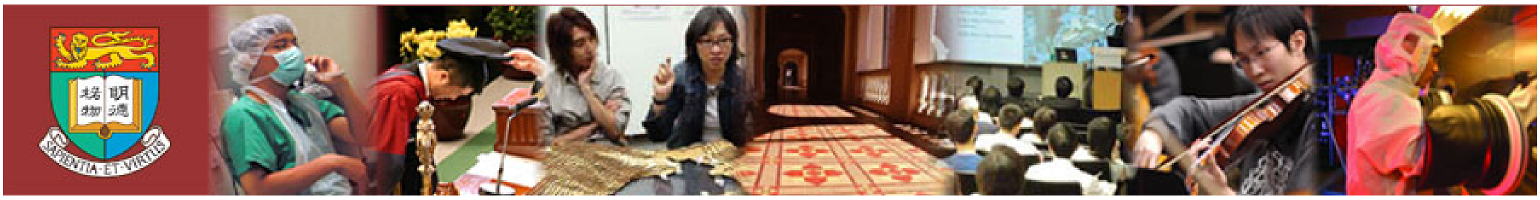

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Incremental-Dissipativity-Based Output Synchronization of Dynamical Networks with Switching Topology

Tao Liu, David J. Hill and Jun Zhao

Abstract— **This paper studies asymptotic output synchronization for a class of dynamical networks with switching topology whose node dynamics are characterized by a quadratic form of incremental-dissipativity. The output synchronization problem of the switched network is first converted into a set stability analysis of a nonlinear dissipative system with a particular selection of input-output pair, which is related to special features of interconnected incremental-dissipative systems. Then, synchronization by designing switching among subnetworks, where none of them is self-synchronizing, is investigated by using the single Lyapunov function method. Algebraic synchronization criteria are established, and the results are applied to investigate synchronization of coupled biochemical oscillators.**

I. Introduction

Synchronization of dynamical networks or interconnected systems – and its related problem of consensus of multi-agent systems – have attracted a great deal of focus due to their extensive applications in physics, biology and engineering. Numerical as well as analytical approaches to dealing with these problems have been reported in the literature (see recent papers and a monograph [1], [2], [3] for details).

An important problem in some real-world dynamical networks is preserving synchronization with a switching topology, which is often due to link failures or new link creation, for example in power grids [4], [5] and communication networks of mobile agents [6]. Due to its hybrid nature, a network with switching topology is much more complicated than networks with fixed topology. Synchronization of a switched network is decided not only by properties of its subnetworks, but also by the switching signal which specifies when and which subnetwork is activated. Switching between self-synchronizing subnetworks may desynchronize a switched network, and meanwhile, switching between subnetworks that are not self-synchronizing may synchronize a switched network. Therefore, synchronization of networks with switching topology has attracted researchers' interest in the past decade [7], [8], [9], [10].

On the other hand, dissipativity theory introduced in [11] and further extended in [12], [13], provides a framework for the analysis and design of complex systems using an input-output or state-space description based on energy-related considerations [14]. It has proved to be an effective tool for studying stability of complex systems in nonlinear system theory. In particular, dissipativity, as well as its special form of passivity, has been extensively used to study stability of interconnected large-scale systems. General sufficient conditions for input-output stability of a large-scale system which consists of linearly coupled dissipative subsystems were obtained in [15] and [16].

Recently, passivity and dissipativity have also been used to investigate state synchronization as well as output synchronization for networks with identical and non-identical nodes, respectively, in [17], [18], [19], [20], [21]. Moreover, the concepts of passivity and dissipativity have been extended to their incremental versions, namely, incremental-passivity and incremental-dissipativity, and have been used to characterize properties with which a network can achieve synchronization.

Incremental-dissipativity with a general nonlinear supply rate was introduced from the Lyapunov approach point of view in [22] where asymptotic output synchronization of a network with identical incrementally-passive oscillators was considered. State synchronization of networks with cyclic feedback systems by using incremental-output-feedback-passivity and limit set detectability was discussed in [23]. In addition, the concept of relaxed-cocoercivity which is an extension of incremental-passivity was introduced from the input-output approach point of view in [24] where a network consisting of interconnected nonlinear input-output operators was discussed, and sufficient conditions for input-output synchronization were developed. These results were further extended to networks with incrementally-dissipative subsystems in [25].

For one thing, most of the existing results developed in incremental-dissipativity theory only exploited properties of its particular forms, namely, incremental-passivity and relaxed-cocoercivity. Apparently, incremental-dissipativity is a more generic system property and can describe a broader class of physical system properties. For another thing, as far as we know, there are no results which utilize this concept for studying networks with switching topology. It is expected that this generic property of systems may benefit the analysis of synchronization for switched networks.

In view of these ideas, we introduce the concept of incremental-dissipativity with a quadratic form from the Lyapunov approach point of view, and study output synchronization of dynamical networks with switching topology.

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First of all, with a property that each node is incrementally-dissipative, we show that the output synchronization problem of such a switched network is equivalent to a set stability problem of a nonlinear dissipative system. Then, we focus on a particular issue for synchronization of switched networks, i.e., when all subnetworks are not self-synchronizing, under what condition we can find a switching signal to synchronize the network, and furthermore how to construct such a synchronizing switching signal. Synchronization criteria are established subsequently.

The contributions of the paper are twofold. First, by introducing the quadratic form of incremental-dissipativity, we build a bridge between output synchronization of dynamical networks with incrementally-dissipative nodes and set stability of dissipative nonlinear systems. Second, the designed switching signal only depends on output of the network, and hence, it is easy to implement in practice.

The rest of the paper is organized as follows. Section II introduces the network model with switching topology, where the concept of (Q, S, R)-dissipativity is briefly reviewed and extended to the incremental case. Section III studies the design of the synchronizing switching signal for the network in which none of its subnetworks is self-synchronizing. Section IV gives a network that consists of identical Goodwin oscillators to illustrate the obtained results. Conclusions are addressed in Section V.

II. Model and Preliminaries

Let $\mathbb{R}$ denote the field of real numbers; $\mathbb{R}_+$ denote all nonnegative real numbers; $\mathbb{R}^n$ be the n-dimensional real vector space; $\mathbb{R}^{n\times m}$ be the set of $n \times m$ real matrices. The superscript "$^\top$" represents the transpose of a vector or a matrix. I_n is an $n \times n$ identity matrix. $1_n \in \mathbb{R}^n$ and $1_{n\times n} \in \mathbb{R}^{n\times n}$ are vector and matrix whose entries are all one, respectively. $\|\cdot\|$ is the Euclidean norm of a vector. If a matrix $P \in \mathbb{R}^{n\times n}$ is symmetric positive (negative) definite, then it is denoted by $P > 0$ ($P < 0$).

In this paper, we consider a dynamical network consisting of N identical nodes. Each isolated node is represented by an n-dimensional nonlinear control system Σ_i:

$$\begin{aligned} \dot{x}_i &= f(x_i, u_i) \\ y_i &= h(x_i, u_i), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ are state variable, input and output signals of the ith node, respectively. The nonlinear function $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is local Lipschitz with respect to the first variable. $h(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuos nonlinear function. We assume that network (1) has unique solution for any initial condition.

The network is formed by choosing control actions which interconnects through the outputs of the nodes with switching topology, as follows:

$$u_i = \sum_{j=1}^{N} a_{ij}^{\sigma(t)} (y_j - y_i). \tag{2}$$

The function $\sigma(t) : \mathbb{R}^+ \to \mathcal{M} = \{1, 2, \ldots, m\}$ is the switching signal, which is a piecewise constant function continuous from the right, i.e., its value keeps constant between any two consecutive discontinuities. It identifies when and which subnetwork will be activated in the switched network. For each fixed $\sigma(t) = k \in \mathcal{M}$, we denote the adjacency matrix by $A_k = (a_{ij}^k) \in \mathbb{R}^{N\times N}$, and call (1) and (2) with the given adjacency matrix A_k as the kth subnetwork of the switched network. The adjacency matrix A_k with $a_{ii}^k = 0$ for all $i = 1, 2, \ldots, N$, represents both the coupling strength and topology of the corresponding subnetwork, and if there is a connection between node i and node j, then $a_{ij}^k \neq 0$, otherwise $a_{ij}^k = 0$.

In this paper, we don't require the nonnegativeness of interconnection coefficients, i.e., a_{ij}^k can be positive or negative if there is a connection between node i and node j in the kth subnetwork. This allows our model to cover a broader class of physical networks. We will discuss the problem that can we still achieve output synchronization by carefully organizing the switching among subnetworks when none of subnetworks is self-synchronizing.

Here, we only consider the case whose switching occurs in a given finite topology set, namely $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, i.e., $m \geq 2$ is a finite number. And we say network (1) achieves output synchronization asymptotically if

$$\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0, \quad \forall\, i, j = 1, 2 \ldots, N. \tag{3}$$

To study output synchronization of network (1) and (2), we need to specify nonlinear dynamical system Σ_i with some particular properties – incremental-dissipativity. We now will give a brief review of dissipativity and incremental-dissipativity for nonlinear dynamical systems.

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u), \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are state, input and output of the system, respectively. $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function such that solutions of the system exists and is unique for any given initial conditions. $h(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous nonlinear function.

Definition 1 ([12]): System (4) is said to be (Q, S, R)-dissipative if there exists a continuously differentiable storage function $V(x) \geq 0$ such that the following dissipation inequality holds:

$$\dot{V}(x) \leq y^\top Q y + 2y^\top S u + u^\top R u \tag{5}$$

along all possible trajectories of the system starting at any initial condition, where $Q = Q^\top \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $R = R^\top \in \mathbb{R}^{m\times m}$ are constant matrices.

Definition 2: System (4) is said to be (Q, S, R)-incrementally-dissipative if there exists a continuously differentiable and positive definite storage function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that for any two solutions $x_1(t)$ and $x_2(t)$ corresponding to inputs $u_1(t)$ and $u_2(t)$, respectively, the following inequality holds

$$\begin{aligned}\dot{V}(x_1,x_2) =&\frac{\partial V}{\partial x_1}f(x_1,u_1)+\frac{\partial V}{\partial x_2}f(x_2,u_2)\\ \leq&(y_1-y_2)^\top Q(y_1-y_2)+2(y_1-y_2)^\top S(u_1-u_2)\\ &+(u_1-u_2)^\top R(u_1-u_2),\end{aligned}$$

where Q, S and R are given in Definition 1.

Similar to [15], we also define incrementally-passive for system (4).

Definition 3: A (Q,S,R)-incrementally-dissipative system (4) is said to be

a). Incrementally-passive if $Q=0$, $S=I$, $R=0$;

b). Input-strongly-incrementally-passive if $Q=0$, $S=I$, $R=-\epsilon I$ for some $\epsilon>0$;

c). Output-strongly-incrementally-passive if $Q=-\epsilon I$, $S=I$, $R=0$ for some $\epsilon>0$;

d). Very-strongly-incrementally-passive if $Q=-\epsilon_1 I$, $S=I$, $R=-\epsilon_2 I$ for some $\epsilon_1>0$, $\epsilon_2>0$.

Remark 1: In [22], the definition of incremental-dissipativity with a general supply rate $\omega(u_1-u_2,y_1-y_2)$ was given. However, this generalized supply rate is hard to analyze for synchronization of networks. To overcome this issue, we specify the supply rate with a quadratic form characterized by (Q,S,R). Compared to incremental-passivity which has been broadly used in the study of synchronization of interconnected systems, our definition is more general and includes the incremental-passivity as a special case.

Remark 2: Strictly speaking, the storage function used in Definition 2 is more general than that used in [22] which depends on the incremental state $\Delta x=x_1-x_2$. Our storage function, which depends on (x_1,x_2), is more like the Lyapunov function used for incremental stability analysis of the auxiliary system in [26]. Moreover, if $y=x$, $Q>0$, $S=0$ and $R=0$, then Definition 2 implies incremental stability discussed in [26]. If $y=x$, $Q>0$, $S=0$ and $R>0$, then Definition 2 also implies that system (4) is incremental input-to-state stability [26].

In this paper, we only consider single-input single-output systems for simplicity. Therefore, matrices Q, S, R in Definition 2 reduce to constants γ_y, γ_{uy}, γ_u, respectively. However, results proposed in the paper can also be extended to multiple-input multiple-output systems similarly.

III. Output Synchronization by Switching Design

In practice, networks may consist of subnetworks with some of them being self-synchronizing and others not. Moreover, in an extreme case, none of the subnetworks is self-synchronizing. Then, the question becomes: is it still possible to synchronize the network by just carefully designing a switching signal $\sigma(t)$.

The answer to the above question is affirmative. For networks with at least one self-synchronizing subnetwork, namely, subnetwork k, we can simply set the switching signal equal to k to synchronize such a network, and hence the design problem is trivial.

Therefore, in this section, we only consider the extreme case where none of the subnetworks is self-synchronizing. We will use the single Lyapunov function method to uncover conditions under which we can design a switching signal that synchronizes the network, and furthermore, we will give a design method to construct such a synchronizing switching signal by using the incrementally-dissipative inequality defined in Definition 2. First, we give a useful lemma which will be used in the proof of the main results.

Let $x=(x_1,x_2,\ldots,x_N)^\top$, $y=(y_1,y_2,\ldots,y_N)^\top$ and $u=(u_1,u_2,\ldots,u_N)^\top$ be vectors of states, outputs and inputs of the network. Then, feedback law (2) can be rewritten as

$$u=-L_{\sigma(t)}y, \tag{6}$$

where L_k, $k=1,2,\ldots,m$, is the Laplacian matrix associated to adjacency matrix A_k with

$$l_{ij}^k=\begin{cases}\sum_{z=1}^N a_{iz}^k, & i=j\\ -a_{ij}^k, & i\neq j.\end{cases} \tag{7}$$

Let $\bar{y}=\frac{1}{N}1_N^\top y=\frac{1}{N}\sum_{j=1}^N y_j$ and $\bar{u}=\frac{1}{N}1_N^\top u=\frac{1}{N}\sum_{j=1}^N u_j$ denote the average output and input of all nodes. Define the error vector y_Δ and u_Δ for y and u, respectively, as follows

$$y_\Delta=(y_1-\bar{y},y_2-\bar{y},\ldots,y_N-\bar{y})^\top, \tag{8}$$

$$u_\Delta=(u_1-\bar{u},u_2-\bar{u},\ldots,u_N-\bar{u})^\top. \tag{9}$$

Moreover, define $\Phi\in\mathbb{R}^{(N-1)\times N}$ by

$$\Phi=\begin{pmatrix}-1+(N-1)\nu & 1-\nu & -\nu & \cdots & -\nu\\ -1+(N-1)\nu & -\nu & 1-\nu & \cdots & -\nu\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1+(N-1)\nu & -\nu & -\nu & \cdots & 1-\nu\end{pmatrix} \tag{10}$$

with $\nu=\frac{N-\sqrt{N}}{N(N-1)}$, and let $\tilde{y}=\Phi y$, $\tilde{u}=\Phi u$. Then, we have the following lemma.

Lemma 1: Consider the network denoted by (1). If each node is $(\gamma_y,\gamma_{uy},\gamma_u)$-incrementally-dissipative, then the network is $(\gamma_y I_{N-1},\gamma_{uy}I_{N-1},\gamma_u I_{N-1})$-dissipative with the input-output pair $(\tilde{u},\tilde{y})$, i.e., there exists a continuously differentiable storage function $V(x)\geq 0$ such that the following inequality holds

$$\dot{V}(x)\leq\gamma_y\|\tilde{y}\|^2+2\gamma_{uy}\tilde{u}^\top\tilde{y}+\gamma_u\|\tilde{u}\|^2. \tag{11}$$

Proof: Since each node in the network is $(\gamma_y,\gamma_{uy},\gamma_u)$-incremental-dissipativity, there exists a continuously differentiable and positive definite function $V_{ij}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ such that for any two nodes i and j in network (1), the following inequality holds

$$\begin{aligned}\dot{V}_{ij}(x_i,x_j)=&\frac{\partial V_{ij}}{\partial x_i}f(x_i,u_i)+\frac{\partial V_{ij}}{\partial x_j}f(x_j,u_j)\\ \leq&\gamma_y\|y_1-y_2\|^2+2\gamma_{uy}(u_1-u_2)(y_1-y_2)\\ &+\gamma_u\|u_1-u_2\|^2,\end{aligned} \tag{12}$$

where x_i, x_j, u_i, u_j and y_i, y_j are state variables, input and output signals of the ith and jth nodes, respectively. Summing $\dot{V}_{ij}(x_i, x_j)$ over all $i, j = 1, 2, \ldots, N$ gives

$$\begin{aligned}\frac{1}{2N}\sum_{i,j}^{N}\dot{V}_{ij}(x_i,x_j) \leq \frac{1}{2N}\sum_{i,j}^{N}\big(\gamma_y\|y_1-y_2\|^2 \\ +2\gamma_{uy}(u_1-u_2)(y_1-y_2)+\gamma_u\|u_1-u_2\|^2\big).\end{aligned} \tag{13}$$

For the first term on the righthand side of (13), we have

$$\begin{aligned}\frac{1}{2N}\sum_{i,j}^{N}\gamma_y\|y_i-y_j\|^2 =&\frac{\gamma_y}{N}\sum_{i,j}^{N}\left(y_i^2-y_iy_j\right) \\ =&\gamma_y y^\top\left(y-1_N\bar{y}\right) \\ =&\gamma_y y^\top y_\Delta.\end{aligned} \tag{14}$$

In addition, it is easy to get Φ satisfies $\Phi 1_N = 0$, $\Phi\Phi^\top = I_{N-1}$, and $\Phi^\top\Phi = I_N - \frac{1}{N}1_{N\times N}$ from (10). These properties with the definition of y_Δ in (8) lead to

$$\begin{aligned}\Phi^\top\tilde{y} =&\Phi^\top\Phi y \\ =&\left(I_N-\frac{1}{N}1_{N\times N}\right)y \\ =&y_\Delta.\end{aligned} \tag{15}$$

Substituting (15) into (14) gives

$$\frac{1}{2N}\sum_{i,j}^{N}\gamma_y\|y_i-y_j\| =\gamma_y y^\top\Phi^\top\Phi y=\gamma_y\|\tilde{y}\|^2. \tag{16}$$

Similarly, we can get following equations for the second and third terms on the righthand side of (13)

$$\begin{aligned}&\frac{1}{2N}\sum_{i,j}^{N}\gamma_{uy}(u_i-u_j)(y_i-y_j)=\gamma_{uy}\tilde{u}^\top\tilde{y}, \\ &\frac{1}{2N}\sum_{i,j}^{N}\gamma_u\|u_i-u_j\|=\gamma_u\|\tilde{u}\|^2.\end{aligned} \tag{17}$$

Substituting (16) and (17) into (13) yields

$$\frac{1}{2N}\sum_{i,j}^{N}\dot{V}_{ij}(x_i,x_j)\leq\gamma_y\|\tilde{y}\|^2+2\gamma_{uy}\tilde{u}^\top\tilde{y}+\gamma_u\|\tilde{u}\|^2. \tag{18}$$

Selecting

$$V(x)=\frac{1}{2N}\sum_{i,j}^{N}V_{ij}(x_i,x_j) \tag{19}$$

completes the proof. ■

Lemma 1 states that if each node of the network is incrementally-dissipative, then the entire network can be considered as a dissipative nonlinear system by choosing a particular input-output pair $(\tilde{u}, \tilde{y})$. Moreover, also thanks to this particular form of $\tilde{y}$, we will see in the sequel that output synchronization of the original network is exactly a set stability problem of the nonlinear system with $(\tilde{u}, \tilde{y})$. This result is addressed in the following theorem.

Theorem 1: Consider the network denoted by (1) and (2). Suppose that each node is $(\gamma_y, \gamma_{uy}, \gamma_u)$-incrementally-dissipative. If the following sets

$$\Omega_k=\left\{\tilde{y}\mid -\tilde{y}^\top\Theta_k\tilde{y}<0\right\} \tag{20}$$

make a partition of $\mathbb{R}^{N-1}$, i.e., $\bigcup_{k=1}^{m}\Omega_k = \mathbb{R}^{N-1}$, where $\Theta_k = -\gamma_y I_{N-1} + \gamma_{uy}\left(\tilde{L}_k^\top + \tilde{L}_k\right) - \gamma_u\tilde{L}_k^\top\tilde{L}_k$ with $\tilde{L}_k = \Phi L_k\Phi^\top$, then network (1) with switching topology achieves output synchronization asymptotically under the switching signal

$$\begin{cases}\sigma(t)=k, & \text{if } \sigma(t^-)=k \text{ and } \tilde{y}\in\Omega_k; \\ \sigma(t)=l, & \text{if } \sigma(t^-)=k \text{ and } \tilde{y}\in\partial\Omega_k\bigcap\Omega_l.\end{cases} \tag{21}$$

where $\partial\Omega_k = \left\{x \mid y^\top\Theta_k\tilde{y} = 0\right\}$.

Proof: For each given $k = 1, 2, \ldots, m$, multiplying Φ on both side of (6) gives

$$\begin{aligned}\tilde{u}=&-\Phi L_k y \\ =&-\Phi L_k\left(I_N-\Phi^\top\Phi\right)y-\Phi L_k\Phi^\top\Phi y \\ =&-\Phi L_k 1_{N\times N}y-\tilde{L}_k\tilde{y} \\ =&-\tilde{L}_k\tilde{y},\end{aligned} \tag{22}$$

where the last equality comes from the fact that $L_k 1_N = 0$. Select Lyapunov function candidate (19). Then based on Lemma (1), we have

$$\begin{aligned}\dot{V}=&\frac{1}{2N}\sum_{i,j}^{N}\dot{V}_{ij}(x_i,x_j) \\ \leq&\gamma_y\|\tilde{y}\|^2+2\gamma_{uy}\tilde{u}^\top\tilde{y}+\gamma_u\|\tilde{u}\|^2 \\ =&\gamma_y\|\tilde{y}\|^2-\gamma_{uy}\tilde{y}^\top(\tilde{L}_k^\top+\tilde{L}_k)\tilde{y}+\gamma_u\tilde{y}^\top\tilde{L}_k^\top\tilde{L}_k\tilde{y} \\ =&-\tilde{y}^\top\Theta_k\tilde{y},\end{aligned} \tag{23}$$

According to (20), when $\sigma(t) = k$, i.e., $\tilde{y} \in \Omega_k$, we have

$$\dot{V}\leq-\tilde{y}^\top\Theta_k\tilde{y}<0 \tag{24}$$

for all $\tilde{y} \neq 0$. Since $V \geq 0$ and $\dot{V} \leq 0$, according to LaSalle invariance principle, the network converges to a set $\mathcal{S}_1 = \left\{x \mid \tilde{y} = 0, x \in \mathbb{R}^{nN}\right\}$ asymptotically as $t \to \infty$. Moreover, from (15), we have

$$\|y_\Delta\|^2=y_\Delta^\top y_\Delta=\tilde{y}^\top\Phi\Phi^\top\tilde{y}=\|\tilde{y}\|^2 \tag{25}$$

Therefore, we have

$$\mathcal{S}_1=\mathcal{S}_2=\left\{x\mid y_\Delta=0,x\in\mathbb{R}^{nN}\right\}, \tag{26}$$

i.e.,

$$\lim_{t\to\infty}\|y_i(t)-y_j(t)\|=0,$$

for all $i, j = 1, 2, \ldots, N$, i.e., the switched network achieves output synchronization asymptotically under designed switching signal (21). ■

Remark 3: Combining Lemma 1 and Theorem 1, we conclude that output synchronization of dynamical network (1) and (2) with incrementally-dissipative nodes is equivalent to set stability of a dissipative nonlinear dynamical system also denoted by (1) and (2) but with input-output pair $(\tilde{u}, \tilde{y})$. Therefore, we establish a relationship between output synchronization of dynamical networks and set stability of dissipative dynamical systems. By doing so, we can apply traditional dissipativity theory to investigate output synchronization of dynamical networks, which may simpify the analysis procedure significantly.

Theorem 1 gives a general statement of the conditions under which asymptotic output synchronization of the network can be achieved, and provides a way to construct a synchronizing switching signal. Then, problems remaining are under what conditions the partition of (20) exists. This question will be answered in the next theorem where we apply the convex combination technique to constructing such a partition.

Theorem 2: Consider the network denoted by (1) and (2). Suppose that each node is $(\gamma_y, \gamma_{uy}, \gamma_u)$-incrementally-dissipative. If there exist nonnegative constants $\theta_k \geq 0$, $k = 1, 2, \ldots, m$ with $\sum_{k=1}^{m} \theta_k = 1$, such that

$$\Theta = -\gamma_y I_{N-1} + \gamma_{uy} \left(\tilde{L}^\top + \tilde{L} \right) - \gamma_u \tilde{L}^\top \tilde{L} > 0 \qquad (27)$$

with $L = \sum_{k=1}^{m} \theta_k L_k$ and $\tilde{L} = \Phi L \Phi^\top$, then the network achieves output synchronization asymptotically under switching signal (21). Moreover, if L_k is symmetric and irreducible, then condition (27) can be simplified as

$$-\gamma_y + 2\gamma_{uy}\lambda_i - \gamma_u \lambda_i^2 > 0, \quad i = 2, 3, \ldots, N \qquad (28)$$

where $\lambda_i > 0$ are nonzero eigenvalues of L.

Proof: The convex combination method guarantees that such a partition of $\mathbb{R}^{N-1}$ described in (20) exists (refer to [27] for more details). Therefore, based on Theorem 1, the switched network achieves output synchronization asymptotically under designed switching signal (21).

Moreover, if L_k is symmetric, i.e., each subnetwork k is undirected, then L and $\tilde{L}$ are also symmetric matrices. Therefore, there exists a unitary matrix $U \in \mathbb{R}^{(N-1)\times(N-1)}$ such that

$$U^\top \tilde{L} U = \Lambda = \mathrm{diag}\left\{\lambda_2, \lambda_3, \ldots, \lambda_N\right\} \qquad (29)$$

with $U^\top U = I_{N-1}$. Multiplying $U^\top$ and U on both side of (27), leads to

$$U^\top \Theta U = -\gamma_y I_{N-1} + 2\gamma_{uy}\Lambda - \gamma_u \Lambda^2, \qquad (30)$$

which is a diagonal matrix with diagonal entries $-\gamma_y + 2\gamma_{uy}\lambda_i - \gamma_u\lambda_i^2$, $i = 2, 3, \ldots, N$. Therefore, the positive definiteness of Θ is equivalent to the positiveness of all its eigenvalues, i.e., condition (28). ∎

Remark 4: It is worth pointing out that the designed switching signal (21) only depends on the output of the network. This is important in practice because sometimes it is not easy or even impossible to get whole information of state, and thus, our method can be implemented easily in practice.

For those particular forms of incremental-dissipativity defined in Definition 3, we have the following corollary.

Corollary 1: The condition established in Theorem 2 reduces to

a). $\Theta = \tilde{L}^\top + \tilde{L} > 0$ for incremental-passivity;

b). $\Theta = \tilde{L}^\top + \tilde{L} - \gamma_u \tilde{L}^\top \tilde{L} > 0$ for input-strong-incremental-passivity;

c). $\Theta = -\gamma_y I_{N-1} + \gamma_{uy} \left(\tilde{L}^\top + \tilde{L} \right) > 0$ for output-strong-incremental-passivity;

d). $\Theta = -\gamma_y I_{N-1} + \gamma_{uy} \left(\tilde{L}^\top + \tilde{L} \right) - \gamma_u \tilde{L}^\top \tilde{L} > 0$ for very-strong-incremental-passivity.

If L_k is symmetric and irreducible, then above conditions can be simplified as

a). $\lambda_i > 0$ for incremental-passivity;

b). $-\gamma_y + 2\gamma_{uy}\lambda_i > 0$ for input-strong-incremental-passivity;

c). $2\gamma_{uy}\lambda_i - \gamma_u\lambda_i^2 > 0$ for output-strong-incremental-passivity;

d). $-\gamma_y + 2\gamma_{uy}\lambda_i - \gamma_u\lambda_i^2 > 0$ for very-strong-incremental-passivity.

IV. An Example

Consider a network which consists of 10 identical Goodwin oscillators [28]. The network model is

$$\dot{x}_i = Ax_i + Bf(x_i) + Bu_i$$

where $f(x_i) = \frac{1}{1+x_{i3}^p}$,

$$A = \begin{pmatrix} -b_1 & 0 & 0 \\ b_2 & -b_2 & 0 \\ 0 & b_3 & -b_3 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

It is shown in [29] that with $b_1 = b_2 = b_3 = 0.5$ and $p = 17$, each isolated Goodwin oscillator is incrementally-dissipative with $\gamma_y = 0.5662$, $\gamma_{uy} = 0.5$ and $\gamma_u = 0$, and has a stable limit cycle oscillation.

Suppose that the network consists of 3 different subnetworks with adjacency matrices

$$A_1 = 10 \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = 10 \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = 10 \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Simulation shows that none of the subnetworks is slef-synchronizing (see Fig. 1). By selecting $\theta_1 = 0.3168$, $\theta_2 = 0.2923$ and $\theta_3 = 0.3909$, condition (27) is satisfied, and hence, the switched network achieves output synchronization under switching signal (21). Fig. 2 gives the output synchronization error $y_\triangle$ and the corresponding switching signal.

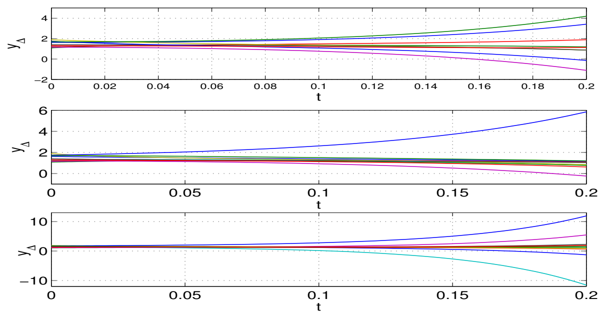

Fig. 1. The output synchronization error $y_\triangle$ of each subnetwork.

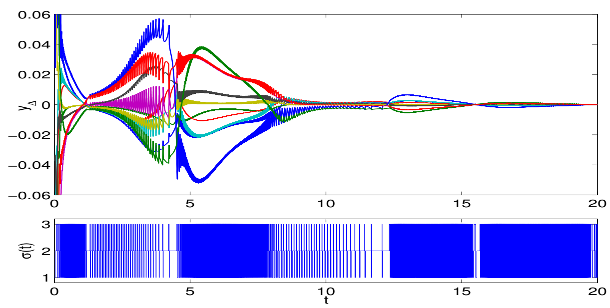

Fig. 2. The output synchronization error $y_\triangle$ and the switching signal $\sigma(t)$

V. Conclusion

Output synchronization for a class of dynamical networks with diffusively coupled identical nodes and switching topology has been studied. The concept of incremental-dissipativity has been used to characterize dynamical features of each node. It has been shown that features of incrementally-dissipative system can simply the analysis significantly. The single Lyapunov function method has been applied to design a switching signal to synchronize a network in which none of its subnetworks is self-synchronizing. Corresponding sufficient conditions have been established. The obtained results have been used to investigate output synchronization of biochemical oscillators, which also shows the effectiveness of proposed results.

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