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# Semiglobal Observer-Based Leader-Following Consensus With Input Saturation

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**Abstract**—This paper studies the observer-based leader-following consensus of a linear multiagent system on switching networks, in which the input of each agent is subject to saturation. Based on a low-gain output feedback method, distributed consensus protocols are developed. Under the assumptions that the networks are connected or jointly connected and that each agent is asymptotically null controllable with bounded controls and detectable, semiglobal observer-based leader-following consensus of the multiagent system can be reached on switching networks. A numerical example is presented to illustrate the theoretical results.

**Index Terms**—Consensus, input saturation, low-gain feedback, multiagent systems, state observer.

## I. INTRODUCTION

THE CURRENT study of multiagent systems pervades a wide range of sciences, including physical, biological, and even economical sciences [1], [2]. Its impact on modern engineering and technology is prominent and will be far reaching. Typical cooperative protocols of multiagent systems include consensus [3], [4], synchronization [5], flocking [6]–[8], formation control [9]–[11], and containment control [12]. In particular, the purpose of consensus of multiagent systems is to make a group of agents to reach a common state value, relying only on their neighbors' local information. Many different agent dynamics have been investigated for the consensus problem, such as single-integrator kinematics [3], double-integrator dynamics [13], [14], linear dynamics [15]–[18], and nonlinear dynamics [19]. In particular, the multiagent systems with agents described by general linear dynamics can be regarded as a generalization

of those with agents described by single-integrator kinematics and double-integrator dynamics and can also be considered as the linearized multiagent systems with agents described by nonlinear dynamics.

It is well known that saturation nonlinearities are ubiquitous in physical and engineering systems, and agents in multiagent systems should be subject to input saturation [20], [21]. Therefore, consensus of multiagent systems with saturation constraints is not only theoretically challenging but also more practical. However, there are only a few works on multiagent systems concerning with saturation constraints [22]–[28].

In this paper, we will address the observer-based leader-following consensus of agents described by general linear systems subject to input saturation. By utilizing low-gain design technique [29], we will design consensus algorithms that achieve semiglobal leader-following consensus of such agents on switching networks. The contributions of this paper are threefold. First, this work extends the existing results on the consensus of linear multiagent systems in [15]–[18] to the case with input saturation and extends the existing results on the consensus with saturation constraints [22]–[27] to the case of agents described by more general linear dynamics. Second, this work relaxes the convergence condition on the output feedback consensus of linear multiagent systems in [15]–[17] to jointly connected networks and extends state feedback consensus in [18] and [28] to output feedback one. Compared with the low-gain state feedback consensus of linear multiagent systems in [28], the proof method for the output feedback case is different, which is more difficult than the state feedback one. Specifically, a novel output feedback consensus algorithm is proposed, and a new Lyapunov function is constructed to analyze the stability of multiagent systems. In particular, the negative semidefiniteness of the derivative of the Lyapunov function is naturally guaranteed in [28], which is no longer guaranteed for the output feedback case due to the presence of some cross terms. A careful examination of the negative semidefiniteness of the derivative of the Lyapunov function is therefore required. For the case of jointly connected networks, it is more complicated since only parts of the cross terms in the derivative of the Lyapunov function can be dealt with. Furthermore, the construction of the output feedback consensus algorithm is different from the existing output feedback consensus algorithms in [15] and [16]. Finally, compared to the existing results on the consensus of linear multiagent systems in [15]–[18], the proposed algorithms do not require any global information on the underlying unweighted network, i.e., the knowledge of eigenvalues of the coupling matrix of the unweighted network.

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## II. PRELIMINARIES

### A. Graph Theory

This paper considers the problem of semiglobal observer-based leader-following consensus on networks of switching topology. Consider an undirected graph  $G(t) = \{V, E(t)\}$  with  $N$  agents, where  $V = \{1, 2, \dots, N\}$  is a nonempty finite set of agents in the systems and  $E(t) = \{(i, j) \in V \times V : i \sim j\}$  is a set of edges, in which an edge contains an unordered pair of vertices representing neighboring relations among the agents. Vertices  $i$  and  $j$  are adjacent when  $(i, j) \in E(t)$ . The adjacency matrix  $\mathcal{A}(t) = (a_{ij}(t))$  is defined as  $a_{ij}(t) = 1$ , if  $(i, j) \in E(t)$ ; otherwise,  $a_{ij}(t) = 0$ . The Laplacian matrix of graph  $G(t)$  is defined as  $L(t) = \Delta(\mathcal{A}(t)) - \mathcal{A}(t)$ , where  $\Delta(\mathcal{A}(t))$  is the degree matrix with  $i$ th diagonal elements  $\sum_{j=1, j \neq i}^N a_{ij}(t)$ . The eigenvalues of Laplacian matrix  $L(t)$  are denoted as  $\lambda_1(L(t)) \leq \lambda_2(L(t)) \leq \dots \leq \lambda_N(L(t))$ . Then,  $\lambda_1(L(t)) = 0$  with a corresponding eigenvector  $\mathbf{1} = [1 \ 1 \ \dots \ 1]^T \in \mathbf{R}^N$ .  $\lambda_2(L(t)) > 0$  when undirected graph  $G(t)$  is a connected graph.  $I_n$  is the identity matrix of order  $n$ , and  $\otimes$  stands for the Kronecker product. In this paper, we use the notation  $M > 0$  for a positive definite matrix  $M$  and  $M \geq 0$  for a nonnegative definite matrix  $M$ . We define that  $\bar{G}(t)$  is a graph consisting of  $N$  agents and a leader,  $L_{\delta(t)}$  is the symmetric Laplacian of the undirected graph  $G(t)$  consisting of  $N$  agents, and the matrix  $H_{\delta(t)} = \text{diag}(h_1(t), h_2(t), \dots, h_N(t))$ , where  $\delta : [0, \infty) \rightarrow \Gamma$  is a switching signal whose value at time  $t$  is the index of the graph at time  $t$  and  $\Gamma$  is finite. In this paper,  $h_i(t) = 1$  when agent  $i$  is a neighbor of the leader at time  $t$ ; otherwise,  $h_i(t) = 0$ .

*Lemma 1 [14]:* We define that  $L$  is the symmetric Laplacian of an undirected graph  $G$  consisting of  $N$  agents and  $\bar{G}$  is the graph consisting of  $N$  agents and a leader, which contains a spanning tree with the leader as the root vertex. Then, we have  $L + H > 0$ .

*Lemma 2 [19]:* Let  $L_1$  and  $L_2$  be the symmetric Laplacians of graphs  $G_1$  and  $G_2$  with  $N$  agents, respectively. Moreover, let  $\bar{G}_1$  be a graph with  $N$  agents and a leader which contains a spanning tree. After adding some edge(s) among the  $N$  agents into the graph  $\bar{G}_1$ , we obtain a new graph  $\bar{G}_2$ . Then, we have  $\lambda_i(L_2 + H) \geq \lambda_i(L_1 + H) > 0, i = 1, 2, \dots, N$ .

*Remark 1:* We define that  $\bar{G}_s$  is a spanning tree consisting of  $K$  agents and a leader and  $\bar{G}_{s1}$  is a graph obtained by adding edge(s) among the  $K$  agents into the graph  $\bar{G}_s$ . Let  $L_s(K)$  and  $L_{s1}(K)$  be the corresponding symmetric Laplacians of graphs  $G_s$  and  $G_{s1}$  with  $K$  agents, respectively. Then, we have  $\lambda_1(L_s(K) + H(K)) \leq \lambda_1(L_{s1}(K) + H(K))$ . The number of possible spanning trees with  $K$  agents and a leader is finite when the number of vertices of the spanning tree is finite and fixed. By using an exhaustive search method, one can obtain the minimum value of  $\lambda_1(L_s(K) + H(K))$ , i.e.,  $\min_{K=N} \{\lambda_1(L_s(K) + H(K))\}$ , when the system consists of  $N$  agents in the graph  $G_s$ . We can also obtain the minimum value of the least positive eigenvalues of the spanning trees  $\min_{2 \leq K \leq N} \{\lambda_1(L_s(K) + H(K))\}$ , in which the number of the agents and the leader of the spanning trees can be an integer  $2 \leq K \leq N$ , using an exhaustive search method.

### B. Problem Statement

Consider a group of  $N$  agents with general linear dynamics, labeled as  $1, 2, \dots, N$ . The motion of each agent is described by

$$\begin{aligned} \dot{x}_i &= Ax_i + B\sigma(u_i) \\ y_i &= Cx_i, \quad i = 1, 2, \dots, N \end{aligned} \quad (1)$$

where  $x_i \in \mathbf{R}^n$  is the state of agent  $i$ ,  $y_i \in \mathbf{R}^p$  is the measurement output of agent  $i$ ,  $u_i \in \mathbf{R}^m$  is the control input acting on agent  $i$ , and  $\sigma : \mathbf{R}^m \rightarrow \mathbf{R}^m$  is a saturation function defined as  $\sigma(u_i) = [\text{sat}(u_{i1}) \text{sat}(u_{i2}) \dots \text{sat}(u_{im})]^T$ ,  $\text{sat}(u_{ij}) = \text{sgn}(u_{ij}) \min\{|u_{ij}|, \varpi\}$ , for some constant  $\varpi > 0$ . The dynamics of the leader, labeled as  $N + 1$ , is described by

$$\begin{aligned} \dot{x}_{N+1} &= Ax_{N+1} \\ y_{N+1} &= Cx_{N+1}. \end{aligned} \quad (2)$$

The problem of semiglobal observer-based leader-following consensus for the agents and leader described previously is the following: For any bounded set  $\mathbf{X} \subset \mathbf{R}^n$  given *a priori*, construct a control law  $u_i$  for each agent  $i$ , which only uses the measurement outputs from neighbor agents, such that

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_{N+1}(t)\| = 0, \quad i = 1, 2, \dots, N$$

as long as  $x_i(0) \in \mathbf{X}$  for all  $i = 1, 2, \dots, N, N + 1$ .

*Assumption 1:* The pair  $(A, B)$  is asymptotically null controllable with bounded controls (ANCBC), i.e.,  $(A, B)$  is stabilizable, and all the eigenvalues of  $A$  are in the closed left-half  $s$ -plane.

*Assumption 2:* The pair  $(A, C)$  is detectable.

## III. MAIN RESULTS

### A. Consensus on Connected Switching Networks

*Assumption 3:* The graph  $\bar{G}(t)$  consists of  $N$  agents and a leader, which contains a spanning tree rooted at the leader at all times.

*Lemma 3 [29]:* By Assumption 1, for each  $\varepsilon \in (0, 1]$ , it follows that there exists a unique matrix  $P(\varepsilon) > 0$  satisfying the algebraic Riccati equation (ARE)

$$A^T P(\varepsilon) + P(\varepsilon)A - P(\varepsilon)BB^T P(\varepsilon) + \varepsilon I = 0.$$

Moreover,  $\lim_{\varepsilon \rightarrow 0} P(\varepsilon) = 0$ .

The low-gain output feedback design for the multiagent system (1) is carried out in two steps.

#### Low-gain-output-feedback-based consensus algorithm

Step 1) Solve the parametric ARE

$$A^T P(\varepsilon) + P(\varepsilon)A - \gamma P(\varepsilon)BB^T P(\varepsilon) + \varepsilon I = 0, \quad \varepsilon \in (0, 1] \quad (3)$$

where  $\gamma \leq \min_{K=N} \{\lambda_1(L_s(K) + H(K))\}$  is a positive constant. The existence of a unique positive definite solution  $P(\varepsilon)$  for the ARE (3) is established in Lemma 3.

Step 2) Construct a linear output feedback law for agent  $i$  as

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i - F(y_i - C\hat{x}_i) - BB^T P(\varepsilon) \\ &\quad \times \left( \sum_{j=1}^N a_{ij}(t)(\hat{x}_i - \hat{x}_j) + h_i(t)(\hat{x}_i - \hat{x}_{N+1}) \right) \\ u_i &= -B^T P(\varepsilon) \left( \sum_{j=1}^N a_{ij}(t)(\hat{x}_i - \hat{x}_j) + h_i(t)(\hat{x}_i - \hat{x}_{N+1}) \right), \\ &\quad i=1, 2, \dots, N \\ \dot{\hat{x}}_{N+1} &= A\hat{x}_{N+1} - F(y_{N+1} - C\hat{x}_{N+1}) \end{aligned} \quad (4)$$

where  $\hat{x}_i \in \mathbf{R}^n$  is the protocol state of agent  $i$  and  $F \in \mathbf{R}^{n \times p}$  is the feedback gain matrix, which is chosen such that  $(A + FC)$  is asymptotically stable. The existence of such an  $F$  is guaranteed by Assumption 2. The fact that  $\lim_{\varepsilon \rightarrow 0} P(\varepsilon) = 0$ , as established in Lemma 3, motivates the ‘‘low-gain feedback.’’

*Remark 2:* Note that the construction of the output feedback consensus algorithm is different from that of the existing output feedback consensus algorithms in [15] and [16]. In the algorithm (4), no information on the network topology is needed, and each agent only acquires the state information of its neighbors. The value of  $\gamma$  can be calculated when the number of agents  $N$  is known.

*Lemma 4* [23]: For any  $\Upsilon \in \mathbf{R}^{m \times m}$  and any  $\varsigma_i, \xi_i \in \mathbf{R}^m$ ,  $i = 1, 2, \dots, N$

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}(t)(\varsigma_i - \varsigma_j)^T (\Upsilon(\varepsilon)(\xi_i - \xi_j)) \\ &= \sum_{i=1}^N \sum_{j=1}^N a_{ij}(t) \varsigma_i^T (\Upsilon(\varepsilon)(\xi_i - \xi_j)). \end{aligned}$$

*Theorem 1:* Consider a multiagent system of  $N$  agents with general linear dynamics (1) and a leader with dynamics (2). Suppose that Assumptions 1, 2, and 3 hold. The control inputs  $u_i$  for the agent (4) achieve semiglobal consensus of the multiagent system. That is, for any given bounded set  $\mathbf{X} \subset \mathbf{R}^n$ , there is an  $\varepsilon^* > 0$  such that, for each  $\varepsilon \in (0, \varepsilon^*]$

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_{N+1}(t)\| = 0, \quad i = 1, 2, \dots, N$$

as long as  $x_i(0) \in \mathbf{X}$  for all  $i = 1, 2, \dots, N, N+1$ .

*Proof:* Let  $\tilde{x}_i = x_i - x_{N+1}$ ,  $\tilde{\hat{x}}_i = \hat{x}_i - \hat{x}_{N+1}$ , and  $e_i = \tilde{x}_i - \tilde{\hat{x}}_i$ . Then, we have

$$\begin{aligned} \dot{\tilde{x}}_i &= A\tilde{x}_i - B\sigma \left( B^T P(\varepsilon) \sum_{j=1}^N a_{ij}(t)(\tilde{x}_i - \tilde{x}_j + e_j - e_i) \right. \\ &\quad \left. + h_i(t)B^T P(\varepsilon)(\tilde{x}_i - e_i) \right) \\ \dot{\tilde{\hat{x}}}_i &= A\tilde{\hat{x}}_i - BB^T P(\varepsilon) \left( \sum_{j=1}^N a_{ij}(t)(\tilde{x}_i - \tilde{x}_j + e_j - e_i) \right. \\ &\quad \left. + h_i(t)(\tilde{x}_i - e_i) \right) - FCe_i \end{aligned}$$

$$\begin{aligned} \dot{e}_i &= (A + FC)e_i - B\sigma \left( B^T P(\varepsilon) \sum_{j=1}^N a_{ij}(t)(\tilde{x}_i - \tilde{x}_j + e_j - e_i) \right. \\ &\quad \left. + h_i(t)B^T P(\varepsilon)(\tilde{x}_i - e_i) \right) \\ &\quad + BB^T P(\varepsilon) \left( \sum_{j=1}^N a_{ij}(t)(\tilde{x}_i - \tilde{x}_j + e_j - e_i) \right. \\ &\quad \left. + h_i(t)(\tilde{x}_i - e_i) \right) \end{aligned} \quad (5)$$

for which let us consider the common Lyapunov function

$$\begin{aligned} V(\tilde{x}, e) &= \sum_{i=1}^N \tilde{x}_i^T P(\varepsilon) \tilde{x}_i + \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \\ &\quad \times \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N e_i^T P_e e_i \end{aligned} \quad (6)$$

where  $P_e > 0$  satisfies  $(A + FC)^T P_e + P_e(A + FC) = -I$  and  $\theta$  is a positive constant chosen such that  $\theta \geq \lambda_N(L(t) + H(t))$ . Since  $(A + FC)$  is asymptotically stable, the existence of such a  $P_e$  can be guaranteed. For notational convenience, we have defined  $\tilde{x} = [\tilde{x}_1^T \tilde{x}_2^T \dots \tilde{x}_N^T]^T$ ,  $\hat{x} = [\hat{x}_1^T \hat{x}_2^T \dots \hat{x}_N^T]^T$ , and  $e = [e_1^T e_2^T \dots e_N^T]^T$ .

For a constant  $c > 0$ , we can find the following inequality:

$$\begin{aligned} c \geq &\sup_{\varepsilon \in (0, 1], (x_i(0), \hat{x}_i(0)) \in \mathbf{X}} \sum_{i=1}^N \left\{ \tilde{x}_i^T(0) P(\varepsilon) \tilde{x}_i(0) \right. \\ &\left. + \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \times \left( \frac{\theta^2}{\gamma} + 1 \right) e_i(0)^T P_e e_i(0) \right\} \end{aligned} \quad (7)$$

for all  $i = 1, 2, \dots, N+1$ , which is guaranteed by the fact that  $\mathbf{X}$  is bounded and  $\lim_{\varepsilon \rightarrow 0} P(\varepsilon) = 0$  by Lemma 3.

Let  $L_V(c) := \{\tilde{x} \in \mathbf{R}^{Nn}, e \in \mathbf{R}^{Nn} : V(\tilde{x}, e) \leq c\}$ , and let  $\varepsilon^* \in (0, 1]$  be such that, for each  $\varepsilon \in (0, \varepsilon^*]$ ,  $(\tilde{x}, e) \in L_V(c)$  implies that

$$\left\| B^T P(\varepsilon) \left( \sum_{j=1}^N a_{ij}(t)(\hat{x}_i - \hat{x}_j) + h_i(t)(\hat{x}_i - \hat{x}_{N+1}) \right) \right\|_{\infty} \leq \varpi, \quad i = 1, 2, \dots, N \quad (8)$$

where  $\|z\|_{\infty} = \max_i |z_i|$  for  $z \in \mathbf{R}^n$ . The existence of such an  $\varepsilon^*$  is also due to the fact that  $\lim_{\varepsilon \rightarrow 0} P(\varepsilon) = 0$ .

Thus, for any  $\varepsilon \in (0, \varepsilon^*]$ , the dynamics of (5) remains linear within  $L_V(c)$ . Thus, by Lemma 4, we can derive the derivative of  $V$  along the trajectories of the agents within the set  $L_V(c)$  as

$$\begin{aligned} \dot{V}(\tilde{x}, e) &= \sum_{i=1}^N \dot{\tilde{x}}_i^T P(\varepsilon) \tilde{x}_i + \sum_{i=1}^N \tilde{x}_i^T P(\varepsilon) \dot{\tilde{x}}_i \\ &\quad + \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) \\ &\quad \times \sum_{i=1}^N (\dot{e}_i^T P_e e_i + e_i^T P_e \dot{e}_i) \end{aligned}$$



$$\begin{aligned}
 &= \sum_{i=1}^N \tilde{x}_i^T (P(\varepsilon)A + A^T P(\varepsilon)) \tilde{x}_i \\
 &\quad - 2 \sum_{i=1}^N \tilde{x}_i^T P(\varepsilon)BB^T P(\varepsilon) \\
 &\quad \times \left( \sum_{j=1}^N a_{ij}(t)(\tilde{x}_i - \tilde{x}_j) + h_i(t)\tilde{x}_i \right) \\
 &\quad + 2 \sum_{i=1}^N \tilde{x}_i^T P(\varepsilon)BB^T P(\varepsilon) \\
 &\quad \times \left( \sum_{j=1}^N a_{ij}(t)(e_i - e_j) + h_i(t)e_i \right) \\
 &\quad - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N e_i^T e_i \\
 &= \sum_{i=1}^N \tilde{x}_i^T (P(\varepsilon)A + A^T P(\varepsilon)) \tilde{x}_i \\
 &\quad - \sum_{i=1}^N \sum_{j=1}^N a_{ij}(t) (\tilde{x}_i^T - \tilde{x}_j^T) P(\varepsilon)BB^T P(\varepsilon) (\tilde{x}_i - \tilde{x}_j) \\
 &\quad + \sum_{i=1}^N \sum_{j=1}^N a_{ij}(t) (\tilde{x}_i^T - \tilde{x}_j^T) P(\varepsilon)BB^T P(\varepsilon) (e_i - e_j) \\
 &\quad - 2 \sum_{i=1}^N h_i(t) \tilde{x}_i^T P(\varepsilon)BB^T P(\varepsilon) \tilde{x}_i \\
 &\quad + 2 \sum_{i=1}^N h_i(t) \tilde{x}_i^T P(\varepsilon)BB^T P(\varepsilon) e_i \\
 &\quad - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N e_i^T e_i. \quad (9)
 \end{aligned}$$

From the definition of the Laplacian of graph, for any  $\xi_i \in \mathbf{R}^m$  and  $\zeta_i \in \mathbf{R}^m, i = 1, 2, \dots, N$

$$\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}(t) (\xi_i - \xi_j)^T (\zeta_i - \zeta_j) = \xi^T (L(t) \otimes I_m) \zeta$$

where  $\xi = [\xi_1 \ \xi_2 \ \dots \ \xi_N]^T$  and  $\zeta = [\zeta_1 \ \zeta_2 \ \dots \ \zeta_N]^T$ , and using the identity  $(A \otimes B)(C \otimes D) = AC \otimes BD$ , we can continue (9) as follows:

$$\begin{aligned}
 \dot{V}(\tilde{x}) &= \tilde{x}^T (I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))) \tilde{x} \\
 &\quad - 2\tilde{x}^T (I_N \otimes P(\varepsilon)B) (L(t) \otimes I_n) (I_N \otimes B^T P(\varepsilon)) \tilde{x} \\
 &\quad - 2\tilde{x}^T (I_N \otimes P(\varepsilon)B) (H(t) \otimes I_n) (I_N \otimes B^T P(\varepsilon)) \tilde{x} \\
 &\quad + 2\tilde{x}^T (I_N \otimes P(\varepsilon)B) (L(t) \otimes I_n) (I_N \otimes B^T P(\varepsilon)) e \\
 &\quad + 2\tilde{x}^T (I_N \otimes P(\varepsilon)B) (H(t) \otimes I_n) (I_N \otimes B^T P(\varepsilon)) e \\
 &\quad - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e \\
 &= \tilde{x}^T (I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon)) - (L(t) + H(t)) \\
 &\quad \otimes (2P(\varepsilon)BB^T P(\varepsilon))) \tilde{x} \\
 &\quad + \tilde{x}^T ((L(t) + H(t)) \otimes (2P(\varepsilon)BB^T P(\varepsilon))) e \\
 &\quad - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e. \quad (10)
 \end{aligned}$$

The symmetry of matrix  $L(t) + H(t)$  implies that there exist orthogonal matrices  $T(t) \in \mathbf{R}^{N \times N}, \hat{x} = (T(t) \otimes I_n)\tilde{x}$ , and  $\hat{e} = (T(t) \otimes I_n)e$ , such that

$$\begin{aligned}
 L(t) + H(t) &= T^T(t) \text{diag} \{ \lambda_1(L(t) + H(t)), \lambda_2(L(t) + H(t)), \dots, \\
 &\quad \lambda_N(L(t) + H(t)) \} T(t) \\
 &:= T^T(t) \text{diag} \{ \lambda_1(t), \lambda_2(t), \dots, \lambda_N(t) \} T(t).
 \end{aligned}$$

Thus, (10) can be further continued as

$$\begin{aligned}
 \dot{V}(\tilde{x}) &= \hat{x}^T (T^T(t) \otimes I_n) (I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))) \\
 &\quad \times (T(t) \otimes I_n) \tilde{x} - \hat{x}^T (T^T(t) \otimes I_n) \\
 &\quad \times (\text{diag} \{ \lambda_1(t), \dots, \lambda_N(t) \} \otimes (2P(\varepsilon)BB^T P(\varepsilon))) \\
 &\quad \times (T(t) \otimes I_n) \tilde{x} + \hat{x}^T (T^T(t) \otimes I_n) \\
 &\quad \times (\text{diag} \{ \lambda_1(t), \dots, \lambda_N(t) \} \otimes (2P(\varepsilon)BB^T P(\varepsilon))) \\
 &\quad \times (T(t) \otimes I_n) e - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T \\
 &\quad \times (T^T(t) \otimes I_n) (T(t) \otimes I_n) e \\
 &= \sum_{i=1}^N \hat{x}_i^T (P(\varepsilon)A + A^T P(\varepsilon) - 2\lambda_i(t)P(\varepsilon)BB^T P(\varepsilon)) \hat{x}_i \\
 &\quad + \sum_{i=1}^N \hat{x}_i^T (2\lambda_i(t)P(\varepsilon)BB^T P(\varepsilon)) \hat{e}_i \\
 &\quad - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N \hat{e}_i^T \hat{e}_i \\
 &\leq \sum_{i=1}^N \hat{x}_i^T (P(\varepsilon)A + A^T P(\varepsilon) - 2\gamma P(\varepsilon)BB^T P(\varepsilon)) \hat{x}_i \\
 &\quad + \sum_{i=1}^N \hat{x}_i^T (2\lambda_i P(\varepsilon)BB^T P(\varepsilon)) \hat{e}_i \\
 &\quad - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N \hat{e}_i^T \hat{e}_i \\
 &\leq -\varepsilon \sum_{i=1}^N \tilde{x}_i^T \tilde{x}_i - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \sum_{i=1}^N e_i^T e_i \\
 &\quad - \sum_{i=1}^N \left( \sqrt{\gamma} B^T P(\varepsilon) \hat{x}_i - \frac{\lambda_i}{\sqrt{\gamma}} B^T P(\varepsilon) \hat{e}_i \right)^T \\
 &\quad \times \left( \sqrt{\gamma} B^T P(\varepsilon) \hat{x}_i - \frac{\lambda_i}{\sqrt{\gamma}} B^T P(\varepsilon) \hat{e}_i \right) \\
 &< 0, \quad \forall (\tilde{x}_i, e_i) \in L_V(c) \setminus \{0\}.
 \end{aligned}$$

This implies that the trajectory of  $(\tilde{x}_i, e_i)$  starting from the level set  $L_V(c)$  will converge to the origin asymptotically as time approaches infinity, which, in turn, implies that

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_{N+1}(t)\| = 0, \quad i = 1, 2, \dots, N.$$

This completes the proof.  $\square$

*Remark 3:* Since the ARE-based method is adopted in this paper, the resulting feedback gain is indirectly dependent on the low gain parameter  $\varepsilon$ . For a different value of  $\varepsilon$ , the solution of a parameterized ARE is required. Therefore, we can use the following bisectioning method to compute the parameter

$\varepsilon^*$  without using any global information of multiagent systems. First, we choose an  $\varepsilon^* \in (0, 1]$ . From (3), we can compute the corresponding  $P(\varepsilon^*)$ . Note that the calculation of  $\gamma$  in (3) only depends on the number of agents  $N$ . For a given bounded set  $\mathbf{X} \subset \mathbf{R}^n$ , we can find the maximum value of  $\|\mu\|_\infty$ , where  $\|\mu\|_\infty = \max_i |\mu_i|$  for  $\mu \in \mathbf{X}$ . Let the maximum value of  $\|\mu\|_\infty$  be  $a$  and  $(x_i(0), \hat{x}_i(0)) \in \mathbf{X}$ ; we have  $\|x_i(0)\|_\infty \leq a$  and  $\|\hat{x}_i(0)\|_\infty \leq a$ . Let  $\mathbf{1} = [1, 1, \dots, 1]^T$ . Since  $\tilde{x}_i = x_i - x_{N+1}$ ,  $\hat{\tilde{x}}_i = \hat{x}_i - \hat{x}_{N+1}$ , and  $e_i = \tilde{x}_i - \hat{\tilde{x}}_i$ , from (6), we have

$$\begin{aligned} V(\tilde{x}(0), e(0)) &= \sum_{i=1}^N \tilde{x}_i^T(0) P(\varepsilon^*) \tilde{x}_i(0) + \lambda_{\max}(P(\varepsilon^*) B B^T P(\varepsilon^*)) \\ &\quad \times \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N e_i^T(0) P_e e_i(0) \\ &\leq 4a^2 \sum_{i=1}^N \mathbf{1}^T P(\varepsilon^*) \mathbf{1} + 16a^2 \lambda_{\max}(P(\varepsilon^*) B B^T P(\varepsilon^*)) \\ &\quad \times \left( \frac{\theta^2}{\gamma} + 1 \right) \sum_{i=1}^N \mathbf{1}^T P_e \mathbf{1}. \end{aligned}$$

From Lemma 2, we have  $\lambda_N(L_c + H(t)) \geq \lambda_N(L(t) + H(t))$ , where  $L_c$  is the corresponding symmetric Laplacian of complete graph consisting of the  $N$  agents. We can choose  $\theta = \max_{K=N} \{\lambda_N(L_c(K) + H(K))\}$ , which can be calculated via an exhaustive search method, when the number of agents  $N$  is known. Therefore, from (7), we can design  $c = 4a^2 \sum_{i=1}^N \mathbf{1}^T P(\varepsilon^*) \mathbf{1} + 16a^2 \lambda_{\max}(P(\varepsilon^*) B B^T P(\varepsilon^*)) ((\theta^2/\gamma) + 1) \sum_{i=1}^N \mathbf{1}^T P_e \mathbf{1}$ . From (6) and  $\dot{V} \leq 0$ , we have  $\tilde{x}_i^T P(\varepsilon^*) \tilde{x}_i \leq c$  and  $\lambda_{\max}(P(\varepsilon^*) B B^T P(\varepsilon^*)) ((\theta^2/\gamma) + 1) e_i^T P_e e_i \leq c$ . Thus, we can obtain the maximum values of  $\|\tilde{x}_i\|_\infty$  and  $\|e_i\|_\infty$ ,  $i = 1, 2, \dots, N$ . Let the maximum value of  $\|\tilde{x}_i\|_\infty$  be  $g$  and the maximum value of  $\|e_i\|_\infty$  be  $f$ ; we have

$$\begin{aligned} &\left\| B^T P(\varepsilon^*) \left( \sum_{j=1}^N a_{ij}(t) (\hat{x}_i - \hat{x}_j) + h_i(t) (\hat{x}_i - \hat{x}_{N+1}) \right) \right\|_\infty \\ &\leq \|(2N-1)(g+f) B^T P(\varepsilon^*) \mathbf{1}\|_\infty. \end{aligned}$$

If  $\|(2N-1)(g+f) B^T P(\varepsilon^*) \mathbf{1}\|_\infty \leq \varpi$ , then the parameter  $\varepsilon^*$  is feasible. Otherwise, we use  $\varepsilon^*/2$  to repeat the aforementioned calculation process until the saturation constraint  $\|(2N-1)(g+f) B^T P(\varepsilon^*) \mathbf{1}\|_\infty \leq \varpi$  is satisfied.

### B. Consensus on Jointly Connected Switching Networks

In this section, we will investigate the consensus problem in a more practical scene, in which the agents get in touch, directly or indirectly, with the leader from time to time.

*Assumption 4:* There exists an infinite sequence of contiguous, nonempty, and uniformly bounded time intervals  $[t_k, t_{k+1})$ ,  $k = 0, 1, \dots$ , and there are  $m_k$  switching topologies in each uniformly bounded time intervals  $[t_k, t_{k+1})$  (i.e., there exists a finite sequence of contiguous and nonempty time subintervals  $[t_k^j, t_k^{j+1})$ ,  $j = 0, 1, \dots, m_k$ , with  $t_k = t_k^0$ ,  $t_{k+1} = t_k^{m_k}$ , and  $t_k^{j+1} - t_k^j \geq \tau$  for some constant  $\tau > 0$ , and the intercon-

nection topology does not change during each of such time subintervals) such that, across each time interval, there exists a joint path from the leader to every agent. In other words, the neighboring graph  $\bar{G}(t)$  has a jointly spanning tree across each uniformly bounded interval  $[t_k, t_{k+1})$ ,  $k = 0, 1, \dots$ , with  $t_0 = 0$  and  $t_{k+1} - t_k \leq T$  for some constant  $T > 0$ .

*Assumption 5:* There exists a  $P(\varepsilon) > 0$  that satisfies

$$A^T P(\varepsilon) + P(\varepsilon) A - \bar{\gamma} P(\varepsilon) B B^T P(\varepsilon) + \varepsilon I = 0, \quad \varepsilon \in (0, 1] \quad (11)$$

where  $\bar{\gamma} \leq \min_{2 \leq K \leq N} \{\lambda_1(L_s(K) + H(K))\}$  is a positive constant and  $\min_{2 \leq K \leq N} \{\lambda_1(L_s(K) + H(K))\}$  denotes the minimum value of the least positive eigenvalues of the spanning trees, in which the number of the agents and the leader of the spanning trees can be an integer  $2 \leq K \leq N$  and

$$A^T P(\varepsilon) + P(\varepsilon) A \leq 0. \quad (12)$$

*Theorem 2:* Consider a multiagent system of  $N$  agents with general linear dynamics (1) and a leader with dynamics (2). Suppose that Assumptions 1, 2, 4, and 5 hold. The control inputs  $u_i$  for the agent (4) achieve semiglobal consensus of the multiagent system. That is, for any given bounded set  $\mathbf{X} \subset \mathbf{R}^n$ , there is an  $\varepsilon^* > 0$  such that, for each  $\varepsilon \in (0, \varepsilon^*]$

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_{N+1}(t)\| = 0, \quad i = 1, 2, \dots, N$$

as long as  $x_i(0) \in \mathbf{X}$  for all  $i = 1, 2, \dots, N, N+1$ .

*Proof:* Let us consider the common Lyapunov function

$$\begin{aligned} V(\tilde{x}, e) &= \sum_{i=1}^N \tilde{x}_i^T P(\varepsilon) \tilde{x}_i + \lambda_{\max}(P(\varepsilon) B B^T P(\varepsilon)) \\ &\quad \times \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) \sum_{i=1}^N e_i^T P_e e_i \quad (13) \end{aligned}$$

for system (5), where  $P(\varepsilon) > 0$  satisfies (11) and (12). Similar to the analysis of Theorem 1, we will show that the set

$$L_V(c) := \{\tilde{x} \in \mathbf{R}^{Nn}, e \in \mathbf{R}^{Nn} : V(\tilde{x}, e) \leq c\} \quad (14)$$

is positively invariant. The derivative of  $V$  along the trajectories of the agents in the set  $L_V(c)$  is given by

$$\begin{aligned} \dot{V}(\tilde{x}, e) &= \tilde{x}^T (I_N \otimes (P(\varepsilon) A + A^T P(\varepsilon))) \tilde{x} \\ &\quad - \tilde{x}^T ((L(t) + H(t)) \otimes (2P(\varepsilon) B B^T P(\varepsilon))) \tilde{x} \\ &\quad + \tilde{x}^T ((L(t) + H(t)) \otimes (2P(\varepsilon) B B^T P(\varepsilon))) e \\ &\quad - \lambda_{\max}(P(\varepsilon) B B^T P(\varepsilon)) \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) e^T e. \quad (15) \end{aligned}$$

For the symmetry of matrix  $L(t) + H(t) \geq 0$ , there exist orthogonal matrices  $T(t) \in \mathbf{R}^{N \times N}$ ,  $\hat{x} = (T(t) \otimes I_n) \tilde{x}$ , and  $\hat{e} = (T(t) \otimes I_n) e$ . Without loss of generality, we assume that the first  $M$  ( $1 \leq M \leq N$ ) eigenvalues of matrix  $L(t) + H(t)$  are zero, i.e.,  $\lambda_i(t) = 0$  for  $i = 1, 2, \dots, M$ ; then, one has

$$\begin{aligned} \dot{V}(\tilde{x}, e) &\leq \sum_{i=1}^M \hat{x}_i^T [P(\varepsilon) A + A^T P(\varepsilon)] \hat{x}_i \\ &\quad - \lambda_{\max}(P(\varepsilon) B B^T P(\varepsilon)) \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{i=1}^M \hat{e}_i^T \hat{e}_i - \varepsilon \sum_{i=M+1}^N \hat{x}_i^T \hat{x}_i \\
 & - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \sum_{i=M+1}^N \hat{e}_i^T \hat{e}_i \\
 & - \sum_{i=M+1}^N \left[ \sqrt{\gamma} B^T P(\varepsilon) \hat{x}_i - \frac{\lambda_i}{\sqrt{\gamma}} B^T P(\varepsilon) \hat{e}_i \right]^T \\
 & \times \left[ \sqrt{\gamma} B^T P(\varepsilon) \hat{x}_i - \frac{\lambda_i}{\sqrt{\gamma}} B^T P(\varepsilon) \hat{e}_i \right] \\
 & \leq 0. \tag{16}
 \end{aligned}$$

Therefore, the set (14) is positively invariant.

From (16),  $\lim_{t \rightarrow \infty} V(\tilde{x}(t), e(t))$  exists. Considering the infinite sequences  $V(\tilde{x}(t_k), e(t_k))$ ,  $k = 0, 1, \dots$ , and using Cauchy's convergence criteria, one has that, for any  $\vartheta > 0$ , there exists a positive number  $M_\vartheta$  such that,  $\forall k \geq M_\vartheta$ ,  $V(\tilde{x}(t_k)) - V(\tilde{x}(t_{k+1})) = \int_{t_k}^{t_{k+1}} [-\dot{V}(\tilde{x}(t))] dt < \vartheta$ .

Therefore,

$$\begin{aligned}
 \vartheta & > \int_{t_k^0}^{t_k^1} [-\dot{V}(\tilde{x}(t))] dt + \dots + \int_{t_k^{m_k-1}}^{t_k^{m_k}} [-\dot{V}(\tilde{x}(t))] dt \\
 & \geq - \int_{t_k^0}^{t_k^1} \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\
 & + \int_{t_k^0}^{t_k^1} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^0)} + H_{\delta(t_k^0)}) \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \right\} dt \\
 & - \int_{t_k^0}^{t_k^1} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^0)} + H_{\delta(t_k^0)}) \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \right\} dt \\
 & + \int_{t_k^0}^{t_k^1} \left\{ \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e \right\} dt \\
 & - \dots - \int_{t_k^{m_k-1}}^{t_k^{m_k}} \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\
 & + \int_{t_k^{m_k-1}}^{t_k^{m_k}} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^{m_k-1})} + H_{\delta(t_k^{m_k-1})}) \right. \right. \\
 & \quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \right\} dt \\
 & - \int_{t_k^{m_k-1}}^{t_k^{m_k}} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^{m_k-1})} + H_{\delta(t_k^{m_k-1})}) \right. \right. \\
 & \quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \right\} dt
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{t_k^{m_k-1}}^{t_k^{m_k}} \left\{ \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e \right\} dt \\
 & \geq - \int_{t_k^0}^{t_k^0+\tau} \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\
 & + \int_{t_k^0}^{t_k^0+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^0)} + H_{\delta(t_k^0)}) \right. \right. \\
 & \quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \right\} dt \\
 & - \int_{t_k^0}^{t_k^0+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^0)} + H_{\delta(t_k^0)}) \right. \right. \\
 & \quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \right\} dt \\
 & + \int_{t_k^0}^{t_k^0+\tau} \left\{ \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e \right\} dt \\
 & - \dots - \int_{t_k^{m_k-1}}^{t_k^{m_k-1}+\tau} \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\
 & + \int_{t_k^{m_k-1}}^{t_k^{m_k-1}+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^{m_k-1})} + H_{\delta(t_k^{m_k-1})}) \right. \right. \\
 & \quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \right\} dt \\
 & - \int_{t_k^{m_k-1}}^{t_k^{m_k-1}+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^{m_k-1})} + H_{\delta(t_k^{m_k-1})}) \right. \right. \\
 & \quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \right\} dt \\
 & + \int_{t_k^{m_k-1}}^{t_k^{m_k-1}+\tau} \left\{ \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \times \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e \right\} dt. \tag{17}
 \end{aligned}$$

From (12) and the symmetric matrices  $L_{\delta(t_k^{j-1})} + H_{\delta(t_k^{j-1})} \geq 0$ ,  $j = 1, \dots, m_k - 1$ , one has

$$\begin{aligned}
 & \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \\
 & - \tilde{x}^T \left( (L_{\delta(t_k^{j-1})} + H_{\delta(t_k^{j-1})}) \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \\
 & + \tilde{x}^T \left( (L_{\delta(t_k^{j-1})} + H_{\delta(t_k^{j-1})}) \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \\
 & - \lambda_{\max}(P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\gamma} + 1 \right) e^T e \\
 & \leq 0.
 \end{aligned}$$

Therefore,

$$\begin{aligned} \vartheta &> - \int_{t_k^j}^{t_k^j+\tau} \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\ &+ \int_{t_k^j}^{t_k^j+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^{j-1})} + H_{\delta(t_k^{j-1})}) \right. \right. \\ &\quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \right\} dt \\ &- \int_{t_k^j}^{t_k^j+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^{j-1})} + H_{\delta(t_k^{j-1})}) \right. \right. \\ &\quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \right\} dt \\ &+ \int_{t_k^j}^{t_k^j+\tau} \left\{ \lambda_{\max} (P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) e^T e \right\} dt \end{aligned}$$

which implies that, for  $i = 0, 1, \dots, m_k - 1$

$$\begin{aligned} &\lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^i)} + H_{\delta(t_k^i)}) \right. \right. \\ &\quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) \tilde{x} \right\} dt \\ &+ \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \tilde{x}^T \left( (L_{\delta(t_k^i)} + H_{\delta(t_k^i)}) \right. \right. \\ &\quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right) e \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \lambda_{\max} (P(\varepsilon)BB^T P(\varepsilon)) \right. \\ &\quad \left. \times \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) e^T e \right\} dt = 0. \end{aligned}$$

From Assumption 4 and Lemma 1, all eigenvalues of the matrix  $L_{\delta(t_k^0)} + \dots + L_{\delta(t_k^{m_k-1})} + H_{\delta(t_k^0)} + \dots + H_{\delta(t_k^{m_k-1})}$  are positive. From Lemma 2 and Assumption 5

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} \int_t^{t+\tau} m_k \left\{ \tilde{x}^T [I_N \otimes (P(\varepsilon)A + A^T P(\varepsilon))] \tilde{x} \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \tilde{x}^T \left[ \left( L_{\delta(t_k^0)} + \dots + L_{\delta(t_k^{m_k-1})} \right. \right. \right. \\ &\quad \left. \left. \left. + H_{\delta(t_k^0)} + \dots + H_{\delta(t_k^{m_k-1})} \right) \right. \right. \\ &\quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right] \tilde{x} \right\} dt \end{aligned}$$

$$\begin{aligned} &+ \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \tilde{x}^T \left[ \left( L_{\delta(t_k^0)} + \dots + L_{\delta(t_k^{m_k-1})} \right. \right. \right. \\ &\quad \left. \left. \left. + H_{\delta(t_k^0)} + \dots + H_{\delta(t_k^{m_k-1})} \right) \right. \right. \\ &\quad \left. \left. \otimes (2P(\varepsilon)BB^T P(\varepsilon)) \right] e \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} m_k \\ &\times \left\{ \lambda_{\max} (P(\varepsilon)BB^T P(\varepsilon)) \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) e^T e \right\} dt \\ &\leq \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \sum_{i=1}^N \hat{x}_i^T (P(\varepsilon)A + A^T P(\varepsilon)) \right. \\ &\quad \left. - \bar{\gamma} P(\varepsilon)BB^T P(\varepsilon) \right\} \hat{x}_i \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \sum_{i=1}^N \hat{x}_i^T (\bar{\gamma} P(\varepsilon)BB^T P(\varepsilon)) \hat{x}_i \right\} dt \\ &+ \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \sum_{i=1}^N \hat{x}_i^T (2\lambda_i P(\varepsilon)BB^T P(\varepsilon)) \hat{e}_i \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \lambda_{\max} (P(\varepsilon)BB^T P(\varepsilon)) \right. \\ &\quad \left. \times \left( \frac{\theta^2}{\bar{\gamma}} + 1 \right) \sum_{i=1}^N \hat{e}_i^T \hat{e}_i \right\} dt \\ &\leq - \lim_{t \rightarrow \infty} \int_t^{t+\tau} \varepsilon \left\{ \sum_{i=1}^N \tilde{x}_i^T \tilde{x}_i \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \lambda_{\max} (P(\varepsilon)BB^T P(\varepsilon)) \sum_{i=1}^N e_i^T e_i \right\} dt \\ &- \lim_{t \rightarrow \infty} \int_t^{t+\tau} \left\{ \sum_{i=1}^N \left( \sqrt{\bar{\gamma}} B^T P(\varepsilon) \hat{x}_i - \frac{\lambda_i}{\sqrt{\bar{\gamma}}} B^T P(\varepsilon) \hat{e}_i \right)^T \right. \\ &\quad \left. \times \left( \sqrt{\bar{\gamma}} B^T P(\varepsilon) \hat{x}_i - \frac{\lambda_i}{\sqrt{\bar{\gamma}}} B^T P(\varepsilon) \hat{e}_i \right) \right\} dt \\ &\leq 0. \end{aligned}$$

Therefore,  $\lim_{t \rightarrow \infty} \varepsilon \sum_{i=1}^N \tilde{x}_i^T \tilde{x}_i = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \tilde{x}_i = 0$ .  $\square$

#### IV. NUMERICAL EXAMPLE

In this section, a numerical example is presented to illustrate the theoretical results. The simulation is performed with four agents and one leader. The system matrices are chosen



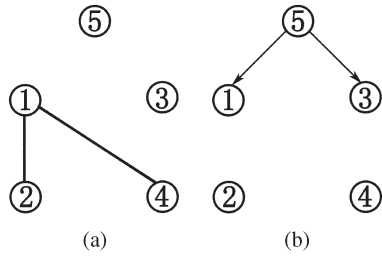


Fig. 1. Interaction networks. (a)  $\bar{G}_1$ . (b)  $\bar{G}_2$ .

as  $A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and  $C = [0 \ 1]$ . It is straightforward to verify that  $(A, B)$  is ANCBC and  $(A, C)$  is detectable. Moreover,  $F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$  is chosen such that  $(A + FC)$  is asymptotically stable. Initial states  $x_{i1}(0)$  and  $x_{i2}(0)$  of all agents are randomly chosen from box  $[-2, 2] \times [-2, 2]$ , respectively, the initial values  $\hat{x}_i(0)$  of all agents are chosen as  $[0.1, 0.1]^T$ , the initial state  $x_{N+1}(0)$  of the leader is chosen as  $[0.5, 0.5]^T$ , and the initial value  $\hat{x}_{N+1}(0)$  of the leader is chosen as  $[0.35, 0.35]^T$ . The two interaction networks, i.e.,  $\bar{G}_1$  and  $\bar{G}_2$ , are chosen as in Fig. 1, and each network is active for half of the time in each time interval. Therefore, there is a jointly connected path from the leader to every agent in each time interval. Since there are four agents and one leader in the group, we can obtain the minimum eigenvalue of the possible spanning trees consisting of four agents and one leader, i.e.,  $\min_{2 \leq K \leq 4} \{\lambda_1(L_s(K) + H(K))\} = 0.1206$ , using an exhaustive method. Therefore, we can choose  $\bar{\gamma} = 0.1 \leq \min_{2 \leq K \leq 4} \{\lambda_1(L_s(K) + H(K))\} = 0.1206$ . For  $\varepsilon = 0.1$  and  $\varepsilon = 0.001$ , by using a standard numerical software, we find that there exist positive definite matrices  $P(0.1) = \begin{bmatrix} 0.0500 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$  and  $P(0.001) = \begin{bmatrix} 0.0005 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix}$ , such that conditions (11) and (12) hold. Fig. 2 shows the consensus of four agents and one leader applying control protocol (4). Fig. 2(a) and (b) depicts the curves of the state difference between the four agents and the leader and the control of the four agents when  $\varepsilon = 0.1$ , respectively. Fig. 2(c) and (d) depicts the curves of the state difference between the four agents and the leader and the control of the four agents when  $\varepsilon = 0.001$ , respectively. It is obvious from Fig. 2 that the control protocol (4) is capable of achieving stable consensus motion, and for the same initial conditions, as the value of  $\varepsilon$  decreases, the state peaks slowly while the control input decreases.

V. CONCLUSION

In this paper, we have investigated semiglobal observer-based leader-following consensus of multiagent linear systems with input saturation on networks of switching topology. We have used a low-gain output feedback strategy to design the new observer-based consensus algorithms, without requiring any knowledge of the interaction network topology. Under the assumption that the system is ANCBC and detectable, all the agents in the group asymptotically synchronize with the leader on connected or jointly connected switching networks by employing bounded control inputs. Future work will be to

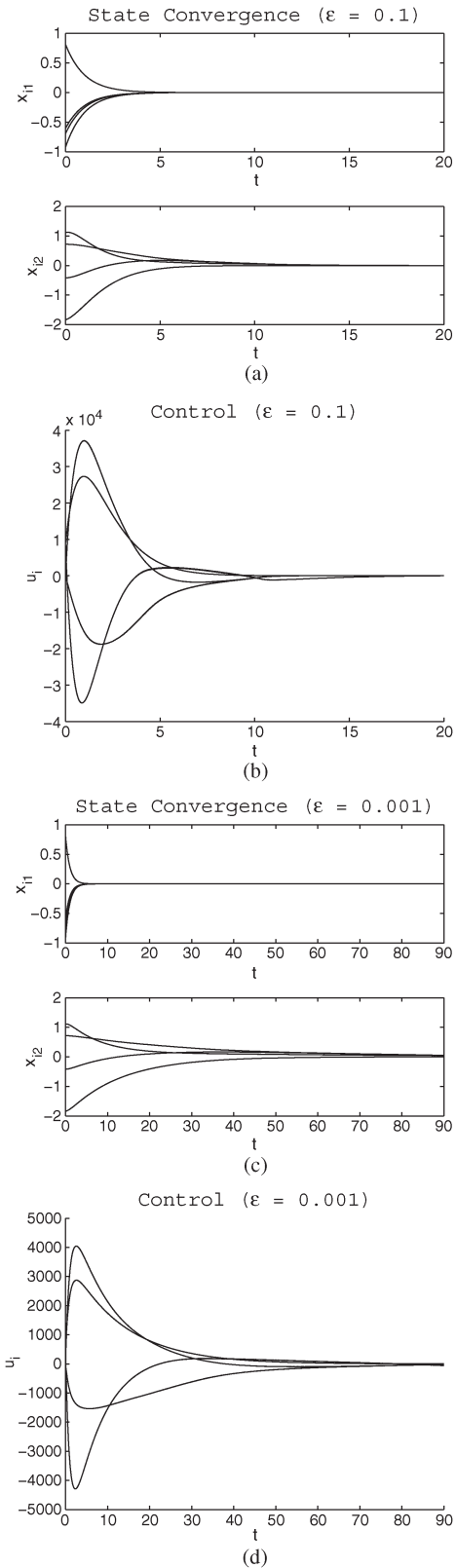


Fig. 2. Leader-following consensus of four agents with one leader under control protocol (4). (a) State convergence ( $\varepsilon = 0.1$ ). (b) Control ( $\varepsilon = 0.1$ ). (c) State convergence ( $\varepsilon = 0.001$ ). (d) Control ( $\varepsilon = 0.001$ ).

further investigate the proposed algorithms on the weighted and directed switching networks and to study the robustness of the proposed algorithms against noise and time delays.

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