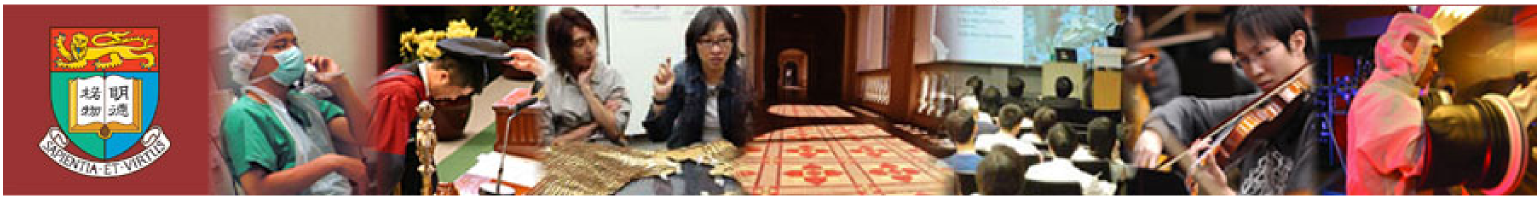

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Finiteness of Fixed Equilibrium Configurations of Point Vortices in the Plane with Background Flow

Pak-Leong Cheung[1*] and Tuen Wai Ng[1†]

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Abstract. For a dynamic system consisting of n point vortices in an ideal plane fluid with a steady, incompressible and irrotational background flow, a more physically significant definition of a fixed equilibrium configuration is suggested. Under this new definition, if the complex polynomial w that determines the aforesaid background flow is nonconstant, we have found an attainable generic upper bound $\frac{(m+n-1)!}{(m-1)!\,n_1!\cdots n_{i_0}!}$ for the number of fixed equilibrium configurations. Here, $m = \deg w$, i_0 is the number of species, and each n_i is the number of vortices in a species. We transform the rational function system arisen from fixed equilibria into a polynomial system, whose form is good enough to apply the BKK theory (named after D. N. Bernshtein [3], A. G. Khovanskii [12] and A. G. Kushnirenko [13]) to show the finiteness of its number of solutions. Having this finiteness, the required bound follows from Bézout's theorem or the BKK root count by T. Y. Li and X.-S. Wang [14].

1 Introduction

Every polynomial w in one complex variable generates a *steady* (i.e. independent of time) fluid flow in the complex plane $\mathbb{C}$ by the map $\zeta \mapsto (\operatorname{Re} w(\zeta), -\operatorname{Im} w(\zeta))$, which is identified with $\overline{w(\zeta)}$. The polynomial w is called the *complex velocity* of this flow; and since w satisfies the Cauchy–Riemann equations, the flow is *incompressible* and

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irrotational. Assume that the fluid in $\mathbb{C}$ is *ideal* (in the sense of [1, Section 1.3, p.6]). If $n \geq 2$ (point) vortices $z_1(t), \ldots, z_n(t) \in \mathbb{C}$ with their respective *circulations*

$$\Gamma_1, \ldots, \Gamma_n \in \mathbb{R}_* := \mathbb{R} \setminus \{0\} \tag{1}$$

(which are constants, due to Helmholtz's theorems (cf. [15, Section 2.2])) are situated in the ideal fluid with the above flow $\overline{w}$ in the background, then the dynamics of these vortices will be governed by

$$\frac{\mathrm{d}z_j(t)}{\mathrm{d}t} = -\frac{1}{2\pi\mathrm{i}} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{\Gamma_k}{\overline{z_j(t)} - \overline{z_k(t)}} + \overline{w(z_j(t))}, \ j = 1, \ldots, n \tag{2}$$

(cf. [11, Equation (2)] (caution: typo on the left-hand side) and [5, Equation (34)]).

The case of no background flow is when $w \equiv 0$. Since the early 1980s, there have been many studies on all types of stationary configurations (in the sense of [16, Definitions 0.2 & 1.1.2], see also [16, Definitions 0.3 & 1.1.3 & Proposition 1.1.4]), as well as other aspects of the vortex dynamics (2). So far, researchers have only found configurations with special patterns or for small n (see [4], [5], [11], etc., for these results). Given the difficulty of determining (analytically or numerically) the stationary configurations, inquiry into their number becomes a natural alternative. O'Neil [16, Theorems 5.1.1, 5.2.1 & 6.5.1] (cf. [17, Propositions 1 & 2], and beware of different terminologies) gave such results for three types:

> *(Let $w \equiv 0$.) For almost every choice of circulations* (1) *that satisfies*
>
> - $\sum_{i<j} \Gamma_i \Gamma_j = 0$, *there are exactly* $(n-2)!$ *equilibrium configurations. (This relation among the circulations* (1) *is necessary for the existence of such configurations.)*
> - $\sum_j \Gamma_j = 0$, *there are exactly* $(n-1)!$ *rigidly translating configurations. (This relation among the circulations* (1) *is necessary for the existence of such configurations.)*
> - $\sum_{i<j} \Gamma_i \Gamma_j \neq 0$ *and* $\sum_j \Gamma_j \neq 0$, *there are no more than* $n!/2$ *collinear relative equilibrium configurations.*

See also Hampton [8] and Hampton and Moeckel [10] for more results about the number of configurations when $n = 4, 5$.

As for the case when a background flow is present (i.e. $w \not\equiv 0$), there appears to be no corresponding knowledge so far apart from the few cases in [11, Sections III.B.2 and III.D] and [5, Section 3.3]. Our main result (Theorem 7) concerning the finiteness of the number of *fixed equilibrium configurations* (to be defined in Definition 4) will fill this deficiency. Also note that this terminology is synonymous with 'equilibrium configurations' in [16] and 'stationary equilibrium configurations' in [17], but the present one seems more common in the literature.

2 Results

Before proceeding, two conventions are to be understood throughout this article: (i) 'number of solutions' of any single polynomial or rational function equation, or any such system, counts multiplicity; (ii) 'finitely many' includes 'none'.

Fixed equilibria are the solutions of the rational function system (4) below in the n unknowns $z_1, \ldots, z_n$. Equivalently, they are the distinct solutions of the polynomial system $\mathscr{S}$ obtained by clearing denominators in (4). The inquiry into finiteness of the number of solutions of polynomial systems is reminiscent of a tool 'reduced system test' (Lemma 1) in the *BKK theory* (as detailed in [9, Section 3]). However, this test does not work at least in some cases of this rather complicated polynomial system $\mathscr{S}$. Therefore, we have found an alternative one (8) (Lemma 2) whose form is good enough for applying the test to confirm the finiteness of the number of solutions of this new polynomial system (8) (Proposition 3). These finitely many solutions $(z_1, \ldots, z_n)$ are then reduced to *fixed equilibrium configurations* by the notion of *equivalent solutions* (Definition 4, which is different from those in [10], [16] and [17]). We will consider a natural situation (Definition 6) where equivalent solutions arise, and arrive at the main result (Theorem 7). The rest of this section is devoted to expanding this paragraph.

For *fixed equilibria*, set $z_j(t) \equiv z_j$ (so that $\frac{\mathrm{d}z_j(t)}{\mathrm{d}t} \equiv 0$) in the system (2) and hence we have

$$-w(z_j) = \frac{1}{2\pi\mathrm{i}} \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{\Gamma_k}{z_j - z_k}, \; j = 1, \ldots, n. \tag{3}$$

The case $w \equiv c \in \mathbb{C}_* := \mathbb{C} \setminus \{0\}$ goes back to O'Neil's results in [16] and [17], therefore we assume that the *degree* $m := \deg w$ *of the background flow* is positive in what follows. Then, by complex conjugation and an appropriate rescaling, it only suffices to consider the normalized system

$$z_j{}^m + W(z_j) = \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{\Gamma_k}{z_j - z_k} =: L_j(z_1, \ldots, z_n), \; j = 1, \ldots, n, \tag{4}$$

where W is a polynomial of degree at most $m-1$ with coefficients determined by w. One might clear denominators to obtain a polynomial system $\mathscr{S}$ of n equations, where each equation is of degree $m+n-1$ and is in the n unknowns $z_1, \ldots, z_n$. Polynomial systems could have infinitely or finitely many solutions. If $\mathscr{S}$ falls into the latter case, then Bézout's theorem ([6, Theorem 2.3.1]) would provide

$$(m+n-1)^n \tag{5}$$

as an upper bound for the number of solutions.

As far as finiteness of the number of solutions of $\mathscr{S}$ is concerned, the following test in the BKK theory may be called upon (for details, the reader is referred to [9, Section 3], and also [10] for an application):

Lemma 1. ('Reduced system test' for finiteness of the number of solutions of a polynomial system in $\mathbb{C}_*^n$) [9, Propositions 2 & 3] *Consider a system of m polynomial equations in n unknowns:*

$$P_k(z_1, \dots, z_n) = \sum_{\mathbf{r}=(r_1,\dots,r_n)\in\mathcal{A}_k} c_{\mathbf{r}} {z_1}^{r_1} \cdots {z_n}^{r_n} = 0, \quad c_{\mathbf{r}} \in \mathbb{C}_*, \quad k = 1, \dots, m, \tag{6}$$

where each finite subset $\mathcal{A}_k$ of $(\mathbb{N} \cup \{0\})^n$ is called the support of P_k. For each $\boldsymbol{\alpha} \in \mathbb{R}^n$, the system

$$P_{k,\boldsymbol{\alpha}}(z_1, \dots, z_n) := \sum_{\mathbf{r}\in\mathcal{A}_k,\ \boldsymbol{\alpha}\cdot\mathbf{r}=\min_{\mathbf{r}'\in\mathcal{A}_k}\boldsymbol{\alpha}\cdot\mathbf{r}'} c_{\mathbf{r}} {z_1}^{r_1} \cdots {z_n}^{r_n} = 0, \quad k = 1, \dots, m, \tag{7}$$

is called the reduced system (of (6)) determined by $\boldsymbol{\alpha}$. Suppose that there exists an $\boldsymbol{\alpha}_0 \in \mathbb{Z}^n$ such that every reduced system (7) with $\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha} \le 0$ has no solution in $\mathbb{C}_*^n$. Then, the original system (6) has only finitely many solutions in $\mathbb{C}_*^n$.

Remark. The seemingly weird condition '$\boldsymbol{\alpha} \cdot \mathbf{r} = \min_{\mathbf{r}'\in\mathcal{A}_k} \boldsymbol{\alpha} \cdot \mathbf{r}'$' in (7) actually admits a beautiful geometric interpretation in terms of *supporting hyperplane*. We refer the reader to the paragraph preceding [9, Proposition 3]. This interpretation will be used when proving Proposition 3 in Section 4.

Despite the availability of such a handy finiteness test, the structure of $\mathscr{S}$ is still too complicated for the test to conclude anything even in some cases with small n (the number of vortices) and m (the degree of the background flow). More precisely, there are reduced systems which do have solutions in $\mathbb{C}_*^n$ but no choice of $\boldsymbol{\alpha}_0$ could avoid all of these reduced systems. This difficulty has motivated us to transform (4) into the following better polynomial system:

Lemma 2. (An equivalent polynomial system) *The system (4) is equivalent to*

$$\begin{cases} \sum_j \Gamma_j {z_j}^m + \sum_j \Gamma_j W(z_j) = 0 \\ \sum_j \Gamma_j {z_j}^{m+1} + \sum_j \Gamma_j z_j W(z_j) = \sum_{i<j} \Gamma_{i,j} \\ \sum_j \Gamma_j {z_j}^{m+2} + \sum_j \Gamma_j {z_j}^2 W(z_j) = \sum_j \Gamma_j \Gamma^j z_j \\ \qquad \vdots \\ \sum_j \Gamma_j {z_j}^{m+k-1} + \sum_j \Gamma_j {z_j}^{k-1} W(z_j) = \sum_j \Gamma_j \Gamma^j {z_j}^{k-2} + \sum_{\substack{r+s=k-2 \\ r,s\neq 0,\ i<j}} \Gamma_{i,j} {z_i}^r {z_j}^s \\ \qquad \vdots \\ \sum_j \Gamma_j {z_j}^{m+n-1} + \sum_j \Gamma_j {z_j}^{n-1} W(z_j) = \sum_j \Gamma_j \Gamma^j {z_j}^{n-2} + \sum_{\substack{r+s=n-2 \\ r,s\neq 0,\ i<j}} \Gamma_{i,j} {z_i}^r {z_j}^s \end{cases} \tag{8}$$

with constraint

$$z_i \neq z_j, \quad i \neq j, \tag{9}$$

where $\Gamma_{i,j} := \Gamma_i \Gamma_j$ *and* $\Gamma^j := \sum_{i\neq j} \Gamma_i$.

Remark.

(i) The n equations of this new system (8) are of degrees m, $m+1$, ... and $m+n-1$ respectively, and the leftmost sums are the only sources of these degrees.

(ii) The right-hand side of the k-th ($k = 2, \ldots, n$) equation of (8) is either a homogeneous polynomial of degree $k-2$ or, in the degenerate case, identically zero.

(iii) Each equation of (8) is invariant under any permutation of the n circulation-vortex pairs $\{(\Gamma_j, z_j) : j = 1, \ldots, n\}$, while in the original system (4), there will be a permutation of the n equations.

(iv) *Admissible* solutions of (8) are those that satisfy constraint (9). This terminology will not appear until Section 5.

(v) Lemma 2 will be proved in Section 3 by transforming (4) into (8) via an explicit matrix (16) which is invertible under (9). This new system (8) can also be obtained by replacing $2\pi i \bar{v}_j$ by ${z_j}^m + W(z_j)$ in O'Neil's [17, Equation (4.1)]. Our and O'Neil's methods are different. In particular, the involvement of the Vandermonde determinant in (18) in our method seems surprising.

As the reader will see in Section 4, the advantage of Lemma 2's transformation is that (8) is in a special form that facilitates using Lemma 1.

The following proposition shows that the condition

$$\sum_{j\in I} \Gamma_j \neq 0 \;\text{ for all }\; I \subset \{1, \ldots, n\} \quad \text{and} \quad \sum_{i<j} \Gamma_i \Gamma_j \neq 0 \tag{10}$$

guarantees that (8) alone (i.e. without the constraint (9)) already has finitely many solutions:

Proposition 3. (Upper bound for the number of solutions of the equivalent polynomial system) *Assume that* $m \geq 1$. *Then, for any choice of the circulations* (1) *that satisfies* (10), *the system* (8) *has at most* $\frac{(m+n-1)!}{(m-1)!}$ *solutions, and so does* (4). *This bound can be attained.*

Remark. There are cases where (4) has infinitely many solutions, such as in [11, Section III.B & III.C] and [5, Section 3.2]. But these existing cases either have no background flow or a background flow of degree 0 (i.e. $m < 1$), thus are not covered by Proposition 3.

In Proposition 3, the finiteness of the number of solutions of (8) under assumption (10) will be shown in Section 4. Then by Remark (i) following Lemma 2, Bézout's theorem provides the required upper bound $\frac{(m+n-1)!}{(m-1)!}$ for the number of solutions which is better than the bound (5). As we shall see in the proof of Lemma 2 in Section 3, every solution of (4) also satisfies (8), so (4) inherits the upper bound for its number of solutions. Finally, Section 5 will provide examples where this bound is attained.

Next, we suggest a perhaps new definition of a *fixed equilibrium configuration* in order to state the main result Theorem 7 of this article. Fix any polynomial w of degree $m \geq 1$ which provides the background flow in $\mathbb{C}$ as in Section 1. Let $\mathbf{z} = (z_1, \ldots, z_n)$ be a solution of (3) (necessarily, $z_i \neq z_j$ for $i \neq j$). Recall that these vortices in the background flow w generate the flow

$$\overline{V}_{\mathbf{z}}(\zeta) := -\frac{1}{2\pi\mathrm{i}} \sum_j \frac{\Gamma_j}{\overline{\zeta} - \overline{z_j}} + \overline{w(\zeta)}$$

or the complex velocity

$$V_{\mathbf{z}}(\zeta) := \overline{\overline{V}_{\mathbf{z}}(\zeta)} = \frac{1}{2\pi\mathrm{i}} \sum_j \frac{\Gamma_j}{\zeta - z_j} + w(\zeta)$$

on $\mathbb{C}_{\mathbf{z}} := \mathbb{C} \setminus \{z_1, \ldots, z_n\}$. Note that $V_{\mathbf{z}}$ is a rational function on $\mathbb{C}$ with simple poles at $z_1, \ldots, z_n$.

Definition 4. (Equivalent solutions. Fixed equilibrium configuration) Two solutions $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{z}' = (z_1', \ldots, z_n')$ of (3) are said to be *equivalent* (denoted by $\mathbf{z} \sim \mathbf{z}'$) if they generate two non-reflectively similar flows with w, in a plane-geometrical sense. More precisely, $\mathbf{z} \sim \mathbf{z}'$ if there exists $(a, b) \in \mathbb{C}_* \times \mathbb{C}$ such that $\overline{a}\overline{V}_{\mathbf{z}'}(a\zeta + b) = \overline{V}_{\mathbf{z}}(\zeta)$ for $\zeta \in \mathbb{C}_{\mathbf{z}}$, or, equivalently (in terms of complex velocity),

$$aV_{\mathbf{z}'}(a\zeta + b) = V_{\mathbf{z}}(\zeta) \quad \text{for} \quad \zeta \in \mathbb{C}_{\mathbf{z}}. \tag{11}$$

Then, each equivalence class $[\mathbf{z}]$ is called a *fixed equilibrium configuration.*

In O'Neil's definition [16, Definitions 0.4 & 1.1.5], two solutions $\mathbf{z}$ and $\mathbf{z}'$ are regarded as equivalent if there exists $(a, b) \in \mathbb{C}_* \times \mathbb{C}$ such that $z_j' = az_j + b$ for all j, so only the shapes of vortex sets are involved. Our definition of equivalent solutions of (3) has an extra physical significance: Besides the shapes of vortex sets, their effects on the rest of the plane are also considered. Here we illustrate the difference between our and O'Neil's definitions:

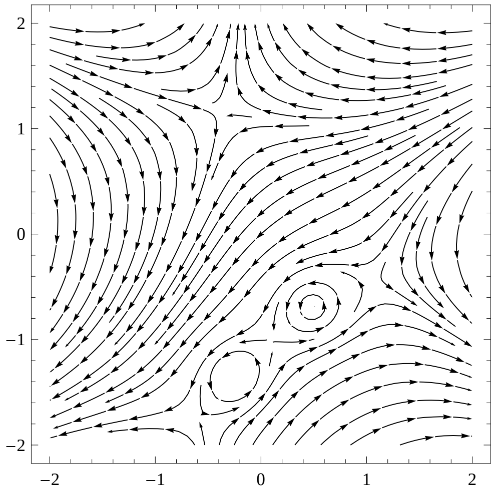
Example 5. Consider the two-vortex case $n = 2$, both with circulation $\Gamma_1 = \Gamma_2 = 1$, and with background flow $w(\zeta) = -\frac{\zeta^2+1}{2\pi\mathrm{i}}$ (thus $m = 2$). By Lemma 2, we are to solve

$$\begin{cases} {z_1}^2 + {z_2}^2 + 2 &= 0 \\ {z_1}^3 + {z_2}^3 + z_1 + z_2 &= 1 \end{cases}$$

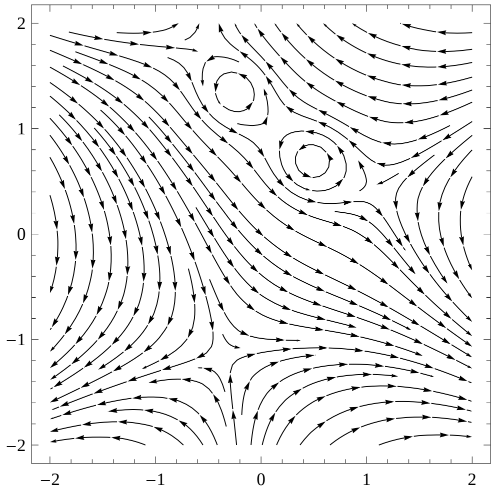
with constraint $z_1 \neq z_2$, and the solutions are

$$\begin{aligned}(z_1, z_2) \approx\ & (-0.250 \pm 1.349\,\mathrm{i}, 0.487 \pm 0.693\,\mathrm{i}),\\ & (0.487 \pm 0.693\,\mathrm{i}, -0.250 \pm 1.349\,\mathrm{i}),\\ & (-0.237 \pm 1.028\,\mathrm{i}, -0.237 \mp 1.028\,\mathrm{i}).\end{aligned} \tag{12}$$

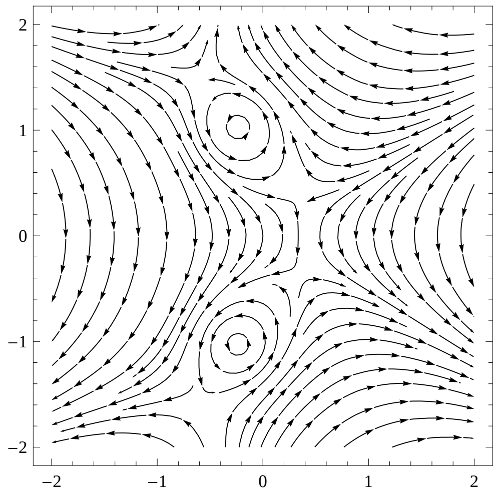
Any pair of two-point sets in the plane differ by translation, rotation and/or scaling, so all the six solutions in (12) are equivalent in O'Neil's sense, thereby constituting exactly one fixed equilibrium configuration. But by Definition 4, they constitute three fixed equilibrium configurations because they generate three flows with w that are not non-reflectively similar as shown in Figure 1. The streamlines in Figure 1 are actually formed by superimpositions in Figure 2. (Figures 1 and 2 are generated by Mathematica.)



(a) $(z_1, z_2) \approx (-0.250 - 1.349\,\mathrm{i}, 0.487 - 0.693\,\mathrm{i})$



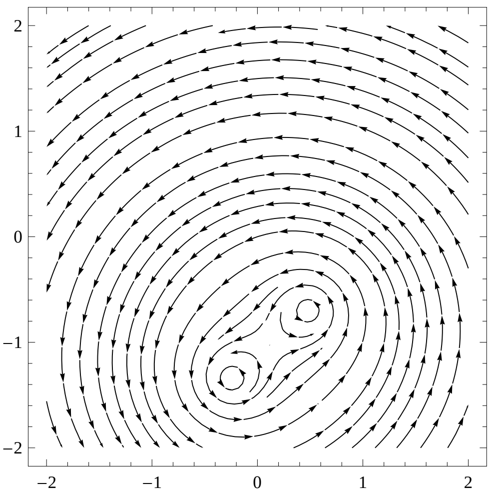
(b) $(z_1, z_2) \approx (-0.250 + 1.349\,\mathrm{i}, 0.487 + 0.693\,\mathrm{i})$



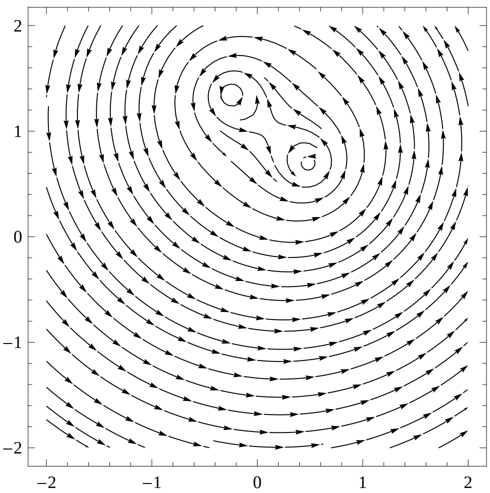
(c) $(z_1, z_2) \approx (-0.237 - 1.028\,\mathrm{i}, -0.237 + 1.028\,\mathrm{i})$

Figure 1: The streamlines of the flows $\overline{V}_{(z_1,z_2)}$ generated by the vortex sets (12) with the background flow $w(\zeta) = -\frac{\zeta^2+1}{2\pi\mathrm{i}}$ in Example 5.

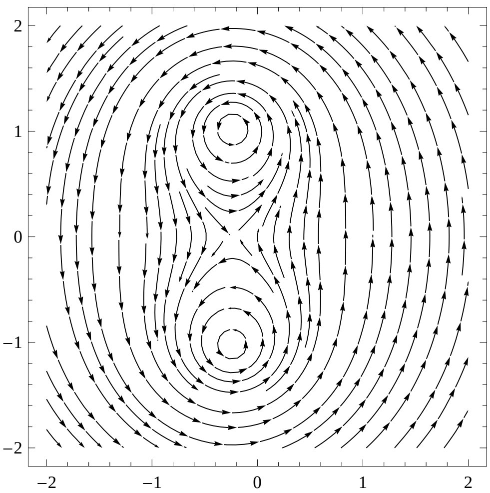
Our definition introduces comparison (11) of rational functions to distinguish between different configurations. We are not going to elaborate this definition to



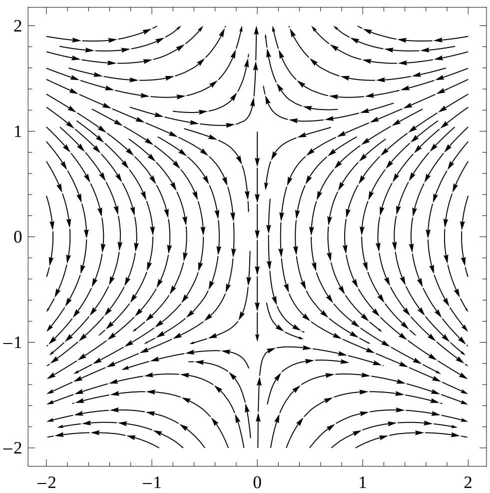
(a) $(z_1, z_2) \approx (-0.250 - 1.349\,\mathrm{i}, 0.487 - 0.693\,\mathrm{i})$



(b) $(z_1, z_2) \approx (-0.250 + 1.349\,\mathrm{i}, 0.487 + 0.693\,\mathrm{i})$



(c) $(z_1, z_2) \approx (-0.237 - 1.028\,\mathrm{i}, -0.237 + 1.028\,\mathrm{i})$



(d) background flow

Figure 2: The streamlines of the flows generated by the vortex sets (12) without the background flow $w(\zeta) = -\frac{\zeta^2+1}{2\pi \mathrm{i}}$ in Example 5. The background flow (d) superimposes with (a)–(c) respectively to form Figures 1(a)–(c).

the full extent, but just to observe a natural situation where equivalent solutions, in our sense, of (3) arise:

Definition 6. (Species) Partition the circulations (1) into their i_0 distinct values:

$$\Gamma_{j_{i,1}} = \cdots = \Gamma_{j_{i,n_i}} =: [\Gamma_i], \quad i = 1, \ldots, i_0, \quad [\Gamma_i] \neq [\Gamma_{i'}], \quad i \neq i', \quad \textstyle\sum_i n_i = n. \qquad (13)$$

For each solution $(z_1, \ldots, z_n)$ of (3), the set of all the vortices

$$[z_i] := \left\{ z_{j_{i,1}}, \ldots, z_{j_{i,n_i}} \right\}$$

which possess the common circulation $[\Gamma_i]$ is called a *species* (cf. [11]).

Note that the concept of species was not involved in defining equivalent solutions in [16]. Here, it acts as follow: If $\Gamma_i = \Gamma_j$ for some $i < j$, then with every solution $\mathbf{z} = (z_1, \dots, z_i, \dots, z_j, \dots, z_n)$ of (3) there associates another solution $\mathbf{z}' = (z_1, \dots, z_j, \dots, z_i, \dots, z_n)$, and then $\mathbf{z} \sim \mathbf{z}'$ because (11) holds with $(a, b) = (1, 0)$. Consequently, a standard combinatorial argument bridges Proposition 3 to our main theorem:

Theorem 7. (Upper bound for the number of fixed equilibrium configurations in a background flow) *Assume that $m \geq 1$. Then, for any choice of the circulations* (1) *that satisfies* (10)*, there are at most $\frac{(m+n-1)!}{(m-1)!\, n_1! \cdots n_{i_0}!}$ fixed equilibrium configurations, where n_i are the sizes of the species as in* (13)*. This bound can be attained.*

Section 5 will provide examples where this bound is and is not attained. It will also suggest another factor in reducing the solutions of (3), (4) or (8) to fixed equilibrium configurations via Definition 4.

3 Proof of Lemma 2

Write (4) in vector form

$$\mathbf{Z}_n + \mathbf{W}_n = \mathbf{L}_n, \tag{14}$$

where

$$\mathbf{Z}_n := \big({z_i}^m\big)_{i=1,\dots,n}, \quad \mathbf{W}_n := \big(W(z_i)\big)_{i=1,\dots,n} \quad \text{and} \quad \mathbf{L}_n := \big(L_i(z_1, \dots, z_n)\big)_{i=1,\dots,n}. \tag{15}$$

Left-multiplying each side by the square matrix

$$\mathbf{T}_n := \big(\Gamma_j {z_j}^{i-1}\big)_{i,j=1,\dots,n}, \tag{16}$$

the left-hand side simply becomes

$$\begin{aligned} \mathbf{T}_n(\mathbf{Z}_n + \mathbf{W}_n) &= \begin{pmatrix} \Gamma_1 & \Gamma_2 & \cdots & \Gamma_n \\ \Gamma_1 z_1 & \Gamma_2 z_2 & \cdots & \Gamma_n z_n \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_1 {z_1}^{n-1} & \Gamma_2 {z_2}^{n-1} & \cdots & \Gamma_n {z_n}^{n-1} \end{pmatrix} \left(\begin{pmatrix} {z_1}^m \\ \vdots \\ {z_n}^m \end{pmatrix} + \begin{pmatrix} W(z_1) \\ \vdots \\ W(z_n) \end{pmatrix} \right) \\ &= \begin{pmatrix} \sum_j \Gamma_j {z_j}^m \\ \sum_j \Gamma_j {z_j}^{m+1} \\ \vdots \\ \sum_j \Gamma_j {z_j}^{m+n-1} \end{pmatrix} + \begin{pmatrix} \sum_j \Gamma_j W(z_j) \\ \sum_j \Gamma_j z_j W(z_j) \\ \vdots \\ \sum_j \Gamma_j {z_j}^{n-1} W(z_j) \end{pmatrix}. \end{aligned} \tag{17}$$

On the right-hand side, the first entry of $\mathbf{T}_n\mathbf{L}_n$ is

$$\begin{pmatrix}\Gamma_1 & \cdots & \Gamma_n\end{pmatrix}\begin{pmatrix}\displaystyle\sum_{i\neq 1}\frac{\Gamma_i}{z_1-z_i}\\ \displaystyle\sum_{i\neq 2}\frac{\Gamma_i}{z_2-z_i}\\ \vdots\\ \displaystyle\sum_{i\neq n}\frac{\Gamma_i}{z_n-z_i}\end{pmatrix}=\sum_j\Gamma_j\sum_{i\neq j}\frac{\Gamma_i}{z_j-z_i}=\sum_{i<j}\left(\frac{\Gamma_j\Gamma_i}{z_j-z_i}+\frac{\Gamma_i\Gamma_j}{z_i-z_j}\right)=0$$

by cancellations. The second entry is

$$\begin{aligned}\begin{pmatrix}\Gamma_1z_1 & \cdots & \Gamma_nz_n\end{pmatrix}\begin{pmatrix}\displaystyle\sum_{i\neq 1}\frac{\Gamma_i}{z_1-z_i}\\ \displaystyle\sum_{i\neq 2}\frac{\Gamma_i}{z_2-z_i}\\ \vdots\\ \displaystyle\sum_{i\neq n}\frac{\Gamma_i}{z_n-z_i}\end{pmatrix}&=\sum_j\Gamma_jz_j\sum_{i\neq j}\frac{\Gamma_i}{z_j-z_i}\\ &=\sum_{i<j}\left(\frac{\Gamma_j\Gamma_iz_j}{z_j-z_i}+\frac{\Gamma_i\Gamma_jz_i}{z_i-z_j}\right)=\sum_{i<j}\Gamma_i\Gamma_j.\end{aligned}$$

The k-th entry, $k=3,\ldots,n$, is

$$\begin{aligned}\begin{pmatrix}\Gamma_1{z_1}^{k-1} & \cdots & \Gamma_n{z_n}^{k-1}\end{pmatrix}\begin{pmatrix}\displaystyle\sum_{i\neq 1}\frac{\Gamma_i}{z_1-z_i}\\ \displaystyle\sum_{i\neq 2}\frac{\Gamma_i}{z_2-z_i}\\ \vdots\\ \displaystyle\sum_{i\neq n}\frac{\Gamma_i}{z_n-z_i}\end{pmatrix}&=\sum_j\Gamma_j{z_j}^{k-1}\sum_{i\neq j}\frac{\Gamma_i}{z_j-z_i}\\ &=\sum_{i<j}\left(\frac{\Gamma_j\Gamma_i{z_j}^{k-1}}{z_j-z_i}+\frac{\Gamma_i\Gamma_j{z_i}^{k-1}}{z_i-z_j}\right)\\ &=\sum_{i<j}\Gamma_i\Gamma_j({z_i}^{k-2}+{z_i}^{k-3}z_j+\cdots+z_i{z_j}^{k-3}+{z_j}^{k-2}).\end{aligned}$$

We have left-multiplied each side of (14) by $\mathbf{T}_n$ to obtain (8), thus the claimed equivalence between (4) and '(8) and (9)' would follow if $\mathbf{T}_n$ is invertible. Indeed,

$$\det\mathbf{T}_n=\begin{vmatrix}\Gamma_1 & \Gamma_2 & \cdots & \Gamma_n\\ \Gamma_1z_1 & \Gamma_2z_2 & \cdots & \Gamma_nz_n\\ \vdots & \vdots & \ddots & \vdots\\ \Gamma_1{z_1}^{n-1} & \Gamma_2{z_2}^{n-1} & \cdots & \Gamma_n{z_n}^{n-1}\end{vmatrix}=\prod_j\Gamma_j\cdot\prod_{i<j}(z_j-z_i) \qquad (18)$$

is just $\prod_j \Gamma_j$ times of the Vandermonde determinant $\prod_{i<j}(z_j - z_i)$, where (1) and (9) guarantees that their product is non-zero.

4 Proof of Proposition 3

4.1 A Little More General System

Consider the following more general system than (8):

$$
\begin{cases}
P_1(z_1,\ldots,z_n) := \sum_j \Gamma_j {z_j}^m + \sum_{j,r} A_j^{1,r} {z_j}^r = 0 \\
P_2(z_1,\ldots,z_n) := \sum_j \Gamma_j {z_j}^{m+1} + \sum_{j,r} A_j^{2,r} {z_j}^{r+1} + C_0 = 0 \\
P_3(z_1,\ldots,z_n) := \sum_j \Gamma_j {z_j}^{m+2} + \sum_{j,r} A_j^{3,r} {z_j}^{r+2} + \sum_j C_j^3 z_j = 0 \\
\quad \vdots \\
P_k(z_1,\ldots,z_n) := \sum_j \Gamma_j {z_j}^{m+k-1} + \sum_{j,r} A_j^{k,r} {z_j}^{r+k-1} \\
\qquad\qquad + \sum_j C_j^k {z_j}^{k-2} + \sum_{\substack{r+s=k-2 \\ r,s\neq 0,\ i<j}} C_{i,j}^{k,r,s} {z_i}^r {z_j}^s = 0 \\
\quad \vdots \\
P_n(z_1,\ldots,z_n) := \sum_j \Gamma_j {z_j}^{m+n-1} + \sum_{j,r} A_j^{n,r} {z_j}^{r+n-1} \\
\qquad\qquad + \sum_j C_j^m {z_j}^{n-2} + \sum_{\substack{r+s=n-2 \\ r,s\neq 0,\ i<j}} C_{i,j}^{m,r,s} {z_i}^r {z_j}^s = 0,
\end{cases}
\tag{19}
$$

with coefficients $\Gamma_j, C_0 \in \mathbb{C}_*$ and $A_{r,j}, C_j^k, C_{i,j}^{k,r,s} \in \mathbb{C}$, where $\sum_r = \sum_{r=0}^{m-1}$. We will prove the following finiteness result for (19) via Sections 4.2 and 4.3:

Proposition 8. (Finiteness of the number of solutions) *The system* (19) *has only finitely many solutions if $\sum_{j\in I} \Gamma_j \neq 0$ for all $I \subset \{1,\ldots,n\}$ and $C_0 \neq 0$.*

The system (19) has Bézout bound $\frac{(m+n-1)!}{(m-1)!}$. This coincides with the generally finer Li and Wang's BKK root count [14], due to the somewhat special structure of (19) (to be seen in (21)):

Proposition 9. (BKK root count in $\mathbb{C}^n$) *Assume that $\Gamma_j, C_0 \in \mathbb{C}_*$. Let $\mathcal{M}(\mathcal{N}_1,\ldots,\mathcal{N}_n)$ denote the mixed volume of the Newton polytopes $\mathcal{N}_k$ of $\mathcal{A}_k \cup \{\mathbf{0}\}$, $k = 1,\ldots,n$, where each $\mathcal{A}_k$ is the support of the polynomial P_k in* (19)*. Then,*

$$\mathcal{M}(\mathcal{N}_1,\ldots,\mathcal{N}_n) = \frac{(m+n-1)!}{(m-1)!}.$$

Proof of Proposition 9. The supports of P_k are respectively

$$\begin{aligned}
\mathcal{A}_1 &= m\mathcal{E}_n \cup \bigcup_{j,r} S_j^{1,r} \\
\mathcal{A}_2 &= (m+1)\mathcal{E}_n \cup \bigcup_{j,r} S_j^{2,r} \cup \{\mathbf{0}\} \\
\mathcal{A}_3 &= (m+2)\mathcal{E}_n \cup \bigcup_{j,r} S_j^{3,r} \cup \bigcup_{j} S_j^{3} \\
&\vdots \\
\mathcal{A}_k &= (m+k-1)\mathcal{E}_n \cup \bigcup_{j,r} S_j^{k,r} \cup \bigcup_{j} S_j^{k} \cup \bigcup_{\substack{r+s=k-2 \\ r,s\neq 0,\ i<j}} S_{i,j}^{k,r,s} \\
&\vdots \\
\mathcal{A}_n &= (m+n-1)\mathcal{E}_n \cup \bigcup_{j,r} S_j^{n,r} \cup \bigcup_{j} S_j^{n} \cup \bigcup_{\substack{r+s=n-2 \\ r,s\neq 0,\ i<j}} S_{i,j}^{n,r,s},
\end{aligned} \tag{20}$$

where $\mathcal{E}_n := \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, $\bigcup_r = \bigcup_{r=0}^{m-1}$,

$$S_j^{k,r} := \begin{cases} \{(r+k-1)\mathbf{e}_j\} & \text{if } A_j^{k,r} \neq 0 \\ \emptyset & \text{if } A_j^{k,r} = 0 \end{cases}, \quad S_j^{k} := \begin{cases} \{(k-2)\mathbf{e}_j\} & \text{if } C_j^{k} \neq 0 \\ \emptyset & \text{if } C_j^{k} = 0 \end{cases}$$

$$\text{and} \quad S_{i,j}^{k,r,s} := \begin{cases} \{r\mathbf{e}_i + s\mathbf{e}_j\} & \text{if } C_{i,j}^{k,r,s} \neq 0 \\ \emptyset & \text{if } C_{i,j}^{k,r,s} = 0 \end{cases}.$$

No matter what value $S_j^{k,r}$, S_j^k and $S_{i,j}^{k,r,s}$ take in (20), the Newton polytopes $\mathcal{N}_k$ of $\mathcal{A}_k \cup \{\mathbf{0}\}$ are

$$\begin{aligned}
\mathcal{N}_1 &= \mathrm{Conv}(m\mathcal{E}_n \cup \{\mathbf{0}\}) = m\Delta_n \\
\mathcal{N}_2 &= \mathrm{Conv}((m+1)\mathcal{E}_n \cup \{\mathbf{0}\}) = (m+1)\Delta_n \\
\mathcal{N}_3 &= \mathrm{Conv}((m+2)\mathcal{E}_n \cup \{\mathbf{0}\}) = (m+2)\Delta_n \\
&\vdots \\
\mathcal{N}_k &= \mathrm{Conv}\left((m+k-1)\mathcal{E}_n \cup \{\mathbf{0}\}\right) = (m+k-1)\Delta_n \\
&\vdots \\
\mathcal{N}_n &= \mathrm{Conv}\left((m+n-1)\mathcal{E}_n \cup \{\mathbf{0}\}\right) = (m+n-1)\Delta_n,
\end{aligned} \tag{21}$$

where

$$\Delta_n := \mathrm{Conv}(\mathcal{E}_n \cup \{\mathbf{0}\}) \tag{22}$$

is the unit simplex in $\mathbb{R}^n$. Such a simplification is due to that $(r+k-1)\mathbf{e}_j$ and $r\mathbf{e}_i + s\mathbf{e}_j$ ($k = 3, \ldots, n$, $r+s = k-2$ and $i < j$) actually lie in $\mathrm{Conv}((m+k-1)\mathcal{E}_n \cup \{\mathbf{0}\})$. By the multilinearity [7, Theorem 7.4.12.b, p.338] of mixed volume and [7, Exercise 7.3.b, p.306, & Exercise 7.7.b, p.338], the required mixed volume is

$$\begin{aligned}
\mathcal{M}(\mathcal{N}_1, \ldots, \mathcal{N}_n) &= \mathcal{M}(m\Delta_n, (m+1)\Delta_n, \ldots, (m+n-1)\Delta_n) \\
&= m \cdot (m+1) \cdot \cdots \cdot (m+n-1) \cdot \mathcal{M}(\Delta_n, \Delta_n, \ldots, \Delta_n) \\
&= \frac{(m+n-1)!}{(m-1)!} \cdot n!\,\mathrm{Vol}_n(\Delta_n) = \frac{(m+n-1)!}{(m-1)!}.
\end{aligned}$$

□

Equation (21) has just revealed that the elements of the supports $\mathcal{A}_k$ of P_k corresponding to the terms with coefficients $A^{\cdot}_{\cdot}$ and $C^{\cdot}_{\cdot}$ can simply be ignored in the formation of the Newton polytopes $\mathcal{N}_k$. This will be useful in the upcoming sections.

4.2 The Initial Case

To prove Proposition 8, we shall apply strong induction on the number of vortices n; and it contains Proposition 3 as a case, and then leads to Theorem 7 as analyzed in Section 2. Lemma 1 will enter to test the finiteness of the number of solutions of polynomial systems in $\mathbb{C}_*^n$. Taking the geometric interpretation of reduced system in Lemma 1 as detailed in the paragraph preceding [9, Proposition 3] for granted, we will need the following notation for brevity of the upcoming discussions: let $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ denote the supporting hyperplane of a polytope $\mathcal{N}$ with inward normal vector $\boldsymbol{\alpha}$.

Here we start with the initial case $n = 2$:

$$\begin{cases} P_1(z_1, z_2) = \Gamma_1 {z_1}^m + \Gamma_2 {z_2}^m + A_1^{1,m-1} {z_1}^{m-1} + A_2^{1,m-1} {z_2}^{m-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \cdots + (A_1^{1,0} + A_2^{1,0}) \qquad = 0 \\ P_2(z_1, z_2) = \Gamma_1 {z_1}^{m+1} + \Gamma_2 {z_2}^{m+1} + A_1^{2,m-1} {z_1}^m + A_2^{2,m-1} {z_2}^m + \cdots + C_0 = 0 \\ \qquad\qquad\qquad \Gamma_1,\ \Gamma_2,\ \Gamma_1 + \Gamma_2,\ C_0 \neq 0 \end{cases} \tag{23}$$

<ins>Case 1: $(z_1, z_2) \in \mathbb{C}_*^2$.</ins> Lemma 1 will show that (23) has only finitely many solutions in $\mathbb{C}_*^2$. To this end, consider, by (20), the Newton polytopes

$$\mathcal{N}_1 = \operatorname{Conv}(\{m\mathbf{e}_1, m\mathbf{e}_2\} \cup \cdots) \quad \text{and}$$
$$\mathcal{N}_2 = \operatorname{Conv}(\{(m+1)\mathbf{e}_1, (m+1)\mathbf{e}_2, \mathbf{0}\} \cup \cdots)$$

of the supports $\mathcal{A}_1$ and $\mathcal{A}_2$ of P_1 and P_2, and their Minkowski sum

$$\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2 = \operatorname{Conv}(\{a_2\mathbf{e}_1, a_2\mathbf{e}_2\} \cup \cdots), \tag{24}$$

where and hereafter,

$$a_n := m + (m+1) + \cdots + (m+n-1) = nm + \tbinom{n}{2}. \tag{25}$$

Note that this '$\cdots$' in (24) does not alter the fact that

$$\mathcal{N} \subset a_2\Delta_2, \tag{26}$$

where Δ_2 is as in (22). Now, consider the reduced systems of (23) determined by all the $\boldsymbol{\alpha}$ or $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ satisfying $(\mathbf{e}_1 + \mathbf{e}_2) \cdot \boldsymbol{\alpha} \leq 0$. Because of (26) and such a choice of $\boldsymbol{\alpha}$, each $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ actually supports $\mathcal{N}$ at a face of the facet $\operatorname{Conv}(\{a_2\mathbf{e}_1, a_2\mathbf{e}_2\}) = a_2\Delta_2$:

- <ins>Case I: $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ supports $\mathcal{N}$ at the 0-face</ins>

$$a_2\mathbf{e}_j, \quad j = 1, 2,$$

  then $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}_1}$ and $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}_2}$ support $\mathcal{N}_1$ and $\mathcal{N}_2$ at the 0-faces

$$m\mathbf{e}_j \quad \text{and} \quad (m+1)\mathbf{e}_j$$

respectively, giving the reduced system

$$\begin{cases} \Gamma_j {z_j}^m &= 0 \\ \Gamma_j {z_j}^{m+1} &= 0 \\ \quad \Gamma_j \neq 0 \end{cases}$$

which has no solution in $\mathbb{C}_*^2$.

- Case II: $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ supports $\mathcal{N}$ at the 1-face

$$\operatorname{Conv}(\{a_2\mathbf{e}_1, a_2\mathbf{e}_2\}),$$

then $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}_1}$ and $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}_2}$ support $\mathcal{N}_1$ and $\mathcal{N}_2$ at the 1-faces

$$\operatorname{Conv}(\{m\mathbf{e}_1, m\mathbf{e}_2\}) \quad \text{and} \quad \operatorname{Conv}(\{(m+1)\mathbf{e}_1, (m+1)\mathbf{e}_2\})$$

respectively, giving the reduced system

$$\begin{cases} \widetilde{P}_1(z_1, z_2) := \Gamma_1 {z_1}^m + \Gamma_2 {z_2}^m &= 0 \\ \widetilde{P}_2(z_1, z_2) := \Gamma_1 {z_1}^{m+1} + \Gamma_2 {z_2}^{m+1} &= 0 \\ \qquad \Gamma_1,\ \Gamma_2,\ \Gamma_1 + \Gamma_2 \neq 0. \end{cases} \tag{27}$$

(i) If $z_1 \neq z_2$, then $\det \mathbf{T}_2 \neq 0$ by (18), so that $\mathbf{T}_2$ is invertible. Left-multiplying both sides of (27) by ${\mathbf{T}_2}^{-1}$, it is transformed to

$$\mathbf{Z}_2 = \mathbf{0}$$

(see (17)), where $\mathbf{Z}_2$ is as in (15). But this contradicts that $z_1 \neq z_2$.

(ii) If $z_1 = z_2$, then the sub-system consisting of the first equation

$$\begin{cases} (\Gamma_1 + \Gamma_2) {z_1}^m = 0 \\ \quad \Gamma_1 + \Gamma_2 \neq 0 \end{cases}$$

already has no solution in $\mathbb{C}_*$.

Therefore, in any case, (27) has no solution in $\mathbb{C}_*^2$.

What we have established so far is that for every $\boldsymbol{\alpha}$ or $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ with $(\mathbf{e}_1 + \mathbf{e}_2) \cdot \boldsymbol{\alpha} \leq 0$, the corresponding reduced system has no solution in $\mathbb{C}_*^2$. Hence, it follows from Lemma 1 that (23) has only finitely many solutions in $\mathbb{C}_*^2$.

Case 2: $(z_1, z_2) \in \mathbb{C}_* \times \{0\}$. The system (23) degrades to

$$\begin{cases} P_1(z_1, 0) = \Gamma_1 {z_1}^m + A_1^{1,m-1} {z_1}^{m-1} + \cdots + (A_1^{1,0} + A_2^{1,0}) = 0 \\ P_2(z_1, 0) = \Gamma_1 {z_1}^{m+1} + A_1^{2,m-1} {z_1}^m + \cdots + C_0 \qquad\qquad\quad = 0 \\ \qquad\qquad \Gamma_1,\ C_0 \neq 0, \end{cases} \tag{28}$$

where the first equation already has only finitely many (at most m) solutions, thus so does (28). Similar for the case where $(z_1, z_2) \in \{0\} \times \mathbb{C}_*$.

Case 3: $(z_1, z_2) \in \{0\}^2$. The system (23) degrades to

$$\begin{cases} P_1(0,0) = A_1^{1,0} + A_2^{1,0} = 0 \\ P_2(0,0) = C_0 \qquad\qquad = 0 \\ \qquad\qquad C_0 \neq 0 \end{cases}$$

which is simply inconsistent.

Combining all the above three cases, (23) has only finitely many solutions in $\mathbb{C}^2$, and Proposition 8 with $n = 2$ is proved.

4.3 Strong Induction

As the reader may have noticed or will see soon, everything actually lies in the non-solvability of the following special reduced system in $\mathbb{C}_*^k$ under the assumption of Proposition 8:

Lemma 10. (Special reduced system) *The system*

$$\begin{cases} \widetilde{P}_1(z_1, \dots, z_k) := \Gamma_1 {z_1}^m + \cdots + \Gamma_k {z_k}^m &= 0 \\ \widetilde{P}_2(z_1, \dots, z_k) := \Gamma_1 {z_1}^{m+1} + \cdots + \Gamma_k {z_k}^{m+1} &= 0 \\ \qquad\qquad \vdots \\ \widetilde{P}_k(z_1, \dots, z_k) := \Gamma_1 {z_1}^{m+k-1} + \cdots + \Gamma_k {z_k}^{m+k-1} &= 0 \\ \qquad \sum_{j \in I} \Gamma_j \neq 0, \quad I \subset \{1, \dots, k\} \end{cases} \tag{29}$$

has no solution in $\mathbb{C}_*^k$.

Proof of Lemma 10. The case $k = 2$ is just (27). Assume that the lemma holds when $k = K$, then consider (29) with $k = K + 1$:

$$\begin{cases} \widetilde{P}_1(z_1, \dots, z_K, z_{K+1}) = \Gamma_1 {z_1}^m + \cdots + \Gamma_K {z_K}^m + \Gamma_{K+1} {z_{K+1}}^m &= 0 \\ \widetilde{P}_2(z_1, \dots, z_K, z_{K+1}) = \Gamma_1 {z_1}^{m+1} + \cdots + \Gamma_K {z_K}^{m+1} + \Gamma_{K+1} {z_{K+1}}^{m+1} &= 0 \\ \qquad\qquad \vdots \\ \widetilde{P}_K(z_1, \dots, z_K, z_{K+1}) = \Gamma_1 {z_1}^{m+K-1} + \cdots + \Gamma_K {z_K}^{m+K-1} \\ \qquad\qquad + \Gamma_{K+1} {z_{K+1}}^{m+K-1} &= 0 \\ \widetilde{P}_{K+1}(z_1, \dots, z_K, z_{K+1}) = \Gamma_1 {z_1}^{m+K} + \cdots + \Gamma_K {z_K}^{m+K} \\ \qquad\qquad + \Gamma_{K+1} {z_{K+1}}^{m+K} &= 0 \\ \qquad \sum_{j \in I} \Gamma_j \neq 0, \quad I \subset \{1, \dots, K+1\} \end{cases} \tag{30}$$

(i) If all z_j are distinct, then $\det \mathbf{T}_n \neq 0$ by (18), so that $\mathbf{T}_n$ is invertible. Left-multiplying both sides of (30) by ${\mathbf{T}_n}^{-1}$, it is transformed to

$$\mathbf{Z}_n = \mathbf{0}$$

(see (17)), where $\mathbf{Z}_n$ is as in (15). But this contradicts that all z_j are distinct.

(ii) If some z_j are equal, say, $z_K = z_{K+1}$, then the sub-system consisting of the first K equations

$$\begin{cases} \widetilde{P}_1(z_1,\ldots,z_K,z_K) &= \Gamma_1 {z_1}^m + \cdots + \Gamma_{K-1}{z_{K-1}}^m + (\Gamma_K+\Gamma_{K+1}){z_K}^m = 0 \\ \widetilde{P}_2(z_1,\ldots,z_K,z_K) &= \Gamma_1 {z_1}^{m+1} + \cdots + \Gamma_{K-1}{z_{K-1}}^{m+1} \\ &\qquad\qquad + (\Gamma_K+\Gamma_{K+1}){z_K}^{m+1} \qquad\qquad = 0 \\ &\vdots \\ \widetilde{P}_K(z_1,\ldots,z_K,z_K) &= \Gamma_1 {z_1}^{m+K-1} + \cdots + \Gamma_{K-1}{z_{K-1}}^{m+K-1} \\ &\qquad\qquad + (\Gamma_K+\Gamma_{K+1}){z_K}^{m+K-1} \qquad = 0 \\ &\sum_{j\in I}\Gamma_j \neq 0, \quad I \subset \{1,\ldots,K+1\} \end{cases}$$

already has no solution in $\mathbb{C}_*^K$ due to the induction hypothesis (with Γ_K replaced by $\Gamma_K+\Gamma_{K+1}$), thus so does (30) in $\mathbb{C}_*^{K+1}$. Similar for the other cases.

Therefore, in any case, (30) has no solution in $\mathbb{C}_*^{K+1}$, and the lemma follows from induction. □

Finally, we are in a position to prove Proposition 8. Recall that the case $n=2$ has already been proved above. Now, assume that it holds for $n=2,\ldots,N-1$, then consider the case $n=N$:

$$\begin{cases} P_1(z_1,\ldots,z_N) = \sum_j \Gamma_j {z_j}^m + \sum_{j,r} A_j^{1,r}{z_j}^r = 0 \\ P_2(z_1,\ldots,z_N) = \sum_j \Gamma_j {z_j}^{m+1} + \sum_{j,r} A_j^{2,r}{z_j}^{r+1} + C_0 = 0 \\ P_k(z_1,\ldots,z_N) = \sum_j \Gamma_j {z_j}^{m+k-1} + \sum_{j,r} A_j^{k,r}{z_j}^{r+k-1} \\ \qquad\qquad + \sum_j C_j^k {z_j}^{k-2} + \sum_{\substack{r+s=k-2\\ r,s\neq 0,\ i<j}} C_{i,j}^{k,r,s}{z_i}^r{z_j}^s = 0, \\ \quad k=3,\ldots,N,\ \sum_{j\in I}\Gamma_j\neq 0, \quad I\subset\{1,\ldots,N\},\ C_0\neq 0 \end{cases} \tag{31}$$

Case 1: $(z_1,\ldots,z_N)\in\mathbb{C}_*^N$. Lemma 1 will show that (31) has only finitely many solutions in $\mathbb{C}_*^N$. To this end, consider, by (20), the Newton polytopes

$$\begin{aligned} \mathcal{N}_1 &= \mathrm{Conv}(\{m\mathbf{e}_1,\ldots,m\mathbf{e}_N\}\cup\cdots) \\ \mathcal{N}_2 &= \mathrm{Conv}(\{(m+1)\mathbf{e}_1,\ldots,(m+1)\mathbf{e}_N,\mathbf{0}\}\cup\cdots) \\ \mathcal{N}_3 &= \mathrm{Conv}(\{(m+2)\mathbf{e}_1,\ldots,(m+2)\mathbf{e}_N\}\cup\cdots) \\ &\vdots \\ \mathcal{N}_N &= \mathrm{Conv}(\{(m+N-1)\mathbf{e}_1,\ldots,(m+N-1)\mathbf{e}_N\}\cup\cdots) \end{aligned}$$

of the supports $\mathcal{A}_1,\ldots,\mathcal{A}_N$ of $P_1,\ldots,P_N$, and their Minkowski sum of

$$\mathcal{N} = \mathcal{N}_1+\cdots+\mathcal{N}_N = \mathrm{Conv}\left(\{a_N\mathbf{e}_1,\ldots,a_N\mathbf{e}_N\}\cup\cdots\right),$$

where a_N is as in (25). Note that this '$\cdots$' does not alter the fact that

$$\mathcal{N}\subset a_N\Delta_N.$$

Now, consider the reduced systems of (31) determined by all the $\boldsymbol{\alpha}$ or $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ satisfying $(\mathbf{e}_1 + \cdots + \mathbf{e}_N) \cdot \boldsymbol{\alpha} \leq 0$. Because of (26) and such a choice of $\boldsymbol{\alpha}$, each $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ actually supports $\mathcal{N}$ at a face of the facet $\mathrm{Conv}(\{a_N\mathbf{e}_1, \ldots, a_N\mathbf{e}_N\}) = a_N\Delta_N$:

- Case I: $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ supports $\mathcal{N}$ at the 0-face

  $$a_N\mathbf{e}_j, \quad j = 1, \ldots, N,$$

  then $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}_1}, \ldots, \ell_{\boldsymbol{\alpha}}^{\mathcal{N}_N}$ support $\mathcal{N}_1, \ldots, \mathcal{N}_N$ at the 0-faces

  $$m\mathbf{e}_j, \ldots, (m+N-1)\mathbf{e}_j$$

  respectively, giving the reduced system

  $$\begin{cases} \Gamma_j {z_j}^m &= 0 \\ \Gamma_j {z_j}^{m+1} &= 0 \\ &\vdots \\ \Gamma_j {z_j}^{m+N-1} &= 0 \\ \quad \Gamma_j \neq 0 \end{cases}$$

  which has no solution in $\mathbb{C}_*^N$.

- Case II: $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ supports $\mathcal{N}$ at the $(k-1)$-face $(k = 2, \ldots, N)$

  $$\mathrm{Conv}(\{a_N\mathbf{e}_{j_1}, \ldots, a_N\mathbf{e}_{j_k}\}), \quad j_1 < \cdots < j_k,$$

  then $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}_1}, \ldots, \ell_{\boldsymbol{\alpha}}^{\mathcal{N}_N}$ support $\mathcal{N}_1, \ldots, \mathcal{N}_N$ at the $(k-1)$-faces

  $$\mathrm{Conv}(\{m\mathbf{e}_{j_1}, \ldots, m\mathbf{e}_{j_k}\}), \ldots, \mathrm{Conv}(\{(m+N-1)\mathbf{e}_{j_1}, \ldots, (m+N-1)\mathbf{e}_{j_k}\})$$

  respectively, giving the reduced system

  $$\begin{cases} \widetilde{P}_1(z_1, \ldots, z_N) &= \Gamma_{j_1} {z_{j_1}}^m + \cdots + \Gamma_{j_k} {z_{j_k}}^m &= 0 \\ \widetilde{P}_2(z_1, \ldots, z_N) &= \Gamma_{j_1} {z_{j_1}}^{m+1} + \cdots + \Gamma_{j_k} {z_{j_k}}^{m+1} &= 0 \\ &\vdots \\ \widetilde{P}_k(z_1, \ldots, z_N) &= \Gamma_{j_1} {z_{j_1}}^{m+k-1} + \cdots + \Gamma_{j_k} {z_{j_k}}^{m+k-1} &= 0 \\ &\vdots \\ \widetilde{P}_N(z_1, \ldots, z_N) &= \Gamma_{j_1} {z_{j_1}}^{m+N-1} + \cdots + \Gamma_{j_k} {z_{j_k}}^{m+N-1} &= 0, \\ \sum_{j \in I} \Gamma_j \neq 0, \quad I \subset \{j_1, \ldots, j_k\} \end{cases}$$

  where the sub-system consisting of the first k equations already has no solution in $\mathbb{C}_*^k$ by Lemma 10.

What we have established so far is that for every $\boldsymbol{\alpha}$ or $\ell_{\boldsymbol{\alpha}}^{\mathcal{N}}$ with $(\mathbf{e}_1+\cdots+\mathbf{e}_N)\cdot\boldsymbol{\alpha} \leq 0$, the corresponding reduced system has no solution in $\mathbb{C}_*^N$. Hence, it follows from Lemma 1 that (31) has only finitely many solutions in $\mathbb{C}_*^N$.

Case 2: $(z_1, \ldots, z_N) \in \mathbb{C}_*^k \times \{0\}^{N-k}$ $(k = 2, \ldots, N-1)$. The system (31) degrades to

$$\begin{cases} P_1(z_1, \ldots, z_k, 0, \cdots, 0) = \sum_{j=1}^{k} \Gamma_j {z_j}^m + \sum_{j=1}^{k} \sum_r A_j^{1,r} {z_j}^r & = 0 \\ P_2(z_1, \ldots, z_k, 0, \cdots, 0) = \sum_{j=1}^{k} \Gamma_j {z_j}^{m+1} + \sum_{j=1}^{k} \sum_r A_j^{2,r} {z_j}^{r+1} + C_0 & = 0 \\ P_k(z_1, \ldots, z_k, 0, \cdots, 0) = \sum_{j=1}^{k} \Gamma_j {z_j}^{m+k-1} + \sum_{j=1}^{k} \sum_r A_j^{k,r} {z_j}^{r+k-1} & \\ \qquad + \sum_{j=1}^{k} C_j^k {z_j}^{k-2} + \sum_{\substack{r+s=k-2 \\ r,s\neq 0,\ i<j}} C_{i,j}^{k,r,s} {z_i}^r {z_j}^s & = 0, \\ k = 3, \ldots, N,\ \sum_{j\in I} \Gamma_j \neq 0, \quad I \subset \{1, \ldots, k\},\ C_0 \neq 0 & \end{cases} \tag{32}$$

where the sub-system consisting of the first k equations already has only finitely many solutions in $\mathbb{C}^k$ by induction hypothesis, thus so does (32). Similar for the other cases where in $(z_1, \ldots, z_N)$ exactly $N-k$ coordinates equal 0.

Case 3: $(z_1, \ldots, z_N) \in \mathbb{C}_* \times \{0\}^{N-1}$. The system (31) degrades to

$$\begin{cases} P_1(z_1, 0, \cdots, 0) := \Gamma_1 {z_1}^m + \sum_r A_1^{1,r} {z_1}^r & = 0 \\ P_2(z_1, 0, \cdots, 0) := \Gamma_1 {z_1}^{m+1} + \sum_r A_1^{2,r} {z_1}^{r+1} + C_0 & = 0 \\ P_k(z_1, 0, \cdots, 0) := \Gamma_1 {z_1}^{m+k-1} + \sum_r A_1^{k,r} {z_1}^{r+k-1} + C_1^k {z_1}^{k-2} & = 0, \\ k = 3, \ldots, N,\ \Gamma_1 \neq 0 & \end{cases} \tag{33}$$

where the first equation already has only finitely many (at most m) solutions, thus so does (33). Similar for the other cases where in $(z_1, \ldots, z_N)$ exactly $N-1$ coordinates equal 0.

Case 4: $(z_1, \ldots, z_N) \in \{0\}^N$. The system (31) degrades to

$$\begin{cases} P_1(0, 0, \cdots, 0) := \sum_j A_j^{1,0} & = 0 \\ P_2(0, 0, \cdots, 0) := C_0 & = 0 \\ P_k(0, 0, \cdots, 0) := 0 & = 0, \\ k = 3, \ldots, N,\ C_0 \neq 0 & \end{cases}$$

is simply inconsistent.

Combining all the above four cases, (31) has only finitely many solutions in $\mathbb{C}^N$, and Proposition 8 follows from strong induction.

5 Background Flow of Degree One

By a background flow *of degree one*, we mean, without any loss of generality, $w(\zeta) = -\frac{\zeta+c}{2\pi\mathrm{i}}$ for some constant $c \in \mathbb{C}$ in (3) or $m = 1$ and $W \equiv c$ in (4) or (8) (the case with $c = 0$ is commonly called a *quadrupole background flow*). For whichever two distinct solutions $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{z}' = (z'_1, \ldots, z'_n)$ of (3) with such w to be equivalent, by (11), one looks for $(a, b) \in \mathbb{C}_* \times \mathbb{C}$ such that

$$aV_{\mathbf{z}'}(a\zeta + b) = V_{\mathbf{z}}(\zeta) \tag{34}$$

$$\sum_j \frac{\Gamma_j}{\zeta - \frac{z'_j - b}{a}} - a^2\zeta - ab - ac = \sum_j \frac{\Gamma_j}{\zeta - z_j} - \zeta - c.$$

The equality between the two analytic parts already forces

$$(a, b) = (1, 0) \quad \text{or} \quad (-1, -2c). \tag{35}$$

In Section 5.1, through the simplest example of two vortices, we will illustrate that the bounds in Proposition 3 and Theorem 7 may or may not be attained. In this example, in addition, all the fixed equilibrium configurations come from the reduction, via Definition 4, of the solutions of (3), (4) or (8) by symmetries of the vortex sets and/or the given background flow only. Section 5.2 will provide an example that a repeated solution of (3), (4) or (8) exists, thus suggesting another factor in the reduction of the solutions to fixed equilibrium configurations via Definition 4. All these results will be summarized in Tables 1 and 2.

5.1 Two Vortices

For two vortices (i.e. $n = 2$), (8) reads

$$\begin{cases} P_1(z_1, z_2) = \Gamma_1 z_1 + \Gamma_2 z_2 + c(\Gamma_1 + \Gamma_2) = 0 \\ P_2(z_1, z_2) = \Gamma_1 {z_1}^2 + \Gamma_2 {z_2}^2 + c\Gamma_1 z_1 + c\Gamma_2 z_2 - \Gamma_1\Gamma_2 = 0 \\ \Gamma_1,\ \Gamma_2 \neq 0 \end{cases} \tag{36}$$

Computing by (the improved) Buchberger's algorithm ([2, Algorithm GRÖBNER-NEW2, p.232, & Subalgorithm UPDATE, p.230]) with respect to the variable ordering $z_1 > z_2$, we obtain a Gröbner basis

$$\mathcal{G} = \{P_1, Q_3\}, \quad \text{where} \quad Q_3 := -\tfrac{\Gamma_1+\Gamma_2}{\Gamma_1}{z_2}^2 - \tfrac{2c(\Gamma_1+\Gamma_2)}{\Gamma_1} z_2 + \Gamma_1 - \tfrac{c^2(\Gamma_1+\Gamma_2)}{\Gamma_1} \not\equiv 0,$$

of $\mathcal{F} = \{P_1, P_2\}$. Thus, (36) is equivalent to $P_1 = Q_3 = 0$, i.e.

$$\begin{cases} \Gamma_1 z_1 + \Gamma_2 z_2 + c(\Gamma_1 + \Gamma_2) = 0 \\ -\tfrac{\Gamma_1+\Gamma_2}{\Gamma_1}{z_2}^2 - \tfrac{2c(\Gamma_1+\Gamma_2)}{\Gamma_1} z_2 + \Gamma_1 - \tfrac{c^2(\Gamma_1+\Gamma_2)}{\Gamma_1} = 0 \\ \Gamma_1,\ \Gamma_2 \neq 0 \end{cases} \tag{37}$$

Case 1: $\Gamma_1 + \Gamma_2 = 0$, then $\Gamma_1 \neq \Gamma_2$ because $\Gamma_1, \Gamma_2 \neq 0$, and then the second equation in (37) reads $\Gamma_1 = 0$ which already has no solution. But this case is beyond the scope of Proposition 3 and Theorem 7.

Case 2: $\Gamma_1 + \Gamma_2 \neq 0$, then the two solutions are

$$(z_1, z_2) = \left(-c \mp \tfrac{\Gamma_2}{\sqrt{(\Gamma_1+\Gamma_2)}}, -c \pm \tfrac{\Gamma_1}{\sqrt{(\Gamma_1+\Gamma_2)}}\right) =: \mathbf{z}^{\pm},$$

and the bound $\frac{(1+2-1)!}{(1-1)!} = 2$ in Proposition 3 is attained. Moreover, $\mathbf{z}^{\pm}$ are admissible (see Remark (iv) following Definition 4) since $\Gamma_1 + \Gamma_2 \neq 0$, and are distinct since $\Gamma_1, \Gamma_2 \neq 0$.

- Case I: $\Gamma_1 = \Gamma_2$ (one species), then

$$\begin{aligned} 2\pi \mathrm{i} V_{\mathbf{z}^-}(\zeta) &= \frac{\Gamma_1}{\zeta + c - \sqrt{\frac{\Gamma_1}{2}}} + \frac{\Gamma_1}{\zeta + c + \sqrt{\frac{\Gamma_1}{2}}} - \zeta - c \\ &= \frac{\Gamma_1}{\zeta + c + \sqrt{\frac{\Gamma_1}{2}}} + \frac{\Gamma_1}{\zeta + c - \sqrt{\frac{\Gamma_1}{2}}} - \zeta - c = 2\pi \mathrm{i} V_{\mathbf{z}^+}(\zeta) \end{aligned}$$

so that (34) with $(a, b) = (1, 0)$ in (35) is satisfied, and $\mathbf{z}^{\pm}$ constitute only one fixed equilibrium configuration. In this case, the bound $\frac{(1+2-1)!}{(1-1)!2!} = 1$ in Theorem 7 is attained.

- Case II: $\Gamma_1 \neq \Gamma_2$ (two species), then

$$\begin{aligned} 2\pi \mathrm{i} V_{\mathbf{z}^-}(\zeta) &= \frac{\Gamma_1}{\zeta + c - \frac{\Gamma_2}{\sqrt{(\Gamma_1+\Gamma_2)}}} + \frac{\Gamma_2}{\zeta + c + \frac{\Gamma_1}{\sqrt{(\Gamma_1+\Gamma_2)}}} - \zeta - c \\ &\neq \frac{\Gamma_1}{\zeta + c + \frac{\Gamma_2}{\sqrt{(\Gamma_1+\Gamma_2)}}} + \frac{\Gamma_2}{\zeta + c - \frac{\Gamma_1}{\sqrt{(\Gamma_1+\Gamma_2)}}} - \zeta - c = 2\pi \mathrm{i} V_{\mathbf{z}^+}(\zeta) \end{aligned}$$

so that (34) with $(a, b) = (1, 0)$ in (35) is not satisfied, but

$$\begin{aligned} -2\pi \mathrm{i} V_{\mathbf{z}^-}(-\zeta - 2c) &= -\frac{\Gamma_1}{-\zeta - 2c + c - \frac{\Gamma_2}{\sqrt{(\Gamma_1+\Gamma_2)}}} - \frac{\Gamma_2}{-\zeta - 2c + c + \frac{\Gamma_1}{\sqrt{(\Gamma_1+\Gamma_2)}}} \\ &\qquad + (-\zeta - 2c) + c \\ &= \frac{\Gamma_1}{\zeta + c + \frac{\Gamma_2}{\sqrt{(\Gamma_1+\Gamma_2)}}} + \frac{\Gamma_2}{\zeta + c - \frac{\Gamma_1}{\sqrt{(\Gamma_1+\Gamma_2)}}} - \zeta - c = 2\pi \mathrm{i} V_{\mathbf{z}^+}(\zeta) \end{aligned}$$

so that (34) with $(a, b) = (-1, -2c)$ in (35) is satisfied, hence $\mathbf{z}^{\pm}$ still constitute only one fixed equilibrium configuration. In this case, the bound $\frac{(1+2-1)!}{(1-1)!} = 2$ in Theorem 7 is not attained.

5.2 Repeated Solution

There is a case of three vortices in a quadrupole background flow (i.e. $n = 3$ and $w(\zeta) = -\frac{\zeta}{2\pi \mathrm{i}}$ in (3) or $m = 1$ and $W \equiv 0$ in (4) or (8)) where (3), (4) or (8) has

	$\Gamma_1+\Gamma_2=0$	$\Gamma_1+\Gamma_2\neq 0$
$\Gamma_1=\Gamma_2$ (1 species)	—	1 / 2 / 2
$\Gamma_1\neq\Gamma_2$ (2 species)	0 / 0 / 0	1 / 2 / 2

Table 1: This table summarizes Section 5.1, i.e. the case $n=2$, $m=1$ and $W\equiv$ constant in (8). 'a / b / c' means that 'the system (8) has c solutions (counting multiplicity), b of which are distinct and admissible, and that these solutions constitute a fixed equilibrium configurations (in the sense of Definition 4)'. '—' means non-existence of the case.

repeated solution. In such a case, the bound in Theorem 7 must not be attained. Consider (8) with the extra assumption that $\Gamma_1=\Gamma_2$ (at most two species):

$$\begin{cases} P_1(z_1,z_2,z_3)=\Gamma_1 z_1+\Gamma_1 z_2+\Gamma_3 z_3 & =0\\ P_2(z_1,z_2,z_3)=\Gamma_1 {z_1}^2+\Gamma_1 {z_2}^2+\Gamma_3 {z_3}^2-{\Gamma_1}^2-2\,\Gamma_1\Gamma_3 & =0\\ P_3(z_1,z_2,z_3)=\Gamma_1 {z_1}^3+\Gamma_1 {z_2}^3+\Gamma_3 {z_3}^3-\Gamma_1(\Gamma_1+\Gamma_3)z_1 & \\ \qquad\qquad -\Gamma_1(\Gamma_1+\Gamma_3)z_2-2\,\Gamma_1\Gamma_3 z_3 & =0\\ \Gamma_1,\ \Gamma_3\neq 0 & \end{cases} \tag{38}$$

Computing by the aforesaid Buchberger's algorithm with respect to the variable ordering $z_1>z_2>z_3$, we obtain a Gröbner basis

$$\mathcal{G}=\{P_1,Q_4,Q_5\}\,,$$

where

$$Q_4:=-2\,\Gamma_1{z_2}^2-2\,\Gamma_3 z_2 z_3-\tfrac{\Gamma_3(\Gamma_1+\Gamma_3)}{\Gamma_1}{z_3}^2+\Gamma_1(\Gamma_1+2\,\Gamma_3)\not\equiv 0\quad\text{and}$$
$$Q_5:=-\tfrac{\Gamma_3(\Gamma_1+\Gamma_3)(2\,\Gamma_1+\Gamma_3)}{2\,{\Gamma_1}^2}{z_3}^3+\tfrac{\Gamma_3(5\,\Gamma_1+4\,\Gamma_3)}{2}z_3\not\equiv 0,$$

of $\mathcal{F}=\{P_1,P_2,P_3\}$. Thus, (38) is equivalent to $P_1=Q_4=Q_5=0$, i.e.

$$\begin{cases} \Gamma_1 z_1+\Gamma_1 z_2+\Gamma_3 z_3=0\\ -2\,\Gamma_1{z_2}^2-2\,\Gamma_3 z_2 z_3-\tfrac{\Gamma_3(\Gamma_1+\Gamma_3)}{\Gamma_1}{z_3}^2+\Gamma_1(\Gamma_1+2\,\Gamma_3)=0\\ -\tfrac{\Gamma_3(\Gamma_1+\Gamma_3)(2\,\Gamma_1+\Gamma_3)}{2\,{\Gamma_1}^2}{z_3}^3+\tfrac{\Gamma_3(5\,\Gamma_1+4\,\Gamma_3)}{2}z_3=0\\ \Gamma_1,\ \Gamma_3\neq 0 \end{cases} \tag{39}$$

Case 1: $(\Gamma_1+\Gamma_3)(2\,\Gamma_1+\Gamma_3)=0$, then $\Gamma_1\neq\Gamma_3$ because $\Gamma_1,\Gamma_3\neq 0$, so that there are two species. And then writing $\Gamma_3=-\alpha\Gamma_1$, where $\alpha=1$ or 2, (39) reads

$$\begin{cases} \Gamma_1 z_1+\Gamma_1 z_2-\alpha\Gamma_1 z_3=0\\ -2\,\Gamma_1{z_2}^2+2\alpha\Gamma_1 z_2 z_3+\alpha(1-\alpha)\Gamma_1{z_3}^2+(1-2\alpha){\Gamma_1}^2=0\\ -\tfrac{\alpha(5-4\alpha){\Gamma_1}^2}{2}z_3=0\\ \Gamma_1,\ \Gamma_3\neq 0 \end{cases}$$

The two solutions

$$(z_1,z_2,z_3)=\left(\mp\sqrt{\tfrac{(1-2\alpha)\Gamma_1}{2}},\pm\sqrt{\tfrac{(1-2\alpha)\Gamma_1}{2}},0\right)=:\mathbf{z}^{\pm}$$

are admissible and distinct, both because of $\Gamma_1 \neq 0$ and $\alpha \neq \frac{1}{2}$. Moreover,

$$\begin{aligned} 2\pi \mathrm{i} V_{\mathbf{z}^-}(\zeta) &= \frac{\Gamma_1}{\zeta - \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} + \frac{\Gamma_1}{\zeta + \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} + \frac{-\alpha\Gamma_1}{\zeta} - \zeta \\ &= \frac{\Gamma_1}{\zeta + \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} + \frac{\Gamma_1}{\zeta - \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} + \frac{-\alpha\Gamma_1}{\zeta} - \zeta = 2\pi \mathrm{i} V_{\mathbf{z}^+}(\zeta) \end{aligned}$$

so that (34) with $(a, b) = (1, 0)$ in (35) is satisfied, and they constitute only one fixed equilibrium configuration. But this case is beyond the scope of Proposition 3 and Theorem 7.

Case 2: $(\Gamma_1 + \Gamma_3)(2\,\Gamma_1 + \Gamma_3) \neq 0$, then, first of all, the third equation of (39) has three solutions

$$z_3 = 0, \quad \pm\Gamma_1\sqrt{\tfrac{5\,\Gamma_1 + 4\,\Gamma_3}{(\Gamma_1+\Gamma_3)(2\,\Gamma_1+\Gamma_3)}}.$$

Each leads to two z_2 via the second equation in (39) and then one z_1 via the first, resulting in six solutions of (39), thus the bound $\frac{(1+3-1)!}{(1-1)!} = 6$ in Proposition 3 is attained. Now, one could observe that if

- $5\,\Gamma_1 + 4\,\Gamma_3 = 0$, then $\Gamma_1 \neq \Gamma_3$ because $\Gamma_1, \Gamma_3 \neq 0$, so that there are two species. Now, $z_3 = 0$ is actually a triple zero, and the six solutions of (39) are $(z_1, z_2, z_3) =$

  $$\mathbf{z}^\pm := \left(\mp\sqrt{\tfrac{\Gamma_1+2\,\Gamma_3}{2}}, \pm\sqrt{\tfrac{\Gamma_1+2\,\Gamma_3}{2}}, 0\right) \quad \text{(each repeated thrice)}.$$

  Note that $5\,\Gamma_1 + 4\,\Gamma_3 = 0 \xRightarrow{\Gamma_1,\Gamma_3 \neq 0} \Gamma_1 + 2\,\Gamma_3 \neq 0$, so $\mathbf{z}^\pm$ are admissible and distinct. Moreover,

  $$\begin{aligned} 2\pi \mathrm{i} V_{\mathbf{z}^-}(\zeta) &= \frac{\Gamma_1}{\zeta - \sqrt{\frac{\Gamma_1+2\,\Gamma_3}{2}}} + \frac{\Gamma_1}{\zeta + \sqrt{\frac{\Gamma_1+2\,\Gamma_3}{2}}} + \frac{-\frac{5}{4}\Gamma_1}{\zeta} - \zeta \\ &= \frac{\Gamma_1}{\zeta + \sqrt{\frac{\Gamma_1+2\,\Gamma_3}{2}}} + \frac{\Gamma_1}{\zeta - \sqrt{\frac{\Gamma_1+2\,\Gamma_3}{2}}} + \frac{-\frac{5}{4}\Gamma_1}{\zeta} - \zeta = 2\pi \mathrm{i} V_{\mathbf{z}^+}(\zeta) \end{aligned}$$

  so that (34) with $(a, b) = (1, 0)$ in (35) is satisfied, and they constitute only one fixed equilibrium configuration. Thus, the bound $\frac{(1+3-1)!}{(1-1)!2!} = 3$ in Theorem 7 is not attained.

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$\Gamma_1 = \Gamma_2$	$(\Gamma_1 + \Gamma_3)(2\,\Gamma_1 + \Gamma_3) = 0$	$5\,\Gamma_1 + 4\,\Gamma_3 = 0$ (necessarily, $(\Gamma_1 + \Gamma_3)(2\,\Gamma_1 + \Gamma_3) \neq 0$)
$\Gamma_1 = \Gamma_3$ (1 species)	—	—
$\Gamma_1 \neq \Gamma_3$ (2 species)	1 / 2 / 2	1 / 2 / 6

Table 2: This table summarizes Section 5.2, i.e. the case $n = 3$, $m = 1$ and $W \equiv 0$ with $\Gamma_1 = \Gamma_2$ in (8). 'a / b / c' means that 'the system (8) has c solutions (counting multiplicity), b of which are distinct and admissible, and that these solutions constitute a fixed equilibrium configurations (in the sense of Definition 4)'. '—' means non-existence of the case.

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