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<b>Author(s)</b>	<b>Gu, J; Ching, WK; Zheng, H</b>
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# A Hidden Markov Reduced-form Risk Model

Jia-Wen Gu and Wai-Ki Ching

AMAC Laboratory,

Department of Mathematics,

The University of Hong Kong,

Pokfulam Road, Hong Kong.

Email: jwgu.hku@gmail.com, wching@hku.hk.

Harry Zheng

Department of Mathematics,

Imperial College, London, SW7 2AZ, UK.

Email: h.zheng@imperial.ac.uk.

**Abstract**—In this paper, we propose a reduced-form credit risk model with a hidden state process. The hidden state process is adopted to model the underlying economic environment with an observable state revealing the delayed and noisy information of the underlying economic state. Our model is a generalization of the work in Gu et al. [1]. Under this framework, we give a computational method to extract the underlying economic state and to find the distribution of multiple default times. Numerical experiment is conducted to illustrate the impact of change in observable state and the contagion effect of defaults.

## I. INTRODUCTION

Modeling default risk has long been an important problem in both theory and practice of banking and finance. In the aftermath of the Global Financial Crisis (GFC), much attention has been paid to investigate the appropriateness of the current practice of default risk modeling in banking, finance and insurance industries. Popular credit risk models currently used in the industries have their origins from two major classes of models. The first class of models was pioneered by Black and Scholes [2] and Merton [3] and is called a structural firm value model. The basic idea of the model is to describe explicitly the relationship between the default of the firm and its asset value. More specifically, the default of the firm is triggered by the event that the asset value of the firm falls below a certain threshold level related to the liabilities of the firm. The structural firm value model provides a theoretical basis for the commercial KMV model which has been widely used for default risk model in the financial industry. The second class of models was developed by Jarrow and Turnbull [4] and Madan and Unal [5] and is called a reduced-form, intensity-based credit risk model. The main idea of the model is to consider defaults as exogenous events and to model their occurrences using Poisson processes and their variants.

Regarding the reduced-form models, intensity-based credit

risk models have been widely used to model portfolio credit risk and to describe dependent default risks. There are two major types of reduced-form, intensity-based models for describing dependent default risk, namely bottom-up models and top-down models. The bottom-up models focus on modeling default intensities of individual reference entities and their aggregation to form a portfolio default intensity. Some works on bottom-up models include Duffie and Gârleanu [6], Jarrow and Yu [7], Schönbucher and Schubert [8], Giesecke and Goldberg [9], Duffie, Saita and Wang [10] and Yu [11] etc. These works differ mainly in their specifications for the parametric forms of default intensities of individual entities and the way these intensities are aggregated. The top-down models concern modeling the occurrence defaults at a portfolio level. A default intensity for the whole portfolio is modeled without reference to the identities of individual entities. Some procedures such as random thinning can be used to recover the default intensities of the individual entities. Some works on top-down models include Davis and Lo [12] [13], Giesecke, Goldberg and Ding [14], Brigo, Pallavicini and Torresetti [15], Longstaff and Rajan [16] and Cont and Minca [17]. In this paper, we focus on the bottom-up model.

Yu [11] extended the Lando's model to incorporate multiple defaults and their correlation. The so-called "total hazard construction" by Norros [18] and Shaked and Shanthikumar [19] was used to generate default times with interacting intensities. Zheng and Jiang [20] proposed a unified factor-contagion model for modeling correlated defaults and provide an analytical solution for modeling default times with "total hazard construction". Gu et al. [1] introduced an "ordered default rate" method to give a recursive formula for the distribution of default times in pricing basket CDSs in the context of a reduced-form, intensity-based model, which significantly enhances the computational efficiency in finding the prices

of CDSs. In this paper, we extend their work and propose a reduced-form, intensity-based credit risk models with hidden state process, where the work in Gu et al.[1] serves as our special case.

The remainder of the paper is structured as follows. In Section II, we present the model setup. In Section III, we give the computational method and present our main results. We illustrate our proposed computational method by presenting a numerical example in Section IV. Finally, Section V concludes the paper.

## II. THE MODEL

Uncertainty is described by a complete probability space  $(\Omega, \mathcal{F}, P)$ , where  $P$  is a risk-neutral probability measure. Under this probability space, we have  $K$  interacting entities. We let  $N_t^i := 1_{\{\tau_i \leq t\}}$ , where  $\tau_i$  represents the default time of name  $i$ . Suppose we have an underlying process  $(X_t)_{t \geq 0}$  representing the dynamics of the economic condition. Let  $\mathcal{F}_t^X := \sigma(X_s, 0 \leq s \leq t) \vee \mathcal{N}$ , where  $\mathcal{N}$  represents all the null sets. We also let

$$\mathcal{H}_t := \mathcal{F}_t^X \vee \mathcal{F}_t^N,$$

where

$$\mathcal{F}_t^N = \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \dots \vee \mathcal{F}_t^K$$

and

$$\mathcal{F}_t^i := \sigma(1_{\{\tau_i \leq s\}}, 0 \leq s \leq t) \vee \mathcal{N}.$$

We assume that there exists a  $\mathcal{H}_t$ -measurable processes  $(\lambda_i(t))_{t \geq 0}, i = 1, 2, \dots, k$ , such that

$$1_{\{\tau_i \leq t\}} - \int_0^{t \wedge \tau_i} \lambda_i(s) ds$$

is a  $\mathcal{H}_t$ -martingale. For the market participants, they do not observe the underlying process  $(X_t)_{t \geq 0}$  directly. Instead, they observe a process  $(Y_t)_{t \geq 0}$ , revealing the delayed and noisy information of  $(X_t)_{t \geq 0}$ , and also the default process  $(N_t^i)_{t \geq 0}$ . Hence, the information set available to the market participants is

$$\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_t^N,$$

where  $\mathcal{F}_t^Y := \sigma(Y_s, 0 \leq s \leq t) \vee \mathcal{N}$ . We further assume that both  $(X_t)_{t \geq 0}$  is an ‘‘exogenous’’ process to  $(N_t^i)_{t \geq 0}, i = 1, 2, \dots, K$  and  $P(\tau_i \neq \tau_j) = 1, i \neq j$ .

Throughout the paper, we suppose  $(X_t)_{t \geq 0}$  is a two-state Markov chain taking value in  $\{x_1, x_2\}$ . We assume the transition rates of the chain for ‘‘ $x_1 \rightarrow x_2$ ’’ and ‘‘ $x_2 \rightarrow x_1$ ’’ are denoted as  $\theta_1$  and  $\theta_2$ , respectively. The observable process  $(Y_t)_{t \geq 0}$  is again a two-state Markov chain taking

values in  $\{y_1, y_2\}$ , with transition rates depending on  $X_t$ , i.e.,  $\eta_1(X_t)(y_1 \rightarrow y_2)$  and  $\eta_2(X_t)(y_2 \rightarrow y_1)$ , where  $\eta_1$  and  $\eta_2$  are real-valued functions. The model can be extended to multiple state case with similar computation procedures as follows, but for simplicity, we just focus on two state case in this paper. At time 0, we suppose that  $X_0$  is at state  $x_M$  and  $Y_0$  is at state  $y_N$ . We suppose that the default intensity process of name  $i$  has the following representation:

$$\lambda_i(t) = a_i X_t \left( 1 + \sum_{j \neq i} b_{ij} 1_{\{\tau_j \leq t\}} \right), \quad i = 1, 2, \dots, K.$$

where  $a_i$  is a positive constant and  $b_{ij}$  is a real-valued constant.

Our aim is to compute the conditional joint distribution of default times  $P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t)$ . Due to the Markov property of  $X_t$  and the structure of  $\lambda_i(t)$ , we have

$$\begin{aligned} & P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t) \\ &= \sum_{i=1,2} P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t^N, X_t = x_i) \\ & \quad \times P(X_t = x_i \mid \mathcal{F}_t). \end{aligned}$$

In what follows, we proceed to evaluate  $P(X_t = x_i \mid \mathcal{F}_t)$  and  $P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t^N, X_t = x_i)$ .

## III. THE COMPUTATIONAL METHOD

For  $\omega \in \mathcal{F}_t$ , we can express  $\omega$  in a more clear way as  $\omega = (N_t^Y, N_t^D, \mathbf{S}_{N_t^Y}, \mathbf{I}_{N_t^D}, \mathbf{T}_{N_t^D})$ , where  $\mathbf{S}_{N_t^Y} = (S_1, S_2, \dots, S_{N_t^Y})$ ,  $\mathbf{I}_{N_t^D} = (B_1, B_2, \dots, B_{N_t^D})$  and  $\mathbf{T}_{N_t^D} = (T_1, T_2, \dots, T_{N_t^D})$ . With this shorthand, up to time  $t$ , we observe  $N_t^Y$  jumps in chain  $Y$  at time  $0 < S_1 < \dots < S_{N_t^Y} \leq t$ , also  $N_t^D$  defaults with name  $B_i$  defaulting at time  $T_i$ , where  $0 < T_1 < \dots < T_{N_t^D} \leq t$ . Given the information up to time  $t$ , i.e.,  $\mathcal{F}_t$ , we divide the time period  $[0, t]$  into  $N_t^Y + N_t^D$  sub-period,  $(0, h_1], (h_1, h_2], \dots, (h_{N_t^Y + N_t^D - 1}, h_{N_t^Y + N_t^D}]$ . In each of them, exactly one default or on jump in  $Y$  is observed. If up to time  $t$ , no jump or default has been observed, please refer to the computational method in Appendix.

Suppose that  $s$  and  $s + \Delta s$  are two endpoints of one sub-period. The following characterizes the computational method for  $P(X_t = x_i \mid \mathcal{F}_t)$ . For  $\omega \in \{T_k = s + t_k \in (s, s + \Delta s]\}$ ,

$$\begin{aligned} & P(X_s = x_i \mid \mathcal{F}_{s+\Delta s}) \\ &= P(X_s = x_i \mid \mathcal{F}_s, T_k = s + t_k, B_k = \beta) \\ &= \frac{P(X_s = x_i \mid \mathcal{F}_s) \cdot \left( \sum_{l=1,2} f_{T_k}^{i,l}(s + t_k; \beta, s, \Delta s) \right)}{\sum_{j=1,2} P(X_s = x_j \mid \mathcal{F}_s) \cdot \left( \sum_{l=1,2} f_{T_k}^{j,l}(s + t_k; \beta, s, \Delta s) \right)} \quad (1) \end{aligned}$$

$$\begin{aligned}
& P(X_{s+\Delta s} = x_i | \mathcal{F}_{s+\Delta s}) \\
&= \sum_{j=1,2} P(X_s = x_j | \mathcal{F}_{s+\Delta s}) P(X_{s+\Delta s} = x_i | \mathcal{F}_{s+\Delta s}, X_s = j) \\
&= \sum_{j=1,2} P(X_s = x_j | \mathcal{F}_{s+\Delta s}) \frac{f_{T_k}^{j,i}(s+t_k; \beta, s, \Delta s)}{\sum_{l=1,2} f_{T_k}^{j,l}(s+t_k; \beta, s, \Delta s)} \quad (2)
\end{aligned}$$

where

$$\frac{f_{T_k}^{j,i}(t; \beta, s, \Delta s)}{dP(T_k \in dt, B_k = \beta, X_{s+\Delta s} = x_i | X_s = x_j, N_s^D, N_s^Y, I_{N_s^D})} = \frac{f_{T_k}^{j,i}(t; \beta, s, \Delta s)}{dt}$$

Similarly, we have for  $\omega \in \{S_k = s + s_k \in (s, s + \Delta s)\}$ ,

$$\begin{aligned}
& \frac{P(X_s = x_i | \mathcal{F}_{s+\Delta s})}{P(X_s = x_i | \mathcal{F}_s)} \left( \sum_{l=1,2} f_{S_k}^{i,l}(s+s_k; s, \Delta s) \right) \\
&= \frac{P(X_s = x_i | \mathcal{F}_{s+\Delta s})}{\sum_{j=1,2} P(X_s = x_j | \mathcal{F}_s)} \left( \sum_{l=1,2} f_{S_k}^{j,l}(s+s_k; s, \Delta s) \right) \quad (3)
\end{aligned}$$

$$\begin{aligned}
& P(X_{s+\Delta s} = x_i | \mathcal{F}_{s+\Delta s}) \\
&= \sum_{j=1,2} P(X_s = x_j | \mathcal{F}_{s+\Delta s}) \frac{f_{S_k}^{j,i}(s+s_k; s, \Delta s)}{\left( \sum_{l=1,2} f_{S_k}^{j,l}(s+s_k; s, \Delta s) \right)}. \quad (4)
\end{aligned}$$

where

$$\frac{f_{S_k}^{j,i}(t; s, \Delta s)}{dP(S_k \in dt, X_{s+\Delta s} = x_i | X_s = x_j, N_s^D, N_s^Y, I_{N_s^D})} = \frac{f_{S_k}^{j,i}(t; s, \Delta s)}{dt}$$

Combining Equalities (1), (2), (3) and (4), we obtain a recursion method for computing  $P(X_t = x_i | \mathcal{F}_t)$  in terms of

$$f_{T_k}^{j,i}(s+t_k; \beta, s, \Delta s)$$

and

$$f_{S_k}^{j,i}(s+s_k; s, \Delta s).$$

Let  $T_{ikj}(\Delta s)$  be the occupation time of the chain  $X$  in state  $k$  in the time interval  $[s, s + \Delta s]$  given the chain starting from  $X_s = x_i$  and ending at  $X_{s+\Delta s} = x_j$ . We wish to determine the joint distribution of  $(T_{i1j}(t), T_{i2j}(t))$ . Note that the joint distribution of  $(T_{i1j}(t), T_{i2j}(t))$  is completely determined by its joint moment generating function. We shall then derive the joint moment generating function in the sequel.

For each  $i, j = 1, 2$ , we let

$$T_{ij}(t) = (T_{i1j}(t), T_{i2j}(t))^T \quad \text{and} \quad u = (u_1, u_2)^T \in \mathcal{R}^M.$$

The moment generating function of  $T_{ij}(t)$  is given by:

$$\Psi_{ij}(u, t) = E(\exp\{u^T T_{ij}(t)\}).$$

Let  $\xi_i$  denote the time of the first jump from state  $x_i$  to another state. Assume  $j \neq i$ , the density of  $\xi_i$  given  $X_0 = x_i, X_t = x_j$  is

$$f_{\xi_i}(s) = \frac{\theta_i e^{-\theta_i s} P_{jj}(t-s)}{P_{ij}(t)}.$$

Hence

$$\begin{aligned}
& \Psi_{ij}(u, t) \\
&= E(\exp\{u^T T_{ij}(t)\}) \\
&= \frac{\theta_i}{P_{ij}(t)} \int_0^t e^{(u_i - \theta)s} P_{jj}(t-s) E(\exp\{u^T T_{jj}(t-s)\}) ds \\
&= \frac{\theta_i}{P_{ij}(t)} \int_0^t e^{(u_i - \theta)s} \Psi_{jj}(u, t-s) ds \\
&= \frac{\theta_i}{P_{ij}(t)} \int_0^t e^{(u_i - \theta)(t-s)} \Psi_{jj}(u, s) ds
\end{aligned}$$

where  $P_{ij}(t)$  denotes the transition probability of the Markov chain from  $x_i$  to  $x_j$  with time  $t$ . Let  $\Phi_{ij}(u, t) = P_{ij}(t) \Psi_{ij}(u, t)$  and we obtain

$$\Phi_{ij}(u, t) = \theta_i \int_0^t e^{(u_i - \theta)(t-s)} \Phi_{jj}(u, s) ds. \quad (5)$$

One the other hand, the density of  $\xi_i$  given  $X_0 = x_i, X_t = x_i$  is

$$f_{\xi_i}(s) = \begin{cases} \frac{\theta_i e^{-\theta_i s} P_{ji}(t-s)}{P_{ii}(t)}, & x < t \\ \frac{e^{-\theta_i s}}{P_{ii}(t)}, & x \geq t. \end{cases}$$

Hence

$$\begin{aligned}
& \Psi_{ii}(u, t) \\
&= E(\exp\{u^T T_{ii}(t)\}) \\
&= E(E(\exp\{u^T T_{ii}(t)\} | \xi_i)) \\
&= \frac{\theta_i}{P_{ii}(t)} \int_0^t e^{(u_i - \theta)s} P_{ji}(t-s) \Psi_{ji}(u, t-s) ds + \frac{e^{(u_i - \theta)t}}{P_{ii}(t)} \\
&= \frac{\theta_i}{P_{ii}(t)} \int_0^t e^{(u_i - \theta)(t-s)} P_{ji}(s) \Psi_{ji}(u, s) ds + \frac{e^{(u_i - \theta)t}}{P_{ii}(t)}
\end{aligned}$$

and

$$\Phi_{ii}(u, t) = \theta_i \int_0^t e^{(u_i - \theta)(t-s)} \Phi_{ji}(u, s) ds + e^{(u_i - \theta)t}. \quad (6)$$

Taking derivatives on both sides of Eq. (5) and Eq. (6), we yield

$$\frac{\partial \Phi_{ij}(u, t)}{\partial t} = \theta_i \Phi_{jj}(u, t) + (u_i - \theta) \Phi_{ij}(u, t) \quad (7)$$

and

$$\frac{\partial \Phi_{ii}(u, t)}{\partial t} = \theta_i \Phi_{ji}(u, t) + (u_i - \theta) \Phi_{ii}(u, t). \quad (8)$$

For simplicity of discussion, we define the matrix

$$A = \begin{bmatrix} u_1 - \theta_1 & \theta_1 \\ \theta_2 & u_2 - \theta_2 \end{bmatrix}$$

and write

$$\Phi_u(t) = \begin{bmatrix} \Phi_{11}(u, t) & \Phi_{12}(u, t) \\ \Phi_{21}(u, t) & \Phi_{22}(u, t) \end{bmatrix}.$$

Combining Eq. (7) and Eq. (8), we obtain

$$\frac{\partial \Phi_u(t)}{\partial t} = A \Phi_u(t).$$

Solving the system of linear ODEs with the initial condition  $\Phi_{ij}(u, 0) = 1, i, j = 1, 2$ , we obtain the following proposition.

*Proposition 1:* The moment generating function of  $T_{ij}(t)$  is given by  $\Psi_{ij}(u, t)$  where

$$\Psi_{ij}(u, t) = \frac{\Phi_{ij}(u, t)}{P_{ij}(t)}$$

and

$$\Phi_u(t) = \begin{bmatrix} \Phi_{11}(u, t) & \Phi_{12}(u, t) \\ \Phi_{21}(u, t) & \Phi_{22}(u, t) \end{bmatrix}$$

has a unique solution as

$$\Phi_u(t) = e^{At} \mathbf{1} \cdot \mathbf{1}^T,$$

where  $\mathbf{1}$  is the two-dimensional column vector with all entries being equal to 1.

Now we are in the position to give the explicit formula for the desired probability.

$$\begin{aligned} f_{S_k}^{j,i}(s + s_k; s, \Delta s) &= \sum_{l=1,2} P_{jl}(s_k) P_{li}(\Delta s - s_k) \eta_{C(N_s^Y)}(x_l) \\ &\times \Psi_{jl} \left( -(\eta_{C(N_s^Y)}(x_1), \eta_{C(N_s^Y)}(x_2))^T, s_k \right) \\ &\times \Psi_{li} \left( -(\eta_{C(N_{s+1}^Y)}(x_1), \eta_{C(N_{s+1}^Y)}(x_2))^T, \Delta s - s_k \right) \\ &\times \Psi_{jl} \left( - \left( \sum_{i \notin I_{N_s^D}} a_i \left( 1 + \sum_{j \in I_{N_s^D}} b_{ij} \right) \right) (x_1, x_2)^T, s_k \right) \\ &\times \Psi_{li} \left( - \left( \sum_{i \notin I_{N_s^D}} a_i \left( 1 + \sum_{j \in I_{N_s^D}} b_{ij} \right) \right) (x_1, x_2)^T, \Delta s - s_k \right) \end{aligned}$$

$$\begin{aligned} f_{T_k}^{j,i}(s + t_k; \beta, s, \Delta s) &= \sum_{l=1,2} P_{jl}(t_k) P_{li}(\Delta s - t_k) a_{\beta} x_l \left( 1 + \sum_{j \in I_{N_s^D}} b_{\beta j} \right) \\ &\times \Psi_{jl} \left( -(\eta_{C(N_s^Y)}(x_1), \eta_{C(N_s^Y)}(x_2))^T, t_k \right) \\ &\times \Psi_{li} \left( -(\eta_{C(N_s^Y)}(x_1), \eta_{C(N_s^Y)}(x_2))^T, \Delta s - t_k \right) \\ &\times \Psi_{jl} \left( - \left( \sum_{i \notin I_{N_s^D}} a_i x_1 \left( 1 + \sum_{j \in I_{N_s^D}} b_{ij} \right) \right) (x_1, x_2)^T, t_k \right) \\ &\times \Psi_{li} \left( - \left( \sum_{i \notin I_{N_s^D}^*} a_i x_1 \left( 1 + \sum_{j \in I_{N_s^D}^*} b_{ij} \right) \right) (x_1, x_2)^T, \Delta s - t_k \right) \end{aligned}$$

where  $I_{N_s^D}^* = I_{N_s^D} \cup \{\beta\}$  and

$$C(x) = \begin{cases} 2, & x + Y_N \equiv 0 \pmod{2} \\ 1, & x + Y_N \equiv 1 \pmod{2}. \end{cases}$$

We adopt the ordered default rate approach proposed in Gu et al.[1] to compute the probability  $P(\tau_1 > t_1, \tau_2 >$

$t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t^N, X_t = x_i)$ . Suppose that  $\{\alpha_1, \alpha_2, \dots, \alpha_K\} = \{1, 2, \dots, K\}$  and  $t_i < t^i < t, i = 1, 2, \dots, m, t_i > t, i = m + 1, \dots, K$ . For  $\omega \in \{\tau_{\alpha_1} = t^1, \tau_{\alpha_2} = t^2, \dots, \tau_{\alpha_m} = t^m, \tau_{\alpha_{m+1}} > t, \dots, \tau_{\alpha_K} > t\}$ , then

$$\begin{aligned} &P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t^N, X_t = x_i) \\ &= P(\tau_{\alpha_{m+1}} > t_{\alpha_{m+1}}, \dots, \tau_{\alpha_K} > t_{\alpha_K} \mid \tau_{\alpha_1} = t^1, \dots, \\ &\quad \tau_{\alpha_m} = t^m, X_t = x_i). \end{aligned}$$

To obtain the desired probability, it suffices to find its density function, i.e.,  $f(t_{\alpha_{m+1}}, \dots, t_{\alpha_K}) = (-1)^{K-m} \frac{d}{dt_{\alpha_{m+1}} \dots dt_K} P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_K > t_K \mid \mathcal{F}_t^N, X_t = x_i)$ . Without loss of generality, we assume that  $t_{\alpha_{m+1}} < \dots < t_{\alpha_K}$ . Let  $\lambda^{i+1}(t)$  denote the  $i$ th default rate given  $\tau_1 = t^1, \dots, \tau_m = t^m, \tau_{m+1} = t_{m+1}, \dots, \tau_i = t_i$  and  $\mathcal{F}_{\infty}^X$ , then,

$$\begin{aligned} \lambda^{i+1}(t) &= \sum_{j=i+1}^K a_{\alpha_j} X_t \left( 1 + \sum_{l=1}^i b_{\alpha_j \alpha_l} \right) \\ &\quad \tau_i \leq t < \min\{\tau_{i+1}, \dots, \tau_K\}. \end{aligned}$$

Then, we can obtain,

$$\begin{aligned} &f(t_{\alpha_{m+1}}, \dots, t_{\alpha_K}) \\ &= E \left[ \prod_{i=m}^{K-1} a_{\alpha_{i+1}} X_{t_{\alpha_{i+1}}} \left( 1 + \sum_{l=1}^i b_{\alpha_{i+1} \alpha_l} \right) \right. \\ &\quad \left. \times \exp \left( - \int_{t_{\alpha_i}}^{t_{\alpha_{i+1}}} \sum_{j=i+1}^K a_{\alpha_j} X_u \left( 1 + \sum_{l=1}^i b_{\alpha_j \alpha_l} \right) du \right) \right] \end{aligned}$$

where  $t_{\alpha_m}$  temporarily denotes  $t$ .

#### IV. NUMERICAL ILLUSTRATION

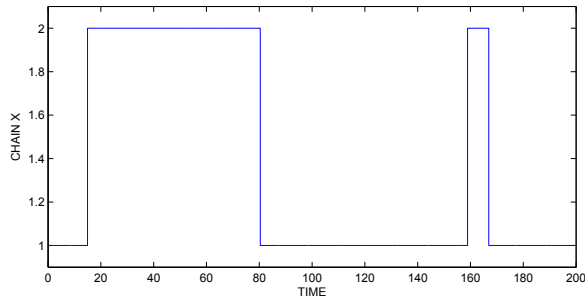
To illustrate the computational method presented in Section III, we conduct numerical experiment as follows. For simplicity, we assume our portfolio contains  $K = 10$  homogeneous entities, i.e., the intensity of name  $i$  is given by

$$\lambda_i(t) = a X_t \left( 1 + \sum_{j \neq i} b 1_{\{\tau_j \leq t\}} \right), \quad i = 1, 2, \dots, K,$$

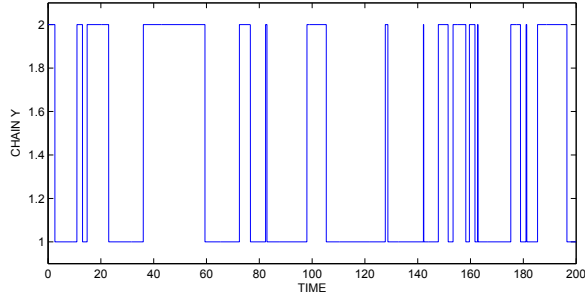
where  $a$  is positive constant and  $b$  nonnegative constant. We consider a basket CDS contract that pays \$1 if  $k$ th-to-default out of a portfolio of reference entities occurs prior to expiry date. To simplify our discussion, we assume that the payment (if any) occurs at expiration, and that the buyer pays a premium at the initiation of the swap contract. With a constant risk-free interest rate  $r$ , the value of the  $k$ th-to-default CDS contract at time  $t$  is given by

$$S_k(t) = \exp\{-r(T - t)\} P(\tau^k \leq T \mid \mathcal{F}_t),$$

where  $\tau^k$  is the  $k$ th-to-default time and  $T$  is the expiry date.



(a)



(b)

Fig. 1. Sample Path of Chain  $X$  and  $Y$  in Simulation I.

The setting of parameters is as follows. We let the contagion factors  $a = 1, b = 0.005$ . For the hidden Markov chain  $X_t$ , we set  $\theta_1 = \theta_2 = 0.01$  and  $(x_1, x_2) = (0.001, 0.01)$ , where state  $x_1$  and  $x_2$  represent the “good” and “bad” economic state, respectively. For the observable chain  $Y_t$ , we define the transition rates

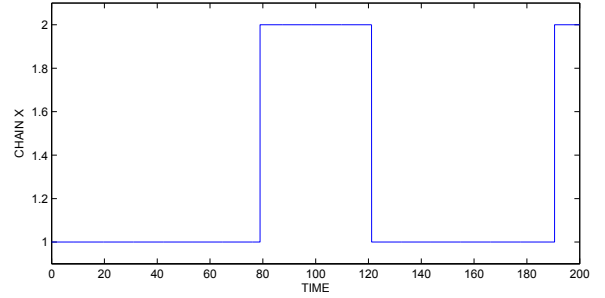
$$\eta_1(x) = \begin{cases} 0.1, & x = x_1 \\ 0.2, & x = x_2 \end{cases}$$

and

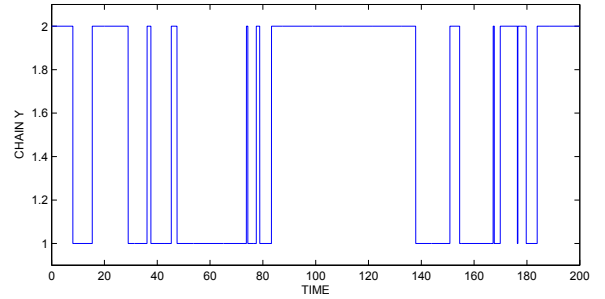
$$\eta_2(x) = \begin{cases} 0.2, & x = x_1 \\ 0.1, & x = x_2. \end{cases}$$

State  $y_1$  represents the delayed information of the “good” economics and  $y_2$  represents the delayed information of “bad” economics. Initially,  $X_0 = x_1$  and  $Y_0 = y_2$ . To give an intuitive understanding of the property of chains  $X$  and  $Y$ , we simulate some of their sample paths in Figures 1, 2 and 3.

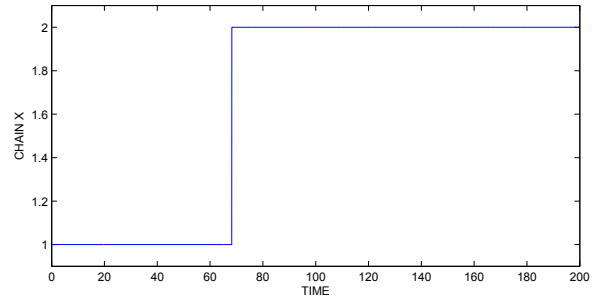
In Table 1, we select trajectories of the information set and present the value of the 4th-to-default basket Credit Default Swap (CDS) along the time (from day 21 to day 60) with the expiry date of contract set to be  $T = 180$  (days). In Scenario 1, we assume no jump in chain  $Y$  or default observed from day 1 to day 60; in Scenario 2, we assume we observe one jump in  $Y$  at sometime between day 32 and day 33; in Scenario 3, we assume we observe one jump in  $Y$  at sometime between



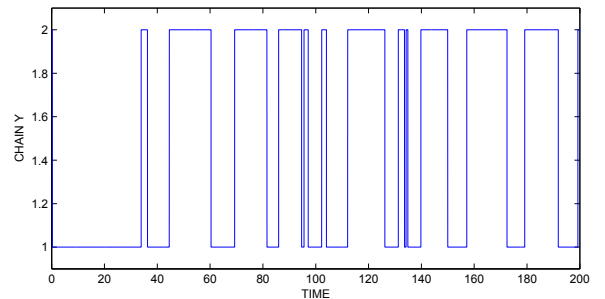
(a)



(b)

Fig. 2. Sample Path of Chain  $X$  and  $Y$  in Simulation II.

(a)



(b)

Fig. 3. Sample Path of Chain  $X$  and  $Y$  in Simulation III.

TABLE I. VALUE OF BASKET CREDIT DEFAULT SWAP ALONG THE TIME

Day	Value of Basket CDS( $\times 10^{-4}$ )		
	Scenario 1	Scenario 2	Scenario 3
26	3.682	3.682	3.682
27	3.854	3.854	3.854
28	4.033	4.033	4.033
29	4.221	4.221	4.221
30	4.417	4.417	4.417
31	4.623	4.623	4.623
32	4.837	4.837	4.837
33	5.062	<b>5.259</b>	<b>5.259</b>
34	5.297	5.370	5.370
35	5.542	5.394	5.394
36	5.799	5.391	5.391
37	6.067	5.399	5.399
38	6.347	5.435	5.435
39	6.640	5.502	5.502
40	6.946	5.599	5.599
41	7.266	5.722	5.722
42	7.601	5.868	5.868
43	7.951	6.035	6.035
44	8.316	6.220	6.220
45	8.697	6.423	6.423
46	9.096	6.641	6.641
47	9.513	6.875	6.875
48	9.948	7.123	7.123
49	10.40	7.385	7.385
50	10.88	7.661	7.661
51	11.37	7.951	<b>9.383</b>
52	11.89	8.255	10.80
53	12.43	8.573	12.35
54	13.00	8.905	13.78
55	13.59	9.251	15.06
56	14.20	9.612	16.24
57	14.84	9.988	17.35
58	15.52	10.38	18.43
59	16.21	10.78	19.50
60	16.95	11.21	20.57

day 32 and day 33 and one default at sometime between day 50 and day 51.

We remark that when observable chain  $Y$  jumps from State  $y_2$  to State  $y_1$ , we observe a gradual decrease in the value of the basket CDS. This is consistent with our intuition, as the jumps in observable chain reveals a better hidden economic state and this reduce the risk of default. When one default observed, the value of the CDS contract increase, as the possibility of default goes up due to the default contagion.

## V. CONCLUDING REMARKS

In this paper, we propose a reduced-form credit risk model with a hidden state process modeling the underlying economic environment. Under this framework, we give a computational method to extract the underlying economic state and the distribution of multiple default times. The main advantage of our model is that it can capture the delayed and noisy information

of the hidden economic state which is more consistent with the real situation.

## APPENDIX

For  $\omega \in \{\text{no jump or default observed in } [0, t]\}$ ,

$$\begin{aligned}
 & P(X_t = x_i | \mathcal{F}_t) \\
 = & \frac{P(X_t = x_i, \text{no jump or default in } [0, t])}{\sum_{j=1,2} P(X_t = x_j, \text{no jump or default in } [0, t])}
 \end{aligned}$$

where

$$\begin{aligned}
 & P(X_t = x_j, \text{no jump or default in } [0, t]) \\
 = & P(X_t = x_j) \Psi_{Mj}(-(\eta_{C(0)}(x_1), \eta_{C(0)}(x_2))^T, t) \\
 & \times \Psi_{Mj}(-(\sum_{l=1}^K a_l x_1, \sum_{l=1}^K a_l x_2)^T, t).
 \end{aligned}$$

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