are respectively given by

$$\mathbf{h}^*_{1,\delta,Z}(x,y|z) = e^{-\delta\left(\frac{x+y}{c}\right)} \mathbf{h}^*_{1,Z}(x,y|z) = \frac{1}{c}\, e^{-\delta\left(\frac{x+y}{c}\right)} \mathbf{b}(x-z)\, \mathbf{P}\, \mathbf{k} \left(\frac{x+y}{c} \right), \qquad x > z; y > 0, \tag{2.5}$$

and

$$\mathbf{h}^*_{2,\delta,Z}(x,y,v|z) = \int_{\frac{\max\{x,z\}+y}{c}}^{\infty} e^{-\delta t} \mathbf{h}^*_{2,Z}(t,x,y,v|z)\, dt, \qquad x > v > 0; y > 0. \tag{2.6}$$

Similar to Cheung and Landriault (2009, Equation (2.9)), one may shift the process $\{Z(t)\}_{t\geq 0}$ by an amount z and apply the discounted densities $\mathbf{h}^*_{1,\delta,Z}(x,y|0)$ and $\mathbf{h}^*_{2,\delta,Z}(x,y,v|0)$ to derive a Markov renewal equation satisfied by $\mathbf{\Phi}_{\delta,Z}(z) = [\phi_{\delta,ij,Z}(z)]^m_{i,j=1}$. This leads to

$$\mathbf{\Phi}_{\delta,Z}(z) = \int_0^z \mathbf{f}_{\delta,Z}(y)\, \mathbf{\Phi}_{\delta,Z}(z-y)\, dy + \boldsymbol{\alpha}_{\delta,Z}(z), \qquad z \geq 0, \tag{2.7}$$

where

$$\mathbf{f}_{\delta,Z}(y) = \int_0^\infty \mathbf{h}^*_{1,\delta,Z}(x,y|0)\, dx + \int_0^\infty \int_0^x \mathbf{h}^*_{2,\delta,Z}(x,y,v|0)\, dv\, dx, \qquad y > 0, \tag{2.8}$$

is the matrix ladder height density, and

$$\begin{aligned} \boldsymbol{\alpha}_{\delta,Z}(z) = &\int_z^\infty \int_0^\infty w(x+z, y-z, z, z)\, \mathbf{h}^*_{1,\delta,Z}(x,y|0)\, dx\, dy \\ &+ \int_z^\infty \int_0^\infty \int_0^x w(x+z, y-z, z, v+z)\, \mathbf{h}^*_{2,\delta,Z}(x,y,v|0)\, dv\, dx\, dy, \qquad z \geq 0, \end{aligned} \tag{2.9}$$

is the outside term of the Markov renewal equation. Note that the (i,j)-th element of $\mathbf{f}_{\delta,Z}(y)$ represents the discounted (with respect to the time of the first drop) density of the first drop amount y together with the event that $\{G_n\}_{n=0}^\infty$ is in state j at the time of the drop, given that $G_0 = i$. Moreover, the matrix $\int_0^\infty \mathbf{f}_{\delta,Z}(y)\, dy$ is strictly substochastic if either $\delta > 0$ or the positive security loading condition

$$\sum_{j=1}^m \pi_j(\beta_j - c\kappa_j) > 0 \tag{2.10}$$

holds, where $(\pi_1, \pi_2, \ldots, \pi_m)$ is the stationary distribution of the Markov chain $\{G_n\}_{n=0}^\infty$. For the rest of the paper, either assumption is made so that solution of the defective Markov renewal equation (2.7) is unique (e.g. Miyazawa (2002)). Such a solution is given by (see also Çinlar (1969, Section 3a) or Asmussen (2003, Chapter VII.4))

$$\mathbf{\Phi}_{\delta,Z}(z) = \boldsymbol{\alpha}_{\delta,Z}(z) + \sum_{n=1}^\infty \int_0^z \mathbf{f}^{*n}_{\delta,Z}(z-y)\, \boldsymbol{\alpha}_{\delta,Z}(y)\, dy, \qquad z \geq 0. \tag{2.11}$$

Here the n-fold convolution $\mathbf{f}^{*n}_{\delta,Z}(\cdot)$ is defined recursively via $\mathbf{f}^{*n}_{\delta,Z}(\cdot) = (\mathbf{f}^{*(n-1)}_{\delta,Z} * \mathbf{f}_{\delta,Z})(\cdot) = (\mathbf{f}_{\delta,Z} * \mathbf{f}^{*(n-1)}_{\delta,Z})(\cdot)$ for $n = 2, 3, \dots$ with $\mathbf{f}^{*1}_{\delta,Z}(\cdot) \equiv \mathbf{f}_{\delta,Z}(\cdot)$, and the convolution operator $*$ for two conformable matrix functions $\mathbf{a}_1(\cdot)$ and $\mathbf{a}_2(\cdot)$ is defined by $(\mathbf{a}_1 * \mathbf{a}_2)(x) = \int_0^x \mathbf{a}_1(x-y)\,\mathbf{a}_2(y)\,dy$ for $x \geq 0$. We also refer interested readers to e.g. Wu (1999), Miyazawa (2002) and Li and Luo (2005) for two-sided bounds and asymptotic behaviour of the solution of a defective Markov renewal equation. Once $\mathbf{\Phi}_{\delta,Z}(\cdot)$ is known, $\mathbf{\Phi}_{\delta,U}(u) = [\phi_{\delta,ij,U}(u)]_{i,j=1}^m$ can be conveniently obtained from (2.2) as

$$\mathbf{\Phi}_{\delta,U}(u) = \int_0^{\frac{u}{c}} e^{-\delta t}\mathbf{k}(t)\,\mathbf{\Phi}_{\delta,Z}(u-ct)\,dt + \int_{\frac{u}{c}}^{\infty} e^{-\delta t} w(u, ct-u, u, u)\,\mathbf{k}(t)\,dt, \qquad u \geq 0. \tag{2.12}$$

It is instructive to note that the Markov renewal equation (2.7) and its solution (2.11) are characterized by $\mathbf{f}_{\delta,Z}(\cdot)$ and $\boldsymbol{\alpha}_{\delta,Z}(\cdot)$, which are in turn characterized by the discounted densities $\mathbf{h}^*_{1,\delta,Z}(x,y|0)$ and $\mathbf{h}^*_{2,\delta,Z}(x,y,v|0)$. While $\mathbf{h}^*_{1,\delta,Z}(x,y|0)$ is explicitly known in (2.5), $\mathbf{h}^*_{2,\delta,Z}(x,y,v|0)$ is yet to be determined. Defining $\mathbf{\Phi}_{124,\delta,Z}(z) = [\phi_{124,\delta,ij,Z}(z)]_{i,j=1}^m$ to be a special case of $\mathbf{\Phi}_{\delta,Z}(z)$ under the penalty function $w(x,y,r,v) = w_{124}(x,y,v)$ that does not depend on the third argument r, in principle (see Remark 3) this comes down to finding $\mathbf{\Phi}_{124,\delta,Z}(0)$ due to the relationship

$$\begin{aligned}\mathbf{\Phi}_{124,\delta,Z}(z) = &\int_0^\infty \int_z^\infty w_{124}(x,y,z)\,\mathbf{h}^*_{1,\delta,Z}(x,y|z)\,dx\,dy \\ &+ \int_0^\infty \int_0^\infty \int_0^x w_{124}(x,y,v)\,\mathbf{h}^*_{2,\delta,Z}(x,y,v|z)\,dv\,dx\,dy, \qquad z \geq 0.\end{aligned} \tag{2.13}$$

Further information about $\mathbf{\Phi}_{124,\delta,Z}(0)$ can usually be obtained by conditioning on the pair (Y_1, W_1) to arrive at, for $i, j \in \mathcal{E}$ and $z \geq 0$,

$$\begin{aligned}\phi_{124,\delta,ij,Z}(z) = &\int_0^\infty b_i(y) \sum_{l=1}^m p_{il} \int_0^{\frac{z+y}{c}} e^{-\delta t} k_l(t)\,\phi_{124,\delta,lj,Z}(z+y-ct)\,dt\,dy \\ &+ \int_0^\infty b_i(y)\,p_{ij} \int_{\frac{z+y}{c}}^\infty e^{-\delta t} k_j(t)\,w_{124}(z+y, ct-z-y, z)\,dt\,dy.\end{aligned} \tag{2.14}$$

In general, the exact evaluation of $\mathbf{\Phi}_{124,\delta,Z}(0)$ or $\mathbf{h}^*_{2,\delta,Z}(x,y,v|0)$ relies on both the Markov renewal equation (2.7) and the integral equation (2.14). This typically requires specific distributional assumptions on either the inter-arrival times or the gain sizes. These will be illustrated in the next two sections.

Remark 3 From (2.13), it is clear that $\mathbf{h}^*_{2,\delta,Z}(x,y,v|z)$ characterizes $\mathbf{\Phi}_{124,\delta,Z}(z)$ (as $\mathbf{h}^*_{1,\delta,Z}(x,y|z)$ is known). Indeed, it is also true that $\mathbf{h}^*_{2,\delta,Z}(x,y,v|z)$ is characterized by $\mathbf{\Phi}_{124,\delta,Z}(z)$. To see this, it is sufficient to assume a penalty in the form $w_{124}(x,y,v) = e^{-s_1 x - s_2 y - s_4 v}$, so that

$$\int_0^\infty \int_0^\infty \int_0^x e^{-s_1 x - s_2 y - s_4 v}\mathbf{h}^*_{2,\delta,Z}(x,y,v|z)\,dv\,dx\,dy = \mathbf{\Phi}_{124,\delta,Z}(z) - \int_0^\infty \int_z^\infty e^{-s_1 x - s_2 y - s_4 z}\mathbf{h}^*_{1,\delta,Z}(x,y|z)\,dx\,dy.$$

Once $\mathbf{\Phi}_{124,\delta,Z}(z)$ has been determined, the right-hand side of the above equation is known. According to the left-hand side, this represents the trivariate Laplace transform of $\mathbf{h}^*_{2,\delta,Z}(x,y,v|z)$ with respect to (x,y,v) under the transform arguments (s_1, s_2, s_4). Hence, one can get $\mathbf{h}^*_{2,\delta,Z}(x,y,v|z)$ by performing Laplace transform inversion (which can be analytic or numerical) due to the one-to-one correspondence between probability distribution and Laplace transform. See Section 4. $\square$

3 Exponential inter-arrival times

When specific distributional assumptions on the inter-arrival time densities $k_i(\cdot)$'s are made while the gain size densities $b_i(\cdot)$'s are left arbitrary, one may proceed by identifying the solution form of $\phi_{124,\delta,ij,Z}(z)$ as a function of z (apart from some unknown constants) via the Markov renewal equation (2.7) with the help of some probabilistic arguments. The unknown constants can be determined using the integral equation (2.14). These ideas have been exploited by e.g. Willmot (2007) and Cheung et al. (2011a) in the context of insurance risk processes, where the roles of inter-arrival times and jumps sizes are essentially interchanged in the analysis.

To begin, we first define $k_i^c(y) = (1/c)k_i(y/c)$ and $\overline{K}_i^c(y) = \overline{K}_i(y/c)$ to be the density and survival function of a scaled random variable corresponding to the density $k_i(\cdot)$. The density of the residual lifetime random variable associated to the density $k_i^c(\cdot)$ is then denoted by $k_{i,x}^c(y) = k_i^c(x+y)/\overline{K}_i^c(x)$. Further define the diagonal matrices $\overline{\mathbf{K}}^c(y) = \mathrm{diag}\{\overline{K}_1^c(y), \ldots, \overline{K}_m^c(y)\}$, $\mathbf{k}^c(y) = \mathrm{diag}\{k_1^c(y), \ldots, k_m^c(y)\}$ and $\mathbf{k}_x^c(y) = \mathrm{diag}\{k_{1,x}^c(y), \ldots, k_{m,x}^c(y)\}$. We argue probabilistically that the discounted density $\mathbf{h}_{2,\delta,Z}^*(x, y, v|z)$ defined by (2.6) admits the representation

$$\mathbf{h}_{2,\delta,Z}^*(x, y, v|z) = e^{-\delta\left(\frac{x+y}{c}\right)} \mathbf{h}_{\delta,Z}^{(2)}(x, v|z)\, \mathbf{k}_x^c(y), \qquad x > v > 0; y > 0. \tag{3.1}$$

This can be interpreted as follows. Indeed, the (i,j)-th element of $\mathbf{h}_{\delta,Z}^{(2)}(x, v|z)$ is the discounted (with respect to $\tau_Z^* - W_{M(\tau_Z)+1}$) joint density of $(Z(\tau_Z^* - W_{M(\tau_Z)+1}), R_{Z,M(\tau_Z)})$ at (x, v) together with the event that $G_{M(\tau_Z)+1} = j$, given that $G_0 = i$ and $Z(0^-) = z$. Being at level x in state j at time $\tau_Z^* - W_{M(\tau_Z+1)}$, the next inter-arrival time (having density $k_j(\cdot)$) should be of length $(x+y)/c$ in order to bring the surplus level to $-y$ at time τ_Z^{*-} so that $|Z(\tau_Z^{*-})| = y$. Since the latter event is conditional on that the surplus has to drop below 0 from x before the next gain, this gives the density $k_{j,x}^c(y)$. Finally, the discount factor $e^{-\delta(x+y)/c}$ takes care of the discounting when the process travels from level x to $-y$ at the end. In matrix form, the above descriptions precisely yield (3.1). It is interesting to note that the mathematical role played by the quantity $Z(\tau_Z^* - W_{M(\tau_Z)+1})$ is like that of the surplus prior to ruin in the usual insurance risk model. Because $\mathbf{k}^c(x+y) = \overline{\mathbf{K}}^c(x)\, \mathbf{k}_x^c(y)$, (2.5) can be rewritten as

$$\mathbf{h}_{1,\delta,Z}^*(x, y|z) = e^{-\delta\left(\frac{x+y}{c}\right)} \mathbf{b}(x-z)\, \mathbf{P}\, \overline{\mathbf{K}}^c(x)\, \mathbf{k}_x^c(y), \qquad x > z; y > 0. \tag{3.2}$$

With the use of (3.1) and (3.2), the matrix ladder height density (2.8) becomes

$$\mathbf{f}_{\delta,Z}(y) = \int_0^\infty e^{-\delta\left(\frac{x+y}{c}\right)} \left(\mathbf{b}(x)\, \mathbf{P}\, \overline{\mathbf{K}}^c(x) + \int_0^x \mathbf{h}_{\delta,Z}^{(2)}(x, v|0)\, dv \right) \mathbf{k}_x^c(y)\, dx, \qquad y > 0. \tag{3.3}$$

For the remainder of this section, we assume exponential inter-arrival time densities $k_i(t) = \lambda_i e^{-\lambda_i t}$ for $i \in \mathcal{E}$. With Remark 3 made at the end of the previous section, we shall study the Gerber-Shiu function $\mathbf{\Phi}_{124,\delta,Z}(z)$ under the penalty $w_{124}(x, y, v) = e^{-s_1 x - s_4 v} w_2(y)$. (The mathematical analysis is not more difficult even we keep $w_2(\cdot)$ general instead of using the choice of $w_2(y) = e^{-s_2 y}$.) Defining $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1 \ldots, \lambda_m\}$, one has $\mathbf{k}_x^c(y) = \mathbf{k}^c(y) = (\mathbf{\Lambda}/c)e^{-(\mathbf{\Lambda}/c)y}$ and $\overline{\mathbf{K}}^c(x) = e^{-(\mathbf{\Lambda}/c)x}$. Let $\boldsymbol{\alpha}_{124,\delta,Z}(z)$ be the special case of $\boldsymbol{\alpha}_{\delta,Z}(z)$ defined by (2.9) under the afore-mentioned choice of penalty function. Hence, application of (3.1) and (3.2) leads (2.9) to

$$\begin{aligned}\boldsymbol{\alpha}_{124,\delta,Z}(z) = &\int_z^\infty \int_0^\infty e^{-\delta\left(\frac{x+y}{c}\right)} \mathbf{b}(x)\, \mathbf{P} \left(\frac{\mathbf{\Lambda}}{c}\, e^{-\frac{\mathbf{\Lambda}}{c}(x+y)} \right) e^{-s_1(x+z) - s_4 z} w_2(y-z)\, dx\, dy \\ &+ \int_z^\infty \int_0^\infty \int_0^x e^{-\delta\left(\frac{x+y}{c}\right)} \mathbf{h}_{\delta,Z}^{(2)}(x, v|0) \left(\frac{\mathbf{\Lambda}}{c}\, e^{-\frac{\mathbf{\Lambda}}{c} y} \right) e^{-s_1(x+z) - s_4(v+z)} w_2(y-z)\, dv\, dx\, dy\end{aligned}$$

$$= \int_0^\infty \int_0^\infty e^{-\delta\left(\frac{x+y+z}{c}\right)} \mathbf{b}(x)\, \mathbf{P} \left(\frac{\boldsymbol{\Lambda}}{c}\, e^{-\frac{\boldsymbol{\Lambda}}{c}(x+y+z)} \right) e^{-s_1(x+z)-s_4 z} w_2(y)\, dx\, dy$$
$$+ \int_0^\infty \int_0^\infty \int_0^x e^{-\delta\left(\frac{x+y+z}{c}\right)} \mathbf{h}_{\delta,Z}^{(2)}(x,v|0) \left(\frac{\boldsymbol{\Lambda}}{c}\, e^{-\frac{\boldsymbol{\Lambda}}{c}(y+z)} \right) e^{-s_1(x+z)-s_4(z+v)} w_2(y)\, dv\, dx\, dy.$$

Because we will focus on $\boldsymbol{\alpha}_{124,\delta,Z}(z)$ as a function of z, it is convenient to write

$$\boldsymbol{\alpha}_{124,\delta,Z}(z) = \boldsymbol{\Sigma}_\delta(s_1,s_4)\, e^{-\frac{\boldsymbol{\Lambda}}{c}z} e^{-\left(s_1+s_4+\frac{\delta}{c}\right)z}, \qquad z \geq 0, \tag{3.4}$$

where

$$\boldsymbol{\Sigma}_\delta(s_1,s_4) = \left(\frac{1}{c} \int_0^\infty \mathbf{b}(x)\, \mathbf{P}\, e^{-\frac{\boldsymbol{\Lambda}}{c}x} e^{-\left(s_1+\frac{\delta}{c}\right)x}\, dx + \int_0^\infty \int_0^x \mathbf{h}_{\delta,Z}^{(2)}(x,v|0) e^{-\left(s_1+\frac{\delta}{c}\right)x - s_4 v}\, dv\, dx \right) \boldsymbol{\Lambda}$$
$$\times \left(\int_0^\infty e^{-\frac{\boldsymbol{\Lambda}}{c}y} e^{-\frac{\delta}{c}y} w_2(y)\, dy \right)$$

is independent of z. In what follows, for a function $g(\cdot)$ defined on $(0,\infty)$ (which is not necessarily a probability density), its Laplace transform is denoted by $\widetilde{g}(s) = \int_0^\infty e^{-sx} g(x)\, dx$ for $\mathrm{Re}(s) \geq 0$. The Laplace transform of a matrix-valued function is taken element-wise. Taking Laplace transforms on both sides of (3.4) with respect to z yields

$$\widetilde{\boldsymbol{\alpha}}_{124,\delta,Z}(s) = \boldsymbol{\Sigma}_\delta(s_1,s_4)\, \mathrm{diag}\left\{ \frac{1}{s+s_1+s_4+\frac{\lambda_1+\delta}{c}}, \ldots, \frac{1}{s+s_1+s_4+\frac{\lambda_m+\delta}{c}} \right\}. \tag{3.5}$$

On the other hand, taking Laplace transforms on both sides of (2.7) followed by rearrangements leads to

$$\widetilde{\boldsymbol{\Phi}}_{124,\delta,Z}(s) = [\mathbf{I} - \widetilde{\mathbf{f}}_{\delta,Z}(s)]^{-1} \widetilde{\boldsymbol{\alpha}}_{124,\delta,Z}(s) = \frac{\mathrm{adj}(\mathbf{I} - \widetilde{\mathbf{f}}_{\delta,Z}(s))}{\det(\mathbf{I} - \widetilde{\mathbf{f}}_{\delta,Z}(s))}\, \widetilde{\boldsymbol{\alpha}}_{124,\delta,Z}(s), \tag{3.6}$$

where $\mathbf{I}$ is the identity matrix of dimension m. Under the present assumption of exponential inter-arrival times, (3.3) becomes

$$\mathbf{f}_{\delta,Z}(y) = \Upsilon_\delta\, e^{-\frac{\boldsymbol{\Lambda}}{c}y} e^{-\frac{\delta}{c}y}, \qquad y > 0,$$

where

$$\Upsilon_\delta = \left\{ \frac{1}{c} \int_0^\infty e^{-\frac{\delta}{c}x} \left(\mathbf{b}(x)\, \mathbf{P}\, e^{-\frac{\boldsymbol{\Lambda}}{c}x} + \int_0^x \mathbf{h}_{\delta,Z}^{(2)}(x,v|0)\, dv \right) dx \right\} \boldsymbol{\Lambda}.$$

Hence, it is clear that

$$\widetilde{\mathbf{f}}_{\delta,Z}(s) = \Upsilon_\delta\, \mathrm{diag}\left\{ \frac{1}{s+\frac{\lambda_1+\delta}{c}}, \ldots, \frac{1}{s+\frac{\lambda_m+\delta}{c}} \right\}. \tag{3.7}$$

Since we aim at identifying the solution form of $\phi_{124,\delta,ij,Z}(\cdot)$ via (3.6), we turn to the denominator of (3.6). As in Cheung et al. (2011a, Theorem 1), it can be proved using de Smit (1995, Theorem 11.3) that all the m roots of the equation (in ξ)

$$\det(\mathbf{I} - \widetilde{\mathbf{f}}_{\delta,Z}(\xi)) = 0 \tag{3.8}$$

have negative real parts. These roots are denoted by $\{-\varepsilon_k\}_{k=1}^m$. By applying (3.5) and (3.7) to (3.6) and assuming that $\{\varepsilon_k\}_{k=1}^m$ and $\{s_1+s_4+(\lambda_k+\delta)/c\}_{k=1}^m$ are all distinct, it is observed that $\widetilde{\phi}_{124,\delta,ij,Z}(s)$ generally admits the partial fractions expansion, for $i,j \in \mathcal{E}$,

$$\widetilde{\phi}_{124,\delta,ij,Z}(s) = \sum_{k=1}^m \frac{\vartheta_{ij,k}}{s+\varepsilon_k} + \frac{\eta_{ij}}{s+s_1+s_4+\frac{\lambda_j+\delta}{c}},$$

where $\vartheta_{ij,k}$'s and η_{ij}'s are some unknown constants. Inversion of Laplace transforms gives, for $i,j \in \mathcal{E}$ and $z \geq 0$,

$$\phi_{124,\delta,ij,Z}(z) = \sum_{k=1}^{m} \vartheta_{ij,k} e^{-\varepsilon_k z} + \eta_{ij} e^{-\left(s_1+s_4+\frac{\lambda_j+\delta}{c}\right)z}. \tag{3.9}$$

Having identified the solution form (3.9), the next step is to determine the constants involved by back substitution into the integral equation (2.14). We consider the case $s_1+s_4 \neq 0$ (see Remark 4). Omitting the straightforward algebra, we arrive at

$$\begin{aligned}
&\sum_{k=1}^{m} \vartheta_{ij,k} e^{-\varepsilon_k z} + \eta_{ij} e^{-\left(s_1+s_4+\frac{\lambda_j+\delta}{c}\right)z} \\
&= \sum_{k=1}^{m} \left\{ \widetilde{b}_i(\varepsilon_k) \sum_{l=1}^{m} \frac{p_{il}\lambda_l}{\lambda_l+\delta-c\varepsilon_k} \, \vartheta_{lj,k} \right\} e^{-\varepsilon_k z} \\
&\quad + \left\{ \widetilde{b}_i\left(s_1+s_4+\frac{\lambda_j+\delta}{c}\right) \sum_{l=1}^{m} \frac{p_{il}\lambda_l}{\lambda_l-(cs_1+cs_4+\lambda_j)} \, \eta_{lj} \right\} e^{-\left(s_1+s_4+\frac{\lambda_j+\delta}{c}\right)z} \\
&\quad - \sum_{l=1}^{m} \widetilde{b}_i\left(\frac{\lambda_l+\delta}{c}\right) p_{il}\lambda_l \left\{ \sum_{k=1}^{m} \frac{\vartheta_{lj,k}}{\lambda_l+\delta-c\varepsilon_k} + \frac{\eta_{lj}}{\lambda_l-(cs_1+cs_4+\lambda_j)} \right\} e^{-\left(\frac{\lambda_l+\delta}{c}\right)z} \\
&\quad + \frac{1}{c} \, p_{ij}\lambda_j \widetilde{w}_2\left(\frac{\lambda_j+\delta}{c}\right) \widetilde{b}_i\left(s_1+\frac{\lambda_j+\delta}{c}\right) e^{-\left(s_1+s_4+\frac{\lambda_j+\delta}{c}\right)z}.
\end{aligned} \tag{3.10}$$

Equating the coefficients of $e^{-\varepsilon_k z}$ on both sides yields

$$\vartheta_{ij,k} = \widetilde{b}_i(\varepsilon_k) \sum_{l=1}^{m} \frac{p_{il}\lambda_l}{\lambda_l+\delta-c\varepsilon_k} \, \vartheta_{lj,k}, \qquad i,j,k = 1,2,\ldots,m. \tag{3.11}$$

For each fixed $j,k = 1,2,\ldots,m$, it is assumed that $\{\vartheta_{ij,k}\}_{i=1}^{m}$ are not all 0. With the system (3.11) of m linear equations in the unknowns $\{\vartheta_{ij,k}\}_{i=1}^{m}$ having a non-trivial solution, it is immediate that $\det(\mathbf{I} - \mathbf{\Pi}_\delta(-\varepsilon_k)) = 0$ for $k = 1,2,\ldots,m$, where $\mathbf{\Pi}_\delta(s)$ is the square matrix defined by

$$\mathbf{\Pi}_\delta(s) = \left[\widetilde{b}_i(-s) \, \frac{p_{ij}\lambda_j}{\lambda_j+\delta+cs} \right]_{i,j=1}^{m} = \widetilde{\mathbf{b}}(-s) \, \mathbf{P} \, \widetilde{\mathbf{k}}(\delta+cs).$$

In other words, $\{-\varepsilon_k\}_{k=1}^{m}$ satisfy the Lundberg's equation (in ξ)

$$\det(\mathbf{I} - \mathbf{\Pi}_\delta(\xi)) = 0. \tag{3.12}$$

Again, application of de Smit (1995, Theorem 11.3) reveals that the above equation has exactly m roots with negative real parts. Hence, one asserts that these roots are $\{-\varepsilon_k\}_{k=1}^{m}$.

In equating the remaining coefficients in (3.10), the terms involving $e^{-(s_1+s_4+(\lambda_j+\delta)/c)z}$ imply that

$$\eta_{ij} = \widetilde{b}_i\left(s_1+s_4+\frac{\lambda_j+\delta}{c}\right) \sum_{l=1}^{m} \frac{p_{il}\lambda_l}{\lambda_l-(cs_1+cs_4+\lambda_j)} \, \eta_{lj} + \frac{1}{c} \, p_{ij}\lambda_j \widetilde{w}_2\left(\frac{\lambda_j+\delta}{c}\right) \widetilde{b}_i\left(s_1+\frac{\lambda_j+\delta}{c}\right)$$

$$i,j = 1,2,\ldots,m; \tag{3.13}$$

whereas the coefficients of $e^{-((\lambda_l+\delta)/c)z}$ lead to

$$\sum_{k=1}^{m} \frac{\vartheta_{lj,k}}{\lambda_l+\delta-c\varepsilon_k} + \frac{\eta_{lj}}{\lambda_l-(cs_1+cs_4+\lambda_j)} = 0, \qquad l,j = 1,2,\ldots,m. \tag{3.14}$$

For each fixed $j=1,2,\ldots,m$, the values of $\{\eta_{ij}\}_{i=1}^m$ can be solved directly from the system (3.13) of m linear equations. It is instructive to note that the assumption that $\{\varepsilon_k\}_{k=1}^m$ and $\{s_1+s_4+(\lambda_k+\delta)/c\}_{k=1}^m$ are all distinct guarantees the uniqueness of the solution $\{\eta_{ij}\}_{i=1}^m$, since the coefficient matrix $\mathbf{I}-\mathbf{\Pi}_\delta(s_1+s_4+(\lambda_j+\delta)/c)$ has non-zero determinant. With $\{-\varepsilon_k\}_{k=1}^m$ and $\{\eta_{ij}\}_{i,j=1}^m$ obtained, the calculation of $\{\vartheta_{ij,k}\}_{i,j,k=1}^m$ indeed comes down to determining $\{\vartheta_{ij,k}\}_{i,k=1}^m$ for each fixed $j=1,2,\ldots,m$. Then for each fixed $k=1,2,\ldots,m$, the m equations (by varying i) in the system (3.11) are linearly dependent because of $\det(\mathbf{I}-\mathbf{\Pi}_\delta(-\varepsilon_k))=0$, and one of these equations is removed to get $m-1$ equations. This gives rise to $m(m-1)$ equations by varying k. Together with the m equations in (3.14) (by varying l), one arrives at a total of m^2 linear equations from which $\{\vartheta_{ij,k}\}_{i,k=1}^m$ can be solved for.

Remark 4 In some special cases, the solution form (3.9) can be further simplified. For example, if $s_1+s_4=0$, by inspecting (3.6) we note that (3.9) still holds under the simplification $\eta_{ii}=0$ for $i=1,2,\ldots,m$. Linear equations can be obtained by replacing s_1+s_4 and $\{\eta_{ii}\}_{i=1}^m$ by 0 in (3.10). Equating the coefficients of $e^{-\varepsilon_k z}$ still leads to (3.11), i.e. $\{-\varepsilon_k\}_{k=1}^m$ are the roots of (3.12) with negative real parts. The coefficients of $e^{-((\lambda_l+\delta)/c)z}$ result in

$$\sum_{k=1}^m \frac{\vartheta_{lj,k}}{\lambda_l+\delta-c\varepsilon_k}+\frac{\eta_{lj}}{\lambda_l-\lambda_j}=0, \qquad l\neq j, \tag{3.15}$$

and

$$\begin{aligned}\eta_{ij}=&\ \widetilde{b}_i\left(\frac{\lambda_j+\delta}{c}\right)\sum_{k=1,k\neq j}^m \frac{p_{ik}\lambda_k}{\lambda_k-\lambda_j}\eta_{kj}-\widetilde{b}_i\left(\frac{\lambda_j+\delta}{c}\right)p_{ij}\lambda_j\sum_{k=1}^m\frac{\vartheta_{jj,k}}{\lambda_j+\delta-c\varepsilon_k}\\&+\frac{1}{c}p_{ij}\lambda_j\widetilde{w}_2\left(\frac{\lambda_j+\delta}{c}\right)\widetilde{b}_i\left(s_1+\frac{\lambda_j+\delta}{c}\right), \qquad i,j=1,2,\ldots,m. \end{aligned}\tag{3.16}$$

Similar to the case $s_1+s_4\neq 0$, it is sufficient to fix $j=1,2,\ldots,m$ throughout. Then there are $m-1$ equations in (3.15) (by varying l) and m equations in (3.16) (by varying i). For each fixed $k=1,2,\ldots,m$, one of the m equations (in i) in the system (3.11) is removed to get $m-1$ equations, resulting in $m(m-1)$ by varying k. These form a total of m^2+m-1 linear equations in $\{\vartheta_{ij,k}\}_{i,k=1}^m$ and $\{\eta_{ij}\}_{i\neq j}$. It is instructive to note that this case of $s_1+s_4=0$ can be useful in obtaining the Laplace transform (with argument s) of the last jump before ruin $X_{M(\tau_U)}=Z(\tau_Z^*-W_{M(\tau_Z)+1})-R_{Z,M(\tau_Z)}$ for the process $\{Z(t)\}_{t\geq 0}$ by letting $s_1=-s_4=s$.

As another example, the very special case where $s_1=s_4=0$ would lead to $\eta_{ij}=0$ for all $i,j=1,2,\ldots,m$ (see Cheung et al. (2011a, Appendix)). Replacing s_1, s_4 and $\{\eta_{ij}\}_{i,j=1}^m$ by 0 in (3.10), one observes that $\{-\varepsilon_k\}_{k=1}^m$ are again the roots of (3.12) with negative real parts. In addition, we arrive at, for $j=1,2,\ldots,m$,

$$\begin{cases}\sum_{k=1}^m \frac{\vartheta_{lj,k}}{\lambda_l+\delta-c\varepsilon_k}=0, & l\neq j.\\ \sum_{k=1}^m \frac{\vartheta_{jj,k}}{\lambda_j+\delta-c\varepsilon_k}=\frac{1}{c}\widetilde{w}_2\left(\frac{\lambda_j+\delta}{c}\right). & \end{cases}\tag{3.17}$$

The procedure to solve for the $\{\vartheta_{ij,k}\}_{i,j,k=1}^m$ from (3.11) and (3.17) is similar to the general case of $s_1+s_4\neq 0$ and is omitted here. $\square$

4 Exponential gain sizes

In this section, we make reverse distributional assumptions in comparison to Section 3, i.e. the gain size densities are assumed to be exponential with $b_i(y)=\mu_i e^{-\mu_i y}$ for $i\in\mathcal{E}$ whereas the inter-arrival times are kept general. From Remark 3 at the end of Section 2, we consider the Gerber-Shiu function $\mathbf{\Phi}_{124,\delta,Z}(z)$ under the choice of penalty function $w_{124}(x,y,v) = e^{-s_1x-s_2y-s_4v}$. As we shall see, we are able to determine $\mathbf{h}^*_{2,\delta,Z}(x,y,v|0)$ by performing Laplace transform inversion of $\mathbf{\Phi}_{124,\delta,Z}(0)$ with respect to (s_1,s_2,s_4). This is sufficient to characterize $\mathbf{\Phi}_{\delta,Z}(\cdot)$ via (2.7) or (2.11), and hence $\mathbf{\Phi}_{\delta,U}(\cdot)$ via (2.12).

First, we define the Dickson-Hipp operator $\mathcal{T}_s$ introduced by Dickson and Hipp (2001) which will be extensively used in our analysis. For $\mathrm{Re}(s) \geq 0$, it is defined as

$$\mathcal{T}_s f(y) = \int_y^\infty e^{-s(x-y)} f(x)\,dx, \qquad y \geq 0.$$

Readers are also referred to Li and Garrido (2004, Section 3) for various properties of Dickson-Hipp operators. With exponential gains and the penalty function $w_{124}(x,y,v) = e^{-s_1x-s_2y-s_4v}$, by a change of variable $x = (z+y)/c$ we can rewrite (2.14) as

$$\begin{aligned}\phi_{124,\delta,ij,Z}(z) &= c\mu_i \sum_{l=1}^m p_{il} \int_{\frac{z}{c}}^\infty e^{-\mu_i(cx-z)} \int_0^x e^{-\delta t} k_l(t)\, \phi_{124,\delta,lj,Z}(c(x-t))\,dt\,dx \\ &\quad + c\mu_i p_{ij} e^{-\left(s_1+s_4+\frac{\delta}{c}\right)z} \int_{\frac{z}{c}}^\infty e^{-\left(\mu_i+s_1+\frac{\delta}{c}\right)(cx-z)} \mathcal{T}_{cs_2+\delta} k_j(x)\,dx.\end{aligned}$$

Upon differentiation of the above integral equation with respect to z, one obtains the integro-differential equation

$$\begin{aligned}\phi'_{124,\delta,ij,Z}(z) &= \mu_i \phi_{124,\delta,ij,Z}(z) - \mu_i \sum_{l=1}^m p_{il} \int_0^{\frac{z}{c}} e^{-\delta t} k_l(t)\, \phi_{124,\delta,lj,Z}(z-ct)\,dt \\ &\quad - \mu_i p_{ij} e^{-\left(s_1+s_4+\frac{\delta}{c}\right)z} \mathcal{T}_{cs_2+\delta} k_j\left(\frac{z}{c}\right) - c\mu_i p_{ij} s_4 e^{-\left(s_1+s_4+\frac{\delta}{c}\right)z} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j\left(\frac{z}{c}\right).\end{aligned}$$

Taking Laplace transforms yields

$$\begin{aligned}&(s-\mu_i)\widetilde{\phi}_{124,\delta,ij,Z}(s) + \mu_i \sum_{l=1}^m p_{il}\, \widetilde{k}_l(cs+\delta)\, \widetilde{\phi}_{124,\delta,lj,Z}(s) \\ &= \phi_{124,\delta,ij,Z}(0) - c\mu_i p_{ij} \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) - c^2 \mu_i p_{ij} s_4 \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0).\end{aligned}$$

This can be conveniently rewritten in matrix form as

$$\mathbf{A}_\delta(s)\, \widetilde{\mathbf{\Phi}}_{124,\delta,Z}(s) = \mathbf{\Phi}_{124,\delta,Z}(0) - \mathbf{\Delta}_\delta(s), \tag{4.1}$$

where

$$\mathbf{A}_\delta(s) = s\mathbf{I} - \boldsymbol{\mu} + \boldsymbol{\mu}\,\mathbf{P}\,\widetilde{\mathbf{k}}(cs+\delta)$$

with $\boldsymbol{\mu} = \mathrm{diag}\{\mu_1,\ldots,\mu_m\}$, and the (i,j)-th element of $\mathbf{\Delta}_\delta(s)$ is given by

$$[\mathbf{\Delta}_\delta(s)]_{ij} = c\mu_i p_{ij} \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) + c^2 \mu_i p_{ij} s_4 \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0). \tag{4.2}$$

Using the result in Li and Garrido (2004, Section 3, Property 2) regarding double Dickson-Hipp operators, we note that

$$\begin{aligned}&cs_4 \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) \\ &= [(cs+cs_1+cs_4+\delta) - (c\mu_i+cs_1+\delta)] \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) \\ &\quad + c(\mu_i - s) \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) \\ &= \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) - \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{cs_2+\delta} k_j(0) + c(\mu_i - s) \mathcal{T}_{cs+cs_1+cs_4+\delta} \mathcal{T}_{c\mu_i+cs_1+\delta} \mathcal{T}_{cs_2+\delta} k_j(0).\end{aligned}$$

Thus, (4.2) becomes

$$[\mathbf{\Delta}_\delta(s)]_{ij} = c\mu_i p_{ij}\mathcal{T}_{c\mu_i+cs_1+\delta}\mathcal{T}_{cs_2+\delta}k_j(0) + c^2\mu_i p_{ij}(\mu_i - s)\mathcal{T}_{cs+cs_1+cs_4+\delta}\mathcal{T}_{c\mu_i+cs_1+\delta}\mathcal{T}_{cs_2+\delta}k_j(0).$$

Since later on we will invert Laplace transforms with respect to (s_1, s_2, s_4), by explicitly writing the Dickson-Hipp operators as multiple integrals we can rewrite the above expression as

$$\begin{aligned}
[\mathbf{\Delta}_\delta(s)]_{ij} = &\frac{1}{c}\int_0^\infty\int_0^\infty e^{-s_1x-s_2y}\left\{\mu_i e^{-\mu_i x-\delta\left(\frac{x+y}{c}\right)}p_{ij}\, k_j\left(\frac{x+y}{c}\right)\right\}dx\,dy \\
&+\frac{1}{c}\int_0^\infty\int_0^\infty\int_0^x e^{-s_1x-s_2y-s_4v}\left\{\mu_i(\mu_i-s)e^{-sv-\mu_i(x-v)-\delta\left(\frac{x+y}{c}\right)}p_{ij}\, k_j\left(\frac{x+y}{c}\right)\right\}dv\,dx\,dy.
\end{aligned} \tag{4.3}$$

To obtain $\mathbf{\Phi}_{124,\delta,Z}(0)$ from (4.1), we apply the eigenvector method (see e.g. Albrecher and Boxma (2005, Section 2), Ren (2007, Section 3), and Zhang et al. (2011, Theorem 2)). Following Adan and Kulkarni (2003, Theorem 2.3) and Albrecher and Boxma (2005, Proposition 2.1), it is known that the Lundberg's equation (in ξ)

$$\det \mathbf{A}_\delta(\xi) = 0$$

has exactly m roots with non-negative real parts. We denote these roots by $\{\rho_k\}_{k=1}^m$, which are assumed to be distinct. For $k = 1, 2, \ldots, m$, define $\boldsymbol{\gamma}_k$ to be the left eigenvector of $\mathbf{A}_\delta(\rho_k)$ corresponding to the eigenvalue 0. Further assuming that every element of $\widetilde{\mathbf{\Phi}}_{124,\delta,Z}(\rho_k)$ is finite, pre-multiplying (4.1) by $\boldsymbol{\gamma}_k$ under $s = \rho_k$ yields, for $k = 1, 2, \ldots, m$,

$$\mathbf{0} = \boldsymbol{\gamma}_k\mathbf{A}_\delta(\rho_k)\,\widetilde{\mathbf{\Phi}}_{124,\delta,Z}(\rho_k) = \boldsymbol{\gamma}_k\mathbf{\Phi}_{124,\delta,Z}(0) - \boldsymbol{\gamma}_k\mathbf{\Delta}_\delta(\rho_k),$$

where $\mathbf{0}$ is a zero column vector of dimension m. Using (4.3), the above equation can be rearranged as

$$\begin{aligned}
\boldsymbol{\gamma}_k\mathbf{\Phi}_{124,\delta,Z}(0) = \boldsymbol{\gamma}_k &\int_0^\infty\int_0^\infty e^{-s_1x-s_2y}\left[\frac{1}{c}\,\mu_i e^{-\mu_i x-\delta\left(\frac{x+y}{c}\right)}p_{ij}\, k_j\left(\frac{x+y}{c}\right)\right]_{i,j=1}^m dx\,dy \\
&+\int_0^\infty\int_0^\infty\int_0^x e^{-s_1x-s_2y-s_4v}e^{-\rho_k v}\boldsymbol{\gamma}_k\left[\frac{1}{c}\,\mu_i^2 e^{-\mu_i(x-v)-\delta\left(\frac{x+y}{c}\right)}p_{ij}\, k_j\left(\frac{x+y}{c}\right)\right]_{i,j=1}^m dv\,dx\,dy \\
&-\int_0^\infty\int_0^\infty\int_0^x e^{-s_1x-s_2y-s_4v}\rho_k e^{-\rho_k v}\boldsymbol{\gamma}_k\left[\frac{1}{c}\,\mu_i e^{-\mu_i(x-v)-\delta\left(\frac{x+y}{c}\right)}p_{ij}\, k_j\left(\frac{x+y}{c}\right)\right]_{i,j=1}^m dv\,dx\,dy.
\end{aligned}$$

Since this is true for $k = 1, 2, \ldots, m$, putting all the pieces together yields

$$\begin{aligned}
\mathbf{\Gamma}\,\mathbf{\Phi}_{124,\delta,Z}(0) = \mathbf{\Gamma}&\int_0^\infty\int_0^\infty e^{-s_1x-s_2y}\left\{\frac{1}{c}\,\boldsymbol{\mu}e^{-\boldsymbol{\mu}x-\delta\mathbf{I}\left(\frac{x+y}{c}\right)}\mathbf{P}\,\mathbf{k}\left(\frac{x+y}{c}\right)\right\}dx\,dy \\
&+\int_0^\infty\int_0^\infty\int_0^x e^{-s_1x-s_2y-s_4v}e^{-\boldsymbol{\rho}v}\mathbf{\Gamma}\left\{\frac{1}{c}\,\boldsymbol{\mu}^2 e^{-\boldsymbol{\mu}(x-v)-\delta\mathbf{I}\left(\frac{x+y}{c}\right)}\mathbf{P}\,\mathbf{k}\left(\frac{x+y}{c}\right)\right\}dv\,dx\,dy \\
&-\int_0^\infty\int_0^\infty\int_0^x e^{-s_1x-s_2y-s_4v}\boldsymbol{\rho}e^{-\boldsymbol{\rho}v}\mathbf{\Gamma}\left\{\frac{1}{c}\,\boldsymbol{\mu}e^{-\boldsymbol{\mu}(x-v)-\delta\mathbf{I}\left(\frac{x+y}{c}\right)}\mathbf{P}\,\mathbf{k}\left(\frac{x+y}{c}\right)\right\}dv\,dx\,dy,
\end{aligned} \tag{4.4}$$

where $\boldsymbol{\rho} = \mathrm{diag}\{\rho_1, \ldots, \rho_m\}$ and $\mathbf{\Gamma}$ is the square matrix containing the left eigenvectors $\boldsymbol{\gamma}_k$'s such that $\mathbf{\Gamma} = [\boldsymbol{\gamma}_1^\top, \ldots, \boldsymbol{\gamma}_m^\top]^\top$. Further define the matrix $\mathbf{\Theta} = \mathbf{\Gamma}^{-1}\boldsymbol{\rho}\mathbf{\Gamma}$. Then it is known that $\mathbf{\Gamma}^{-1}e^{-\boldsymbol{\rho}v}\mathbf{\Gamma} = e^{-\mathbf{\Theta}v}$ and $\mathbf{\Gamma}^{-1}\boldsymbol{\rho}e^{-\boldsymbol{\rho}v}\mathbf{\Gamma} = \mathbf{\Theta}\mathbf{\Gamma}^{-1}e^{-\boldsymbol{\rho}v}\mathbf{\Gamma} = \mathbf{\Theta}e^{-\mathbf{\Theta}v} = e^{-\mathbf{\Theta}v}\mathbf{\Theta}$. Hence, pre-multiplying (4.4) by $\mathbf{\Gamma}^{-1}$ along with the use of (2.5) (under $\mathbf{b}(x) = \boldsymbol{\mu}e^{-\boldsymbol{\mu}x}$) yields

$$\begin{aligned}\boldsymbol{\Phi}_{124,\delta,Z}(0) = &\int_0^\infty \int_0^\infty e^{-s_1x-s_2y}\mathbf{h}^*_{1,\delta,Z}(x,y|0)\,dx\,dy\\ &+\int_0^\infty \int_0^\infty \int_0^x e^{-s_1x-s_2y-s_4v}e^{-\boldsymbol{\Theta}v}(\boldsymbol{\mu}-\boldsymbol{\Theta})\mathbf{h}^*_{1,\delta,Z}(x,y|v)\,dv\,dx\,dy.\end{aligned}$$

Comparing with (2.13) under $z=0$ and $w_{124}(x,y,v)=e^{-s_1x-s_2y-s_4v}$ leads us to

$$\mathbf{h}^*_{2,\delta,Z}(x,y,v|0) = e^{-\boldsymbol{\Theta}v}(\boldsymbol{\mu}-\boldsymbol{\Theta})\mathbf{h}^*_{1,\delta,Z}(x,y|v), \qquad x>v>0; y>0,$$

via uniqueness of Laplace transforms.

5 Applications

5.1 Fair price of a perpetual insurance

In the context of the usual insurance risk process, ruin-related problems involving perpetual (re)insurance have been studied by e.g. Pafumi (1998) and Dickson and Waters (2004, Section 6.3). Generally speaking, the idea of perpetual (re)insurance is that a company (which may be an insurance company) pays a single premium up-front to another (re)insurer who guarantees to make any necessary capital injections to keep the company's surplus non-negative, so that the company will be able to continue its business forever. In the case of the dual risk model (1.1), this means that the company will have its expenses paid by the insurer whenever its surplus reaches zero. Mathematically, the dual risk process modified by the above perpetual insurance, denoted by $\{U^I(t)\}_{t\geq 0}$, follows the dynamics

$$U^I(t) = U(t) - \min\left(0, \inf_{0\leq s\leq t} U(s)\right), \qquad t\geq 0.$$

See Figures 3a&b.

INSERT FIGURE 3

Figures 3a&b: Original and modified sample paths for $\{U(t)\}_{t\geq 0}$ and $\{U^I(t)\}_{t\geq 0}$

Before finding the fair price at time 0 of the perpetual insurance contract, we need to introduce a few intermediate functions. We define $\mathbf{L}^*_{\delta,U}(u) = [L^*_{\delta,ij,U}(u)]_{i,j=1}^m$ to be the special case of $\boldsymbol{\Phi}_{\delta,U}(u)$ with $w(\cdot,\cdot,\cdot,\cdot)\equiv 1$, i.e. $\mathbf{L}^*_{\delta,U}(u)$ represents the matrix Laplace transform of τ^*_U. Also define $\mathbf{L}_{\delta,U}(u) = [L_{\delta,ij,U}(u)]_{i,j=1}^m$ to be matrix Laplace transform of the ruin time τ_U. Recalling the relationship $\tau_U = \tau^*_U - |U(\tau^{*-}_U)|/c$ from Section 1, one asserts that $\mathbf{L}_{\delta,U}(u)$ can be retrieved from $\boldsymbol{\Phi}_{\delta,U}(u)$ by letting $w(x,y,v,r) = e^{(\delta/c)y}$. Similarly, the special cases of $\boldsymbol{\Phi}_{\delta,Z}(z)$ under $w(\cdot,\cdot,\cdot,\cdot)\equiv 1$ and $w(x,y,v,r)=e^{(\delta/c)y}$ are denoted by $\mathbf{L}^*_{\delta,Z}(z) = [L^*_{\delta,ij,Z}(z)]_{i,j=1}^m$ and $\mathbf{L}_{\delta,Z}(z) = [L_{\delta,ij,Z}(z)]_{i,j=1}^m$ respectively. Note that $\mathbf{L}^*_{\delta,Z}(z)$ and $\mathbf{L}_{\delta,Z}(z)$ can both be computed using Remark 4 if the inter-arrival times are exponential, and hence $\mathbf{L}^*_{\delta,U}(u)$ and $\mathbf{L}_{\delta,U}(u)$ follow from (2.12) as well.

The upcoming analysis is similar to that in Cheung (2012, Section 3.2) who considered the dual dependent Sparre Andersen model. The difference is that we need to additionally take into account the states of the Markov chain at both time 0 and the first time $\{U(t)\}_{t\geq 0}$ reaches zero. Throughout we assume a force of interest $\delta > 0$. Given the initial state $G_0 = i$ and the initial capital $U^I(0) = u$, we denote the 'price' of the perpetual insurance in $\{U^I(t)\}_{t\geq 0}$ by $\mathrm{PI}_{i,U}(u)$. Analogous to Cheung (2012, Equation (3.3)), we arrive at, for $i \in \mathcal{E}$ and $u \geq 0$,

$$\begin{aligned}\mathrm{PI}_{i,U}(u) = c\sum_{j=1}^{m} E\left[e^{-\delta\tau_U}\overline{a}_{\overline{\tau_U^*-\tau_U}|\delta}1\{\tau_U < \infty\}1\{G_{N(\tau_U)} = j\}\big|G_0 = i, U(0) = u\right] \\ + \sum_{j=1}^{m} E\left[e^{-\delta\tau_U^*}1\{\tau_U < \infty\}1\{G_{N(\tau_U)} = j\}\big|G_0 = i, U(0) = u\right]\mathrm{PI}_{j,Z}. \qquad (5.1)\end{aligned}$$

Here the actuarial symbol $\overline{a}_{\overline{t}|\delta} = (1 - e^{-\delta t})/\delta$ denotes the present value (discounted at $\delta > 0$) of a continuous stream of payment at rate 1 between time 0 and time t; whereas $\mathrm{PI}_{j,Z}$ is the 'price' of the perpetual insurance corresponding to a similarly modified version of $\{Z(t)\}_{t\geq 0}$ given initial state $G_0 = j$ and zero initial level. The expression (5.1) can be interpreted as follows.

1. In the first term, $c\,E[e^{-\delta\tau_U}\overline{a}_{\overline{\tau_U^*-\tau_U}|\delta}1\{\tau_U < \infty\}1\{G_{N(\tau_U)} = j\}|G_0 = i, U(0) = u]$ represents the present value of the first payment stream at rate c from time τ_U to τ_U^*, if the first ruin of $\{U(t)\}_{t\geq 0}$ occurs in state $G_{N(\tau_U)} = j$.
2. For the second term, $E[e^{-\delta\tau_U^*}1\{\tau_U < \infty\}1\{G_{N(\tau_U)} = j\}|G_0 = i, U(0) = u]$ can be regarded as the discount factor from time τ_U^* to time 0, if the first ruin occurs with $G_{N(\tau_U)} = j$. Under such a case, $\mathrm{PI}_{j,Z}$ is simply the present value (at time τ_U^*) of potential future payments if the process $\{U^I(t)\}_{t\geq 0}$ ever reaches zero again.

In the above two contributions, summing over $j \in \mathcal{E}$ yields the desired result (5.1) as the state j is arbitrary. The same arguments also lead to a similar expression for $\mathrm{PI}_{j,Z}$, namely, for $j \in \mathcal{E}$,

$$\begin{aligned}\mathrm{PI}_{j,Z} = c\sum_{k=1}^{m} E\left[e^{-\delta\tau_Z}\overline{a}_{\overline{\tau_Z^*-\tau_Z}|\delta}1\{\tau_Z < \infty\}1\{G_{M(\tau_Z)+1} = k\}\big|G_0 = j, Z(0^-) = 0\right] \\ + \sum_{k=1}^{m} E\left[e^{-\delta\tau_Z^*}1\{\tau_Z < \infty\}1\{G_{M(\tau_Z)+1} = k\}\big|G_0 = j, Z(0^-) = 0\right]\mathrm{PI}_{k,Z}. \qquad (5.2)\end{aligned}$$

Using matrix notations defined previously, we can rewrite (5.1) and (5.2) respectively as

$$\mathbf{PI}_U(u) = \frac{c}{\delta}\left\{\mathbf{L}_{\delta,U}(u) - \mathbf{L}^*_{\delta,U}(u)\right\}\mathbf{1} + \mathbf{L}^*_{\delta,U}(u)\mathbf{PI}_Z, \qquad u \geq 0, \qquad (5.3)$$

and

$$\mathbf{PI}_Z = \frac{c}{\delta}\left\{\mathbf{L}_{\delta,Z}(0) - \mathbf{L}^*_{\delta,Z}(0)\right\}\mathbf{1} + \mathbf{L}^*_{\delta,Z}(0)\mathbf{PI}_Z. \qquad (5.4)$$

where $\mathbf{PI}_U(u) = \{\mathrm{PI}_{1,U}(u), \ldots, \mathrm{PI}_{m,U}(u)\}^\top$, $\mathbf{PI}_Z = \{\mathrm{PI}_{1,Z}, \ldots, \mathrm{PI}_{m,Z}\}^\top$, and $\mathbf{1}$ is an m-dimensional column vector of ones. Rearrangements of (5.4) yield

$$\mathbf{PI}_Z = \frac{c}{\delta}\left[\mathbf{I} - \mathbf{L}^*_{\delta,Z}(0)\right]^{-1}\left\{\mathbf{L}_{\delta,Z}(0) - \mathbf{L}^*_{\delta,Z}(0)\right\}\mathbf{1},$$

which is an explicit expression for $\mathbf{PI}_Z$. Then $\mathbf{PI}_U(u)$ can be computed from (5.3). However, it is instructive to note that $\mathbf{PI}_U(u)$ is still not the actual price of the perpetual insurance contract. This is because the company needs to use part of its surplus to buy the contract, and the decrease in surplus will in turn drive up the price of the contract. Given $G_0 = i$ and initial surplus u before purchase of insurance, we denote the actual fair price of the perpetual insurance by $\mathrm{API}_i(u)$. If the fair price $\mathrm{API}_i(u)$ exists, then it satisfies

$$\mathrm{API}_i(u) = \mathrm{PI}_{i,U}(u - \mathrm{API}_i(u)). \tag{5.5}$$

With expression for $\mathrm{PI}_{i,U}(\cdot)$ available, one can solve the above equation for $\mathrm{API}_i(u)$ numerically using common software packages such as *Mathematica*. It is also clear that $\mathrm{API}_i(u)$ does not exist if $u \leq \mathrm{PI}_{i,U}(u)$, as the whole surplus is not enough to buy the perpetual insurance in this case. Moreover, if the solution $\mathrm{API}_i(u)$ to (5.5) exists, it must be no less than $\mathrm{PI}_{i,U}(u)$.

Example 1 This example aims at illustrating how certain model parameters of the dual semi-Markovian risk model affect the fair price of the perpetual insurance. For simplicity, we study a two-state model (i.e. $m = 2$). Two different transition probability matrices will be considered, namely

$$\mathbf{P}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{P}_2 = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{10} & \frac{7}{10} \end{bmatrix}.$$

Under $\mathbf{P}_1$, the stationary probabilities of the Markov chain $\{G_n\}_{n=0}^{\infty}$ are given by $\pi_1 = \pi_2 = 1/2$. On the other hand, the transition matrix $\mathbf{P}_2$ would result in the stationary probabilities $\pi_1 = 1/3$ and $\pi_2 = 2/3$. It is assumed that the process has exponential gain sizes and exponential inter-arrival times in both environmental states, i.e. $b_i(y) = \mu_i e^{-\mu_i y}$ and $k_i(t) = \lambda_i e^{-\lambda_i t}$ for $i = 1, 2$, so that one has $\beta_i = 1/\mu_i$ and $\kappa_i = 1/\lambda_i$. Two sets of parameters for $\boldsymbol{\lambda} = \mathrm{diag}\{\lambda_1, \lambda_2\}$ will be used, namely $\boldsymbol{\lambda}_1 = \mathrm{diag}\{4/5, 4/3\}$ and $\boldsymbol{\lambda}_2 = \mathrm{diag}\{4/5, \sqrt{16/61}\}$. Throughout we assume $\boldsymbol{\mu} = \mathrm{diag}\{\mu_1, \mu_2\} = \mathrm{diag}\{1, 5/4\}$, expense rate $c = 0.4$ and force of interest $\delta = 0.05$. By varying $\mathbf{P}$ and $\boldsymbol{\lambda}$, we study four cases that are summarized in Table 1 below.

Case	$\boldsymbol{\lambda}$	$\mathbf{P}$	π_1	π_2	$E_1[Y_1 - cV_1]$ $= \beta_1 - c\kappa_1$	$E_2[Y_1 - cV_1]$ $= \beta_2 - c\kappa_2$	$\mathrm{Var}_1(Y_1 - cV_1)$ $= \beta_1^2 + c^2\kappa_1^2$	$\mathrm{Var}_2(Y_1 - cV_1)$ $= \beta_2^2 + c^2\kappa_2^2$	$\sum_{j=1}^{2} \pi_j(\beta_j - c\kappa_j)$
A	$\boldsymbol{\lambda}_1$	$\mathbf{P}_1$	1/2	1/2	0.500	0.500	1.250	0.730	0.500
B	$\boldsymbol{\lambda}_1$	$\mathbf{P}_2$	1/3	2/3	0.500	0.500	1.250	0.730	0.500
C	$\boldsymbol{\lambda}_2$	$\mathbf{P}_1$	1/2	1/2	0.500	0.019	1.250	1.250	0.259
D	$\boldsymbol{\lambda}_2$	$\mathbf{P}_2$	1/3	2/3	0.500	0.019	1.250	1.250	0.179

Table 1: Parameter values and related attributes in the four cases

In the above table, for $i = 1, 2$ the quantity $E_i[Y_1 - cV_1] = E[Y_1 - cV_1 | G_0 = i]$ (resp. $\mathrm{Var}_i(Y_1 - cV_1) = \mathrm{Var}(Y_1 - cV_1 | G_0 = i)$) represents the expected increment (resp. variance of the increment) of the process $\{U(t)\}_{t \geq 0}$ when it is in state i. In all four cases, the positive security loading condition (2.10) is satisfied as $E_i[Y_1 - cV_1]$ is always positive. In particular, the value on the left-hand side of (2.10) is also presented in the last column of Table 1. With the use of *Mathematica*, the API values are obtained via (5.5) and the results are summarized in Table 2 for different initial states and various values of initial surplus. The entries showing 'NA' correspond to the cases where API value does not exist because of insufficient initial surplus.

In each row of Table 2, it is noted that the value of API decreases as the initial surplus u increases. This is because the process stays further away from level zero when u is larger, and therefore both the chance of ever having an insurance payment and the amount of required payment will be less. In Case A, we observe that the API values under $G_0 = 1$ are higher than the corresponding ones under $G_0 = 2$. An intuitive explanation is that the variance of increment in state 1 is larger than that in state 2 (with the expected increment being same), implying state 1 has a higher risk. Hence, starting in the riskier state 1 results in higher price for the perpetual insurance. Moving from Case A to Case B, the same phenomenon is observed. However, the values in Case B are slightly smaller than those in Case A. Although Cases A and B have identical parameters concerning the gain sizes and the inter-arrival times in each state, under Case B the process only has 1/3 chance of being in the risker state 1 in the long run compared to the probability of 1/2 for Case A. This explains the smaller API values in Case B.

Case	G_0	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$
A	1	NA	NA	0.122	0.020	0.004	6.65×10^{-4}	1.21×10^{-4}	2.18×10^{-5}
A	2	NA	NA	0.083	0.013	0.002	4.24×10^{-4}	7.54×10^{-5}	1.34×10^{-5}
B	1	NA	NA	0.119	0.019	0.003	6.10×10^{-4}	1.09×10^{-4}	1.93×10^{-5}
B	2	NA	NA	0.055	0.009	0.001	2.52×10^{-4}	4.32×10^{-5}	7.41×10^{-6}
C	1	NA	NA	NA	0.501	0.143	0.052	0.020	0.008
C	2	NA	NA	NA	NA	0.564	0.155	0.056	0.021
D	1	NA	NA	NA	NA	0.292	0.103	0.041	0.017
D	2	NA	NA	NA	NA	0.779	0.210	0.081	0.033

Table 2: Values of API under four cases

Now we turn to Cases C and D. In these two cases, the parameter $\boldsymbol{\lambda} = \boldsymbol{\lambda}_2$ is indeed chosen such that the process $\{U(t)\}_{t\geq 0}$ has the same variance of increment in both states. However, the expected increment in state 1 is larger than that in state 2, meaning that state 1 has higher expected profit and is the less risky state. Hence, within each case, the API values when $G_0 = 1$ are smaller than those when $G_0 = 2$. Note that the API values in Case D under $\mathbf{P}_2$ are larger than those in Case C because the process in Case D has higher chance to be in the riskier state 2 in the long run.

Next, one can also compare Cases A and C in which the transition probability matrix $\mathbf{P}_1$ is the same but the parameter $\boldsymbol{\lambda}$ is different. From Table 1, it is clear that state 2 of Case C is riskier than that of Case A as its increment has lower expectation and higher variance. Consequently, the API values in Case C are higher. The fact that Case D has higher API values than Case B can be interpreted in the same manner.

Case	G_0	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$
A	1	1.780	0.405	0.086	0.017	0.003	6.11×10^{-4}	1.14×10^{-4}	2.08×10^{-5}
A	2	2.041	0.390	0.072	0.013	0.002	4.23×10^{-4}	7.54×10^{-5}	1.34×10^{-5}
B	1	1.779	0.406	0.083	0.016	0.003	5.58×10^{-4}	1.02×10^{-4}	1.83×10^{-5}
B	2	1.685	0.295	0.050	0.009	0.001	2.52×10^{-4}	4.32×10^{-5}	7.41×10^{-6}
C	1	4.404	1.877	0.775	0.313	0.125	0.049	0.019	0.008
C	2	9.012	4.199	1.847	0.769	0.317	0.128	0.051	0.020
D	1	6.503	2.963	1.277	0.540	0.226	0.094	0.040	0.017
D	2	8.947	4.514	2.092	0.934	0.408	0.176	0.075	0.032

Table 3: Values of PI under four cases

Finally, for reference we also include Table 3 showing the values of $\mathrm{PI}_{i,U}(u)$. It is noted that the ranking of the values in Table 3 follows closely that in Table 2, and the probabilistic interpretation in relation to the concept of risk is identical to that for Table 2. Moreover, whenever $\mathrm{PI}_{i,U}(u)$ is small (of order 10^{-4} or less) under initial surplus levels of $u = 6, 7, 8$, the value of $\mathrm{API}_i(u)$ in Table 2 is very close to the corresponding $\mathrm{PI}_{i,U}(u)$ in Table 3. In these cases, the purchase of the perpetual insurance itself has virtually no impact on the initial surplus. $\square$

5.2 Ordering properties of the last inter-arrival time containing ruin

In a dependent Sparre Andersen insurance risk model where a given inter-arrival time affects the distribution of the resulting claim severity, Cheung et al. (2011b, Section 2) provided sufficient conditions under which the size of the claim causing ruin (resp. the last inter-arrival time before ruin) is stochastically larger (resp. smaller) than the generic claim size (resp. inter-arrival time) random variable. In this subsection, we aim at studying whether similar ordering properties hold true in the context of a dual risk model. For example, (given that ruin occurs) it is natural to expect that the last inter-arrival time containing the time of ruin tends to be long, since a prolonged period without a gain would be detrimental to the company and can be a reason for ruin.

First, we recall from Section 1 that the inter-arrival time containing the time of ruin is related to other variables by $V_{N(\tau_U)+1} = (U(T_{N(\tau_U)}) + |U(\tau_U^{*-})|)/c$. Therefore, by letting $w(x, y, v, r) = e^{-s(x+y)/c}$ and $\delta = 0$ in (1.6), we retrieve the Laplace transform (with argument s) of $V_{N(\tau_U)+1}$. This will be denoted by, for $i, j \in \mathcal{E}$ and $u \geq 0$,

$$\widetilde{q}_{ij,U}(s, u) = E\left[e^{-sV_{N(\tau_U)+1}} 1\{\tau_U < \infty\} 1\{G_{N(\tau_U)} = j\} | G_0 = i, U(0) = u\right]. \tag{5.6}$$

Upon Laplace transform inversion with respect to s, one obtains $q_{ij,U}(\cdot, u)$ which is the defective density of $V_{N(\tau_U)+1}$ together with the events $\{\tau_U < \infty\}$ and $\{G_{N(\tau_U)} = j\}$, given $G_0 = i$ and $U(0) = u$. Moreover, setting $s = 0$ in (5.6) (or setting $w(\cdot, \cdot, \cdot, \cdot) \equiv 1$ and $\delta = 0$ in (1.6)) leads to $\Pr\{\tau_U < \infty, G_{N(\tau_U)} = j | G_0 = i, U(0) = u\}$. Therefore, the normalized density of $V_{N(\tau_U)+1}$ is given by, for $i, j \in \mathcal{E}$; $u \geq 0$ and $t > 0$,

$$q^*_{ij,U}(t, u) = \frac{q_{ij,U}(t, u)}{\Pr\{\tau_U < \infty, G_{N(\tau_U)} = j | G_0 = i, U(0) = u\}}.$$

Further define the associated proper survival function $\overline{Q}^*_{ij,U}(t, u) = \int_t^\infty q^*_{ij,U}(x, u)\, dx$. With the occurrences of gains and the resulting amounts following a semi-Markovian structure, it is reasonable to compare $\overline{Q}^*_{ij,U}(\cdot, u)$ for ruin in state j with the survival function $\overline{K}_j(\cdot)$ of the generic inter-arrival time in state j to see whether a stochastic ordering holds. However, unlike Cheung et al. (2011b), it is difficult to derive ordering properties analytically in the present model which is more complex. Therefore, in what follows we shall provide a numerical example to support the idea that $\overline{Q}^*_{ij,U}(\cdot, u)$ is larger than $\overline{K}_j(\cdot)$.

Example 2 In this example, we follow identical parameters used in Case A in Example 1 (except that δ is now 0). For $i, j = 1, 2$, Figures 4a-d first show the plots of the densities $q^*_{ij,U}(t, u)$ (when $U(0) = 2, 4, 6, 8$) and $k_j(t)$ against t. It can be seen from Figures 4b&c (i.e. when $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$) that the density $q^*_{ij,U}(t, u)$ is always continuous in t. However, the same is not true for Figures 4a&d: $q^*_{ij,U}(t, u)$ is discontinuous at the point $t = u/c$ with an upward jump. This is because when $i = j$, there is an additional contribution to the density in the domain $t > u/c$ for ruin occurring without any gains, which is evident from (2.2).

INSERT FIGURE 4

Figures 4a-d: Plots of $q^*_{ij,U}(t, u)$ and $k_j(t)$

Next, for $i, j = 1, 2$, Figures 5a-d depict the behaviour of the survival functions $\overline{Q}^*_{ij,U}(t, u)$ and $\overline{K}_j(t)$ against t. Regardless of the initial states and initial surplus levels under consideration, it can be seen that $\overline{Q}^*_{ij,U}(t,u)$ is always larger than $\overline{K}_j(t)$, which is expected. Note that $\overline{Q}^*_{ij,U}(t,u)$ is not smooth at $t = u/c$ when $i = j$ because of the discontinuity of density.

INSERT FIGURE 5

Figures 5a-d: Plots of $\overline{Q}^*_{ij,U}(t,u)$ and $\overline{K}_j(t)$

We have also performed the above analyses using the parameters of Cases B-D in Example 1, and the same phenomena are observed. The related plots are not reproduced here for the sake of brevity. □

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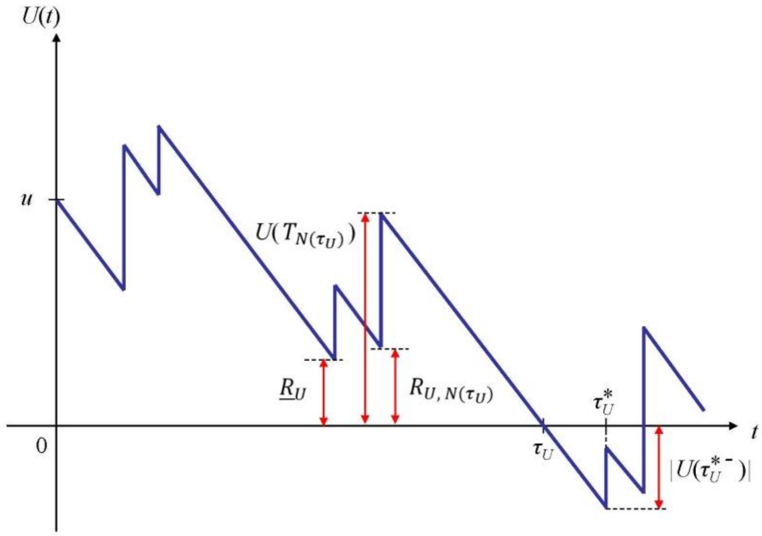

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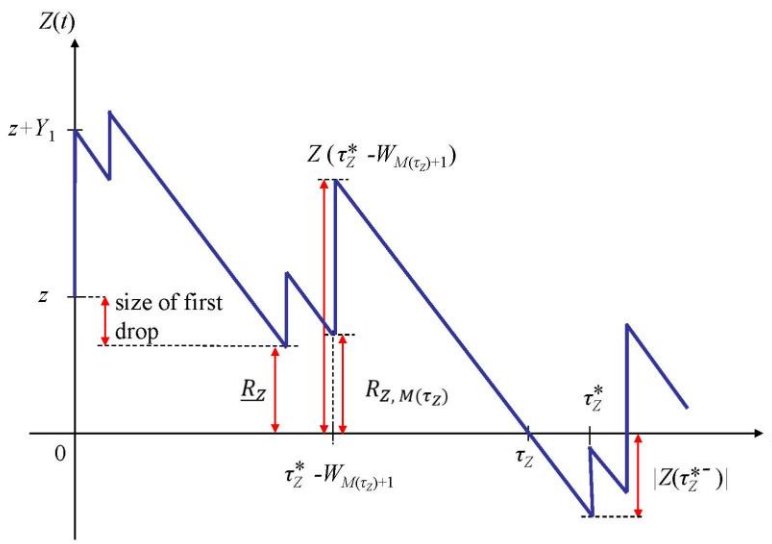


Figure 2

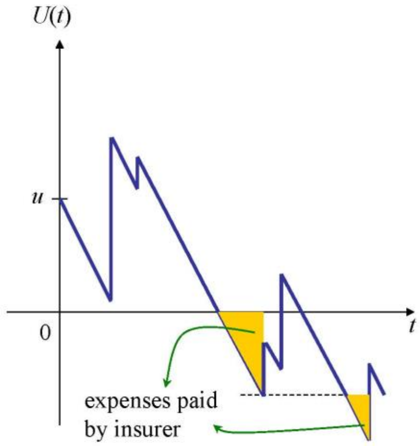
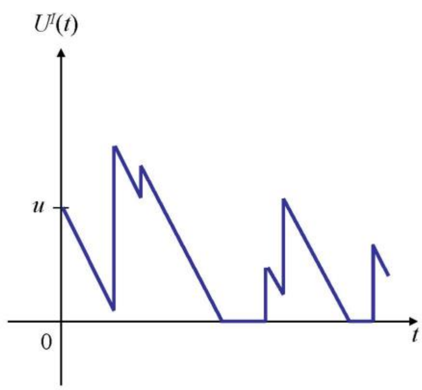


Figure 3a



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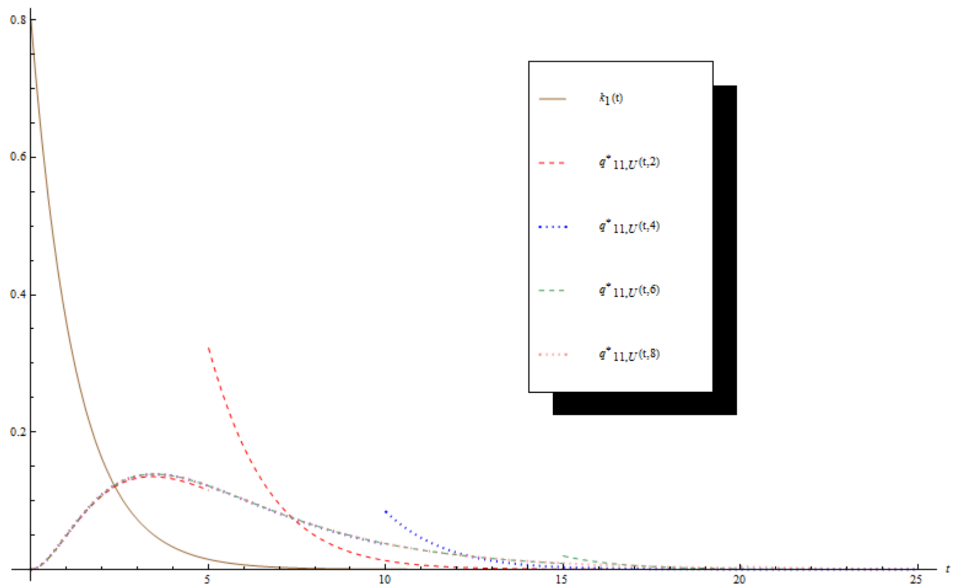

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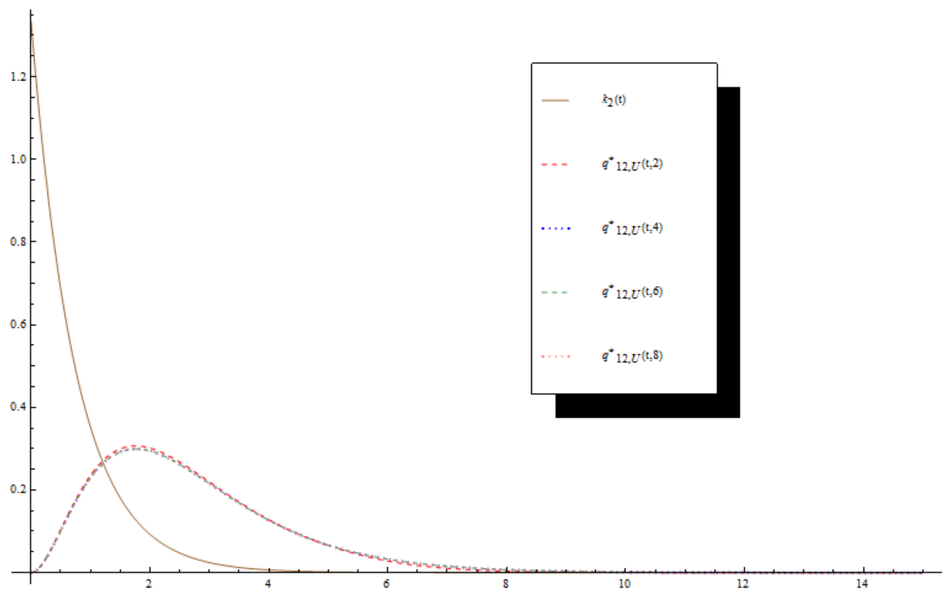

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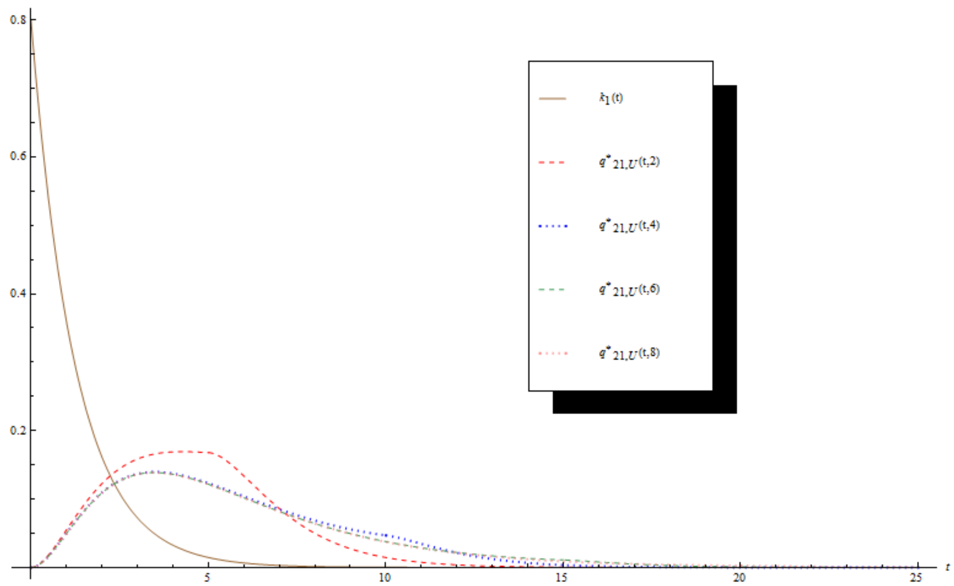


Figure 4c

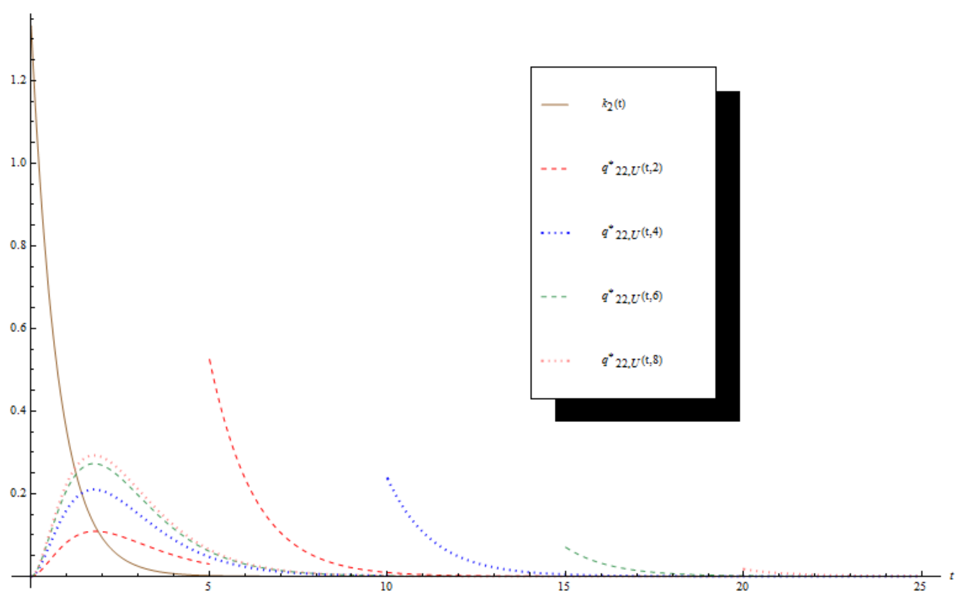
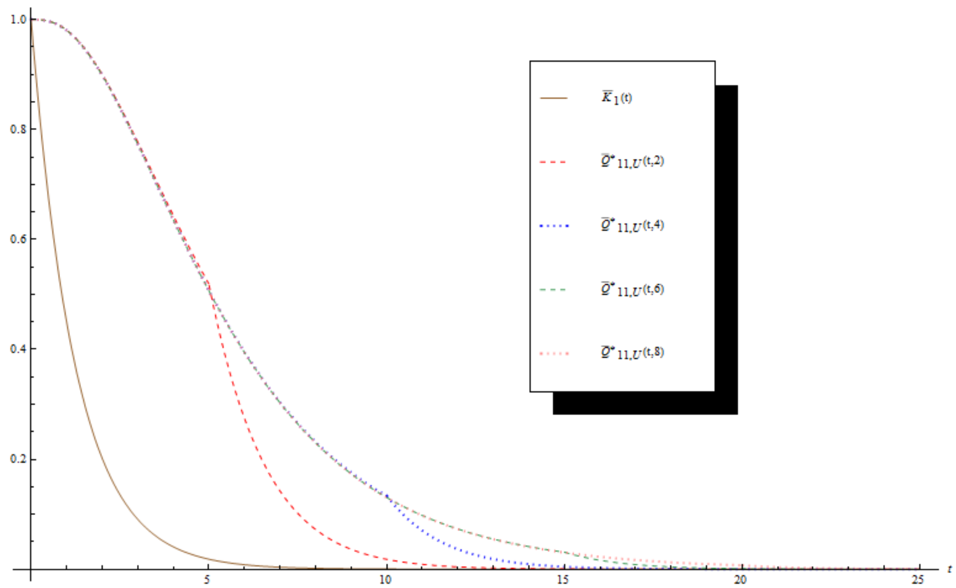


Figure 4d



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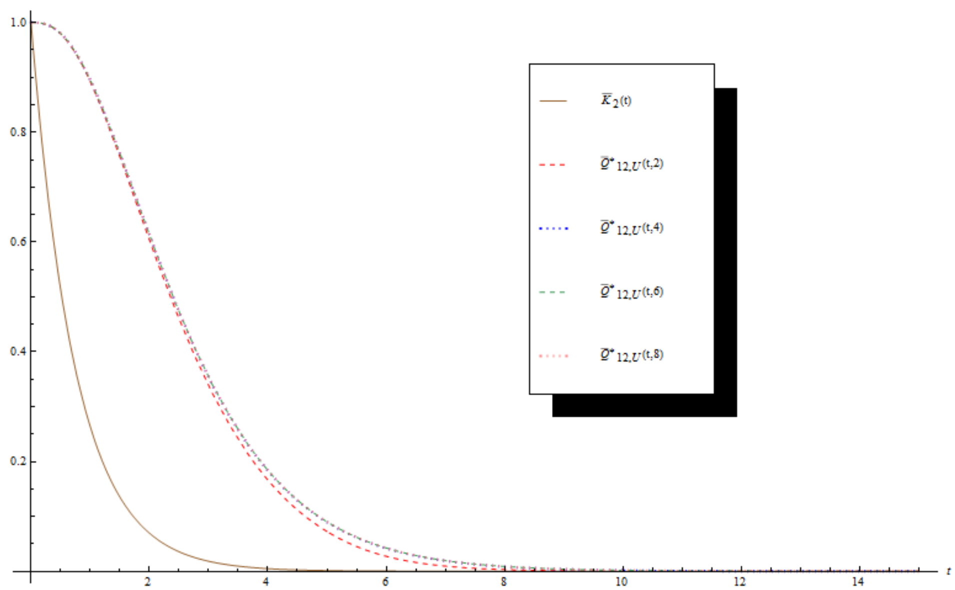

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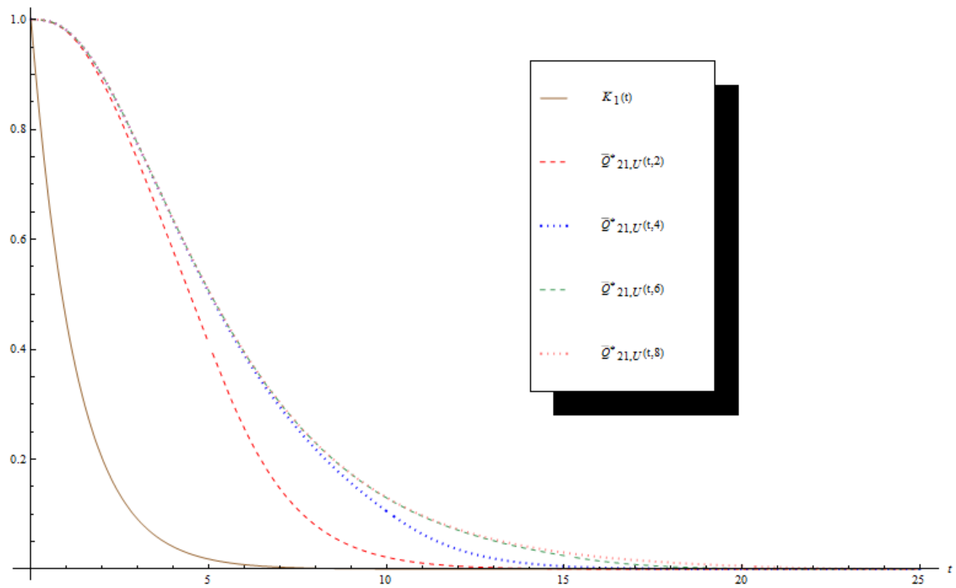

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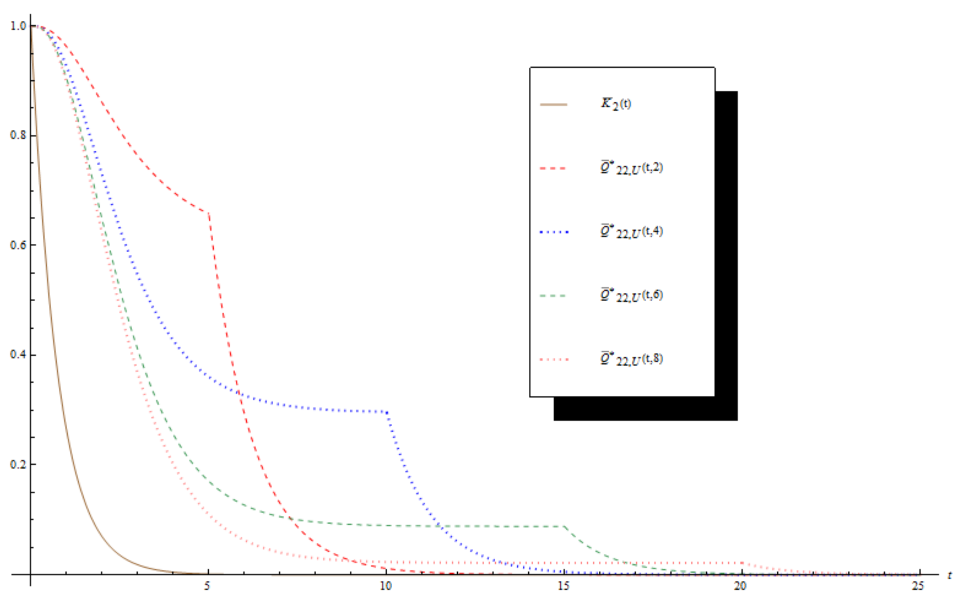

Figure 5d