Title	On the unstable of continuous-time linearized nonlinear systems
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Citation	The 53rd IEEE Annual Conference on Decision and Control (CDC 2014), Los Angeles, CA., 15-17 December 2014. In Conference Proceedings, 2014, p. 2316-2321
Issued Date	2014
URL	http://hdl.handle.net/10722/211402
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53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

On the Unstable of Continuous-Time Linearized Nonlinear Systems

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Abstract— **It has been shown that quantifying the unstable in linear systems is important for establishing the existence of stabilizing feedback controllers. This paper addresses the problem of quantifying the unstable in continuous-time linearized systems obtained from nonlinear systems for a family of constant inputs, i.e., the largest instability measure for all admissible equilibrium points for all admissible constant inputs. It is supposed that the dynamics of the nonlinear system is polynomial in both state and input, and that the set of constant inputs is a semialgebraic set. Two cases are considered: first, when the equilibrium points are known polynomial functions of the input, and, second, when the equilibrium points are unknown (polynomial or non-polynomial) functions of the input. It is shown that upper bounds of the sought instability measure can be established through linear matrix inequalities (LMIs), whose conservatism can be decreased by increasing the size of such LMIs. Some numerical examples illustrate the proposed results.**

I. INTRODUCTION

An important issue in control systems consists of quantifying the unstable. Indeed, it has been shown that this allows one to establish the existence of stabilizing feedback controllers in linear systems. For instance, [1] considers stochastic systems with noise and derive that a stabilizing controller can be designed if and only if the data rate of the channel exceeds a certain function of the instability measure defined as the sum (continuous-time case) or the product (discrete-time case) of the unstable eigenvalues. Analogous results are proposed in [2] where the lowest quantization density of infinite logarithmic quantizers in a single-input linear system is investigated, in [3] where the case of multiple sensors that partially observe the system is considered, in [4] where a virtual system approach for digital finite communication bandwidth control is described, and in [5] for the case of channel modeled by a finite logarithmic quantizer. The reader is also referred to [6], [7] for more information.

The linear system that has to be considered when quantifying the unstable is very often a linearized nonlinear system. Indeed, real plants are generally characterized by nonlinear dynamics. Moreover, considering the linearized system obtained for a certain equilibrium point for a certain constant input is generally sufficiently for designing a locally stabilizing feedback controller. Unfortunately, the constant input is very often unknown, for instance because the user is allowed to change it in order to choose a desired performance. Consequently, the linearized system is unknown as well, and its dependence on the unknown constant input can be very complex since also the equilibrium point depends on the unknown constant input. In fact, the set of admissible equilibrium points cannot be parametrized in general by the constant input since it is the solution of a system of nonlinear equations.

This paper addresses the problem of quantifying the unstable in continuous-time linearized systems obtained from nonlinear systems for a family of constant inputs, i.e., the largest instability measure for all admissible equilibrium points for all admissible constant inputs. It is supposed that the dynamics of the nonlinear system is polynomial in both state and input, and that the set of constant inputs is a semialgebraic set. Two cases are considered: first, when the equilibrium points are known polynomial functions of the input, and, second, when the equilibrium points are unknown (polynomial or non-polynomial) functions of the input. It is shown that upper bounds of the sought instability measure can be established through linear matrix inequalities (LMIs), whose conservatism can be decreased by increasing the size of such LMIs. Some numerical examples illustrate the proposed results.

The paper is organized as follows. Section II introduces the problem formulation. Section III describes the proposed results. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PROBLEM FORMULATION

The notation used throughout the paper is as follows:

- $\mathbb{R}$: space of real numbers;
- $\mathbb{C}$: space of complex numbers;
- 0_n: $n \times 1$ null vector;
- $\mathbb{R}^n_0$: $\mathbb{R}^n \setminus \{0_n\}$;
- I: identity matrix (of size specified by the context);
- A': conjugate transpose of matrix A;
- $A > 0$, $A \geq 0$: hermitian positive definite and semidefinite matrix A;
- $\Re(a)$, $\Im(a)$: real and imaginary parts of $a \in \mathbb{C}$;
- $\ker(A)$: right null space of matrix A;
- $\mathrm{spec}(A)$: spectrum of matrix A;
- $\lambda_{min}(A)$: minimum real eigenvalue of matrix A;
- $\mathrm{conv}\{a, b, \ldots\}$: convex hull of vectors $a, b, \ldots$.

Let us consider the continuous-time nonlinear plant of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial function.

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Let $\phi \in \mathbb{R}^m$ be a reference value of interest of the input vector. We denote with $\Theta(\phi) \subseteq \mathbb{R}^n$ the set of equilibrium points of the nonlinear plant (1) corresponding to ϕ, i.e.,

$$\Theta(\phi) = \{\theta \in \mathbb{R}^n : \ f(\theta, \phi) = 0\}. \tag{2}$$

Let $\theta \in \Theta(\phi)$ be an equilibrium point of interest of the nonlinear plant (1) corresponding to ϕ. For local stabilization in a neighbourhood of the pair (θ, ϕ), the nonlinear plant (1) is generally approximated with its linearized form

$$\dot{\tilde{x}}(t) = A(\theta, \phi)\tilde{x}(t) + B(\theta, \phi)\tilde{u}(t) \tag{3}$$

where

$$\left\{ \begin{array}{rcl} \tilde{x}(t) & = & x(t) - \theta \\ \tilde{u}(t) & = & u(t) - \phi \end{array} \right. \tag{4}$$

are the variations of the state vector and input vector with respect to θ and ϕ, respectively, and

$$\left\{ \begin{array}{rcl} A(\theta, \phi) & = & \left. \dfrac{df(x,u)}{dx} \right|_{(x,u)=(\theta,\phi)} \\ B(\theta, \phi) & = & \left. \dfrac{df(x,u)}{du} \right|_{(x,u)=(\theta,\phi)} \end{array} \right. \tag{5}$$

are the matrices that describe the local behaviour of the plant.

In the literature, several conditions for the existence of a stabilizing controller for the linearized plant (3) under communication constraints have been given based on a certain instability measure of the matrix $A(\theta, \phi)$, which has to be smaller than a specific value depending on the communication constraints considered. Specifically, let $M \in \mathbb{R}^{n\times n}$. Such an instability measure is defined as $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ where

$$\mu(M) = \sum_{i=1}^{n} \max\{0, \Re(\lambda_i(M))\} \tag{6}$$

and $\lambda_i : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the i-th eigenvalue of M.

If the reference value ϕ of the input vector and the equilibrium point θ of the nonlinear plant (1) are known, then the instability measure to consider is known as well, since the matrix $A(\theta, \phi)$ is a constant for such values of θ and ϕ.

However, the pair (θ, ϕ) is very often unknown due to the following reasons:

- the reference value ϕ might change, for instance as the result of a choice of the user;
- the equilibrium point θ might be uncertain in the set $\Theta(\phi)$, for instance due to the initial condition of the nonlinear plant (1).

Due to this uncertainty on the pair (θ, ϕ), it appears clear that one should determine the largest value of the instability measure for all admissible pairs (θ, ϕ) in order to ensure the existence of a stabilizing controller for the linearized plant (3) under communication constraints.

Let us denote with $\Phi \subseteq \mathbb{R}^m$ the set of admissible reference values of the input vector, i.e.,

$$\phi \in \Phi. \tag{7}$$

We suppose that Φ is expressed as

$$\Phi = \{\phi \in \mathbb{R}^m : \ a_i(\phi) \geq 0, \ i = 1, \ldots, n_a\} \tag{8}$$

where $a_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n_a$, are polynomials. The problem addressed in this paper is formulated as follows.

Problem. Determine the largest instability measure $\mu(A(\theta, \phi))$ over the admissible pairs (θ, ϕ), i.e.,

$$\mu^* = \sup_{\substack{\phi\in\Phi \\ \theta\in\Theta(\phi)}} \mu(A(\theta, \phi)). \tag{9}$$

III. PROPOSED METHOD

This section describes the method proposed in this paper for determining μ^* in (9). Let us start by introducing the following two cases.

- Case I: for all $\phi \in \Phi$, the equilibrium points of the nonlinear plant (1) can be expressed by polynomial functions of ϕ. That is, there exist polynomial functions $g_l : \mathbb{R}^m \to \mathbb{R}^n$, $l = 1, \ldots, n_g$, such that the set of equilibrium points $\Theta(\phi)$ in (2) can be written as

$$\Theta(\phi) = \{g_1(\phi), \ldots, g_{n_g}(\phi)\} \quad \forall \phi \in \Phi. \tag{10}$$

- Case II: any possibility (including Case I).

A. CASE I

Let us suppose that (10) holds for some polynomial functions $g_l : \mathbb{R}^m \to \mathbb{R}^n$, $l = 1, \ldots, n_g$. The following example illustrates such a situation.

Example 1. Let us consider the nonlinear system (1) with $n = 3$, $m = 1$ and

$$f(x, u) = \begin{pmatrix} -2x_3 - x_2^2 \\ x_2 - x_1^2 \\ x_1 + x_2 + x_1 u \end{pmatrix}. \tag{11}$$

The set Φ and its expression in (8) are chosen as

$$\left\{ \begin{array}{rcl} \Phi & = & [-1, 1] \\ a_1(\phi) & = & 1 - \phi^2. \end{array} \right. \tag{12}$$

It follows that the set of equilibrium points $\Theta(\phi)$ in (2) can be written as in (10) for polynomial functions $g_l(\phi)$, $l = 1, 2$, given by

$$g_1(\phi) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad g_2(\phi) = \begin{pmatrix} -(1+\phi) \\ (1+\phi)^2 \\ -0.5(1+\phi)^4 \end{pmatrix}. \tag{13}$$

Figure 1 shows $g_1(\phi)$ and $g_2(\phi)$ for some values of ϕ.

□

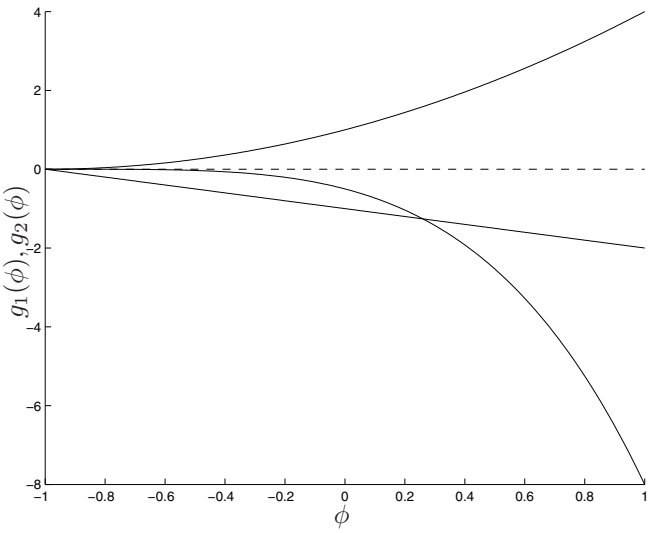

Fig. 1. Example 1. Equilibrium points $g_1(\phi)$ (dashed) and $g_2(\phi)$ (solid) for some values of ϕ in Φ.

In order to determine μ^* in (9) in this case, let us proceed as follows. For $k \in \{1, \ldots, n\}$ define the binomial coefficient

$$c_k = \frac{n!}{(n-k)!k!} \tag{14}$$

and the matrix function $\Omega_k : \mathbb{R}^{n\times n} \to \mathbb{R}^{c_k \times c_k}$ satisfying

$$\operatorname{spec}(\Omega_k(M)) = \left\{ \sum_{i=1}^{k} \lambda_{z_i}(M),\ z \in \mathcal{I}_k \right\} \tag{15}$$

where $M \in \mathbb{R}^{n\times n}$ and $\mathcal{I}_k$ is the set of k-tuples in $\{1, \ldots, n\}$, i.e.,

$$\mathcal{I}_k = \{(z_1, \ldots, z_k) :\ z_i \in \{1, \ldots, n\},\ z_i > z_j \text{ for } i < j\}. \tag{16}$$

The matrix function $\Omega_k(M)$ is linear, and can be directly built from X following the idea described in [8]. For instance:

- for $n = 2$ one has

$$\left\{ \begin{array}{rcl} \mathcal{I}_1 &=& \{z(1) = 1, z(2) = 2\} \\ \mathcal{I}_2 &=& \{z(1) = (1, 2)\} \end{array} \right.$$

  and

$$\left\{ \begin{array}{rcl} \Omega_1(M) &=& M \\ \Omega_2(M) &=& \operatorname{trace}(M); \end{array} \right.$$

- for $n = 3$,

$$\left\{ \begin{array}{l} \mathcal{I}_1 = \{z(1) = 1, z(2) = 2, z(3) = 3\} \\ \mathcal{I}_2 = \{z(1) = (1, 2), z(2) = (1, 3), z(3) = (2, 3)\} \\ \mathcal{I}_3 = \{z(1) = (1, 2, 3)\} \end{array} \right.$$

  and

$$\left\{ \begin{array}{l} \Omega_1(M) = M \\ \Omega_2(M) = \begin{pmatrix} M_{1,1} + M_{2,2} & M_{2,3} & -M_{1,3} \\ M_{3,2} & M_{1,1} + M_{3,3} & M_{1,2} \\ -M_{3,1} & M_{2,1} & M_{2,2} + M_{3,3} \end{pmatrix} \\ \Omega_3(M) = \operatorname{trace}(M). \end{array} \right.$$

The matrix function $\Omega_k(M)$ can be exploited as follows. For $k \in \{1, \ldots, n\}$, $l \in \{1, \ldots, n_g\}$ and $w \in \mathbb{R}$, let us define

$$D_{k,l}(\phi) = \Omega_k(A(g_l(\phi), \phi)) - wI \tag{17}$$

which is a matrix polynomial since $g_l(\phi)$ is polynomial, $A(\theta, \phi)$ is polynomial, and $\Omega_k(M)$ is linear. Moreover, for a symmetric matrix polynomial $P_{k,l} : \mathbb{R}^m \to \mathbb{R}^{c_k \times c_k}$, let us define

$$Q_{k,l}(\phi) = -P_{k,l}(\phi)D_{k,l}(\phi) - D_{k,l}(\phi)'P_{k,l}(\phi). \tag{18}$$

Lastly, let us introduce the following definition: a symmetric matrix polynomial $H : \mathbb{R}^m \to \mathbb{R}^{h\times h}$ is said to be sum of squares (SOS) if there exist matrix polynomials $H_i : \mathbb{R}^m \to \mathbb{R}^{h\times h}$, $i = 1, \ldots, i_{max}$, such that

$$H(\phi) = \sum_{i=1}^{i_{max}} H_i(\phi)'H_i(\phi) \tag{19}$$

(see for instance [9] and references therein about SOS matrix polynomials). The following result provides a condition for establishing an upper bound of μ^* based on a convex optimization problem.

Theorem 1: Let $w \in (0, \infty)$. Let us suppose that there exist $\varepsilon > 0$ and symmetric matrix polynomials $P_{k,l}, R_{k,l,i} : \mathbb{R}^m \to \mathbb{R}^{c_k \times c_k}$, $k \in \{1, \ldots, n\}$, $l \in \{1, \ldots, n_g\}$ and $i \in \{1, \ldots, n_a\}$, such that

$$\left. \begin{array}{l} P_{k,l}(\phi) - I \\ R_{k,l,i}(\phi) \\ S_{k,l}(\phi) - \varepsilon I \end{array} \right\} \text{ are SOS} \tag{20}$$

for all $k \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, n_g\}$, where

$$S_{k,l}(\phi) = Q_{k,l}(\phi) - \sum_{i=1}^{n_a} a_i(\phi)R_{k,l,i}(\phi). \tag{21}$$

Then,

$$\mu^* < w. \tag{22}$$

Proof. Suppose that (20) holds. From (19) this implies that, for all $k \in \{1, \ldots, n\}$, $l \in \{1, \ldots, n_g\}$ and $i \in \{1, \ldots, n_a\}$,

$$\left. \begin{array}{rcl} P_{k,l}(\phi) - I &\geq& 0 \\ R_{k,l,i}(\phi) &\geq& 0 \\ S_{k,l}(\phi) - \varepsilon I &\geq& 0 \end{array} \right\} \ \forall \phi \in \mathbb{R}^m.$$

Since $R_{k,l,i}(\phi) \geq 0$ and $a_i(\phi) \geq 0$ for all $\phi \in \Phi$, it follows that, for all $\phi \in \Phi$,

$$\begin{array}{rcl} \varepsilon I &\leq& S_{k,l}(\phi) \\ &=& Q_{k,l}(\phi) - \sum_{i=1}^{n_a} a_i(\phi)R_{k,l,i}(\phi) \\ &\leq& Q_{k,l}(\phi), \end{array}$$

i.e.,

$$Q_{k,l}(\phi) \geq \varepsilon I \quad \forall \phi \in \Phi.$$

Similarly, one has that $P_{k,l}(\phi) \geq I$ for all $\phi \in \Phi$. Since $\varepsilon > 0$, it follows from (18) that

$$\mathrm{spec}(D_{k,l}(\phi)) \subset \{\lambda \in \mathbb{C} : \ \Re(\lambda) < 0\} \quad \forall \phi \in \Phi.$$

Hence, (17) implies that

$$\mathrm{spec}(\Omega_k(A(g_l(\phi), \phi))) \subset \{\lambda \in \mathbb{C} : \ \Re(\lambda) < w\} \quad \forall \phi \in \Phi.$$

From (15) it follows that

$$\mu(A(g_l(\phi), \phi)) < w \quad \forall \phi \in \Phi$$

and, therefore, $\mu^* < w$. □

Theorem 1 provides a condition for establishing whether a given scalar w is an upper bound of the sought μ^* in Case I. This condition requires to check the existence of $\varepsilon > 0$ and symmetric matrix polynomials $P_{k,l}(\phi)$ and $R_{k,l,i}(\phi)$ satisfying (20). For chosen degrees of such matrix polynomials, (20) is an LMI feasibility test (and, hence, a convex optimization problem): indeed, establishing whether a symmetric matrix polynomial depending affine linearly on some decision variables is SOS is equivalent to establishing feasibility of an LMI as explained for instance in [9]–[13].

The condition provided by Theorem 1 is based on a special case of the Positivstellensatz [9], [14], and is sufficient for any chosen degrees of the symmetric matrix polynomials $P_{k,l}(\phi)$ and $R_{k,l,i}(\phi)$. The conservatism of the condition can be decreased by increasing the degrees of these matrix polynomials, see for instance [15] for a related case in the framework of polytopic systems.

For chosen degrees of the symmetric matrix polynomials $P_{k,l}(\phi)$ and $R_{k,l,i}(\phi)$, let us define

$$\mu_I = \max_{\substack{k=1,\ldots,n \\ l=1,\ldots,n_g}} w_{k,l} \tag{23}$$

where

$$\begin{aligned} w_{k,l} \quad = \quad & \inf_{w \in (0,\infty)} \ w \\ & \text{s.t.} \quad \exists \varepsilon > 0, P_{k,l}(\phi), R_{k,l,i}(\phi), i = 1, \ldots, n_g : \\ & \quad \text{(20) holds.} \end{aligned} \tag{24}$$

From Theorem 1 it follows that

$$\mu^* \leq \mu_I. \tag{25}$$

Indeed, μ_I is the best upper bound of μ^* provided by Theorem 1 for chosen degrees of the symmetric matrix polynomials $P_{k,l}(\phi)$ and $R_{k,l,i}(\phi)$. Let us observe that the quantities $w_{k,l}$ in (24) can be computed through a bisection search on w where the LMI condition (20) is checked for any fixed value of w.

B. CASE II

In this subsection we consider any possibility, i.e., either (10) holds or not for some polynomial functions $g_l : \mathbb{R}^m \to \mathbb{R}^n$, $l = 1, \ldots, n_g$. The following example illustrates a situation where such polynomial functions do not exist.

Example 2. Let us consider the nonlinear system (1) with $n = 2$, $m = 1$ and

$$f(x, u) = \begin{pmatrix} x_1^2 + x_2^2 - u \\ 1 + x_1 + 2x_2 + 3x_1 x_2 \end{pmatrix}. \tag{26}$$

The set Φ and its expression in (8) are chosen as

$$\begin{cases} \Phi & = & [0, 2] \\ a_1(\phi) & = & 2\phi - \phi^2. \end{cases} \tag{27}$$

It follows that the set of equilibrium points $\Theta(\phi)$ in (2) cannot be written as in (10) for polynomial functions $g_l(\phi)$. Figure 2 shows the zero level sets of $f(x, u)$ for some values of u.

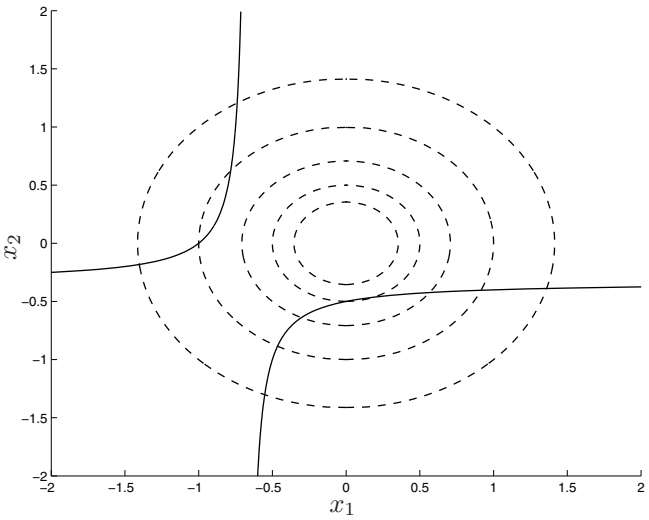

Fig. 2. Example 2. Zero level set of $f_2(x, u)$ (solid) and zero level set of $f_1(x, u)$ for some values of u in Φ (dashed). The intersections between these zero level sets are the set of equilibrium points $\Theta(\phi)$ in (2).

□

In order to determine μ^* in (9) in this case, let us proceed as follows. For $k \in \{1, \ldots, n\}$ and $w \in \mathbb{R}$ let us define the matrix polynomials

$$E_k(\theta, \phi) = \Omega_k(A(\theta, \phi)) - wI. \tag{28}$$

Moreover, for symmetric matrix polynomials $T_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{c_k \times c_k}$, let us define

$$U_k(\theta, \phi) = -T_k(\theta, \phi) E_k(\theta, \phi) - E_k(\theta, \phi)' T_k(\theta, \phi). \tag{29}$$

Lastly, let us introduce the following definition: a symmetric matrix polynomial $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{h \times h}$ is said to be SOS if there exist matrix polynomials $H_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{h \times h}$, $i = 1, \ldots, i_{max}$, such that

$$H(\theta, \phi) = \sum_{i=1}^{i_{max}} H_i(\theta, \phi)' H_i(\theta, \phi). \tag{30}$$

The following result provides a condition for establishing an upper bound of μ^* based on a convex optimization problem.

Theorem 2: Let $w \in (0,\infty)$. Let us suppose that there exist $\varepsilon > 0$ and symmetric matrix polynomials $T_k, V_{k,l}, W_{k,i} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{c_k \times c_k}$, $k,l \in \{1,\ldots,n\}$ and $i \in \{1,\ldots,n_a\}$, such that

$$\left.\begin{array}{l} T_k(\theta,\phi) - I \\ W_{k,i}(\theta,\phi) \\ X_k(\theta,\phi) - \varepsilon I \end{array}\right\} \text{ are SOS} \tag{31}$$

for all $k \in \{1,\ldots,n\}$, where

$$\begin{array}{rcl} X_k(\theta,\phi) & = & U_k(\theta,\phi) - \sum_{l=1}^n f_l(\theta,\phi) V_{k,l}(\theta,\phi) \\ & & - \sum_{i=1}^{n_a} a_i(\phi) W_{k,i}(\theta,\phi). \end{array} \tag{32}$$

Then,

$$\mu^* < w. \tag{33}$$

Proof. Suppose that (31) holds. From (30) this implies that, for all $k \in \{1,\ldots,n\}$ and $i \in \{1,\ldots,n_a\}$,

$$\left.\begin{array}{rcl} T_k(\theta,\phi) - I & \geq & 0 \\ W_{k,i}(\theta,\phi) & \geq & 0 \\ X_k(\theta,\phi) - \varepsilon I & \geq & 0 \end{array}\right\} \forall \theta \in \mathbb{R}^n \ \forall \phi \in \mathbb{R}^m.$$

Since $W_{k,i}(\theta,\phi) \geq 0$, $a_i(\phi) \geq 0$ and $f_l(\theta,\phi) V_{k,l}(\theta,\phi) = 0$ for all $\theta \in \Theta(\phi)$ for all $\phi \in \Phi$, it follows that, for all $\theta \in \Theta(\phi)$ and for all $\phi \in \Phi$,

$$\begin{array}{rcl} \varepsilon I & \leq & X_k(\theta,\phi) \\ & = & U_k(\theta,\phi) - \sum_{l=1}^n f_l(\theta,\phi) V_{k,l}(\theta,\phi) \\ & & - \sum_{i=1}^{n_a} a_i(\phi) W_{k,i}(\theta,\phi) \\ & \leq & U_k(\theta,\phi), \end{array}$$

i.e.,

$$U_k(\theta,\phi) \geq \varepsilon I \quad \forall \theta \in \Theta(\phi) \ \forall \phi \in \Phi.$$

Similarly, one has that $T_k(\theta,\phi) \geq I$ for all $\theta \in \Theta(\phi)$ for all $\phi \in \Phi$. Since $\varepsilon > 0$, it follows from (29) that

$$\mathrm{spec}(E_k(\theta,\phi)) \subset \{\lambda \in \mathbb{C} : \ \Re(\lambda) < 0\} \quad \forall \theta \in \Theta(\phi) \ \forall \phi \in \Phi.$$

Hence, (28) implies that

$$\begin{array}{c} \mathrm{spec}(\Omega_k(A(\theta,\phi))) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) < w\} \\ \forall \theta \in \Theta(\phi) \ \forall \phi \in \Phi. \end{array}$$

From (15) it follows that

$$\mu(A(\theta,\phi)) < w \quad \forall \theta \in \Theta(\phi) \ \forall \phi \in \Phi$$

and, therefore, $\mu^* < w$. □

Theorem 2 provides a condition for establishing whether a given scalar w is an upper bound of the sought μ^* in Case II. This condition requires to check the existence of $\varepsilon > 0$ and symmetric matrix polynomials $T_k(\theta,\phi)$, $V_{k,l}(\theta,\phi)$ and $W_{k,i}(\theta,\phi)$ satisfying (31). For chosen degrees of such matrix polynomials, (31) is an LMI feasibility test analogously to (20) in Theorem 1.

The condition provided by Theorem 2 is sufficient for any chosen degrees of the symmetric matrix polynomials $T_k(\theta,\phi)$, $V_{k,l}(\theta,\phi)$ and $W_{k,i}(\theta,\phi)$. The conservatism of the condition can be decreased by increasing the degrees of these matrix polynomials as discussed for Theorem 1.

For chosen degrees of the symmetric matrix polynomials $T_k(\theta,\phi)$, $V_{k,l}(\theta,\phi)$ and $W_{k,i}(\theta,\phi)$, let us define

$$\mu_{II} = \max_{k=1,\ldots,n} w_k \tag{34}$$

where

$$\begin{array}{rcl} w_k & = & \inf\limits_{w \in (0,\infty)} \ w \\ & & \text{s.t.} \quad \exists \varepsilon > 0, T_k(\theta,\phi), V_{k,l}(\theta,\phi), W_{k,i}(\theta,\phi), \\ & & \quad l = 1,\ldots,n, \ i = 1,\ldots,n_a : \ (31) \text{ holds.} \end{array} \tag{35}$$

From Theorem 2 it follows that

$$\mu^* \leq \mu_{II}. \tag{36}$$

Indeed, μ_{II} is the best upper bound of μ^* provided by Theorem 2 for chosen degrees of the symmetric matrix polynomials $T_k(\theta,\phi)$, $V_{k,l}(\theta,\phi)$ and $W_{k,i}(\theta,\phi)$. Let us observe that the quantities w_k in (35) can be computed through a bisection search on w where the LMI condition (31) is checked for any fixed value of w.

IV. EXAMPLES

In this section we present some illustrative examples of the proposed results. The computations have been done in Matlab using the toolbox SeDuMi [16]. The degree of $R_{k,l,i}(\phi)$ in the LMI (20) is automatically chosen as the largest degree for which $S_{k,l}(\phi)$ has its minimum degree. Similarly, the degrees of $V_{k,l}(\theta,\phi)$ and $W_{k,i}(\theta,\phi)$ in the LMI (31) are automatically chosen as the largest degrees for which $U_k(\theta,\phi)$ has its minimum degree.

A. EXAMPLE 1 (CONTINUED)

Let us consider for Case I the nonlinear system with $f(x,u)$, Φ and $g_i(\phi)$ as in (11)–(13). We have that the matrix $A(\theta,\phi)$ of the linearized system (3) is given by

$$A(\theta,\phi) = \begin{pmatrix} 0 & -2\theta_2 & -2 \\ -2\theta_1 & 1 & 0 \\ 1+\phi & 1 & 0 \end{pmatrix}.$$

Let us determine the upper bound μ_I in (23). This requires to determine the quantities $w_{k,l}$ in (24) for $k \in \{1,2,3\}$ and $l \in \{1,2\}$.

Let us consider first $l = 1$. We have

$$A(g_1(\phi),\phi) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1+\phi & 1 & 0 \end{pmatrix}.$$

By searching for a symmetric matrix polynomial $P_{k,l}(\phi)$ of degree 1, we obtain

$$\left\{\begin{array}{rcl} w_{1,1} & = & 1.000 \\ w_{2,1} & = & 1.000 \\ w_{3,1} & = & 1.000. \end{array}\right.$$

Then, we consider $l = 2$. We have

$$A(g_2(\phi),\phi) = \begin{pmatrix} 0 & -2-4\phi-2\phi^2 & -2 \\ 2+2\phi & 1 & 0 \\ 1+\phi & 1 & 0 \end{pmatrix}.$$

By proceeding as in the previous case, we obtain

$$\begin{cases} w_{1,2} &= 1.000 \\ w_{2,2} &= 1.463 \\ w_{3,2} &= 1.000. \end{cases}$$

Hence, $\mu_I = 1.463$.

It turns out that the found upper bound is tight. Indeed, brute force search shows that, for $\phi = 0.561$,

$$\mu(A(g_2(\phi), \phi)) = \mu_I.$$

This means that μ_I is achieved by $\mu(A(\theta, \phi))$ for an admissible value of ϕ (i.e., 0.561) and θ (i.e., $g_2(0.561)$). Therefore, $\mu_I = \mu^*$.

Let us observe that, since Case II includes Case I, the nonlinear system with $f(x, u)$, Φ and $g_i(\phi)$ as in (11)–(13) can also be studied under Case II. However, it is interesting that, in order to obtain $\mu_{II} = \mu^*$, one must search for a symmetric matrix polynomial $T_k(\theta, \phi)$ of degree 3, while we simply obtain $\mu_I = \mu^*$ by searching for a symmetric matrix polynomial $P_{k,l}(\phi)$ of degree 1: the computational burden required by Case I is significantly smaller than that required by Case II.

B. EXAMPLE 2 (CONTINUED)

Let us consider for Case II the nonlinear system with $f(x, u)$ and Φ as in (26)–(27). We have that the matrix $A(\theta, \phi)$ of the linearized system (3) is given by

$$A(\theta, \phi) = \begin{pmatrix} 2\theta_1 & 2\theta_2 \\ 1 + 3\theta_2 & 2 + 3\theta_1 \end{pmatrix}.$$

Let us determine the upper bound μ_{II} in (34). This requires to determine the quantities w_k in (35) for $k \in \{1, 2\}$.

By searching for a symmetric matrix polynomial $T_k(\theta, \phi)$ of degree 0, we obtain

$$\begin{cases} w_1 &= 6.117 \\ w_2 &= 8.800. \end{cases}$$

Hence, $\mu_{II} = 8.800$.

It turns out that the found upper bound is tight. Indeed, brute force search shows that, for $\theta = (1.360, -0.388)'$ and $\phi = 2$,

$$\mu(A(\theta, \phi)) = \mu_{II}.$$

This means that μ_{II} is achieved by $\mu(A(\theta, \phi))$ for an admissible value of ϕ (i.e., 2) and θ (i.e., $(1.360, -0.388)'$). Therefore, $\mu_{II} = \mu^*$.

V. CONCLUSION

This paper has addressed the problem of determining the largest instability measure in continuous-time linearized nonlinear systems for all admissible equilibrium points for all admissible constant inputs. It has been shown that upper bounds of the sought instability measure can be established through LMIs, whose conservatism can be decreased by increasing the size of such LMIs. Two cases have been considered in the derivation of these results: first, when the equilibrium points are known polynomial functions of the input, and, second, when the equilibrium points are unknown (polynomial or non-polynomial) functions of the input.

Several directions can be considered for future work. In particular, one can investigate the possibility of deriving conditions for establishing the tightness of the found upper bounds, for instance by exploiting the technique in [17]. Also, it will be interesting to understand under which conditions the upper bounds are tight a priori.

ACKNOWLEDGEMENTS

The author would like to thank the Associate Editor and the Reviewers for their useful comments. This work is supported in part by the Research Grants Council of Hong Kong under Grant HKU711213E.

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