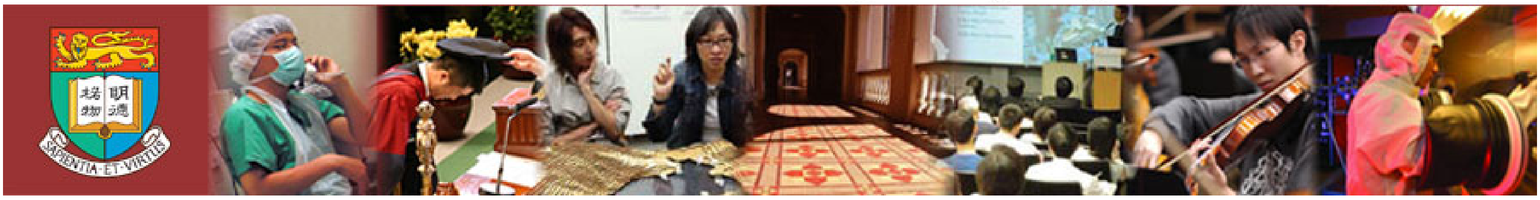

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On the Robust $\mathcal{H}_\infty$ Norm of 2D Mixed Continuous-Discrete-Time Systems with Uncertainty

Graziano Chesi

***Abstract*— This paper addresses the problem of determining the robust $\mathcal{H}_\infty$ norm of 2D mixed continuous-discrete-time systems affected by uncertainty. Specifically, it is supposed that the matrices of the model are polynomial functions of an unknown vector constrained into a semialgebraic set. It is shown that an upper bound of the robust $\mathcal{H}_\infty$ norm can be obtained via a semidefinite program (SDP) by introducing complex Lyapunov functions candidates with rational dependence on a frequency and polynomial dependence on the uncertainty. A necessary and sufficient condition is also provided to establish whether the found upper bound is tight. Some numerical examples illustrate the proposed approach.**

I. Introduction

The study of 2D mixed continuous-discrete-time systems has a long history, with some early works such as [10], [19] introducing basic models, systems theory and stability properties. Applications of these systems can be found in repetitive processes [20], disturbance propagation in vehicle platoons [11], and irrigation channels [14].

Researchers have investigated several fundamental properties of 2D mixed continuous-discrete-time systems, in particular stability, for which key contributions include [2], [7], [12], [13], [21]. Another fundamental property that has been investigated in 2D mixed continuous-discrete-time systems is the $\mathcal{H}_\infty$ norm, for which important contributions include [8], [17], [18] where conditions based on linear matrix inequalities (LMIs) have been provided for establishing upper bounds on the $\mathcal{H}_\infty$ norm.

However, these conditions cannot be used whenever the matrices of the model are affected by uncertainty. In fact, in such a case, one should repeat the existing conditions addressing the uncertainty-free case for all the admissible values of the uncertainty. Clearly, this is impossible since the number of values in a continuous set is infinite and one cannot just consider a finite subset of values such as the vertices in the case of polytopes.

This paper addresses the problem of determining the robust $\mathcal{H}_\infty$ norm of 2D mixed continuous-discrete-time systems affected by uncertainty. Specifically, it is supposed that the matrices of the model are polynomial functions of an unknown vector constrained into a semialgebraic set. It is shown that an upper bound of the robust $\mathcal{H}_\infty$ norm can be obtained via a semidefinite program (SDP) by introducing complex Lyapunov functions candidates with rational dependence on a frequency and polynomial dependence on the uncertainty. A necessary and sufficient condition is also provided to establish whether the found upper bound is tight. Some numerical examples illustrate the proposed approach.

The paper is organized as follows. Section II provides the problem formulation and some preliminaries about sums-of-squares (SOS) matrix polynomials. Section III describes the proposed results. Section IV presents an illustrative example. Lastly, Section V concludes the paper with some final remarks.

G. Chesi is with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Hong Kong. Email: chesi@eee.hku.hk.

II. Preliminaries

A. Problem Formulation

Notation:
- $\mathbb{N}, \mathbb{R}, \mathbb{C}$: natural, real, and complex number sets;
- j: imaginary unit;
- I: identity matrix (of size specified by the context);
- $\Re(\cdot)$, $\Im(\cdot)$: real and imaginary parts;
- $|\cdot|$: magnitude;
- $\|\cdot\|_2$: Euclidean norm;
- $adj(\cdot)$: adjoint;
- $\det(\cdot)$: determinant;
- $trace(\cdot)$: trace;
- $\bar{A}$: complex conjugate;
- A^T, A^H: transpose and complex conjugate transpose;
- $A \otimes B$: Kronecker product;
- Hermitian matrix A: a complex square matrix satisfying $A^H = A$;
- $\star$: corresponding block in Hermitian matrices;
- $A > 0$, $A \geq 0$: Hermitian positive definite and Hermitian positive semidefinite matrix A;
- $\deg(\cdot)$: degree;
- $\|\cdot\|_{\mathcal{L}_2}$: $\mathcal{L}_2$ norm;
- $\|\cdot\|_{Z-\mathcal{H}_\infty}$: Z $\mathcal{H}_\infty$ norm;
- $\|\cdot\|_{LZ-\mathcal{H}_\infty}$: Laplace-Z $\mathcal{H}_\infty$ norm.

Let us consider the 2D mixed continuous-discrete-time system with uncertainty described by

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}x_c(t,k) & = & A_{cc}(p)x_c(t,k) + A_{cd}(p)x_d(t,k) \\ & & +B_c(p)u(t,k) \\ x_d(t,k+1) & = & A_{dc}(p)x_c(t,k) + A_{dd}(p)x_d(t,k) \\ & & +B_d(p)u(t,k) \\ y(t,k) & = & C_c(p)x_c(t,k) + C_d(p)x_d(t,k) \\ & & +D(p)u(t,k) \end{array}\right. \tag{1}$$

where $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, respectively, the scalars t and k are the continuous and discrete times, respectively, $u \in \mathbb{R}^{n_u}$ and

$y \in \mathbb{R}^{n_y}$ are the input and output, respectively, and $p \in \mathbb{R}^q$ is a time-invariant uncertain vector. It is supposed that p is constrained as

$$p \in \mathcal{P} \tag{2}$$

where $\mathcal{P}$ is the set of admissible uncertainties modeled by

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ a_i(p) \geq 0 \ \forall i = 1 \ldots, n_a\} \tag{3}$$

where $a_i(p)$ $i = 1, \ldots, n_a$, are polynomials. The matrices $A_{cc} : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_c}$, $A_{cd} : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_d}$, $A_{dc} : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_c}$, $A_{dd} : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_d}$, $B_c : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_u}$, $B_d : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_u}$, $C_c : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_c}$, $C_d : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_d}$ and $D : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_u}$ are polynomial functions of degree not greater than d_A.

Extending the classical definition of exponential stability of 2D mixed continuous-discrete-time systems [16], we say that the system (1)–(3) is robustly exponentially stable if, for a null input $u(t, k)$, there exist $\beta, \gamma \in \mathbb{R}$ such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\|_2 \leq \beta \varrho e^{-\gamma \min\{t,k\}} \tag{4}$$

for all $t \geq 0$ and $k \geq 0$, for all initial conditions $x_c(0, k)$ and $x_d(t, 0)$, and for all $p \in \mathcal{P}$, where

$$\begin{gathered} \varrho = \max\{\varrho_1, \varrho_2\} \\ \varrho_1 = \sup_{t \geq 0} \|x_d(t,0)\|_2 \,, \quad \varrho_2 = \sup_{k \geq 0} \|x_c(0,k)\|_2 \,. \end{gathered} \tag{5}$$

Similarly, let us introduce the robust $\mathcal{H}_\infty$ norm of (1)–(3), i.e.,

$$\gamma_\infty^* = \sup_{p \in \mathcal{P}} \gamma_\infty(p) \tag{6}$$

where $\gamma_\infty(p)$ is the $\mathcal{H}_\infty$ norm of (1) for the fixed value p of the uncertainty given by

$$\gamma_\infty(p) = \sup_{u: \ \|u\|_{\mathcal{L}_2} \neq 0} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \tag{7}$$

and $\| \cdot \|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$ norm defined as

$$\|u\|_{\mathcal{L}_2} = \sqrt{\sum_{k=0}^{\infty} \int_0^{\infty} \|u(t,k)\|_2^2 dt}. \tag{8}$$

Problem. The problem addressed in this paper consists of determining the robust $\mathcal{H}_\infty$ norm of (1)–(3), i.e., γ_∞^*. □

B. SOS Matrix Polynomials

Here we provide some information about establishing whether a matrix polynomial is SOS via an LMI feasibility test. For reasons that will become clear in the next section, let us consider a matrix polynomial $J : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^{2n_d \times 2n_d}$, $J(\omega, p) = J(\omega, p)^T$, $\omega \in \mathbb{R}$ and $p \in \mathbb{R}^q$.

The matrix polynomial $J(\omega, p)$ is said to be SOS if there exist matrix polynomials $J_i : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^{2n_d \times 2n_d}$, $i = 1, \ldots, k$, such that

$$J(\omega, p) = \sum_{i=1}^{k} J_i(\omega, p)^T J_i(\omega, p). \tag{9}$$

A necessary and sufficient condition for establishing whether $J(\omega, p)$ is SOS can be obtained via an LMI feasibility test.

Indeed, $J(\omega, p)$ can be expressed as

$$J(\omega, p) = (b(\omega, p) \otimes I)^T (K + L(\alpha)) (b(\omega, p) \otimes I) \tag{10}$$

where $b(\omega, p)$ is a vector whose entries are the monomials in ω and p of degree less than or equal to d, K is a symmetric matrix satisfying

$$J(\omega, p) = (b(\omega, p) \otimes I)^T K (b(\omega, p) \otimes I) \,, \tag{11}$$

$L(\alpha)$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{ L = L^T : \ (b(\omega, p) \otimes I)^T L (b(\omega, p) \otimes I) = 0 \right\} \tag{12}$$

and α is a free vector. The representation (10) is known as square matrix representation (SMR) and extends the Gram matrix method for (scalar) polynomials to the matrix case. One has that $J(\omega, p)$ is SOS if and only if there exists α satisfying the LMI

$$K + L(\alpha) \geq 0. \tag{13}$$

See [4] and references therein for details on SOS matrix polynomials.

III. Robust $\mathcal{H}_\infty$ Norm

In this section we address the problem of determining the robust $\mathcal{H}_\infty$ norm of (1)–(3), i.e., γ_∞^* in (6).

Let us start by observing that, for the case of 2D mixed continuous-discrete-time systems without uncertainty, a necessary condition for exponential stability is that the matrices A_{cc} and A_{dd} are Hurwitz (i.e., with all eigenvalues having negative real parts) and Schur (i.e., with all eigenvalues having magnitude less than one), respectively. This means that, without loss of generality, we can introduce the following assumption, which can be checked with existing methods such as [1], [3], [6], [15], [22].

Assumption 1. The matrices $A_{cc}(p)$ and $A_{dd}(p)$ are Hurwitz and Schur, respectively, for all $p \in \mathcal{P}$. □

Let us denote with $U_L(s, k)$ and $Y_L(s, k)$ the Laplace transforms of $u(t, k)$ and $y(t, k)$, respectively, where $s \in \mathbb{C}$. Let us denote with $U_{LZ}(s, z)$ and $Y_{LZ}(s, z)$ the Z-transforms of $U_L(s, k)$ and $Y_L(s, k)$, respectively, where $z \in \mathbb{C}$. The transfer function from $u(t, k)$ and $y(t, k)$ can be expressed as

$$F(s, z, p) = \frac{Y_{LZ}(s, z)}{U_{LZ}(s, z)} \tag{14}$$

which depends not only on s and z but also on the uncertain vector p. Indeed, standard manipulations show that

$$F(s, z, p) = F_3(s, p) (zI - F_1(s, p))^{-1} F_2(s, p) + F_4(s, p) \tag{15}$$

where

$$\left\{ \begin{array}{lcl} F_1(s,p) & = & A_{dc}(p)(sI-A_{cc}(p))^{-1}A_{cd}(p)+A_{dd}(p) \\ F_2(s,p) & = & A_{dc}(p)(sI-A_{cc}(p))^{-1}B_c(p)+B_d(p) \\ F_3(s,p) & = & C_c(p)(sI-A_{cc}(p))^{-1}A_{cd}(p)+C_d(p) \\ F_4(s,p) & = & C_c(p)(sI-A_{cc}(p))^{-1}B_c(p)+D(p). \end{array} \right. \tag{16}$$

We express $F_i(s,p)$, $i=1,\ldots,4$, as

$$F_i(s,p)=\frac{G_i(s,p)}{g(s,p)} \tag{17}$$

where $G_i(s,p)$, $i=1,\ldots,4$, are matrix polynomials of suitable size, and $g(s,p)$ is defined as

$$g(s,p)=\det(sI-A_{cc}(p)). \tag{18}$$

The quantity $\gamma_\infty(p)$ in (7) can be written as

$$\gamma_\infty(p)=\|F(\cdot,\cdot,p)\|_{LZ-\mathcal{H}_\infty} \tag{19}$$

where $\|F(\cdot,\cdot,p)\|_{LZ-\mathcal{H}_\infty}$ is the Laplace-Z $\mathcal{H}_\infty$ norm of $F(s,z,p)$ defined as

$$\|F(\cdot,\cdot,p)\|_{LZ-\mathcal{H}_\infty}=\sup_{\substack{\omega\in\mathbb{R}\\ \theta\in[-\pi,\pi]}}\left\|F(j\omega,e^{j\theta},p)\right\|_2. \tag{20}$$

Hence, it follows that

$$\gamma_\infty(p)=\sup_{\omega\in\mathbb{R}}\|F(j\omega,\cdot,p)\|_{Z-\mathcal{H}_\infty} \tag{21}$$

where $\|F(j\omega,\cdot,p)\|_{Z-\mathcal{H}_\infty}$ is the Z $\mathcal{H}_\infty$ norm of $F(j\omega,z,p)$ defined as

$$\|F(j\omega,\cdot,p)\|_{Z-\mathcal{H}_\infty}=\sup_{\theta\in[-\pi,\pi]}\left\|F(j\omega,e^{j\theta},p)\right\|_2. \tag{22}$$

Since the matrices of (1) are real, one has

$$\left\{ \begin{array}{lcl} G_i(j\omega,p) & = & \overline{G_i(-j\omega,p)} \quad \forall i=1,\ldots,4 \\ g(j\omega,p) & = & \overline{g(-j\omega,p)} \end{array} \right. \tag{23}$$

for all $\omega\in\mathbb{R}$ for all $p\in\mathbb{R}^q$. This suggests that one can focus on Lyapunov function candidates having a similar symmetry property with respect to ω. To this end, let us introduce the following definitions. For a complex matrix function $M:\mathbb{R}\times\mathbb{R}^q\to\mathbb{C}^{n_1\times n_2}$, we say that $M(\omega,p)$ is *even with respect to* ω if

$$M(-\omega,p)=\overline{M(\omega,p)} \quad \forall\omega\in\mathbb{R}\ \forall p\in\mathbb{R}^q \tag{24}$$

and we say that $M(\omega)$ is *odd with respect to* ω if

$$M(-\omega,p)=-\overline{M(\omega,p)} \quad \forall\omega\in\mathbb{R}\ \forall p\in\mathbb{R}^q. \tag{25}$$

Let us define the sets

$$\begin{array}{lcl} \mathcal{M}(n) & = & \{M:\mathbb{R}\times\mathbb{R}^q\to\mathbb{C}^{n\times n}, \\ & & M(\omega,p) \text{ is a Hermitian matrix polynomial}\} \end{array} \tag{26}$$

and

$$\begin{array}{lcl} \mathcal{M}_{even}(n) & = & \{M\in\mathcal{M}(n),\ M(\omega,p) \text{ is even} \\ & & \text{with respect to } \omega\}. \end{array} \tag{27}$$

Let us introduce the Lyapunov function candidate

$$\left\{ \begin{array}{rcl} V_{RAT}(\omega,p) & = & \dfrac{V(\omega,p)}{v(\omega)} \\ V & \in & \mathcal{M}_{even}(n_d) \\ \deg(V) & \le & 2d \end{array} \right. \tag{28}$$

where $d\in\mathbb{N}\cup\{0\}$,

$$v(\omega)=(1+\omega^2)^d, \tag{29}$$

and $\deg(V)$ denotes the maximum degree of the entries of $V(\omega,p)$ in the extended variable $(\omega,p)'$. Define

$$Q(\omega,p)=\begin{pmatrix} q_1(\omega,p) & q_2(\omega,p) \\ \star & q_3(\omega,p) \end{pmatrix} \tag{30}$$

where

$$\left\{ \begin{array}{lcl} q_1(\omega,p) & = & |g(j\omega,p)|^2\,V(\omega,p) \\ & & -G_1(j\omega,p)V(\omega,p)G_1(j\omega,p)^H \\ & & -v(\omega)G_2(j\omega,p)G_2(j\omega,p)^H \\ q_2(\omega,p) & = & -G_1(j\omega,p)V(\omega,p)G_3(j\omega,p)^H \\ & & -v(\omega)G_2(j\omega,p)G_4(j\omega,p)^H \\ q_3(\omega,p) & = & \xi v(\omega)\,|g(j\omega,p)|^2\,I \\ & & -G_3(j\omega,p)V(\omega,p)G_3(j\omega,p)^H \\ & & -v(\omega)G_4(j\omega,p)G_4(j\omega,p)^H \end{array} \right. \tag{31}$$

and $\xi\in\mathbb{R}$. It follows that $Q\in\mathcal{M}_{even}(n_q)$ where

$$n_q=n_d+n_u. \tag{32}$$

Let us define the matrix function

$$\Phi(W)=\begin{pmatrix} W_R & W_I \\ -W_I & W_R \end{pmatrix} \tag{33}$$

where $W_R,W_I\in\mathbb{R}^{n\times n}$ are the real and imaginary parts of $W\in\mathbb{C}^{n\times n}$, i.e., $W=W_R+jW_I$. Let us observe that

$$W \text{ is Hermitian} \iff \Phi(W)=\Phi(W)^T. \tag{34}$$

The following result provides an upper bound on the robust $\mathcal{H}_\infty$ norm of (1)–(3) via a semidefinite program (SDP).

Theorem 1: Define

$$\hat{\gamma}_\infty=\sqrt{\hat{\xi}} \tag{35}$$

where $\hat{\xi}$ is the solution of the SDP

$$\begin{array}{lcl} \hat{\xi} & = & \displaystyle\inf_{\substack{V\in\mathcal{M}_{even}(n_d)\\ R_i\in\mathcal{M}_{even}(n_q)\\ \xi,\varepsilon\in\mathbb{R}}} \xi \\ & & \text{s.t.} \left\{ \begin{array}{l} \Phi(R_i(\omega,p)) \text{ is SOS } \forall i=1,\ldots,n_a \\ \Phi(S(\omega,p)) \text{ is SOS} \\ \varepsilon>0 \\ \deg(V)\le 2d \\ \deg(R_i)\le 2d \end{array} \right. \end{array} \tag{36}$$

where

$$S(\omega,p)=Q(\omega,p)-\sum_{i=1}^{n_a}a_i(p)R_i(\omega,p)-\varepsilon v(\omega)\,|g(j\omega,p)|^2\,I. \tag{37}$$

Then,

$$\hat{\gamma}_\infty \geq \gamma_\infty^*. \tag{38}$$

Proof. Suppose that the constraints in (36) hold. It follows that

$$\forall \omega \in \mathbb{R}\ \forall p \in \mathbb{R}^q \ \left\{ \begin{array}{l} R_i(\omega,p) \geq 0\ \forall i = 1,\ldots,n_a \\ S(\omega,p) \geq 0. \end{array} \right.$$

From (37) it follows that

$$Q(\omega,p) \geq \varepsilon v(\omega) \left|g(j\omega,p)\right|^2 I \quad \forall \omega \in \mathbb{R}\ \forall p \in \mathcal{P}.$$

Let us observe that

$$Q(\omega,p) = v(\omega) \left|g(j\omega,p)\right|^2 E(\omega,p)$$

where $E(\omega,p)$ is obtained from $Q(\omega,p)$ replacing $q_1(\omega,p), q_2(\omega,p), q_3(\omega,p)$ with $e_1(\omega,p), e_2(\omega,p), e_3(\omega,p)$, where

$$\left\{ \begin{array}{rcl} e_1(\omega,p) &=& V_{RAT}(\omega,p) \\ && -F_1(j\omega,p)V_{RAT}(\omega,p)F_1(j\omega,p)^H \\ && -F_2(j\omega,p)F_2(j\omega,p)^H \\ e_2(\omega,p) &=& -F_1(j\omega,p)V_{RAT}(\omega,p)F_3(j\omega,p)^H \\ && -F_2(j\omega,p)F_4(j\omega,p)^H \\ e_3(\omega,p) &=& \xi I - F_3(j\omega,p)V_{RAT}(\omega,p)F_3(j\omega,p)^H \\ && -F_4(j\omega,p)F_4(j\omega,p)^H. \end{array} \right.$$

Since Assumption 1 implies that there exists $\varepsilon_1 > 0$ such that

$$|g(j\omega,p)| \geq \varepsilon_1 \quad \forall \omega \in \mathbb{R}\ \forall p \in \mathbb{R}^q,$$

and since

$$v(\omega) \geq 1 \quad \forall \omega \in \mathbb{R},$$

one can write

$$E(\omega,p) \geq \varepsilon I \quad \forall \omega \in \mathbb{R}\ \forall p \in \mathcal{P}.$$

Since $\varepsilon > 0$, from the bounded real lemma and Schur complement it follows that (see, e.g., [9])

$$\sqrt{\xi} > \|F(j\omega,\cdot,p)\|_{Z-\mathcal{H}_\infty} \quad \forall \omega \in \mathbb{R}\ \forall p \in \mathcal{P}.$$

From (6) and (21), this implies that (38) holds. □

Theorem 1 provides an upper bound on γ_∞^* via an SDP. Indeed, the constraints in (36) are equivalent to LMIs according to Section II-B since $\Phi(R_i(\omega,p))$ and $\Phi(S(\omega,p))$ are affine linear in the decision variables $V(\omega,p)$, $R_i(\omega,p)$, ξ and ε. Let us observe that $V_{RAT}(\omega,p)$ defines a complex Lyapunov function candidate with rational dependence in ω and polynomial dependence in p.

Once that the upper bound $\hat{\gamma}_\infty$ has been obtained, a question arises: is this upper bound tight? The following result provides a sufficient and necessary condition for answering this question.

Theorem 2: Suppose that $\hat{\gamma}_\infty < \infty$. Then,

$$\hat{\gamma}_\infty = \gamma_\infty^* \tag{39}$$

if at least one of the following two sub-conditions holds:

1) there exists $\hat{\omega} \in \mathbb{R}$ and $\hat{p} \in \mathcal{P}$ such that

$$\|F(j\hat{\omega},\cdot,\hat{p})\|_{Z-\mathcal{H}_\infty} = \hat{\gamma}_\infty \tag{40}$$

and

$$\det\left(\Phi\left(\hat{S}(\hat{\omega},\hat{p})\right)\right) = 0 \tag{41}$$

where $\hat{S}(\omega,p)$ is $S(\omega,p)$ evaluated for the optimal values of the decision variables in (36);

2) there exists $\hat{p} \in \mathcal{P}$ such that

$$\lim_{\omega\to\infty} \|F(j\omega,\cdot,\hat{p})\|_{Z-\mathcal{H}_\infty} = \hat{\gamma}_\infty. \tag{42}$$

Moreover, if $\mathcal{P}$ is bounded, this condition is not only sufficient but also necessary.

Proof. "$\Leftarrow$" Suppose that (40) or (42) holds. Then, it follows that $\hat{\gamma}_\infty \leq \gamma_\infty^*$ since γ_∞^* is the supremum of $\|F(j\omega,\cdot,p)\|_{Z-\mathcal{H}_\infty}$ for $\omega \in \mathbb{R}$ and $p \in \mathcal{P}$, while Theorem 1 guarantees that $\hat{\gamma}_\infty \geq \gamma_\infty^*$. Therefore, (39) holds.

"$\Rightarrow$" Suppose that (39) holds and that $\mathcal{P}$ is bounded. This implies that $\mathcal{P}$ is compact. There are two possibilities. The first is that there exist $\hat{\omega} \in \mathbb{R}$ and $\hat{p} \in \mathcal{P}$ such that

$$\gamma_\infty^* = \|F(j\hat{\omega},\cdot,\hat{p})\|_{Z-\mathcal{H}_\infty},$$

which also satisfy (41). In fact, if one supposes for contradiction that (41) does not hold, from the fact that $\Phi(S(\omega,p))$ is SOS it would follow that

$$\Phi\left(\hat{S}(\hat{\omega},\hat{p})\right) > 0,$$

hence implying that the existence of $V(\omega,p) = \hat{V}(\omega,p)$, $R_i(\omega,p) = \hat{R}_i(\omega,p)$, ξ and ε such that the constraints in (36) hold and

$$\xi < \hat{\xi},$$

which is impossible for definition of $\hat{\xi}$. The second possibility is that there exists $\hat{p} \in \mathcal{P}$ such that (42) holds. □

In order to check the first sub-condition of Theorem 2, one can determine the pairs $(\hat{\omega},\hat{p})$ that satisfy (41) since they are typically in a finite number, and then check whether (40) holds for any of these pairs. One way to determine the pairs $(\hat{\omega},\hat{p})$ that satisfy (41) is via the following result.

Theorem 3: The condition (41) holds if and only if there exists $\hat{x} \in \mathbb{R}^{2n_q}$, $\hat{x} \neq 0$, such that

$$b(\hat{\omega},\hat{p}) \otimes \hat{x} \in \ker(T) \tag{43}$$

where T is a positive semidefinite SMR matrix of $\Phi(\hat{S}(\omega,p))$, and $b(\hat{\omega},\hat{p})$ is the corresponding vector of monomials.

Proof. Since $\Phi(\hat{S}(\omega,p))$ is SOS, it follows that $\Phi\left(\hat{S}(\omega,p)\right)$ is positive semidefinite for all $\omega \in \mathbb{R}$ for all $p \in \mathbb{R}^q$. Hence, (41) holds if and only if there exists $\hat{x} \in \mathbb{R}^{2n_q}$, $\hat{x} \neq 0$, such that

$$\Phi\left(\hat{S}(\hat{\omega},\hat{p})\right)\hat{x} = 0.$$

This implies that

$$\begin{aligned} 0 &= \hat{x}'\Phi\left(\hat{S}(\hat{\omega},\hat{p})\right)\hat{x} \\ &= \hat{x}'\left(b(\hat{\omega},\hat{p})\otimes I\right)^T T\left(b(\hat{\omega},\hat{p})\otimes I\right)\hat{x} \\ &= \left(b(\hat{\omega},\hat{p})\otimes \hat{x}\right)^T T\left(b(\hat{\omega},\hat{p})\otimes \hat{x}\right) \end{aligned}$$

where T is a positive semidefinite matrix, whose existence is ensured by the fact that $\Phi(\hat{S}(\omega,p))$. Hence, (43) holds. □

Theorem 3 provides a condition equivalent to (41) based on the existence of $\hat{\omega}\in\mathbb{R}$, $\hat{p}\in\mathcal{P}$ and $\hat{x}\in\mathbb{R}^{2n_q}$ satisfying (43). It turns out that such quantities can be determined through linear algebra operations as explained in [5], [6]. Once the pairs $(\hat{\omega},\hat{p})$ that satisfy (41) have been determined, one checks whether (40) holds for any of these. The positive semidefinite matrix T in (43) is directly provided by the SDP solver used for (36).

In order to check the second sub-condition of Theorem 2, one can adopt a strategy similar to that just described and simplified by the fact that ω is known. Specifically, let us define

$$\begin{aligned} \tilde{\mathcal{M}}(n) = \{&M:\mathbb{R}^q\to\mathbb{R}^{n\times n},\ M(p) \text{ is a symmetric} \\ &\text{matrix polynomial}\}. \end{aligned} \tag{44}$$

Let $\tilde{Q}(p)$ be the matrix polynomial obtained from $Q(\omega,p)$ replacing $q_1(\omega,p),q_2(\omega,p),q_3(\omega,p)$ with $\tilde{q}_1(p),\tilde{q}_2(p),\tilde{q}_3(p)$, where

$$\left\{\begin{aligned} \tilde{q}_1(p) &= \tilde{V}(p)-A_{dd}(p)\tilde{V}(p)A_{dd}(p)^T \\ &\quad -B_d(p)B_d(p)^T \\ \tilde{q}_2(p) &= -A_{dd}(p)\tilde{V}(p)C_d(p)^T \\ &\quad -B_d(p)D(p)^T \\ \tilde{q}_3(p) &= \xi I-C_d(p)\tilde{V}(p)C_d(p)^T \\ &\quad -D(p)\tilde{V}(p)D(p)^T \end{aligned}\right. \tag{45}$$

and $\tilde{V}\in\tilde{\mathcal{M}}(n_d)$. Let us define

$$\tilde{\gamma}_\infty=\sqrt{\tilde{\xi}} \tag{46}$$

where $\tilde{\xi}$ is the solution of the SDP

$$\begin{aligned} \tilde{\xi} = &\inf_{\substack{\tilde{V}\in\tilde{\mathcal{M}}(n_d)\\ \tilde{R}_i\in\tilde{\mathcal{M}}(n_q)\\ \xi,\varepsilon\in\mathbb{R}}} \xi \\ &\text{s.t.}\left\{\begin{array}{l} \Phi(\tilde{R}_i(p)) \text{ is SOS } \forall i=1,\ldots,n_a \\ \Phi(\tilde{S}(p)) \text{ is SOS} \\ \varepsilon>0 \\ \deg(\tilde{V})\le 2d \\ \deg(\tilde{R}_i)\le 2d \end{array}\right. \end{aligned} \tag{47}$$

where

$$\tilde{S}(p)=\tilde{Q}(p)-\sum_{i=1}^{n_a}a_i(p)\tilde{R}_i(p)-\varepsilon I. \tag{48}$$

Theorem 4: Any $\hat{p}\in\mathcal{P}$ that satisfies (42) also satisfies

$$\det\left(\hat{\tilde{S}}(\hat{p})\right)=0 \tag{49}$$

where $\hat{\tilde{S}}(p)$ is $\tilde{S}(p)$ evaluated for the optimal values of the variables in (47). Moreover, (49) holds if and only if there exists $\tilde{x}\in\mathbb{R}^{n_q}$, $\tilde{x}\neq 0$, such that

$$b(\hat{p})\otimes\tilde{x}\in\ker(\tilde{T}) \tag{50}$$

where $\tilde{T}$ is a positive semidefinite SMR matrix of $\hat{\tilde{S}}(p)$ evaluated for the optimal values of the variables in (47), and $b(\hat{p})$ is the corresponding vector of monomials.

Proof. Let us observe that

$$\tilde{\gamma}_\infty\ge\gamma_\infty^{\#}$$

where

$$\gamma_\infty^{\#}=\sup_{p\in\mathcal{P}}\lim_{\omega\to\infty}\|F(j\omega,\cdot,p)\|_{Z-\mathcal{H}_\infty}.$$

Moreover,

$$\gamma_\infty^*\ge\gamma_\infty^{\#}.$$

If (42) holds, then

$$\gamma_\infty^*=\gamma_\infty^{\#}$$

and (49) follows based on the same arguments used in the proof of Theorem 2. Lastly, the equivalence between (49) and (50) follows based on the same arguments used in the proof of Theorem 3. □

Theorem 4 provides a strategy for establishing whether (42) holds for some $\hat{p}\in\mathcal{P}$. Specifically, one determines the values of $\hat{p}$ such that (50) holds similarly to Theorem 3, and then checks whether (42) holds for any of these. The positive semidefinite matrix $\tilde{T}$ in (50) is directly provided by the SDP solver used for (47).

IV. EXAMPLES

In this section we present two illustrative examples of the proposed results. The SDPs are solved with the toolbox SeDuMi [23] for Matlab.

A. Example 1

Let us consider

$$\left\{\begin{array}{llllll} A_{cc}(p) &=& \begin{pmatrix}0&1\\-4&-2\end{pmatrix}, & A_{cd}(p) &=& \begin{pmatrix}-0.6\\0.4p\end{pmatrix} \\ A_{dc}(p) &=& \begin{pmatrix}2&0.5\end{pmatrix}, & A_{dd}(p) &=& 0.5 \\ B_c(p) &=& \begin{pmatrix}0\\2+p\end{pmatrix}, & B_d(p) &=& 1 \\ C_c(p) &=& \begin{pmatrix}0&-1\end{pmatrix}, & C_d(p) &=& 1 \\ D(p) &=& 1, & \mathcal{P} &=& [0,1]. \end{array}\right.$$

Let us use Theorem 1. The set $\mathcal{P}$ is expressed as in (3) with $a(p)=p-p^2$. We solve the SDP (36) with $2d=0$. We find $\hat{\gamma}_\infty=4.157$. This upper bound can be improved by using $2d=2$, which provides

$$\hat{\gamma}_\infty=3.076.$$

Next, let us use Theorem 2 to establish whether the found upper bound is tight. We find that (43) holds with

$$\begin{cases} \hat{\omega} &= 2.018 \\ \hat{p} &= 0.000. \end{cases}$$

Hence, from Theorem 3 it follows that (41) holds for such values of $\hat{\omega}$ and $\hat{p}$. Moreover, for such values of $\hat{\omega}$ and $\hat{p}$, one has that (40) holds. Consequently, from Theorem 2 we conclude that $\hat{\gamma}_\infty$ is tight, i.e., $\gamma^*_\infty = 3.076$.

B. Example 2

Let us consider

$$\begin{cases} A_{cc}(p) &= -1, & A_{cd}(p) &= \begin{pmatrix} 0.4 & 0.4 \end{pmatrix} \\ A_{dc}(p) &= \begin{pmatrix} 0.3 \\ -0.5 \end{pmatrix}, & A_{dd}(p) &= \begin{pmatrix} 0.4p & 0 \\ 0 & 0.3 \end{pmatrix} \\ B_c(p) &= 0, & B_d(p) &= \begin{pmatrix} p \\ 1 \end{pmatrix} \\ C_c(p) &= 0, & C_d(p) &= \begin{pmatrix} 0 & 1-p \end{pmatrix} \\ D(p) &= 1, & \mathcal{P} &= [-1, 1]. \end{cases}$$

Let us use Theorem 1. The set $\mathcal{P}$ is expressed as in (3) with $a(p) = 1 - p^2$. We solve the SDP (36) with $2d = 0$. We find

$$\hat{\gamma}_\infty = 3.857.$$

Next, let us use Theorem 2 to establish whether the found upper bound is tight. We find that (43) does not hold for any $\hat{\omega}$ and $\hat{p}$. Hence, we solve the SDP (47) with $2d = 0$. We find $\tilde{\gamma}_\infty = 3.857$. Moreover, (50) holds with

$$\hat{p} = -1.000.$$

Such a value of $\hat{p}$ also satisfies (42). Consequently, from Theorem 2 we conclude that $\hat{\gamma}_\infty$ is tight, i.e., $\gamma^*_\infty = 3.857$.

V. Conclusion

The problem of determining the robust $\mathcal{H}_\infty$ norm of 2D mixed continuous-discrete-time systems polynomially affected by uncertainty constrained into a semialgebraic set has been considered. It has been shown that an upper bound of the robust $\mathcal{H}_\infty$ norm can be obtained via an SDP by introducing complex Lyapunov functions candidates with rational dependence on a frequency and polynomial dependence on the uncertainty. Moreover, a necessary and sufficient condition has been provided to establish whether the found upper bound is tight.

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