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Technical Notes and Correspondence

On Stabilizability of Nonlinearly Parameterized Discrete-Time Systems

Chanying Li and Michael Z. Q. Chen

Abstract—Most existing works on adaptive control of discrete-time systems focus on the case of linear parametrization. For nonlinearly parameterized systems, the stabilizability turns out to be an intractable issue. This technical note is devoted to seeking the essential factors that determine the stabilizability of nonlinearly parameterized discrete-time systems. A sufficient condition imposed on the structures of the system functions is established. Analysis shows that the sensitivity function of unknown parameters plays a crucial role in characterizing the uncertainties of parameterized systems. One of the implications of this result is that arbitrarily growing nonlinearities in the uncertain model may be allowed for global stabilization.

Index Terms—Adaptive control, discrete-time systems, nonlinear parametrization, sensitivity function, stabilizability.

I. INTRODUCTION

Adaptive control has achieved significant progress over the past several decades and grown to be one of the richest and most lively research areas in control theory. Despite the relative maturity in theory and successful applications in industry, adaptive control has been mainly studied for linear systems ([13], [17]) or nonlinear systems with linearly parameterized uncertainties ([16]). This is because most of the feasible adaptive control laws proposed in the literature require unknown parameters to enter the systems linearly. However, it is inevitable in practice for engineers to deal with nonlinear parameterizations.

Recently, some interesting results have been established for studying nonlinearly parameterized continuous-time systems using adaptive control. Different methodologies have been proposed and developed in this challenging area. As for discrete-time systems with nonlinearly parameterized uncertainties, however, very limited research has been reported in the literature. There are at least two problems in constructing adaptive controllers for nonlinearly parameterized discrete-time systems. One is that the traditional prediction-error-minimization-based estimation (e.g., least squares and gradient methods) combined with the certainty equivalent principle encounters some essential difficulties, both analytically and numerically. The other is that the high-

gain and nonlinear damping methods, which are so powerful for the continuous-time case, are no longer effective in the discrete-time case ([7]–[9]). Indeed, there are only a few results addressing the adaptive control problems for some special classes of nonlinearly parameterized discrete-time systems (see [6], [11], [14]).

This technical note is primarily concerned with the following question: What are the essential factors determining the global stabilizability of nonlinearly parameterized uncertain discrete-time systems? This technical note investigates a basic class of discrete-time systems with unknown, nonlinearly parameterized parameters and a series of constraints are provided on the structures of the system functions to ensure global stabilizability. In particular, it turns out that the linear growth rate of sensitivity function with respect to the unknown parameters plays a crucial role above all other structural constraints. The concept of the sensitivity function on parameters was brought forward to explore the feedback capability. It has a simple form and characterizes the uncertainty of parameterized systems, which is very important in the stabilizability study. In comparison with the existing results on the maximum capability and limitations of the feedback mechanism ([4], [10], [12], [19]), an interesting implication of the presented result is that an arbitrarily fast nonlinear growth rate of the parameterized uncertain system may be allowed for global adaptive stabilization. Another contribution of this technical note is that a novel methodology is proposed to circumvent the aforementioned difficulties for nonlinear parameterizations in the discrete-time case. This methodology is based on the idea of switching control, which has been broadly applied to many areas ([1], [2], [5], [20]). In a similar approach, Angeli and Mosca [2] proposed a Lyapunov-based falsification criterion and successfully stabilized a certain kind of nonlinearly parameterized uncertain systems. In addition, a convenient framework to deal with switching control of uncertain systems has been reported recently ([3], [15], [18]). By using a suitable cost function to orchestrate switching, these works indicate that the stability can be guaranteed, provided that at least one stabilizing controller exists in the family of candidate controllers.

II. MAIN RESULT

Consider the following nonlinearly parameterized discrete-time system:

$$y_{t+1} = f(\theta, \phi_t, u_t, w_{t+1}) \quad (1)$$

where $\theta \in \mathbb{R}^p$ is an unknown parameter vector, $\phi_t = (y_t, y_{t-1}, \dots, y_{t-q+1})^T \in \mathbb{R}^q$ with $y_t = 0$ for any $t < 0$, u_t and w_t are the system regression vector, the input, and the noise signals, respectively. Moreover, $f(\theta, \phi, u, w) : \mathbb{R}^{p+q+2} \rightarrow \mathbb{R}$ is a known continuous nonlinear function with partial derivatives $f_u(\theta, \phi, u, 0) \triangleq (\partial f(\theta, \phi, u, 0)/\partial u)$, $f_\theta(\theta, \phi, u, 0) \triangleq (\partial f(\theta, \phi, u, 0)/\partial \theta)$ and $f_w(\theta, \phi, u, w) \triangleq (\partial f(\theta, \phi, u, w)/\partial w)$ existing and continuous for

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any given $\phi \in \mathbb{R}^q$ with $\|\phi\| \geq M$, where $M \geq 0$ is a constant. Assume that the uncertain parameter vector and the noise signals satisfy the following assumptions:

- (A1) The unknown parameter vector $\theta = (\theta_1, \theta_2, \dots, \theta_p)^T$ belongs to a rectangle in \mathbb{R}^p : $\Theta_0 = \{\theta : |\theta_i| \leq R, 1 \leq i \leq p\} \subset \mathbb{R}^p$, where $R > 0$ is a constant.
 (A2) The noise sequence is bounded in the average sense

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t w_i^2 < \infty. \quad (2)$$

To stabilize the nonlinearly parameterized system (1) with both unknown parameters and noise disturbances under Assumptions (A1) and (A2), one needs to introduce a sensitivity function of the system with respect to the unknown parameters. First, note that for any $w \in \mathbb{R}$, there exists a w^* between 0 and w such that

$$f(\theta, \phi, u, w) = f(\theta, \phi, u, 0) + f_w(\theta, \phi, u, w^*)w \quad (3)$$

and hence system (1) can be rewritten as

$$y_{t+1} = f(\theta, \phi_t, u_t, 0) + f_w(\theta, \phi_t, u_t, w_{t+1}^*) w_{t+1}. \quad (4)$$

For any $\theta \in \mathbb{R}^p$ and $\phi \in \mathbb{R}^q$, define

$$a(\theta, \phi) \triangleq \inf_{u \in \mathbb{R}} |f(\theta, \phi, u, 0)| \quad (5)$$

which in some sense describes a bound on the disturbance-free system function $f(\theta, \phi, u, 0)$ with the known θ , when a stabilizing feedback controller is applied. As a matter of fact, given $\theta \in \mathbb{R}^p$ and $\phi \in \mathbb{R}^q$, by the definition of infimum, there always exists a function $g(\theta, \phi) \in \mathbb{R}$ such that

$$|f(\theta, \phi, g(\theta, \phi), 0)| \leq a(\theta, \phi) + 1. \quad (6)$$

Assume $g(\theta, \phi) : \Theta_0 \rightarrow \mathbb{R}$ satisfying (6) is a differentiable function with respect to the variable θ for any $\phi \in \mathbb{R}^q$ with $\|\phi\| \geq M$ (this assumption can be guaranteed under some mild constraints). Denote the derivative of $g(\theta, \phi)$ by $g_\theta(\theta, \phi)$, which is a row vector. For $g(\theta, \phi)$, the sensitivity function of the unknown parameters for system (1) is defined as

$$S(\theta, \theta', \phi, u) \triangleq -g_\theta^T(\theta, \phi) f_u(\theta', \phi, u, 0) \quad (7)$$

where parameters $\theta, \theta' \in \Theta_0$. The above formula in fact characterizes the sensitivity of the system function with respect to the parameter vector θ . The assumptions imposed on the structure of the system function are stated below:

- (B1) There is a constant $C_1 > 0$ such that for all $\phi \in \mathbb{R}^q$ with $\|\phi\| \geq M$

$$\sup_{(\theta, \theta', u) \in \Theta_0^2 \times \mathbb{R}} \|S(\theta, \theta', \phi, u)\| \leq C_1 \|\phi\|. \quad (8)$$

- (B2) There is a constant $0 < C_2 < \sqrt{1/q}$ such that for all $\phi \in \mathbb{R}^q$ with $\|\phi\| \geq M$

$$\sup_{\theta \in \Theta_0} a(\theta, \phi) \leq C_2 \|\phi\|.$$

- (B3) The function $f_w(\theta, \phi, u, w)$ is uniformly bounded

$$\sup_{(\theta, \phi, u, w) \in \Theta_0 \times \mathbb{R}^{q+2}} |f_w(\theta, \phi, u, w)| < \infty.$$

The standard definitions of the feedback law and stabilizability are presented before the main results.

Definition 2.1: A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t , $\{y_i, i \leq t\}$, that is

$$u_t = h_t(y_0, \dots, y_t)$$

where $h_t(\cdot) : \mathbb{R}^{t+1} \rightarrow \mathbb{R}^1$ can be any (nonlinear) mapping.

Definition 2.2: System (1) under Assumptions (A1) and (A2) is said to be globally stabilizable by feedback, if there exists a feedback control law $\{u_t\}$ such that for any $\phi_{q-1} \in \mathbb{R}^q$, any θ and $\{w_t\}$ satisfying (A1) and (A2), the averaged outputs of the closed-loop system are bounded

$$\limsup_{t \rightarrow \infty} \frac{1}{t+1} \sum_{i=0}^t y_i^2 < \infty. \quad (9)$$

Now, the main theorem of this technical note is stated as follows.

Theorem 2.1: Under Assumptions (A1) and (A2), the nonlinearly parameterized system (1) with Assumptions (B1)–(B3) is globally stabilizable.

Remark 2.1: By Assumptions (A2) and (B3), there exists some constant $C > 0$ such that for any $\theta \in \Theta_0$, $\{\phi_i\}$, $\{u_i\}$ and $\{w_i\}$

$$\sup_{t \geq 1} \sup_{w_i^* \in \mathbb{R}, 1 \leq i \leq t} \frac{1}{t} \sum_{i=1}^t f_w^2(\theta, \phi_{i-1}, u_{i-1}, w_i^*) w_i^2 \leq C. \quad (10)$$

The constant C is supposed to be known and available in the process of the control design.

Remark 2.2: In most applications of interest, for any θ and ϕ , the variable u can be uniquely solved from the equation $f(\theta, \phi, u, 0) = 0$ and hence $a(\theta, \phi) \equiv 0$. Assumption (B2) naturally holds. In this case, there exists a differentiable $g(\theta, \phi)$ with $f(\theta, \phi, g(\theta, \phi), 0) = 0$. Accordingly

$$S(\theta, \theta', \phi, u) = \frac{f_\theta^T(\theta, \phi, u, 0)}{f_u(\theta, \phi, u, 0)} \Big|_{u=g(\theta, \phi)} f_u(\theta', \phi, u, 0). \quad (11)$$

An interesting phenomenon indicated by Theorem 2.1 is illustrated by the following example.

Example 2.1: It is instructive to look at the following system with an unknown scalar parameter θ satisfying (A1) and some exponent $b > 1$:

$$y_{t+1} = \begin{cases} \exp\left\{\frac{\theta}{y_t^{b-1}}\right\} y_t^b + u_t + w_{t+1}, & |y_t| \geq 1, \\ \exp(\theta) y_t + u_t + w_{t+1}, & |y_t| < 1 \end{cases} \quad (12)$$

where the noise signals $\{w_t\}$ are bounded. Note that the order of $f(\theta, \phi, u, 0)$ is the same as ϕ^b when $\phi \rightarrow \infty$. Hence, the growth rate can be arbitrarily large by appropriately choosing the exponent b . However, system (12) is still globally stabilizable by Corollary 2.1 since the sensitivity function $S(\theta, \theta', \phi, u) = O(\phi)$ as $\phi \rightarrow \infty$. This example implies that arbitrarily growing nonlinearities in the uncertain model may be allowed for global stabilization.

Now, consider the following uncertain system:

$$y_{t+1} = f(\theta, \phi_t) + u_t + w_{t+1} \quad (13)$$

where the unknown parameter vector θ and the noise sequence $\{w_t\}$ satisfy Assumptions (A1) and (A2), respectively. It is easy to check that $a(\theta, \phi) \equiv 0$ in this case and by (11) in Remark 2.2, one has

$$S(\theta, \theta', \phi, u) = \left(\frac{\partial f(\theta, \phi)}{\partial \theta} \right)^T.$$

Therefore, from Theorem 2.1, one obtains the following corollary directly.

Corollary 2.1 ([11]): Consider system (13) under Assumptions (A1) and (A2). If there is a constant $C'_1 > 0$ such that

$$\sup_{\theta \in \Theta_0} \left\| \frac{\partial f(\theta, \phi)}{\partial \theta} \right\| \leq C'_1 \|\phi\|, \quad \forall \|\phi\| \geq M$$

then system (13) is stabilizable.

III. DESIGN OF SWITCHING CONTROLLER

If the parameter θ can be estimated online sufficiently close to the true value, then one can design a controller $\{u_t\}$ based on the estimation to stabilize system (1). Now, the controller design is presented in detail.

First of all, one splits the rectangle Θ_0 into a series of small rectangles. Let

$$K > \frac{\sqrt{q(1+\epsilon)}C_1}{1 - \sqrt{q(1+\epsilon)}C_2} \quad (14)$$

be some positive constant, and ϵ be a constant with

$$\epsilon \in \left(0, \frac{1}{C_2^2 q} - 1\right). \quad (15)$$

Here, C_1 and C_2 are defined by Assumptions (B1) and (B2). Denote

$$n \triangleq \lceil 2RK\sqrt{p} \rceil^p \quad (16)$$

where $\lceil x \rceil$ is the smallest integer larger than or equal to x and R is defined by Assumption (A1).

Lemma 3.1: Let Θ_0 be a rectangle in \mathbb{R}^p defined in Assumption (A1), then it can be uniformly split into n rectangles of volume no larger than $1/(K\sqrt{p})^p$, in which the distance between any two points is less than or equal to $1/K$.

Proof: The first conclusion is trivial by (16) since the volume of Θ_0 is $(2R)^p$ and one can have n small equilateral rectangles of side length $2R/\lceil 2RK\sqrt{p} \rceil$. To prove the second assertion, it suffices to show that the diameter of any small rectangle is no more than $1/K$, which is obviously true by the definition of Euclidean distance. ■

Lemma 3.1 is actually a tessellation of rectangle Θ_0 . One can always label these small rectangles as $S_i^n, i = 1, 2, \dots, n$. The estimation idea is as follows: Randomly pick n points $\{\hat{\theta}_i, i = 1, 2, \dots, n\}$ from the n small rectangles $\{S_i^n, i = 1, 2, \dots, n\}$, respectively. This set of points will then be used to estimate the parameter θ online. The estimation error is denoted by

$$\tilde{\theta}_i = \theta - \hat{\theta}_i. \quad (17)$$

To be shown in Section IV, the set of chosen points $\{\hat{\theta}_i\}$ will be indeed helpful in stabilizing system (1), even though the estimation error may not converge to zero. This is attributed to the fine resolution of the tessellation for set Θ_0 .

Now, one proceeds to design the adaptive controller. The controller is designed based on a switching rule, which is characterized by the time function $L_{\phi_s}(t)$ defined below. For $t \geq s \geq q-1$, let

$$L_{\phi_s}(t) \triangleq \max \left\{ \|\phi_s\|^2, C_f \right\} + \frac{2M_\epsilon q(C+1)t(\lambda + (1-\lambda)^2)}{(t-s+1)(1-\lambda)^2} \quad (18)$$

where $\lambda \triangleq (1+\epsilon)q(C_2 + (C_1/K))^2$, C is defined by (10)

$$M_\epsilon \triangleq 2 + \frac{2}{\epsilon} \quad (19)$$

and

$$C_f \triangleq \begin{cases} \sup_{\theta \in \Theta_0, \|\phi\| < M} f^2(\theta, \phi, 0, 0), & M > 0, \\ 0, & M = 0. \end{cases} \quad (20)$$

Note that (14) implies

$$0 < \lambda < 1. \quad (21)$$

Let us take $t_0 = q-1$, and recursively define the switching time $\{t_i\}$ as follows:

$$t_i \triangleq \inf \left\{ t > t_{i-1} : \frac{1}{t-t_{i-1}+1} \sum_{s=t_{i-1}}^t y_s^2 > L_{\phi_{t_{i-1}}}(t) \right\}, 1 \leq i \leq n \quad (22)$$

where by definition $\inf\{\emptyset\} = \infty$ and n is the number of the refined rectangles.

Finally, the switching controller $\{u_t\}$ is defined by

$$u_t = \begin{cases} 0, & 0 \leq t < t_0 \text{ or } t \geq t_n \text{ or } \|\phi_t\| < M, \\ g(\hat{\theta}_1, \phi_t), & t_0 \leq t < t_1 \text{ and } \|\phi_t\| \geq M, \\ g(\hat{\theta}_2, \phi_t), & t_1 \leq t < t_2 \text{ and } \|\phi_t\| \geq M, \\ \dots \\ g(\hat{\theta}_n, \phi_t), & t_{n-1} \leq t < t_n \text{ and } \|\phi_t\| \geq M \end{cases} \quad (23)$$

which can serve as the stabilizing adaptive controller in Theorem 2.1, as shown in next section.

IV. PROOF OF THEOREM 2.1

The proof is divided into several lemmas.

Lemma 4.1: If there exists some $i \in [0, n)$ such that $t_i = \infty$, then $t_j = \infty$ for any $j \in (i, n]$.

Proof: The lemma is trivial according to the definition of $\{t_i\}$ in (22). ■

Lemma 4.2: Let $t_{i-1} < \infty$ for some fixed $i \geq 1$. If for some time $t > t_{i-1}$, the following inequality:

$$|y_{s+1}| \leq \begin{cases} (C_2 + \frac{C_1}{K})\|\phi_s\| + |f_w(\theta, \phi_s, u_s, w_{s+1}^*)w_{s+1}| + 1, & \|\phi_s\| \geq M \\ \sqrt{\frac{C_f}{2}} + |f_w(\theta, \phi_s, u_s, w_{s+1}^*)w_{s+1}|, & \|\phi_s\| < M \end{cases} \quad (24)$$

holds for all $s \in [t_{i-1}, t)$, then

$$\frac{1}{t-t_{i-1}+1} \sum_{s=t_{i-1}}^t y_s^2 \leq L_{\phi_{t_{i-1}}}(t).$$

Proof: See Appendix A. ■

Lemma 4.3: If $t_i < \infty$, $1 \leq i \leq n$, then $\theta \notin S_i^n$.

Proof: Since $t_i < \infty$, $1 \leq i \leq n$, by Lemma 4.1, one has $t_{i-1} < \infty$. Therefore, for any $s \in [t_{i-1}, t_i)$, by definition (23), the controller is

$$u_s = \begin{cases} g(\hat{\theta}_i, \phi_s), & \|\phi_s\| \geq M, \\ 0, & \|\phi_s\| < M. \end{cases} \quad (25)$$

Now, suppose $\theta \in S_i^n$. Then, by Lemma 3.1, one has

$$\|\tilde{\theta}_i\| \leq \frac{1}{K}.$$

Thus, for any $s \in [t_{i-1}, t_i)$ with $\|\phi_s\| \geq M$, by Assumptions (B1)–(B2), (6) and (25), there exist a $\theta_i^* \in S_i^n$ and a u_i^* between

$g(\theta, \phi_s)$ and $g(\hat{\theta}_i, \phi_s)$ such that

$$\begin{aligned} & |f(\theta, \phi_s, u_s, 0)| \\ &= |f(\theta, \phi_s, g(\hat{\theta}_i, \phi_s), 0)| \\ &\leq |f(\theta, \phi_s, g(\theta, \phi_s), 0)| + |f_u(\theta, \phi_s, u_s^*, 0) g_\theta(\theta_i^*, \phi_s) \tilde{\theta}_i| \\ &\leq a(\theta, \phi_s) + 1 + \|S(\theta_i^*, \theta, \phi_s, u_s^*)\| \|\tilde{\theta}_i\| \\ &\leq \left(C_2 + \frac{C_1}{K}\right) \|\phi_s\| + 1. \end{aligned} \quad (26)$$

Consequently, by (4)

$$\begin{aligned} |y_{s+1}| &\leq |f(\theta, \phi_s, u_s, 0)| + |f_w(\theta, \phi_s, u_s, w_{s+1}^*) w_{s+1}| \\ &\leq \left(C_2 + \frac{C_1}{K}\right) \|\phi_s\| \\ &\quad + |f_w(\theta, \phi_s, u_s, w_{s+1}^*) w_{s+1}| + 1 \end{aligned} \quad (27)$$

holds for all $s \in [t_{i-1}, t_i - 1]$ with $\|\phi_s\| \geq M$.

Since the second inequality of (24) is trivial by (20) and (23), together with (27), it implies that (24) holds. As a result, by Lemma 4.2, one has

$$\frac{1}{t_i - t_{i-1} + 1} \sum_{s=t_{i-1}}^{t_i} y_s^2 \leq L_{\phi_{t_{i-1}}}(t_i)$$

which contradicts the definition of t_i . Therefore, $\theta \notin S_i^n$, which completes the proof. \blacksquare

Lemma 4.4: If $\theta \in S_i^n$, then there exists an integer $j \in [0, i - 1]$ such that $t_j < \infty$ and the averaged outputs of system (1) are bounded as follows:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t+1} \sum_{i=0}^t y_i^2 &\leq \max \left\{ \|\phi_{t_j}\|^2, C_f \right\} \\ &\quad + \frac{2M_\epsilon q(C+1)(\lambda + (1-\lambda)^2)}{(1-\lambda)^2}. \end{aligned}$$

Proof: First, by Lemma 4.3, the assumption $\theta \in S_i^n$ implies $t_i = \infty$. Let $j \in [0, i - 1]$ be the first integer such that $t_{j+1} = \infty$ and $t_j < \infty$. This integer exists by the fact that $t_0 < \infty$ but $t_i = \infty$. Then, for any $t \geq t_j + 1$, one has by (22)

$$\begin{aligned} \frac{1}{t - t_j + 1} \sum_{s=t_j}^t y_s^2 &\leq \max \left\{ \|\phi_{t_j}\|^2, C_f \right\} \\ &\quad + \frac{2M_\epsilon q(C+1)t(\lambda + (1-\lambda)^2)}{(t - t_j + 1)(1-\lambda)^2} \end{aligned}$$

which implies

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t+1} \sum_{i=0}^t y_i^2 &\leq \max \left\{ \|\phi_{t_j}\|^2, C_f \right\} \\ &\quad + \frac{2M_\epsilon q(C+1)(\lambda + (1-\lambda)^2)}{(1-\lambda)^2}. \end{aligned}$$

This completes the proof. \blacksquare

Proof of Theorem 2.1: For any ϕ_{q-1} , since $\theta \in \bigcup_{i=1}^n S_i^n$, Lemma 4.4 implies

$$\limsup_{t \rightarrow \infty} \frac{1}{t+1} \sum_{i=0}^t y_i^2 < \infty.$$

The proof is completed. \blacksquare

APPENDIX A

THE PROOF OF LEMMA 4.2

Denote the set of all integers by \mathbb{Z} . Let t_{i-1} be a switching time, $1 \leq i \leq n$. For any time $s \in [t_{i-1} - 1, t)$, let $r_s \in [0, s] \cap \mathbb{Z}$, where time $t \geq t_{i-1}$. Consider a series of integer sets

$$S_i \triangleq \{s_h \in [0, s] \cap \mathbb{Z} : h \in [0, r_s] \cap \mathbb{Z}, s \in [t_{i-1} - 1, t) \cap \mathbb{Z}\}$$

where $1 \leq i \leq n$. Assume that for all $1 \leq i \leq n$, S_i satisfies the following properties: for each s , $s_0 = s$ and for $h \in [1, r_s]$, if there is an integer $s' \in [t_{i-1} - 1, t)$ with $s' \leq s - qh$ and $r_{s'} \geq h$, then

$$s'_h < s_h. \quad (28)$$

The lemma presented below is devoted to proving Lemma 4.2, which plays a key role in the proof of Theorem 2.1.

Lemma A.1: Consider an integer set S_i , $1 \leq i \leq n$. Let integer $s^* \in [t_{i-1} - 1, t)$ and $h \in [1, r_{s^*}]$. The number of $s \in [t_{i-1} - 1, t)$ with $s_h = s_h^*$ is less than $2qh$.

Proof: For $h \geq 1$, by assumption (28), for any

$$s \in [t_{i-1} - 1, s^* - qh] \cup [s^* + qh, t)$$

one has $s_h \neq s_h^*$. Consequently, the possible integers $s \in [t_{i-1} - 1, t)$ with $s_h = s_h^*$ can only be taken from the set $[s^* - qh + 1, s^* + qh - 1] \cap \mathbb{Z}$, which consists of $(2qh - 1)$ integers. The lemma is thus proved. \blacksquare

Proof of Lemma 4.2: Note that for any $s \geq q - 1$

$$\begin{aligned} \|\phi_s\| &= (y_s^2 + y_{s-1}^2 + \cdots + y_{s-q+1}^2)^{\frac{1}{2}} \\ &\leq \sqrt{q} \max_{s-q+1 \leq k \leq s} |y_k| \end{aligned}$$

then, by (24), for all $s \in [t_{i-1}, t)$ with $\|\phi_s\| \geq M(t_{i-1} \geq t_0 \geq q - 1)$, one has

$$\begin{aligned} |y_{s+1}| &\leq \left(C_2 + \frac{C_1}{K}\right) \|\phi_s\| + |f_{w,s}| + 1 \\ &\leq \left(C_2 + \frac{C_1}{K}\right) \sqrt{q} |y_{k_s}| + |f_{w,s}| + 1 \end{aligned}$$

where $k_s \triangleq \arg \max_{s-q+1 \leq k \leq s} |y_k|$ and

$$f_{w,s} \triangleq f_w(\theta, \phi_s, u_s, w_{s+1}^*) w_{s+1}.$$

Note that for ϵ and M_ϵ defined by (15) and (19), respectively, the following inequality:

$$(x + y)^2 \leq (1 + \epsilon)x^2 + \frac{M_\epsilon}{2}y^2 \quad (29)$$

always holds whenever $x, y \geq 0$. Consequently, for any $\|\phi_s\| \geq M$

$$\begin{aligned} y_{s+1}^2 &\leq (1 + \epsilon) \left(C_2 + \frac{C_1}{K}\right)^2 q y_{k_s}^2 + M_\epsilon (f_{w,s}^2 + 1) \\ &= \lambda y_{k_s}^2 + M_\epsilon (f_{w,s}^2 + 1) \end{aligned} \quad (30)$$

where $\lambda = (1 + \epsilon)q(C_2 + (C_1/K))^2$ is defined earlier.

Now, the induction method is used to prove that for any $s \in [t_{i-1}, t)$

$$y_{s+1}^2 \leq \begin{cases} \lambda C_{f,t_{i-1}} + W_s, & \|\phi_s\| \geq M \\ C_f + 2f_{w,s}^2, & \|\phi_s\| < M \end{cases} \quad (31)$$

where $C_{f,t_{i-1}} \triangleq \max\{y_{k_{t_{i-1}}}^2, C_f\}$ and $W_s \triangleq M_\epsilon \sum_{h=0}^{r_s} \lambda^h (f_{w,s_h}^2 + 1)$. Here, $s_0 = s$, $r_s \in [0, s]$ is an integer satisfying

$$r_s = 0, \text{ if } k_s \leq t_{i-1} \quad (32)$$

and if $k_s \geq t_{i-1} + 1$, $\{s_h, 0 \leq h \leq r_s\}$ is a strictly decreasing sequence taking values from the set $[0, s] \cap \mathbb{Z}$ with

$$s_h = (k_s - 1)_{h-1}, 1 \leq h \leq r_s = r_{k_s-1} + 1 \quad (33)$$

where $\{(k_s - 1)_h, 0 \leq h \leq r_{k_s-1}\}$ is also a strictly decreasing sequence. Moreover, when $h \in [1, r_s]$ is fixed, for any integer $s' \geq t_{i-1}$ with $r_{s'} \geq h$

$$s'_h < s_h \quad \text{if} \quad s' \leq s - qh. \quad (34)$$

One can easily check the validity of (31) at the initial time $s = t_{i-1}$ by (24) and (30) with $r_{t_{i-1}} = 0$ and $(t_{i-1})_0 = t_{i-1}$. Moreover, the constraints (32)–(34) automatically hold. Next, suppose that (31)–(34) hold for all $s \in [t_{i-1}, j - 1]$, where $t_{i-1} + 1 \leq j \leq t - 1$. To verify (31)–(34) at time $s = j$, since the second inequality of (31) is true by (24), it suffices to consider the case where $\|\phi_j\| \geq M$. By (30), one has

$$y_{j+1}^2 \leq \lambda y_j^2 + M_\epsilon (f_{w,j}^2 + 1). \quad (35)$$

Now, consider the two cases where $k_j \geq t_{i-1} + 1$ and $k_j \leq t_{i-1}$, respectively. For $k_j \geq t_{i-1} + 1$, since $k_j \leq j$, by our assumption, y_{k_j} satisfies (31) for $s = k_j - 1 \in [t_{i-1}, j - 1]$. Noting that $\lambda < 1$ by (21), one has

$$\max \{\lambda C_{f,t_{i-1}}, C_f\} \leq \max \{\lambda y_{k_{t_{i-1}}}^2, C_f\} \leq C_{f,t_{i-1}}.$$

Then, the two inequalities of (31) can be merged into one at time $s = k_j - 1$ as follows:

$$y_{k_j}^2 \leq C_{f,t_{i-1}} + M_\epsilon \sum_{h=0}^{r_{k_j-1}} \lambda^h (f_{w,(k_j-1)_h}^2 + 1) \quad (36)$$

where the vector ϕ_{k_j-1} can be any value and $\{(k_j - 1)_h, 0 \leq h \leq r_{k_j-1}\}$ with $(k_j - 1)_0 = k_j - 1$ is a sequence taking values from $[0, k_j - 1] \cap \mathbb{Z}$, and $r_{k_j-1} \leq k_j - 1 \leq j - 1$. Consequently, by (21), (35) and (36), it is easy to deduce that

$$y_{j+1}^2 \leq \lambda C_{f,t_{i-1}} + M_\epsilon \sum_{h=0}^{r_j} \lambda^h (f_{w,j_h}^2 + 1) \quad (37)$$

where $r_j \triangleq r_{k_j-1} + 1 \leq j$, $j_0 \triangleq j$ and for any $1 \leq h \leq r_j$, $j_h \triangleq (k_j - 1)_{h-1}$, which yields (33) for $s = j$. Now, fix $h \in [1, r_j]$. For any integer $s \in [t_{i-1}, j - qh]$ with $r_s \geq h$ and $j - qh \geq t_{i-1}$, since $k_j \geq j - q + 1$ and $k_s \leq s$, it is easy to see that

$$k_j - k_s \geq j - q + 1 - s > (h - 1)q \quad (38)$$

which immediately leads to

$$(k_s - 1) < (k_j - 1) - q(h - 1).$$

If $h \geq 2$, noting that $k_s - 1 < k_j - 1 \leq j - 1$, $h - 1 \in [1, r_j - 1] = [1, r_{k_j-1}]$ and $r_{k_s-1} = r_s - 1 \geq h - 1$, by (33), (34), one has

$$j_h = (k_j - 1)_{h-1} > (k_s - 1)_{h-1} = s_h. \quad (39)$$

For $h = 1$, since $k_j > k_s$ by (38), it is easy to see that

$$j_1 = (k_j - 1)_0 = k_j - 1 > k_s - 1 = (k_s - 1)_0 = s_1 \quad (40)$$

which, together with (39), implies that for any $h \in [1, r_j]$, (34) also holds for $s = j$. As a result, (31)–(34) are all valid for $s = j$ when $k_j \geq t_{i-1} + 1$.

For the case of $k_j \leq t_{i-1}$, note that

$$k_j \geq j - q + 1 > t_{i-1} - q + 1$$

that is, $k_j \in [t_{i-1} - q + 1, t_{i-1}]$, hence $|y_{k_j}| \leq |y_{k_{t_{i-1}}}|$, which obviously implies (37) by (35) again with $r_j \triangleq 0$ and $j_0 \triangleq j$. This means that the first inequality in (31) and its constraints (32)–(34) are also true at time $s = j$ for this case, and hence for both cases. Therefore, by induction, (31)–(34) hold for all $s \in [t_{i-1}, t]$.

Finally, since $\lambda < 1$, similar to (36), it can be calculated from (31) that for any ϕ_s with $s \in [t_{i-1}, t)$

$$y_{s+1}^2 \leq \max \{y_{k_{t_{i-1}}}^2, C_f\} + M_\epsilon \left(\sum_{h=0}^{r_s} \lambda^h (f_{w,s_h}^2 + 1) \right). \quad (41)$$

Moreover, $|y_{t_{i-1}}| \leq |y_{k_{t_{i-1}}}|$, which immediately implies (41) for $s = t_{i-1} - 1$ with $r_{t_{i-1}-1} \triangleq 0$ and $(t_{i-1} - 1)_0 \triangleq t_{i-1} - 1$. This in fact shows that (41) with (32)–(34) hold for all $s \in [t_{i-1} - 1, t)$. Hence

$$\frac{1}{t - t_{i-1} + 1} \sum_{s=t_{i-1}}^t y_s^2 \leq \max \{y_{k_{t_{i-1}}}^2, C_f\} + \frac{M_\epsilon \sum_{s=t_{i-1}-1}^{t-1} \sum_{h=0}^{r_s} \lambda^h (f_{w,s_h}^2 + 1)}{t - t_{i-1} + 1}. \quad (42)$$

To calculate (42), one first estimates the following term:

$$\sum_{s=t_{i-1}-1}^{t-1} \sum_{h=0}^{r_s} \lambda^h (f_{w,s_h}^2 + 1) = \sum_{h=0}^{t-1} \lambda^h \sum_{s \in [t_{i-1}-1, t]: r_s \geq h} (f_{w,s_h}^2 + 1). \quad (43)$$

Note that $s_0 = s$, then for $h = 0$

$$\sum_{s \in [t_{i-1}-1, t-1]} (f_{w,s_0}^2 + 1) = \sum_{s=t_{i-1}-1}^{t-1} (f_{w,s}^2 + 1) \leq \sum_{m=0}^{t-1} (f_{w,m}^2 + 1). \quad (44)$$

Furthermore, for $h \geq 1$, Lemma A.1 shows that for each $m \in [0, t)$, the number of $s \in [t_{i-1} - 1, t)$ with $s_h = m$ is less than $2qh$. Then, by (44) and (10) in Remark 2.1, one has

$$\begin{aligned} & \sum_{h=0}^{t-1} \lambda^h \sum_{s \in [t_{i-1}-1, t]: r_s \geq h} (f_{w,s_h}^2 + 1) \\ & \leq \sum_{s=t_{i-1}-1}^{t-1} (f_{w,s}^2 + 1) + \sum_{h=1}^{t-1} \lambda^h \sum_{m=0}^{t-1} 2qh (f_{w,m}^2 + 1) \\ & < 2q \left(\sum_{h=1}^{t-1} \lambda^h h + 1 \right) \sum_{m=0}^{t-1} (f_{w,m}^2 + 1) \\ & \leq 2q(C + 1)t \left(\sum_{h=1}^{t-1} \lambda^h h + 1 \right). \end{aligned} \quad (45)$$

Since

$$\begin{aligned} \left(\sum_{h=1}^{t-1} \lambda^h h + 1 \right) & = 1 + \frac{1}{1 - \lambda} \left(\sum_{h=1}^{t-1} \lambda^h - \lambda^t (t - 1) \right) \\ & < \frac{\lambda + (1 - \lambda)^2}{(1 - \lambda)^2} \end{aligned} \quad (46)$$

by substituting (43)–(46) into (42), one has

$$\begin{aligned} & \frac{1}{t - t_{i-1} + 1} \sum_{s=t_{i-1}}^t y_s^2 \\ & \leq \max \left\{ y_{k_{t_{i-1}}}^2, C_f \right\} + \frac{2M_\epsilon q(C+1)t}{t - t_{i-1} + 1} \left(\sum_{h=1}^{t-1} \lambda^h h + 1 \right) \\ & \leq \max \left\{ \left\| \phi_{t_{i-1}} \right\|^2, C_f \right\} + \frac{2M_\epsilon q(C+1)t(\lambda + (1-\lambda)^2)}{(t - t_{i-1} + 1)(1-\lambda)^2} \\ & = L_{\phi_{t_{i-1}}}(t). \end{aligned}$$

This completes the proof.

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