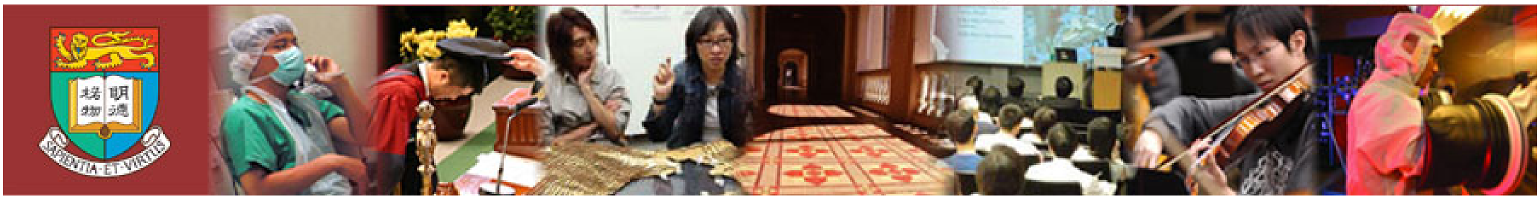

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|---|---|
| Title | On the Solvability of Three-Pair Networks With Common Bottleneck Links |
| Author(s) | Cai, K; Han, G |
| Citation | The IEEE Information Theory Workshop (ITW 2014), Hobart, Australia, 2-5 November 2014. In the Proceedings of the I E E E International Information Theory Workshop, 2014, p. 546-550 |
| Issued Date | 2014 |
| URL | http://hdl.handle.net/10722/204180 |
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# On the Solvability of Three-Pair Networks With Common Bottleneck Links

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***Abstract*—We consider the solvability problem under network coding and derive a sufficient and necessary condition for $3$-pair networks with common "bottleneck links" being solvable. We show that, for such networks: (1) the solvability can be determined in polynomial time; (2) being solvable is equivalent to being linear solvable; (3) finite fields of size $2$ or $3$ are sufficient to construct linear solutions.**

***Index Terms*—Network coding, $k$-pair network, $\mathcal{A}$-set.**

## I. Introduction

One of the most natural and important problems in the theory of network coding is to determine whether a network coding solution exists for a given network and the corresponding rate requirements between its sources and sinks, or in short, the solvability problem under network coding. The case when the network is a single multicast has been well understood: it has been shown [1] that in this case 1) a linear network coding solution always exists; 2) the solution can achieve the max-flow min-cut bound; 3) as a result, an explicit characterization of the capacity region under network coding can be obtained.

Unfortunately, the level of difficulty of the problem has taken a quantum leap for any other networks with multiple sources and sinks, even if they are only "slightly" more general than multicast. In this direction, successes to date are sporadic and most known results were obtained under the rather stringent condition that the information sent/received by sources/sinks are of unit rate, e.g, the sum-network [2]-[3] and the two-multicast network [4]-[5]. Of great relevance to this work are the results obtained in [6], where a simple solvability characterization using the so-called $\mathcal{A}$-set equation has been obtained for a 2-pair network with rate $(1,1)$, and in [7], which gives a sufficient and necessary condition for a family of networks with rate $(1,2)$ being solvable.

In this paper, we are mainly concerned with 3-pair networks having the so-called common "bottleneck links" with rate requirement $(1,1,1)$. For such networks, we will establish a sufficient and necessary condition to determine whether the rate $(1,1,1)$ is achievable, and moreover, we show that 1) the solvability can be determined in polynomial time; 2) being solvable is equivalent to being linear solvable; 3) finite fields of size 2 or 3 are sufficient to construct linear solutions.

Of greater importance than the above-mentioned technical contributions is the methodology we employed, which has been found rather successful for simpler case in [6], in terms of giving explicit conditions for networks being solvable. As elaborated later in the paper, we tend to first have a panoramic view of all the considered networks through a classification based on the so-called $\mathcal{A}$-sets (very elementary topological structures), which naturally leads to an explicit and complete solution to the solvability problem in our setup.

The rest of this paper is organized as follows. In Section II, we introduce our network model, some basic definitions and notations. We analyze the basic network structure in Section III and then illustrate the main result in Section IV. Finally, the paper is concluded in Section V.

## II. Preliminaries

### A. Network Model

*A communication network* $\mathcal{N} = (V, E, S, T)$ consists of a directed acyclic graph (DAG) $G = (V, E)$ with node (vertex) set $V$ and link (edge) set $E$, a source set $S = \{s_i, s_2, \cdots, s_{|S|}\} \subseteq V$ and a sink set $T = \{t_1, t_2, \cdots, t_{|T|}\} \subseteq V$. When $S = \{s\}$ and $T = \{t\}$, the network is called *point-to-point* and simply denoted by $(V, E, s, t)$. For each link $e \in E$, we assume it has the unit capacity, denoted $c(e) = 1$.

Let $e = (a, b)$ be a link from node $a$ to node $b$. We call $a$ the tail of $e$ (denoted $tail(e)$) and $b$ the head of $e$ (denoted $head(e)$). Define $In(e) = \{e' \in E : head(e') = tail(e)\}$ and $Out(e) = \{e' \in E : tail(e') = head(e)\}$.

Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network and let $V = W \biguplus \overline{W}$ ($\biguplus$ means disjoint union) be a vertex partition such that $s \in W$ and $t \in \overline{W}$. *An s-t cut* $C$ is the collection of all the edges from $W$ to $\overline{W}$. The capacity of $C$ is defined as $\sum_{e\in C} c(e)$. The minimum of the cut capacities for all $s$-$t$ cuts is called the *minimum cut capacity of* $\mathcal{N}$ and denoted by $C_{\mathcal{N}}(s,t)$ (or $C(s,t)$ when there is no ambiguity). *A minimum cut of* $\mathcal{N}$ is a cut with capacity $C_{\mathcal{N}}(s,t)$. Note that the minimum cut is not unique in general.

*Definition 2.1 ($\mathcal{A}$-set [6]):* For a point-to-point network $\mathcal{N} = (V, E, s, t)$, the $\mathcal{A}$-set of $\mathcal{N}$ is defined as the union of all its minimum cuts.

Given a communication network $(V, E, S, T)$, there are totally $|S| \times |T|$ point-to-point network $\mathcal{N}_{i,j} = (V, E, s_i, t_j)$ by considering all the source nodes but $s_i$ and all the sink nodes but $t_j$ as internal nodes. For any feasible $i, j$, we use $\mathcal{A}_{i,j}$ to denote the $\mathcal{A}$-set of $\mathcal{N}_{i,j} = (V, E, s_i, t_j)$. And we define

$$\mathcal{A}(i,j) \triangleq \mathcal{A}_{i,i} \cap \mathcal{A}_{j,j},$$

and further,

$$\mathcal{A}(1,2,3) \triangleq \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \cap \mathcal{A}_{3,3}.$$

Note that in this paper, we assume $\mathcal{A}(1,2,3) \neq \emptyset$, which means $\mathcal{N}$ has common "bottleneck links".

### B. Paths in the Network

A $u$-$v$ path $P$ is a string of ordered edges $(e_1, e_2, \cdots, e_n)$ such that $u = tail(e_1)$, $v = head(e_n)$ and $head(e_i) = tail(e_{i+1})$ for $i = 1, 2, \cdots, n-1$, where $u$ ($v$) is called the tail (head) of $P$ and denoted by $tail(P)$ ($head(P)$) and $e_i$ is called an *uplink* (*downlink*) of $e_j$ on $P$ if $i < j$ ($i > j$). For an edge $e$, $e \in P$ means $e$ lies in a path $P$; for two edges $e$ and $e'$, $e \prec e'$ ($e \succ e'$) means there is a path $P$ such that $e$ is an uplink (downlink) of $e'$ on $P$. For two edge sets $A, B$, we write $A \prec B$ ($A \succ B$) if $a \prec b$ ($a \succ b$), $\forall a \in A$ and $\forall b \in B$. A family of $k$ edge-disjoint paths with a same tail and a same head is called an *edge-disjoint $k$-path* and denoted by $P^{(k)}$.

For a DAG, it is well known that there exists a *topological order* according to the relation "$\prec$", that is, if $e_i \prec e_j$ in some path $P$, then $e_i \prec e_j$ in any path $Q$ for $e_i, e_j \in Q$. This topology order is critical for this paper.

We denote by $P[a, b]$ the section of path $P$ from $a$ to $b$, where $a$ and $b$ can be edges or vertexes, or a vertex and an edge.

Let $P_1 = (e_1, e_2, \cdots, e_n)$ and $P_2 = (e'_1, e'_2, \cdots, e'_m)$ be two paths such that $head(P_1) = tail(P_2)$. Denote path $P = (e_1, e_2, \cdots, e_n, e'_1, \cdots, e'_m)$ by $P_1$-$P_2$. Similarly, $P$-$P^{(k)}$ (or/and $P^{(k)}$-$P$) denotes the configuration formed by joining a path $P$ and an edge-disjoint $k$-path $P^{(k)}$.

A path (a family of paths) is usually regarded as a collection of edges. For example, we use $e \in P \cap Q$ to represent that path $P$ and path $Q$ share a common edge $e$.

### C. Three-pair Network Coding Problem

Let $\mathcal{N} = (V, E, \{s_1, s_2, s_3\}, \{t_1, t_2, t_3\})$ is a communication network, where each $t_i$ needs a unit rate information flow from $s_i$ for all $i$.

The desired flows, which are generated in $s_i$ and to be recovered in $t_i$, denoted by $X_i$, $i = 1, 2, 3$, are considered as *independent random variables with unit entropy.* The transmission of the information is assumed delay-free and error-free. The information (random variables) transmitted over an edge $e$ and an edge set $A$ are denoted by $X_e$ and $X_A$, respectively.

*Remark 2.2:* Because the desired flows have unit rate, *throughout the paper,* we assume[1] that for all feasible $i$, $s_i$ has a single out-edge denoted $S(i)$ and $t_i$ has a single in-edge, denoted $T(i)$. All $S(i)$ and $T(i)$ are called the *information edges*.

A *network code* is defined as a collection of functions $\{f_e : e \in E\}$ such that (1) for all $i$, $X_{S(i)} = X_i$; (2) $X_e = f_e(X_{In(e)})$. A *network coding solution* is a network code such that $H(X_{S(i)}|X_{T(i)}) = 0$ for all $i$. A 3-pair network is said to be *solvable* when a network coding solution exists, and *unsolvable* otherwise.

We always suppose there exists at least one path form $s_i$ to $t_i$ for each $i = 1, 2, 3$, otherwise it is obviously unsolvable. We further suppose $s_i \neq s_j$ and $t_i \neq t_j$ for $i \neq j$ and $source \neq sink$ throughout the paper.

[1] Otherwise, we add an auxiliary source node with a single out-edge to each source node and add an auxiliary sink node with a single in-edge to each sink node. Since the desired flow has unit rate, this network are solvable equivalent to the original network.

## III. Network Structure

In this section, we analyze the structure of 3-pair networks with common bottleneck links and give a topological classification of such networks.

We first give some properties of the $\mathcal{A}$-set of point-to-point networks, which requires the following definition [6].

*Definition 3.1 (Containment):* Let $\mathcal{N} = (V, E, s, t)$ and $\mathcal{N}_0 = (V', E', s', t')$ be two point-to-point networks. We say $\mathcal{N}$ *contains* $\mathcal{N}_0$ if there exists a function $f$ from the edges of $\mathcal{N}_0$ to the paths of $\mathcal{N}$ satisfying:

(1) For $e' \in E'$, if $tail(e') = s'$, then $tail(f(e')) = s$;
(2) For $e' \in E'$, if $head(e') = t'$, then $head(f(e')) = t$;
(3) For any $e'_1, e'_2 \in E'$, if $head(e'_1) = tail(e'_2)$, then $head(f(e'_1)) = tail(f(e'_2))$;
(4) For any $e'_1, e'_2 \in E'$, if $e'_1 \neq e'_2$, then $f(e'_1)$ and $f(e'_2)$ are edge-disjoint.

*Lemma 3.2:* [6] Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network such that $s$ has a unique out-edge and $t$ has a unique in-edge and let $\mathcal{A}$ denote the $\mathcal{A}$-set of $\mathcal{N}$. Then,

1) For any edge $e \in \mathcal{A}$ and any $s$-$t$ path $P$, $e \in P$;
2) For any $e \notin \mathcal{A}$, $\exists$ an $s$-$t$ path $P$ such that $e \notin P$;
3) $\mathcal{N}$ contains $\mathcal{N}_0 = P_1$-$P_1^{(2)}$-$P_2$-$P_2^{(2)}$-$\cdots$-$P_n^{(2)}$-$P_{n+1}$ (as shown in Fig. 3(1)) such that $\mathcal{A} = \cup_{i=1}^{n+1} P_i$, where $tail(P_1) = s$, $head(P_{n+1}) = t$, and path $P_i$ is regarded as a collection of edges.

### A. Relation among the $\mathcal{A}$-sets

Let $\mathcal{N} = (V, E, \{s_1, s_2, s_3\}, \{t_1, t_2, t_3\})$ be a 3-pair network. Throughout this section, we assume that for any $i, j$,

$$\mathcal{A}_{i,i} = \{e_1^i, e_2^i, \cdots, e_{\ell_i}^i\}, \mathcal{A}(i,j) = \{e_1^{i,j}, e_2^{i,j}, \cdots, e_{\ell_{i,j}}^{i,j}\},$$

where $e_r^i \prec e_s^i$ and $e_r^{i,j} \prec e_s^{i,j}$ for $r < s$, and

$$\mathcal{A}(1,2,3) = \{e_1, e_2, \cdots, e_\ell\},$$

where $e_r \prec e_s$ for $r < s$.

*Lemma 3.3:* Given $r < s$. If $e_r^i, e_s^i \in \mathcal{A}(i,j)$, then $e_\ell^i \in \mathcal{A}(i,j)$ for all $r < \ell < s$.

*Proof:* Suppose $e_\ell^i \notin \mathcal{A}(i,j)$, which implies $e_\ell^i \notin \mathcal{A}_{j,j}$. Then by Lemma 3.2, there exists an $s_j$-$t_j$ path $P_{j,j}$ such that $e_\ell^i \notin P_{j,j}$. Note that $e_r^i, e_s^i \in \mathcal{A}(i,j)$, we can pick an arbitrary $s_i$-$t_i$ path, namely $P_{i,i}$, and obtain an $s_i$-$t_i$ path $P = P_{i,i}[s_i, e_r^i]$-$P_{j,j}[head(e_r^i), tail(e_s^i)]$-$P_{i,i}[e_s^i, t_i]$ such that $e_\ell^i \notin P$, which is contradictory to 1) of Lemma 3.2. ■

Noticing that $\mathcal{A}(1,2,3) \subseteq \mathcal{A}(i,j)$, we immediately have

*Corollary 3.4:* Given $r < s$. If $e_r^i, e_s^i \in \mathcal{A}(1,2,3)$, then $e_\ell^i \in \mathcal{A}(1,2,3)$ for all $r < \ell < s$.

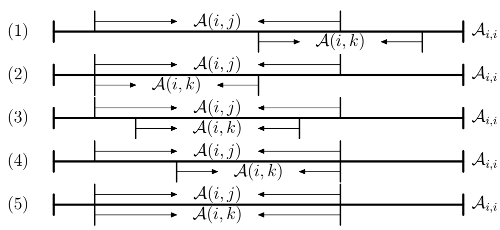


Fig. 1. Relations among $\mathcal{A}(i,j), \mathcal{A}(i,k), \mathcal{A}(k,j)$ and $\mathcal{A}(1,2,3)$.

The following theorem characterizes the relationship of $\mathcal{A}(i,j)$ and $\mathcal{A}(1,2,3)$ in 3-pair networks with common bottleneck links.

*Theorem 3.5:* For any 3-pair network with $\mathcal{A}(1,2,3) \neq \emptyset$, one of the following statements holds: for all distinct $i,j,k \in \{1,2,3\}$,

1) $\mathcal{A}(1,2,3) \subsetneq \mathcal{A}(i,j)$, $\mathcal{A}(1,2,3) \subsetneq \mathcal{A}(i,k)$ and $\mathcal{A}(1,2,3) = \mathcal{A}(j,k)$;
2) $\mathcal{A}(1,2,3) \subsetneq \mathcal{A}(i,j)$ and $\mathcal{A}(1,2,3) = \mathcal{A}(i,k) = \mathcal{A}(j,k)$;
3) $\mathcal{A}(1,2,3) = \mathcal{A}(i,j) = \mathcal{A}(j,k) = \mathcal{A}(i,k)$.

*Proof:* Without loss of generality, suppose $|\mathcal{A}(i,j)| \geq |\mathcal{A}(i,k)| \geq |\mathcal{A}(k,j)|$. We then consider the following cases:

1) $\mathcal{A}(i,j) \not\supseteq \mathcal{A}(i,k)$. In this case, as shown in Fig.1(1), by the fact that $\mathcal{A}(i,j) \cap \mathcal{A}(i,k) = \mathcal{A}(1,2,3) \neq \emptyset$ and Lemma 3.3, we have $\mathcal{A}(k,j) = \mathcal{A}(1,2,3)$.
2) $\mathcal{A}(i,j) \supseteq \mathcal{A}(i,k)$ and $|\mathcal{A}(i,j)| \neq |\mathcal{A}(i,k)|$. In this case, $\mathcal{A}(1,2,3) = \mathcal{A}(i,j) \cap \mathcal{A}(i,k) \subsetneq \mathcal{A}(i,j)$. By Lemma 3.3 and the assumption that $|\mathcal{A}(i,k)| \geq |\mathcal{A}(k,j)|$, we have $\mathcal{A}(1,2,3) = \mathcal{A}(i,k) = \mathcal{A}(k,j)$, as illustrated in Fig. 1(2), Fig. 1(3) and Fig. 1(4).
3) $\mathcal{A}(i,j) \supseteq \mathcal{A}(i,k)$ and $|\mathcal{A}(i,j)| = |\mathcal{A}(i,k)|$. In this case, as shown in Fig. 1(5), we have $\mathcal{A}(1,2,3) = \mathcal{A}(i,j) \cap \mathcal{A}(i,k) = \mathcal{A}(i,j) = \mathcal{A}(j,k) = \mathcal{A}(i,k)$, by Lemma 3.3 and the assumption that $|\mathcal{A}(i,k)| \geq |\mathcal{A}(k,j)|$.

■

### B. Network Structure

Let $\mathcal{N}$ be a 3-pair network with common bottleneck links. Denote by $\mathcal{P}[a,b]$ the collection of all paths from $a$ to $b$, where $a$ and $b$ can be edges or vertexes, or a vertex and an edge. For any $i \neq j$, we define the following subnetworks of $\mathcal{N}$.

- $G(1,2,3) \triangleq \mathcal{P}[e_1, e_\ell]$;
- $G^+(i) \triangleq \mathcal{P}[S(i), tail(e_1)]$;
- $G^-(i) \triangleq \mathcal{P}[head(e_\ell), T(i)]$.

When there exist $i \neq j$ such that $e_1^{i,j} \neq e_1$, we define

- $G^+(i,j) \triangleq \mathcal{P}[e_1^{i,j}, tail(e_1)]$.

When there exists $i \neq j$ such that $e_{\ell_{i,j}}^{i,j} \neq e_\ell$, we define

- $G^-(i,j) \triangleq \mathcal{P}[head(e_\ell), e_{\ell_{i,j}}^{i,j})]$.

For any $i,j,k$, we refer to each $G^+(i)$ or $G^+(j,k)$ ($G^-(i)$ or $G^-(j,k)$) as a $G^+$-subnetwork ($G^+$-subnetwork). Then, by Lemma 3.2, it is easy to see that $G(1,2,3)$ must contain a subnetwork as shown in Fig. 3(1); any $G^+$-subnetwork must

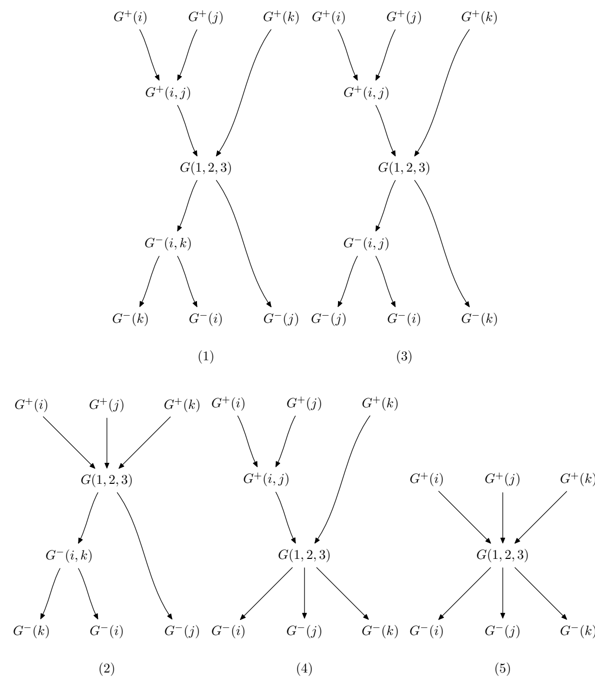


Fig. 2. The basic configurations, where the arrowed lines represent the topological order between the subnetworks.

contain a subnetwork as shown in Fig. 3(2) and any $G^-$-subnetwork must contain a subnetwork as shown in Fig. 3(3). It can also be verified that

$$\text{any } G^+\text{-subnetwork} \prec G(1,2,3) \prec \text{any } G^-\text{-subnetwork},$$

which implies that there are no common edges or vertexes shared by $G^+$- and $G^-$-subnetworks. Together with Theorem 3.5, the structure of all 3-pair networks with common bottleneck links can be classified as follows.

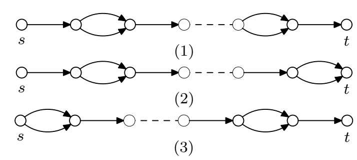


Fig. 3. The subnetworks contained in $G(1,2,3)$, $G^+$ and $G^-$.

*Theorem 3.6:* For a 3-pair network with $\mathcal{A}(1,2,3) \neq \emptyset$, it has one and only one of the *basic configurations* shown in Fig.2, where each Fig.2$(i)$ corresponds to the case shown in Fig.1$(i)$, $i = 1,2,3,4,5$.

Note: in a 3-pair network, any two $G^+$-subnetworks ($G^-$-subnetworks) may share edges; and there may exist paths from $G^+$-subnetworks to $G^-$-subnetworks. The analysis of these edges and paths plays a key role in deriving the main result.

## IV. Main Result

### A. The Condition for Solvability

Let $\mathcal{N}$ be a 3-pair network with $\mathcal{A}(1,2,3) \neq \emptyset$, and for all feasible $i,j$, define

$$\mathcal{A}_{i,j}^{(2)} \triangleq \{\{a,b\} : \{a,b\} \text{ is a cut of } \mathcal{N}_{i,j}; a,b \notin \mathcal{A}_{i,j}\}.$$

Using these notations, our main result can be stated as follows.

*Theorem 4.1:* $\mathcal{N}$ is unsolvable if and only if there exist distinct $i, j, k \in \{1, 2, 3\}$ such that one of the following statements holds:

1) $\mathcal{A}_{i,j} \cap \mathcal{A}(1, 2, 3) \neq \emptyset$;
2) $\exists a \in \mathcal{A}_{i,i} \cap \mathcal{A}_{j,j} \cap \mathcal{A}_{i,k}$ and $b \in \mathcal{A}_{k,k}$ such that $\{a, b\} \in \mathcal{A}^{(2)}_{j,i} \cap \mathcal{A}^{(2)}_{j,k}$.
3) $\exists a \in \mathcal{A}_{i,i} \cap \mathcal{A}_{k,k} \cap \mathcal{A}_{j,k}$ and $b \in \mathcal{A}_{j,j}$ such that $\{a, b\} \in \mathcal{A}^{(2)}_{j,i} \cap \mathcal{A}^{(2)}_{k,i}$.
4) $\exists a \in \mathcal{A}_{i,i} \cap \mathcal{A}_{j,j} \cap \mathcal{A}_{i,k}$ and $b \in \mathcal{A}_{k,k} \cap \mathcal{A}_{j,i}$ such that $\{a, b\} \in \mathcal{A}^{(2)}_{j,k}$.
5) $\exists a \in \mathcal{A}_{j,j} \cap \mathcal{A}_{i,k}$ and $b \in \mathcal{A}_{i,i} \cap \mathcal{A}_{k,k} \cap \mathcal{A}_{j,i}$ such that $\{a, b\} \in \mathcal{A}^{(2)}_{j,k}$.
6) $\exists a \in \mathcal{A}_{i,i} \cap \mathcal{A}_{i,j}$ and $b \in \mathcal{A}_{j,j}$ such that $\{a, b\} \in \mathcal{A}^{(2)}_{k,i} \cap \mathcal{A}^{(2)}_{k,j} \cap \mathcal{A}^{(2)}_{k,k}$.
7) $\exists a \in \mathcal{A}_{j,j} \cap \mathcal{A}_{k,j}$ and $b \in \mathcal{A}_{k,k}$ such that $\{a, b\} \in \mathcal{A}^{(2)}_{i,i} \cap \mathcal{A}^{(2)}_{j,i} \cap \mathcal{A}^{(2)}_{k,i}$.

Note that for any 3-pair network $\mathcal{N}$, all $\mathcal{A}_{i,j}$ can be computed in polynomial time, and so does determining whether $\{a, b\}$ is a cut. Since there are at most $|E|^2$ pairs of $\{a, b\}$, Theorem 4.1 implies the existence of a polynomial time algorithm to decide the solvability of a 3-pair network with common bottleneck links.

*Example 4.2:* Fig.4 gives 8 unsolvable networks: it can be verified that networks (1), (2) and (3) satisfy Conditions 1), 2) and 3) of Theorem 4.1, respectively; and network (4) satisfies Conditions 4) and 5); and networks (5) and (6) satisfy Conditions 6) and 7), respectively; network (7) satisfies Conditions 3) and 6); and network (8) satisfies Conditions 2) and 7), respectively.

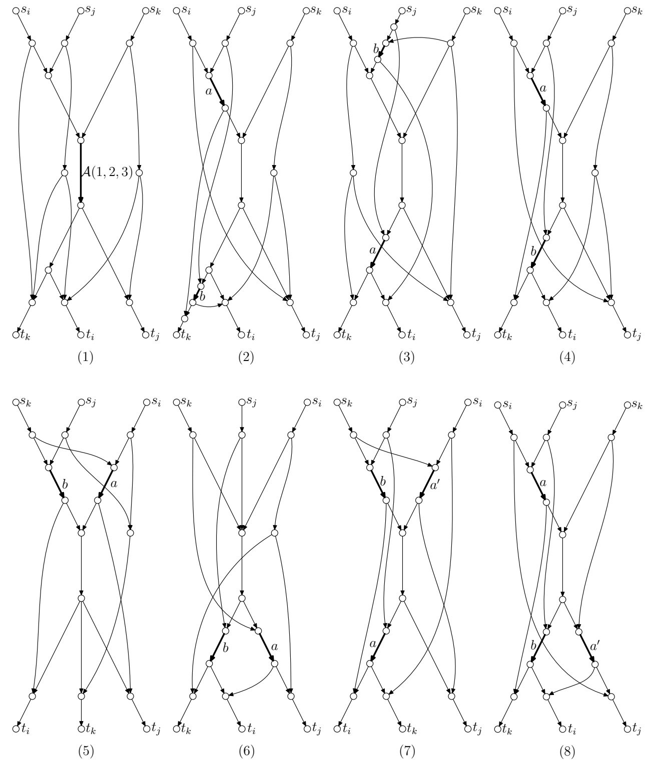


Fig. 4. Example of Unsolvable Networks.

### *B. Proof of the Sufficiency*

The main tool to prove the sufficiency is the following so-called "informational domination" defined in [8].

*Definition 4.3:* Let $A, B$ be two edge sets of a 3-pair network $\mathcal{N}$. We say $A$ informationally dominates $B$, denoted by $A \overset{i}{\rightsquigarrow} B$, if $X_B$ is a function of $X_A$ (or equivalently, $H(X_B|X_A) = 0$) *for all network coding solutions.*

Informational domination has the following properties [8]:

1) For all feasible $i$, $\{T(i)\} \overset{i}{\rightsquigarrow} \{S(i)\}$.
2) If $B \subseteq A$, then $A \overset{i}{\rightsquigarrow} B$.
3) If $A \overset{i}{\rightsquigarrow} B$ and $A \overset{i}{\rightsquigarrow} C$, then $A \overset{i}{\rightsquigarrow} B \cup C$.
4) If $A \overset{i}{\rightsquigarrow} B$ and $B \overset{i}{\rightsquigarrow} C$, then $A \overset{i}{\rightsquigarrow} C$.
5) If $B$ is downstream of $A$, then $A \overset{i}{\rightsquigarrow} B$, where $B$ is *downstream* of $A$ if there is no path from $S = \{s_1, s_2, s_3\}$ to $B$ in $\mathcal{N} \setminus A$.

The following generalization of information domination is required in our proof.

*Definition 4.4 (Sequential Informational Domination):* We say edge set $A$ sequentially informationally dominate edge sets $B_1, B_2, \cdots, B_n$, denoted $A \overset{si}{\rightsquigarrow} (B_1, B_2 \cdots, B_n)$, if for each $2 \leq k \leq n$, $A \overset{i}{\rightsquigarrow} B_k$ implies $A \overset{i}{\rightsquigarrow} B_1, A \cup B_1 \overset{i}{\rightsquigarrow} B_2, \cdots, A \cup_{i=1}^{k-1} B_i \overset{i}{\rightsquigarrow} B_k$.

Obviously, by definition, if $A \overset{si}{\rightsquigarrow} (B_1, B_2, \cdots, B_n)$, then $A \overset{i}{\rightsquigarrow} \cup_{i=1}^{n} B_i$. Now, we are ready to prove the sufficiency.

*Proof of the sufficiency:* Suppose, by way of contradiction, that $\mathcal{N}$ is solvable and 1) holds, i.e., $\exists e^* \in \mathcal{A}_{i,j} \cap \mathcal{A}(1, 2, 3)$. It can be easily checked that

$$\begin{aligned}
\{e^*, S(k)\} &\overset{i}{\rightsquigarrow} \{e^*, S(k), T(j)\} \text{ (by Properties 2,3,5))} \\
&\overset{i}{\rightsquigarrow} \{e^*, S(k), S(j)\} \text{ (by Properties 1,2,3))} \\
&\overset{i}{\rightsquigarrow} \{e^*, S(k), S(j), T(i)\} \text{ (by Properties 2,3,5))} \\
&\overset{i}{\rightsquigarrow} \{e^*, S(k), S(j), S(i)\} \text{ (by Properties 1,2,3))} \\
&\overset{i}{\rightsquigarrow} \{S(k), S(j), S(i)\} \text{ (by Property 2))}
\end{aligned}$$

In other words, $\{e^*, S(k)\} \overset{si}{\rightsquigarrow} (\{S(k)\}, \{S(j)\}, \{S(i)\})$, which, however, gives us a contradiction, since each link is of unit capacity.

Similarly, it can be verified that if $\mathcal{N}$ is solvable and 2) holds, we have $\{a, b\} \overset{si}{\rightsquigarrow} (\{S(k)\}, \{S(i)\}, \{S(j)\})$; if $\mathcal{N}$ is solvable and 3) holds, we have $\{a, b\} \overset{si}{\rightsquigarrow} (\{S(i)\}, \{S(k)\}, \{S(j)\})$; if $\mathcal{N}$ is solvable and 4) holds, we have $\{a, b\} \overset{si}{\rightsquigarrow} (\{S(k)\}, \{S(i)\}, \{S(j)\})$; if $\mathcal{N}$ is solvable and 5) holds, we have $\{a, b\} \overset{si}{\rightsquigarrow} (\{S(k)\}, \{S(i)\}, \{S(j)\})$; if $\mathcal{N}$ is solvable and 6) holds, we have $\{a, b\} \overset{si}{\rightsquigarrow} (\{S(j)\}, \{S(i)\}, \{S(k)\})$; if $\mathcal{N}$ is solvable and 7) holds, we have $\{a, b\} \overset{si}{\rightsquigarrow} (\{S(i)\}, \{S(j)\}, \{S(k)\})$. But each of the above-mentioned cases will give us a contradiction, which completes the proof. ∎

TABLE I
CLASSIFICATION OF 3-PAIR NETWORKS WITH $\mathcal{A}(1,2,3) \neq \emptyset$ AND $\mathcal{A}_{i,j} \cap \mathcal{A}(1,2,3) = \emptyset$

| Classes | (1.1) | (1.2) | (2.1) | (2.2) |
|---|---|---|---|---|
| (1.1′) | Fig.2(3)/$\mathbb{F}_2$-solvable | Fig.2(3)/$\mathbb{F}_2$-solvable | Fig.2(1),(3)/$\mathbb{F}_2$-solvable | Fig.2(2)/$\mathbb{F}_2$-solvable |
| (1.2′) | Fig.2(3)/$\mathbb{F}_2$-solvable | Fig.2(1),(3)/unsolvable case $4,5$) or $\mathbb{F}_2$-solvable | Fig.2(1),(3)/unsolvable case 3) or $\mathbb{F}_2$-solvable | Fig.2(2)/unsolvable case $3,6$) or $\mathbb{F}_2$-solvable |
| (2.1′) | Fig.2(1),(3)/$\mathbb{F}_2$-solvable | Fig.2(1),(3)/unsolvable case 2) or $\mathbb{F}_2$-solvable | Fig.2(1),(3)/$\mathbb{F}_2(\mathbb{F}_3)$-solvable | Fig.2(2)/unsolvable case 6) or $\mathbb{F}_2(\mathbb{F}_3)$-solvable |
| (2.2′) | Fig.2(4)/$\mathbb{F}_2$-solvable | Fig.2(4)/unsolvable case $2,7$) or $\mathbb{F}_2$-solvable | Fig.2(4)/unsolvable case 7) or $\mathbb{F}_2(\mathbb{F}_3)$-solvable | Fig.2(5)/unsolvable case $6,7$) or $\mathbb{F}_2(\mathbb{F}_3)$-solvable |

### C. *Proof Sketch of the Necessity*

In this subsection, we assume that $\mathcal{N}$ is a 3-pair network such that $\mathcal{A}(1,2,3) \neq \emptyset$, and $\mathcal{A}_{i,j} \cap \mathcal{A}(1,2,3) = \emptyset$ for all $i \neq j$. We will prove that if $\mathcal{N}$ violates all Conditions $2)-7)$ of Theorem 4.1, then it is solvable. We will need several lemmas before reaching the proof of necessity.

*Lemma 4.5:* For any feasible $i \neq j$, there exists an $s_i$-$t_j$ path $P$ such that $P \cap G(1,2,3) = \emptyset$.

*Proof:* Suppose, by way of contradiction, the conclusion is not true, i.e., $P \cap G(1,2,3) \neq \emptyset$, for all $s_i$-$t_j$ path $P$. Let $\mathcal{A}(1,2,3) = \{e_1, e_2, \cdots, e_\ell\}$. If $e_1 \in P \cap G(1,2,3)$ for all $s_i$-$t_j$ path $P$, then one can conclude that $e_1 \in \mathcal{A}_{i,j}$, which contradicts the assumption that $\mathcal{A}_{i,j} \cap \mathcal{A}(1,2,3) = \emptyset$. So, suppose there exists an $s_i$-$t_j$ path $P_0$ such that $e_1 \notin P_0 \cap G(1,2,3)$. Pick $e^* \in P_0 \cap G(1,2,3)$ and take an $e^*$-$t_i$ path, say, $Q$ (the existence of $Q$ is guaranteed by $e^* \in \mathcal{P}[e_1, e_\ell]$ and $e_\ell \in \mathcal{A}_{i,i}$). Then one verifies that $P = P_0[s_i, e^*]$-$Q[head(e^*), t_i]$ is an $s_i$-$t_i$ path such that $e_1 \notin P$, which is a contradiction. ■

*Lemma 4.6:* Given $k \neq l$, $k' \neq l'$. For any $s_k$-$t_\ell$ path $P_{k,\ell}$ such that $P_{k,\ell} \cap G(1,2,3) = \emptyset$ and any $s_{k'}$-$t_{\ell'}$ path $P_{k',\ell'}$ such that $P_{k',\ell'} \cap G(1,2,3) = \emptyset$, we have

$$P_{k,\ell} \cap P_{k',\ell'} \neq \emptyset \Leftrightarrow k = k' \text{ or } \ell = \ell'.$$

*Proof:* "$\Leftarrow$" is obviously by noticing that $S(k), T(\ell) \in \mathcal{A}_{k,\ell}$, $\forall k, \ell = 1,2,3$. Now we prove "$\Rightarrow$." Note that $\{k,\ell\} \cap \{k',\ell'\} \neq \emptyset$. If $k \neq k'$ and $\ell \neq \ell'$, then we have either $k = \ell'$ or $\ell = k'$. Without loss of generality, suppose $k = \ell'$, i.e., there exists an edge $e_0 \in P_{k,\ell} \cap P_{k',k}$. Then $P = P_{k,\ell}[s_k, e_0]$-$P_{k',k}[head(e_0), t_k]$ is an $s_k$-$t_k$ path such that $P \cap \mathcal{A}(1,2,3) = \emptyset$, which is a contradiction. ■

*Corollary 4.7:* Suppose $k \neq \ell$ and $k' \neq \ell'$. Then,

$$\mathcal{A}_{k,\ell} \cap \mathcal{A}_{k',\ell'} \neq \emptyset \Leftrightarrow k = k' \text{ or } \ell = \ell'.$$

The above lemmas, together with the results in Section III, motivate a more detailed classification of $\mathcal{N}$. Denote the $\mathcal{A}$-set of $G^+(i)$ by $\mathcal{A}^+_{i,i}$, we have the following cases:

(1) $\exists$ distinct $i,j,k$, s.t., $\mathcal{A}^+_{i,i} \cap \mathcal{A}^+_{j,j} \cap (\mathcal{A}_{i,k} \cup \mathcal{A}_{j,k}) \neq \emptyset$;
- (1.1) $\mathcal{A}^+_{i,i} \cap \mathcal{A}^+_{j,j} \cap \mathcal{A}_{i,k} \cap \mathcal{A}_{j,k} \neq \emptyset$;
- (1.2) $\mathcal{A}^+_{i,i} \cap \mathcal{A}^+_{j,j} \cap \mathcal{A}_{i,k} \cap \mathcal{A}_{j,k} = \emptyset$;

(2) $\forall$ distinct $i,j,k$, $\mathcal{A}^+_{i,i} \cap \mathcal{A}^+_{j,j} \cap (\mathcal{A}_{i,k} \cup \mathcal{A}_{j,k}) = \emptyset$;
- (2.1) $\mathcal{A}^+_{i,i} \cap \mathcal{A}^+_{j,j} \neq \emptyset$;
- (2.2) $\mathcal{A}^+_{i,i} \cap \mathcal{A}^+_{j,j} = \emptyset$;

Denote the $\mathcal{A}$-set of $G^-(i)$ by $\mathcal{A}^-_{i,i}$, we have the cases:

(1′) $\exists$ distinct $i,j,k$, s.t., $\mathcal{A}^-_{i,i} \cap \mathcal{A}^-_{j,j} \cap (\mathcal{A}_{k,i} \cup \mathcal{A}_{k,j}) \neq \emptyset$;
- (1.1′) $\mathcal{A}^-_{i,i} \cap \mathcal{A}^-_{j,j} \cap \mathcal{A}_{k,i} \cap \mathcal{A}_{k,j} \neq \emptyset$;
- (1.2′) $\mathcal{A}^-_{i,i} \cap \mathcal{A}^-_{j,j} \cap \mathcal{A}_{k,i} \cap \mathcal{A}_{k,j} = \emptyset$;

(2′) $\forall$ distinct $i,j,k$, $\mathcal{A}^-_{i,i} \cap \mathcal{A}^-_{j,j} \cap (\mathcal{A}_{k,i} \cup \mathcal{A}_{k,j}) = \emptyset$;
- (2.1′) $\mathcal{A}^-_{i,i} \cap \mathcal{A}^-_{j,j} \neq \emptyset$;
- (2.2′) $\mathcal{A}^-_{i,i} \cap \mathcal{A}^-_{j,j} = \emptyset$;

To obtain the necessity, we enumerate each of the product cases (of those on $\mathcal{A}^+_{ii}$ and $\mathcal{A}^-_{ii}$) and exhaustively examine all the possible scenarios with the help of Lemmas 4.5 and 4.6. Omitting the tedious details, we summarize the results in Table I, which complete the proof of necessity. Here, in Table I, expression like "Fig.2(1),(3)/unsolvable case 4,5) or $\mathbb{F}_2$-solvable" means that the network belongs to Fig.2(1) or (3) and it is either $\mathbb{F}_2$-solvable or unsolvable corresponding to cases 4) or 5) in Theorem 4.1.

## V. CONCLUSIONS AND FUTURE WORK

Theorem 4.1 further reviews that a 3-pair network with $\mathcal{A}(1,2,3) \neq \emptyset$ is solvable if and only if $\nexists e_1, e_2 \in E$, s.t., $\{e_1, e_2\} \overset{i}{\rightsquigarrow} \{S(1), S(2), S(3)\}$. However, this is shown to be false for general 3-pair networks after some further studies.

## ACKNOWLEDGMENT

This work is partially supported by a grant from the University Grants Committee of the Hong Kong Special Administrative Region, China (Project No. AoE/E-02/08).

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