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A Necessary and Sufficient LMI Condition for Stability of 2D Mixed Continuous-Discrete-Time Systems

Graziano Chesi and Richard H. Middleton

Abstract— **This paper addresses the problem of establishing stability of 2D mixed continuous-discrete-time systems. Traditional stability analysis for 2D systems gives a sufficient condition based on 2D version of a Lyapunov equation. Here, a linear matrix inequality (LMI) condition is proposed that extends these results by introducing complex Lyapunov functions depending polynomially on a parameter and by exploiting the Gram matrix method. It is shown that this condition is sufficient for 2D exponential stability for any chosen degree of the Lyapunov function candidate, and it is also shown that this condition is also necessary for a sufficiently large degree. Moreover, an a priori bound on the degree required for achieving necessity is given. Some numerical examples illustrate the proposed methodology.**

I. INTRODUCTION

This paper addresses the problem of establishing stability of 2D mixed continuous-discrete-time systems, as discussed in the monograph [13]. The study of 2D systems has a long history, with some early works such as [7] introducing basic models, systems theory and stability properties. A number of tests for stability of 2D systems have been based around a 2D characteristic polynomial (or more accurately, a multinomial). Some examples of this are [11] which treats exponential stability of 2D discrete-discrete systems using the 2D characteristic polynomial.

Another approach for stability analysis is to use 2D Lyapunov functions, and the related LMI tests to search for a quadratic Lyapunov function (see for example works such as [9], [10]). This approach has the advantages of fast numerical algorithms for solving LMIs. In addition, LMI techniques permit extensions to various 2D synthesis problems such as 2D $\mathcal{H}_\infty$ design, see e.g. [6].

However, despite these advantages, it has long been known (see for example [2]) that existing LMI results are sufficient, but not necessary for 2D stability. More recently, in a closely allied line of work, it has been shown [3] that less conservative LMI based stability tests may be constructed by introducing complex polynomial based Lyapunov functions. The extension to this method here is to allow more general[1] polynomial based Lyapunov functions, exploiting the Gram matrix method. This then gives further sufficient conditions for stability, for any degree of the parameter dependence of the Lyapunov function candidate. This paper establishes that for some finite polynomial degree, LMI based stability tests are tight, that is, for 2D exponential stability it is necessary that there exists a polynomially dependent 2D Lyapunov function. Moreover, an a priori bound on the degree required for achieving necessity is given. Some numerical examples illustrate the proposed condition.

The paper is organized as follows. Section II provides some preliminaries and the problem formulation. Section III describes the proposed methodology. Section IV reports the numerical examples. Lastly, Section V concludes the paper with some final remarks.

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[1] [3] only considers the first order extension to constant Lyapunov functions.

II. PRELIMINARIES

A. Problem Formulation

Let us introduce the notation used throughout the paper:

- $\mathbb{N}, \mathbb{R}, \mathbb{C}$: natural, real, and complex number sets;
- j: imaginary unit, i.e. $j^2 = -1$;
- I: identity matrix (of size specified by the context);
- $\bar{a}$: complex conjugate of $a \in \mathbb{C}$;
- A': conjugate transpose of A, i.e. $(A')_{ij} = \bar{A_{ji}}$;
- Hermitian matrix A: a complex square matrix satisfying $A' = A$;
- $A > 0$, $A \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix A;
- $|\lambda|$: magnitude of $\lambda \in \mathbb{C}$;
- $\|v\|$: Euclidean norm of vector v, i.e. $\|v\| = \sqrt{v'v}$;
- $adj(A)$: adjoint of matrix A;
- $\det(A)$: determinant of matrix A;
- $trace(A)$: trace of matrix A.

We consider the 2D continuous-discrete Roesser space model in [12] given by

$$\begin{pmatrix} \dfrac{d}{dt}x_c(t,k) \\ x_d(t,k+1) \end{pmatrix} = \begin{pmatrix} A_{cc} & A_{cd} \\ A_{dc} & A_{dd} \end{pmatrix} \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \quad (1)$$

where $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ represent the continuous and discrete states, respectively, the scalars $t \in \mathbb{R}$ and $k \in \mathbb{N}$ are the continuous and discrete times, respectively, and $A_{cc} \in \mathbb{R}^{n_c \times n_c}$, $A_{cd} \in \mathbb{R}^{n_c \times n_d}$, $A_{dc} \in \mathbb{R}^{n_d \times n_c}$ and $A_{dd} \in \mathbb{R}^{n_d \times n_d}$ are given matrices.

Problem. The problem addressed in this paper consists of establishing whether (1) is exponentially stable, i.e. there exist $\beta, \gamma \in \mathbb{R}$ such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\| \leq \beta \varrho e^{-\gamma \min\{t,k\}} \quad (2)$$

for all initial conditions $x_c(0,k)$ and $x_d(t,0)$ and for all $t \geq 0$ and $k \geq 0$, where

$$\varrho = \max\{\varrho_1, \varrho_2\} \tag{3}$$

and

$$\begin{array}{rcl} \varrho_1 & = & \sup_{t\geq 0} \|x_d(t,0)\| \\ \varrho_2 & = & \sup_{k\geq 0} \|x_c(0,k)\|. \end{array} \tag{4}$$

B. SOS Matrix Polynomials

Let $M : \mathbb{R}^p \to \mathbb{R}^{q\times q}$ be a real symmetric matrix polynomial of degree $2r$. Then, $M(y)$ can be expressed as

$$M(y) = \left(y^{\{r\}} \otimes I\right)' (N + L(\alpha)) \left(y^{\{r\}} \otimes I\right) \tag{5}$$

where $y^{\{r\}} \in \mathbb{R}^{\sigma(p,r)}$ is a vector whose entries are the monomials in y of degree less than or equal to r whose dimension is given by

$$\sigma(p,r) = \frac{(p+r)!}{p!r!}, \tag{6}$$

N is a real symmetric matrix satisfying

$$M(y) = \left(y^{\{r\}} \otimes I\right)' N \left(y^{\{r\}} \otimes I\right), \tag{7}$$

$L : \mathbb{R}^{\tau(p,r,q)} \to \mathbb{R}^{q\times q}$ is a linear parametrization of the real subspace

$$\mathcal{L} = \left\{ L = L' \in \mathbb{R}^{q\times q} : \ \left(y^{\{r\}} \otimes I\right)' L \left(y^{\{r\}} \otimes I\right) = 0 \right\} \tag{8}$$

whose dimension is given by

$$\tau(p,r,q) = \frac{1}{2} q \left(\sigma(p,r)\left(q\sigma(p,r)+1\right) - (q+1)\sigma(p,2r)\right) \tag{9}$$

and $\alpha \in \mathbb{R}^{\tau(p,r,q)}$ is a free vector. The representation (5) is known as square matrix representation (SMR) of matrix polynomials, and extends the Gram matrix method for (scalar) polynomials.

This representation was introduced in [5] and references therein for establishing whether a matrix polynomial is sum of squares of matrix polynomials (SOS) with an LMI. Specifically, the symmetric matrix polynomial $M(y)$ is SOS if and only if there exist real matrix polynomials $M_i(y)$, $i = 1,\ldots,k$, such that

$$M(y) = \sum_{i=1}^{k} M_i'(y) M_i(y). \tag{10}$$

Equivalently, $M(y)$ is SOS if and only if there exists α satisfying the LMI

$$N + L(\alpha) \geq 0. \tag{11}$$

See e.g. [4] and references therein for details about SOS matrix polynomials.

III. Stability Condition

Let us observe that

$$x_d(t,k+1) = G(s) x_d(t,k) \tag{12}$$

where

$$G(s) = A_{dd} + A_{dc}(sI - A_{cc})^{-1} A_{cd}. \tag{13}$$

We express $G(s)$ as

$$G(s) = \frac{G_N(s)}{g(s)} \tag{14}$$

where $G_N : \mathbb{C} \to \mathbb{C}^{n_d\times n_d}$ is a matrix polynomial of degree n_c and $g(s)$ is a polynomial of degree n_c, in particular

$$g(s) = \det(sI - A_{cc}). \tag{15}$$

The following result is known from the literature (see e.g. [8] and [1] for the discrete-discrete and continuous-continuous cases respectively).

Lemma 1: Assume that A_{cc} is Hurwitz. The system (1) is exponentially stable if and only if

$$|\lambda_i(G(j\omega))| < 1 \ \ \forall i = 1,\ldots,n_d \ \forall \omega \in \mathbb{R} \tag{16}$$

where $\lambda_i(G(j\omega))$ is the i-th eigenvalue of $G(j\omega)$.

An equivalent condition based on parameter-dependent Lyapunov functions exists in the literature and is as follows.

Lemma 2: Assume that A_{cc} is Hurwitz. The system (1) is exponentially stable if and only if there exists a Hermitian matrix function $P : \mathbb{R} \to \mathbb{C}^{n_d\times n_d}$ such that

$$\left\{ \begin{array}{lcl} 0 & < & P(\omega) \\ 0 & < & P(\omega) - G(j\omega)' P(\omega) G(j\omega) \end{array} \right. \quad \forall \omega \in \mathbb{R}. \tag{17}$$

The first contribution of this paper is to show that such a parameter-dependent Lyapunov function can be chosen polynomial in ω and to provide an upper bound on the degree as explained in the following result.

Theorem 1: Assume that A_{cc} is Hurwitz. The system (1) is exponentially stable if and only if there exists a Hermitian matrix polynomial $P : \mathbb{R} \to \mathbb{C}^{n_d\times n_d}$ of degree $2d \leq 2\mu$ where

$$\mu = n_c n_d^2 \tag{18}$$

satisfying (17).

Proof. "$\Leftarrow$" Suppose that there exists a Hermitian matrix polynomial $P(\omega)$ satisfying (17). Then, from Lemma 2 it directly follows that (1) is exponentially stable.

"$\Rightarrow$" Suppose that (1) is exponentially stable. From Lemma 1 one has that the eigenvalues of $G(j\omega)$ strictly lie in the complex unit disc for all $\omega \in \mathbb{R}$. Consequently, the discrete Lyapunov equation

$$P(\omega) - G(j\omega)' P(\omega) G(j\omega) = Q(\omega) \tag{19}$$

with $Q(\omega)$ satisfying

$$Q(\omega) > 0 \ \ \forall \omega \in \mathbb{R} \tag{20}$$

has a unique solution $P(\omega)$ which satisfies

$$P(\omega) > 0 \quad \forall \omega \in \mathbb{R}. \tag{21}$$

Let us gather the free entries of $P(\omega)$ and $Q(\omega)$ into vectors $p(\omega)$ and $q(\omega)$ of length

$$\begin{array}{rcl} l & = & l_1 + l_2 = n_d^2 \\ l_1 & = & \frac{1}{2} n_d (n_d + 1) \\ l_2 & = & \frac{1}{2} n_d (n_d - 1) \end{array} \tag{22}$$

where l_1 and l_2 are the numbers of free entries in the real part and in the imaginary part, respectively, of a $n_d \times n_d$ Hermitian matrix. It follows that the discrete Lyapunov equation (19) can be rewritten as

$$E(\omega) p(\omega) = q(\omega) \tag{23}$$

where $E : \mathbb{R} \to \mathbb{R}^{l \times l}$ is nonsingular for all $\omega \in \mathbb{R}$. The solution $p(\omega)$ is hence obtained as

$$p(\omega) = E(\omega)^{-1} q(\omega). \tag{24}$$

Let us observe that $E(\omega)$ can be written as

$$E(\omega) = \frac{E_N(\omega)}{|g(j\omega)|^2} \tag{25}$$

where $E_N(\omega)$ is a matrix polynomial of degree not larger than $2n_c$ and $|g(j\omega)|^2$ is a polynomial of degree not larger than $2n_c$. Hence,

$$E(\omega)^{-1} = |g(j\omega)|^2 \frac{adj(E_N(\omega))}{\det(E_N(\omega))} \tag{26}$$

where $adj(E_N(\omega))$ is a matrix polynomial of degree not larger than $2n_c(l-1)$ and $\det(E_N(\omega))$ is a polynomial of degree not larger than $2ln_c$. Let us simply select $Q(\omega) = I$, which satisfies (20). It follows that $q(\omega) \equiv q$ is constant, in particular all the entries of q belong to $\{0, 1\}$, and hence $p(\omega)$ in (24) is given by

$$\begin{array}{rcl} p(\omega) & = & |g(j\omega)|^2 \frac{adj(E_N(\omega))}{\det(E_N(\omega))} q \\ & = & \frac{p_N(\omega)}{\det(E_N(\omega))} \end{array} \tag{27}$$

where $p_N(\omega)$ is a vector polynomial of degree not larger than

$$2n_c + 2n_c(l-1) = 2ln_c. \tag{28}$$

Equation (27) states that the solution $P(\omega)$ of the discrete Lyapunov equation (19) with $Q(\omega) = I$ is a matrix rational function. Clearly, this implies that also

$$\hat{P}(\omega) = sgn(\det(E_N(0))) \det(E_N(\omega)) P(\omega) \tag{29}$$

can be used to prove asymptotical stability of (1). In fact, since the solution of the discrete Lyapunov equation (19) is unique, one has that

$$\det(E_N(\omega)) \neq 0 \quad \forall \omega \in \mathbb{R}. \tag{30}$$

Moreover, $E_N(\omega)$ is continuous for all $\omega \in \mathbb{R}$, which implies that $\det(E_N(\omega))$ does not change sign, and hence

$$sgn(\det(E_N(0))) \det(E_N(\omega)) > 0 \quad \forall \omega \in \mathbb{R}. \tag{31}$$

Therefore, the proof is concluded by observing that $\hat{P}(\omega)$ is a matrix polynomial of degree not larger than $2ln_c$, which turns out to be

$$2ln_c = 2n_c n_d^2 = 2\mu. \tag{32}$$

□

Theorem 1 states that exponential stability of (1) is equivalent to the existence of a Hermitian matrix polynomial $P(\omega)$ of degree $2d$ not greater than 2μ satisfying (17). It is useful to observe that, if there exists a Hermitian matrix polynomial $P(\omega)$ of degree $2d$ satisfying (17), than there also exists a Hermitian matrix polynomial $P(\omega)$ of degree $2(d+1)$, denoted by $\hat{P}(\omega)$, satisfying (17), which can be obtained as

$$\hat{P}(\omega) = (1 + \omega^2) P(\omega). \tag{33}$$

At this point, the problem is how to check whether (17) holds whenever $P(\omega)$ is a Hermitian matrix polynomial. We will show in the sequel of this paper that (17) can be equivalently checked through LMIs.

Specifically, given a Hermitian matrix $S \in \mathbb{C}^{n_d \times n_d}$, let us express S as

$$S = S_R + jS_I \tag{34}$$

where $S_R, S_I \in \mathbb{R}^{n_d \times n_d}$ satisfy

$$\left\{ \begin{array}{rcl} S_R & = & S_R' \\ S_I & = & -S_I'. \end{array} \right. \tag{35}$$

We define the symmetric matrix function

$$F(S) = \left(\begin{array}{cc} S_R & S_I \\ S_I' & S_R \end{array} \right). \tag{36}$$

The following result provides the second contribution of this paper, which is a necessary and sufficient LMI condition for positive semidefiniteness and definiteness of $P(\omega)$.

Theorem 2: Let $P : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ be a Hermitian matrix polynomial. Then,

$$P(\omega) \geq 0 \text{ (resp., } P(\omega) > 0) \quad \forall \omega \in \mathbb{R} \tag{37}$$

if and only if there exists $c \in \mathbb{R}$ satisfying the LMIs

$$\left\{ \begin{array}{l} F(P(\omega)) - cI \text{ is SOS} \\ c \geq 0 \text{ (resp., } c > 0). \end{array} \right. \tag{38}$$

Proof. "$\Leftarrow$" Suppose that there exists $c \in \mathbb{R}$ satisfying the LMIs (38). Then, it follows that there exist symmetric matrix polynomials $F_i(\omega)$, $i = 1, \ldots, k$, such that

$$F(P(\omega)) - cI = \sum_{i=1}^{k} F_i(\omega)' F_i(\omega) \tag{39}$$

which clearly implies that

$$F(P(\omega)) - cI \geq 0 \quad \forall \omega \in \mathbb{R}. \tag{40}$$

Let us define the complex vector

$$z = a + jb \tag{41}$$

where $a, b \in \mathbb{R}^{n_d}$, and let $P_R(\omega)$ and $P_I(\omega)$ are the real and imaginary parts of $P(\omega)$. Let us pre- and post-multiply $P(\omega) - cI$ times z' and z, respectively. From (40) we obtain:

$$\begin{array}{rcl} z'\left(P(\omega) - cI\right)z & = & a'(P_R(\omega) - c_I)a \\ & & +b'(P_R(\omega) - c_I)b \\ & & +a'P_I b - b'P_I a \\ & = & \left(\begin{array}{c} a \\ b \end{array}\right)' (F(P(\omega)) - cI) \left(\begin{array}{c} a \\ b \end{array}\right) \\ & \geq & 0 \end{array} \tag{42}$$

which means that

$$P(\omega) - cI \geq 0 \quad \forall \omega \in \mathbb{R} \tag{43}$$

i.e. (37) holds.

"$\Rightarrow$" Suppose that (37) holds. Then, it follows that (43) holds for some $c \geq 0$ (resp., $c > 0$) since $P(\omega)$ is a polynomial function of a scalar variable. For definition of positive semidefinite Hermitian matrix, (43) holds if and only if

$$\begin{array}{rcl} 0 & \leq & z'\left(P(\omega) - cI\right)z \\ & = & a'(P_R(\omega) - c_I)a + b'(P_R(\omega) - c_I)b \\ & & +a'P_I b - b'P_I a \\ & = & \left(\begin{array}{c} a \\ b \end{array}\right)' (F(P(\omega)) - cI) \left(\begin{array}{c} a \\ b \end{array}\right) \end{array} \tag{44}$$

for all $z \in \mathbb{C}^{n_d}$, $z \neq 0$, which holds if and only if (40) holds. The condition (40) states that the symmetric matrix polynomial $F(P(\omega)) - cI$ is positive semidefinite for all $\omega \in \mathbb{R}$. Since ω is a scalar, this is true if and only if $F(P(\omega)) - cI$ is SOS (see e.g. [4] and references therein). Consequently, (38) holds. $\square$

Theorem 2 states that semidefiniteness and definiteness of a Hermitian matrix polynomial $P(\omega)$ can be equivalently established via a SOS test, which is an LMI feasibility test as explained in Section II-B. In particular, from Theorem 2 one has that $P(\omega) \geq 0$ for all $\omega \in \mathbb{R}$ if and only if $F(P(\omega))$ is SOS, while $P(\omega) > 0$ for all $\omega \in \mathbb{R}$ if and only if $F(P(\omega)) - cI$ is SOS for some $c > 0$.

Theorems 1 and 2 can be exploited to investigate exponential stability of (1) through LMIs. To this end, let us define the Hermitian matrix polynomial

$$R(\omega) = |g(j\omega)|^2 P(\omega) - G_N(j\omega)' P(\omega) G_N(j\omega). \tag{45}$$

The following result provides the sought sufficient and necessary LMI condition for establishing exponential stability of (1).

Theorem 3: Assume that A_{cc} is Hurwitz. The system (1) is exponentially stable if and only if there exist a Hermitian matrix polynomial $P : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ of degree $2d \leq 2\mu$ and $c \in \mathbb{R}$ satisfying the LMIs

$$\left\{ \begin{array}{l} F(P(\omega)) - cI \text{ is SOS} \\ F(R(\omega)) - cI \text{ is SOS} \\ c > 0. \end{array} \right. \tag{46}$$

Proof. From Theorem 1 one has that the system (1) is exponentially stable if and only if there exists a Hermitian matrix polynomial $P(\omega)$ of degree $2d \leq 2\mu$ satisfying (17). From Theorem 2 it follows that the first inequality in (17) holds if and only if the first and the third conditions in (46) hold. Then, since

$$G(j\omega)' = \frac{G_N(j\omega)'}{\bar{g}(j\omega)}, \quad G(j\omega) = \frac{G_N(j\omega)}{g(j\omega)} \tag{47}$$

it follows that

$$R(\omega) = |g(j\omega)|^2 \left(P(\omega) - G(j\omega)' P(\omega) G(j\omega)\right). \tag{48}$$

Since A_{cc} is Hurwitz, one has that

$$g(j\omega) \neq 0 \quad \forall \omega \in \mathbb{R} \tag{49}$$

and, hence,

$$R(\omega) > 0 \iff P(\omega) - G(j\omega)' P(\omega) G(j\omega) > 0. \tag{50}$$

Therefore, from Theorem 2 it follows that the second inequality in (17) holds if and only if the second and the third conditions in (46) hold. $\square$

Theorem 3 states that exponential stability of (1) is equivalent to the existence of a Hermitian matrix polynomial $P(\omega)$ of degree $2d$ not greater than 2μ satisfying the SOS condition (46), which is an LMI feasibility test as explained in Section II-B.

Let us observe that $P(\omega)$ and c are defined up to a positive scale factor in (46), i.e. if (46) holds for $P(\omega)$ and c, then (46) also holds for $\beta P(\omega)$ and βc for all $\beta > 0$. A simple way of normalizing $P(\omega)$ and c is to impose the linear equality constraint

$$trace(P(1)) = 1 \tag{51}$$

since the trace of $P(\omega)$ is clearly positive if $P(\omega)$ is positive definite.

In order to quantify the feasibility of (46), we introduce the index

$$\begin{array}{rcl} \zeta & = & \sup\limits_{P(\omega), c} \; c \\ & & \text{s.t.} \left\{ \begin{array}{l} F(P(\omega)) - cI \text{ is SOS} \\ F(R(\omega)) - cI \text{ is SOS} \\ trace(P(1)) = 1. \end{array} \right. \end{array} \tag{52}$$

The LMI variables in (52) are given by the free coefficients of $P(\omega)$, the scalar c, and the vectors α appearing in the SMRs of the matrix polynomials $F(P(\omega)) - cI$ and $F(R(\omega)) - cI$. Taking into account the reduction of one LMI scalar variable due to $trace(P(1)) = 1$, the total number of LMI scalar variables in (52) is hence given by

$$\eta = (2d+1)n_d^2 + \tau(1, d, 2n_d) + \tau(1, d + n_c, 2n_d). \tag{53}$$

Exploiting the expressions of $\tau(1, d, 2n_d)$ and $\tau(1, d + n_c, 2n_d)$, one finally obtains

$$\begin{aligned} \eta &= n_d\left((2d+1)n_d + d(2dn_d - 1) + \right. \\ &\quad \left.(d+n_c)(2(d+n_c)n_d - 1)\right). \end{aligned} \tag{54}$$

Table I shows η in the case $n_c = n_d = n$ for some values of n and d.

	$2d = 0$	2	4	6
$n = 1$	2	10	26	50
2	32	84	168	284
3	162	318	546	846
4	512	856	1328	1928

TABLE I

TOTAL NUMBER OF LMI SCALAR VARIABLES IN (52) IN THE CASE $n_c = n_d = n$ FOR SOME VALUES OF n AND d.

IV. EXAMPLES

In this section we present some illustrative examples of the proposed results. The LMI test (46) is solved with the toolbox SeDuMi [14].

A. Example 1

Let us consider (1) with

$$\begin{aligned} A_{dd} &= \begin{pmatrix} 0 & 0.3 \\ -0.6 & 0 \end{pmatrix} \\ A_{dc} &= \begin{pmatrix} -0.2 & 0.4 \\ 0 & 0.2 \end{pmatrix} \\ A_{cc} &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \\ A_{cd} &= \begin{pmatrix} 0.4 & 0 \\ -0.2 & 0.4 \end{pmatrix}. \end{aligned}$$

The matrix function $G(s)$ is given by (14) with

$$G_N(s) = \begin{pmatrix} -0.16s - 0.2 & 0.3s^2 + 0.46s + 0.22 \\ -0.6s^2 - 0.64s - 0.68 & 0.08s \end{pmatrix}$$

and

$$g_d(s) = s^2 + s + 1.$$

We find that (46) is feasible with a Hermitian matrix polynomial $P(\omega)$ of degree $2d = 0$, and hence the system (1) is exponentially stable according to Theorem 3. In particular, the index ζ in (52) is given by

$$\zeta = 0.218$$

which is achieved with

$$P(\omega) = \begin{pmatrix} 0.565 & -0.056 \\ -0.056 & 0.435 \end{pmatrix}.$$

The total number of LMI scalar variables in (52) is given by (54) and is equal to 32.

B. Example 2

Let us consider (1) with

$$\begin{aligned} A_{dd} &= \begin{pmatrix} 0.4 & -0.5 \\ 0.3 & 0.6 \end{pmatrix} \\ A_{dc} &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \\ A_{cc} &= \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \\ A_{cd} &= \begin{pmatrix} 0.5 & 0.4 \\ -0.6 & 0.3 \end{pmatrix}. \end{aligned}$$

The matrix function $G(s)$ is given by (14) with

$$G_N(s) = \begin{pmatrix} 0.4s^2 + 0.2s - 0.2 & -0.5s^2 - 0.7s - 1.8 \\ 0.3s^2 - 0.5s - 0.8 & 0.6s^2 + 1.1s - 0.7 \end{pmatrix}$$

and

$$g_d(s) = s^2 + 2s + 2.$$

We find that (46) is not feasible with a Hermitian matrix polynomial $P(\omega)$ of degree $2d = 0$, in particular the index ζ in (52) is given by

$$\zeta = -0.596.$$

From the earlier results, it can be guaranteed that we do not need more than a Hermitian matrix polynomial $P(\omega)$ of degree $2d = 2 \cdot 2 \cdot 2^2 = 16$. However, by simple checking, when we increase the degree of $P(\omega)$ incrementally, and find that (46) is feasible with a Hermitian matrix polynomial $P(\omega)$ of degree $2d = 2$, and hence the system (1) is exponentially stable according to Theorem 3. In particular, the index ζ in (52) is given by

$$\zeta = 0.324.$$

which is achieved with $P(\omega)$ given by

$$\Re(P(\omega)) = \begin{pmatrix} 1.254 - 1.786\omega + 0.978\omega^2 & 0.899 - 1.290\omega + 0.458\omega^2 \\ 0.899 - 1.290\omega + 0.458\omega^2 & 2.109 - 2.503\omega + 0.948\omega^2 \end{pmatrix}$$

and

$$\Im(P(\omega)) = \begin{pmatrix} 0 & -0.148 + 0.555\omega - 0.253\omega^2 \\ 0.148 - 0.555\omega + 0.253\omega^2 & 0 \end{pmatrix}.$$

The total number of LMI scalar variables in (52) is given by (54) and is equal to 84.

C. Example 3

Let us consider (1) with

$$A_{dd} = \begin{pmatrix} -0.5 & 0 & 0.2 \\ 0.3 & -0.3 & 0 \\ 0 & -0.4 & 0.3 \end{pmatrix}$$
$$A_{dc} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$A_{cc} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & -2 \\ -1 & 2 & -1 \end{pmatrix}$$
$$A_{cd} = \begin{pmatrix} 0.3 & -0.3 & 0 \\ 0 & 0.5 & 0 \\ 0.2 & 0 & -0.4 \end{pmatrix}.$$

The matrix function $G(s)$ is given by (14) with

$$G_N(s) = \begin{pmatrix} -0.5s^3 - 2.7s^2 - 5.6s - 1.6 & -0.6s - 1.2 + 0.5s^2 & 0.2s^3 + 1.4s^2 + 4.4s + 2.4 \\ 0.3s^3 + 1.8s^2 + 4.3s + 2.4 & -0.3s^3 - 2.3s^2 - 4.4s - 2.4 & -0.8s \\ 0.5s^2 + 1.7s + 1.8 & -0.4s^3 - 2.3s^2 - 3.4s - 2.4 & 0.3s^3 + 1.1s^2 + 1.4s + 1.2 \end{pmatrix}$$

and

$$g_d(s) = s^3 + 5s^2 + 10s + 4.$$

We find that (46) is feasible with a Hermitian matrix polynomial $P(\omega)$ of degree $2d = 0$, and hence the system (1) is exponentially stable according to Theorem 3. In particular, the index ζ in (52) is given by

$$\zeta = 0.282$$

which is achieved with

$$P(\omega) = \begin{pmatrix} 0.367 & -0.050 & 0.011 \\ -0.050 & 0.331 & -0.025 \\ 0.011 & -0.025 & 0.301 \end{pmatrix}.$$

The total number of LMI scalar variables in (52) is given by (54) and is equal to 162.

V. Conclusion

An LMI condition has been proposed for establishing stability of 2D discrete-time systems by introducing complex Lyapunov functions depending polynomially on a parameter and by exploiting the Gram matrix method. It has been shown that this condition is sufficient for any degree of the Lyapunov function candidate on the parameter, and that this condition is also necessary for a sufficiently large degree. Moreover, an a priori bound on the degree required for achieving necessity has been given.

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