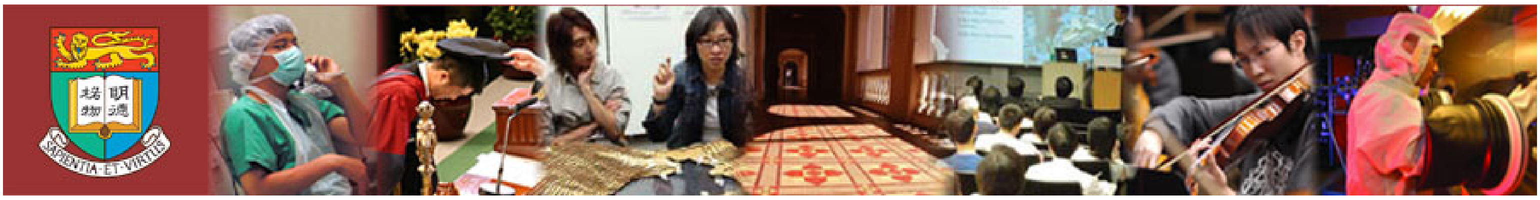

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On the Analysis of the Bifurcation Sets of Equilibrium Points in Parameter Space

G. Chesi, G. Tanaka, Y. Hirata, K. Aihara

Abstract— **This paper addresses the problems of characterizing and estimating the bifurcation sets of equilibrium points in multi-parameter space of a class of nonlinear dynamical systems. Specifically, we investigate the sets of parameters that lead to saddle-node bifurcations and Hopf bifurcations at an equilibrium point of interest. First, a characterization of these sets is provided in terms of the zeros of some functions. Second, this characterization is exploited to estimate such sets through convex programming for the case of polynomial dynamical systems. In particular, two conditions are proposed for establishing whether a sublevel set of a given polynomial does not contain parameters that lead to bifurcations. By using these conditions, the largest of such sublevel sets can be estimated by solving an eigenvalue problem. Some numerical examples illustrate the proposed results.**

I. INTRODUCTION

Nonlinear dynamical systems are commonly used to mathematically model time-varying behavior in the real-world, based on determinism [3]. Originating from Newtonian mechanics, differential and difference equations have been standard mathematical tools to describe the dynamics of physical, biological, chemical, and engineering systems [6]. Variation of system parameters in nonlinear dynamical systems often causes changes in qualitative or topological structure of the solutions, which are called bifurcation phenomena. The studies of bifurcation phenomena have deepened our understanding of transitive events caused by parameter variations, such as sudden disappearance of a stable equilibrium state and emergence of a disordered or chaotic state. Bifurcation theory is a concept which enables to characterize such a transition between qualitatively different regimes by formulating the conditions for the transition [4], [8].

Developing mathematical methodology to specify bifurcation sets and determine the presence or absence of bifurcation sets in a certain parameter set is useful to establish practical control theory for robustly maintaining a favorable state and avoiding an unfavorable state in nonlinear dynamical systems. Although numerical shooting methods to locate bifurcation sets have been successfully applied to analyses of nonlinear systems [5], less attention has been paid to the problem of finding a (largest) parameter set in which it is guaranteed that the system does not show any bifurcation. Approaching this problem is significant as a first step to consider a robust control method for nonlinear dynamical systems with parameter uncertainty.

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This paper addresses the problems of characterizing and estimating the bifurcation sets of equilibrium points in multi-parameter space of a class of nonlinear dynamical systems. Specifically, we investigate the sets of parameters that lead to saddle-node bifurcations and Hopf bifurcations at an equilibrium point of interest. First, a characterization of these sets is provided in terms of the zeros of some functions. Second, this characterization is exploited to estimate such sets through convex programming for the case of polynomial dynamical systems. In particular, two conditions are proposed for establishing whether a sublevel set of a given polynomial does not contain parameters that lead to bifurcations. By using these conditions, the largest of such sublevel sets can be estimated by solving an eigenvalue problem. Some numerical examples illustrate the proposed results.

The paper is organized as follows. Section II introduces some preliminaries. Section III describes the proposed characterization and estimation of the bifurcation sets. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

First, let us introduce the notation used throughout the paper:

- s.t.: subject to;
- $\mathbb{R}$: space of real numbers;
- $\mathbb{C}$: space of complex numbers;
- j: imaginary unit, i.e. $j = \sqrt{-1}$;
- $\Re(a)$, $\Im(a)$: real and imaginary parts of a, i.e. $a = \Re(a) + j\Im(a)$;
- A': transpose of A;
- $\det(A)$: determinant of A;
- $\mathrm{spec}(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$;
- $A > 0$, $A \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix A.

We consider the class of continuous-time nonlinear dynamical systems defined by

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x,p), \\ x(0) & = & x_{init}, \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $x_{init} \in \mathbb{R}^n$ is the initial condition, $p \in \mathbb{R}^q$ is a parameter vector, and $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is a nonlinear function such that the solution of (1) exists.

Further assumptions on $f(x,p)$ will be made throughout the paper wherever required. In particular, in Section III-B it will be assumed that $f(x, p)$ is nonlinear in x and polynomial in p.

Let $x^*(p)$ be an equilibrium point of interest of the system (1). Clearly, $x^*(p)$ satisfies

$$f(x^*(p), p) = 0. \quad (2)$$

We consider the following problems.

Problem 1. To characterize the bifurcation set in the parameter space of $x^*(p)$, i.e. the set

$$\mathcal{B} = \{\bar{p} \in \mathbb{R}^q : \ x^*(p) \text{ has a bifurcation at } p = \bar{p}\}. \quad (3)$$

Problem 2. To determine guaranteed subsets of the parameter space where $x^*(p)$ has no bifurcation. In particular, we consider the computation of the largest sublevel set of a given function $g(p)$ where $x^*(p)$ has no bifurcation, i.e. the optimization problem

$$\begin{array}{rcl} c^* & = & \sup\limits_{c} \ c \\ & & \text{s.t.} \ \ \mathcal{G}(c) \cap \mathcal{B} = \emptyset, \end{array} \quad (4)$$

where

$$\mathcal{G}(c) = \{p \in \mathbb{R}^q : \ g(p) \le c\}. \quad (5)$$

The function $g(p)$ is supposed polynomial, nonnegative and such that

$$c_1 \le c_2 \quad \Rightarrow \quad \mathcal{G}(c_1) \subseteq \mathcal{G}(c_2). \quad (6)$$

Let us observe that Problems 1 and 2 are important problems since dynamical systems are often affected by parameters whose change can lead to bifurcations of the equilibrium points. In particular, Problem 1 aims to determine the set of parameters for which a given equilibrium point has a bifurcation. Problem 2 aims to determine sets of parameters for which a given equilibrium point has no bifurcations, which is important in order to avoid bifurcations.

B. SOS Polynomials

A polynomial is said SOS if is the sum of squares of polynomials. It turns out that establishing whether a polynomial is SOS amounts to checking feasibility of an LMI, see e.g. [2] and references therein.

Indeed, let $p(x)$ be a polynomial of degree not greater than $2m$ with $x \in \mathbb{R}^n$. We can express $p(x)$ as

$$p(x) = b(x)' \left(P + L(\alpha)\right) b(x), \quad (7)$$

where $b(x) \in \mathbb{R}^{\sigma(n,m)}$ (called power vector) is a vector containing all the monomials of degree not greater than m, whose number is given by

$$\sigma(n, m) = \frac{(n+m)!}{n!m!}, \quad (8)$$

for instance according to

$$b(x) = (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots,)', \quad (9)$$

$P \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is a symmetric matrix satisfying

$$p(x) = b(x)'Pb(x), \quad (10)$$

$L(\alpha) \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{L = L' \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)} : \ b(x)'Lb(x) = 0\right\}, \quad (11)$$

whose dimension is given by

$$\tau(n, m) = \frac{1}{2}\sigma(n, m) \left(\sigma(n, m) + 1\right) - \sigma(n, 2m), \quad (12)$$

and $\alpha \in \mathbb{R}^{\tau(n,m)}$ is a free vector. This representation is known as Gram matrix method and square matrix representation (SMR).

The polynomial $p(x)$ is said SOS if and only if there exist polynomials $p_1(x), p_2(x), \ldots$ such that

$$p(x) = \sum_i p_i(x)^2, \quad (13)$$

and this condition holds if and only if there exists α satisfying the LMI

$$P + L(\alpha) \ge 0. \quad (14)$$

Hence, establishing whether $p(x)$ is SOS amounts to establishing whether the LMI (14) is feasible, and this problem can be solved through a convex optimization problem.

III. PROPOSED RESULTS

A. Characterization

In this section we address Problem 1, i.e. the characterization of the bifurcation set $\mathcal{B}$ in (3). Let us start by recalling the following results, which establish a connection between bifurcations of $x^*(p)$ and the Jacobian matrix of $f(x, p)$, which is defined as

$$J(x, p) = \frac{df(x, p)}{dx}. \quad (15)$$

Specifically, the next theorem considers the case of saddle-node bifurcations.

Theorem 1 ([3]): If there is a saddle-node bifurcation of $x^*(p)$ at $p = \bar{p}$, then $J(x^*(\bar{p}), \bar{p})$ has a simple zero eigenvalue.

The following theorem establishes a connection between Hopf bifurcations of $x^*(p)$ and the Jacobian matrix of $f(x, p)$.

Theorem 2 ([3]): If there is a Hopf bifurcation of $x^*(p)$ at $p = \bar{p}$, then $J(x^*(\bar{p}), \bar{p})$ has a simple pair of imaginary eigenvalues.

Remark. Let us observe that Theorems 1 and 2 provide necessary conditions for the existence of saddle-node bifurcations and Hopf bifurcations, respectively. Sufficient and necessary conditions can be obtained by introducing further requirements on the Jacobian matrix of $f(x, p)$ as explained in [3], which are not considered in this paper for simplicity.

The first idea proposed in this paper consists of obtaining a characterization of the bifurcation set in the parameter space by deriving an analytical expression of the set of parameters for which the Jacobian matrix $J(x^*(\bar{p}), \bar{p})$ has null or imaginary eigenvalues. In particular, we define the sets

$$\begin{aligned}\mathcal{B}_{sn} &= \{p \in \mathbb{R}^q : \ J(x^*(p), p) \text{ has a simple} \\ &\quad \text{zero eigenvalue}\}\,,\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathcal{B}_{h} &= \{p \in \mathbb{R}^q : \ J(x^*(p), p) \text{ has a simple pair} \\ &\quad \text{of imaginary eigenvalues}\}\,.\end{aligned} \tag{17}$$

From Theorems 1 and 2 it follows that

$$\mathcal{B} \subseteq \mathcal{B}_{sn} \cup \mathcal{B}_h. \tag{18}$$

The following result provides a characterization of the bifurcation set $\mathcal{B}_{sn}$ in terms of the zeros of two functions of p.

Theorem 3: Let us define

$$f_0(p) = \det\left(J(x^*(p), p)\right), \tag{19}$$

and

$$f_1(p) = \left.\frac{dr(\lambda, p)}{d\lambda}\right|_{\lambda=0}, \tag{20}$$

where

$$r(\lambda, p) = \det\left(\lambda I - J(x^*(p), p)\right). \tag{21}$$

Then,

$$\mathcal{B}_{sn} = \{p \in \mathbb{R}^q : \ f_0(p) = 0, \ f_1(p) \neq 0\}. \tag{22}$$

Proof. From (16) one has that $p \in \mathcal{B}_{sn}$ if and only if $J(x^*(p), p)$ has a simple zero eigenvalue. Since the determinant of a square matrix is the product of its eigenvalues, it follows that $J(x^*(p), p)$ has a zero eigenvalue if and only if

$$f_0(p) = 0$$

i.e. $f_0(p) = 0$ is a necessary condition for p to belong to $\mathcal{B}_{sn}$. In order to make this condition also sufficient, let us observe that the characteristic polynomial of $J(x^*(p), p)$ should have a root in zero with multiplicity not greater than one. Let us observe that $r(\lambda, p)$ is the characteristic polynomial of $J(x^*(p), p)$, and $r(\lambda, p)$ has a root $\lambda = 0$ with multiplicity not greater than one if and only if

$$f_1(p) \neq 0$$

which completes the proof. □

Theorem 3 provides an equivalent expression of the bifurcation set $\mathcal{B}_{sn}$ through the functions $f_0(p)$ and $f_1(p)$. In particular, such a set is given by the values of p that are zeros of $f_0(p)$ but not zeros of $f_1(p)$.

The following result provides a characterization of the bifurcation set $\mathcal{B}_h$ in terms of the zeros of two functions of p and an additional scalar variable.

Theorem 4: Let us define

$$f_2(\omega, p) = r(j\omega, p), \tag{23}$$

and

$$f_3(\omega, p) = \left.\frac{dr(\lambda, p)}{d\lambda}\right|_{\lambda=j\omega}. \tag{24}$$

Then,

$$\mathcal{B}_h = \{p \in \mathbb{R}^q : \ f_2(\omega, p) = 0, \ f_3(\omega, p) \neq 0, \ \omega \in \mathbb{R} \setminus \{0\}\}. \tag{25}$$

Proof. From (17) one has that $p \in \mathcal{B}_h$ if and only if $J(x^*(p), p)$ has a simple pair of imaginary eigenvalues. Since the roots of the characteristic polynomial of a square matrix are the eigenvalues of the matrix, it follows that such a polynomial should have a simple pair of imaginary roots. Since $r(\lambda, p)$ is the characteristic polynomial of $J(x^*(p), p)$ and since $J(x^*(p), p)$ is real, it follows that $J(x^*(p), p)$ has a pair of imaginary eigenvalues if and only if there exists $\omega \in \mathbb{R} \setminus \{0\}$ such that

$$r(j\omega, p) = f_2(\omega, p) = 0,$$

i.e. $f_2(\omega, p) = 0$ for some $\omega \in \mathbb{R} \setminus \{0\}$ is a necessary condition for p to belong to $\mathcal{B}_h$. In order to make this condition also sufficient, let us observe that the root $j\omega$ of $r(\lambda, p)$ should have multiplicity not greater than one, which is the case if and only if

$$\left.\frac{dr(\lambda, p)}{d\lambda}\right|_{\lambda=j\omega} = f_3(\omega, p) \neq 0.$$

Therefore, the theorem holds. □

Theorem 4 provides an equivalent expression of the bifurcation set $\mathcal{B}_h$ through the functions $f_2(\omega, p)$ and $f_3(\omega, p)$. In particular, such a set is given by the values of p for which there exists $\omega \in \mathbb{R} \setminus \{0\}$ such that $f_2(\omega, p)$ is zero and $f_3(\omega, p)$ is nonzero.

B. Estimation

In this section we address Problem 2, i.e. the computation of the largest sublevel set defined by c^* in (4) of a given polynomial $g(p)$ where $x^*(p)$ has no bifurcation. For this, we assume that $f(x, p)$ is nonlinear in x and polynomial in p.

First of all, let us define the largest sublevel sets of $g(p)$ where $x^*(p)$ has no saddle-node bifurcations and no Hopf bifurcations by introducing the quantities

$$\begin{aligned} c_{sn} &= \sup_c \ c \\ &\quad \text{s.t.} \ \ \mathcal{G}(c) \cap \mathcal{B}_{sn} = \emptyset, \end{aligned} \tag{26}$$

and

$$\begin{aligned} c_{h} &= \sup_c \ c \\ &\quad \text{s.t.} \ \ \mathcal{G}(c) \cap \mathcal{B}_{h} = \emptyset. \end{aligned} \tag{27}$$

From (18) one has that

$$\left\{ \begin{array}{rcl} \mathcal{G}(c) \cap \mathcal{B}_{sn} & = & \emptyset, \\ \mathcal{G}(c) \cap \mathcal{B}_{h} & = & \emptyset, \end{array} \right. \Rightarrow \ \mathcal{G}(c) \cap \mathcal{B} = \emptyset, \tag{28}$$

which implies that the sought quantity c^* is not greater than c_{sn} and c_h, i.e.

$$c^* \leq \min\{c_{sn}, c_h\}. \tag{29}$$

Let us start by introducing the following result, which provides a condition for establishing whether the sublevel set $\mathcal{G}(c)$ has no intersection with $\mathcal{B}_{sn}$.

Theorem 5: Suppose there exist a polynomial $s(p)$ and scalars c and $\varepsilon > 0$ such that

$$v(p) \text{ is SOS}, \tag{30}$$

where $u(p)$ is the polynomial

$$v(p) = s(p) f_0(p) + (1 + \|p\|^2)^k \left(g(p) - c - \varepsilon\right), \tag{31}$$

and k is a chosen integer. Then,

$$\mathcal{G}(c) \cap \mathcal{B}_{sn} = \emptyset. \tag{32}$$

Proof. Suppose that $v(p)$ is SOS. Then,

$$v(p) \geq 0 \quad \forall p \in \mathbb{R}^q.$$

Let us consider any $p \in \mathcal{B}_{sn}$. From Theorem 3 this implies that $f_0(p) = 0$ and, hence,

$$\begin{array}{rcl} 0 & \leq & v(p) \\ & = & s(p) f_0(p) + (1 + \|p\|^2)^k \left(g(p) - c - \varepsilon\right) \\ & = & (1 + \|p\|^2)^k \left(g(p) - c - \varepsilon\right). \end{array}$$

Since $(1 + \|p\|^2)^k$ is positive, the previous condition implies that

$$g(p) - c - \varepsilon \geq 0,$$

and, due to the positivity of ε,

$$g(p) > c,$$

i.e. any point p of $\mathcal{B}_{sn}$ lies outside $\mathcal{G}(c)$. □

Theorem 5 provides a condition for establishing whether $\mathcal{G}(c) \cap \mathcal{B}_{sn} = \emptyset$. This condition exploits SOS polynomials and amounts to solving an LMI feasibility test by exploiting the Gram matrix method described in Section II-B.

The condition of Theorem 5 can be used to obtain a lower bound of c_{sn} via a convex optimization problem, in particular

$$\begin{array}{rcl} \hat{c}_{sn} & = & \sup\limits_{c,s} c \\ & & \text{s.t.} \ \ v(p) \text{ is SOS}. \end{array} \tag{33}$$

Indeed, Theorem 5 implies that

$$\hat{c}_{sn} \leq c_{sn}. \tag{34}$$

The optimization problem (33) consists of maximizing a linear function subject to an LMI, and hence belongs to the class of eigenvalue problems, see e.g. [1].

The following result extends Theorem 5 to the estimation of $\mathcal{B}_h$ by providing a condition for establishing whether the sublevel set $\mathcal{G}(c)$ has no intersection with $\mathcal{B}_h$.

Theorem 6: Suppose there exist polynomials $t(\omega, p)$ and $u(\omega, p)$ and scalars c and $\varepsilon > 0$ such that

$$y(\omega, p) \text{ is SOS}, \tag{35}$$

where

$$\begin{array}{rcl} y(\omega, p) & = & t(\omega, p) \Re(f_2(\omega, p)) + u(\omega, p) \Im(f_2(\omega, p)) \\ & & + \omega^2 (1 + \|p\|^2 + \omega^2)^k \left(g(p) - c - \varepsilon\right). \end{array} \tag{36}$$

Then,

$$\mathcal{G}(c) \cap \mathcal{B}_h = \emptyset. \tag{37}$$

Proof. Suppose that $y(\omega, p)$ is SOS. Then,

$$y(\omega, p) \geq 0 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathbb{R}^q.$$

Let us consider any $p \in \mathcal{B}_h$, and let $\omega \in \mathbb{R} \setminus \{0\}$ be such that $f_2(\omega, p) = 0$ (the existence of such a ω is ensured by Theorem 4). We have that

$$\begin{array}{rcl} 0 & \leq & y(\omega, p) \\ & = & t(\omega, p) \Re(f_2(\omega, p)) + u(\omega, p) \Im(f_2(\omega, p)) \\ & & + \omega^2 (1 + \|p\|^2 + \omega^2)^k \left(g(p) - c - \varepsilon\right) \\ & = & \omega^2 (1 + \|p\|^2 + \omega^2)^k \left(g(p) - c - \varepsilon\right). \end{array}$$

Since $(1 + \|p\|^2)^k$ and ω^2 are positive, the previous condition implies that

$$g(p) - c - \varepsilon \geq 0,$$

and, due to the positivity of ε,

$$g(p) > c,$$

i.e. any point p of $\mathcal{B}_h$ lies outside $\mathcal{G}(c)$. □

Theorem 6 provides a condition for establishing whether $\mathcal{G}(c) \cap \mathcal{B}_h = \emptyset$. As the condition of Theorem 5, the condition of Theorem 6 exploits SOS polynomials and amounts to solving an LMI feasibility test by exploiting the Gram matrix method described in Section II-B. Let us observe that the polynomials in the condition of Theorem 6 are polynomials in the vector p and in the scalar ω.

The condition of Theorem 6 can be used to obtain a lower bound of c_h via an eigenvalue problem similarly to (33), in particular

$$\begin{array}{rcl} \hat{c}_{h} & = & \sup\limits_{c,t,u} c \\ & & \text{s.t.} \ \ y(\omega, p) \text{ is SOS}. \end{array} \tag{38}$$

Indeed, Theorem 6 implies that

$$\hat{c}_h \leq c_h. \tag{39}$$

The lower bounds $\hat{c}_{sn}$ and $\hat{c}_h$ can be used to obtain a lower bound of the sought quantity c^*. Indeed, from (29) one obtains

$$c^* \geq \hat{c}, \tag{40}$$

where

$$\hat{c} = \min\{\hat{c}_{sn}, \hat{c}_h\}. \tag{41}$$

This means that the inner points of the sublevel set $\mathcal{G}(\hat{c})$ are guaranteed not to belong to the bifurcation set $\mathcal{B}$, i.e.

$$\mathcal{G}(c) \cap \mathcal{B} = \emptyset \quad \forall c \in [0, \hat{c}). \tag{42}$$

IV. EXAMPLES

In this section we present some illustrative examples of the proposed results. The eigenvalue problems (33) and (38) are solved with the toolbox SeDuMi [7].

A. Example 1

Let us consider (1) with

$$f(x,p) = \begin{pmatrix} -2x_2 + (1+2p_1)x_3 - x_2^2 \\ 7x_1 - (1-p_1)x_2 + x_1x_3 \\ (1+p_1)x_1 - x_3 - x_1^3 \end{pmatrix},$$

and

$$x^*(p) = (0,0,0)'.$$

First, let us use Theorems 3 and 4 to obtain a description of the sets $\mathcal{B}_{sn}$ and $\mathcal{B}_h$. From Theorem 3 it follows that $\mathcal{B}_{sn}$ is given by (22) with

$$f_0(p) = -13 - p_1^2 + 2p_1 - 2p_1^3,$$

and

$$f_1(p) = 14 - 4p_1 - 2p_1^2.$$

From Theorem 4 it follows that $\mathcal{B}_h$ is given by (25) with

$$\begin{aligned} f_2(\omega,p) \quad = \quad & 13 - 2p_1 + p_1^2 - 2\omega^2 + 2p_1^3 + \omega^2 p_1 \\ & + j\left(14\omega - 4\omega p_1 - 2\omega p_1^2 - \omega^3\right), \end{aligned}$$

and

$$f_3(\omega,p) = 14 - 4p_1 - 2p_1^2 - 3\omega^2 + j\left(4\omega - 2\omega p_1\right).$$

Second, let us estimate the largest sublevel set of the function

$$g(p) = p^2,$$

that does not contain values of p leading to bifurcations, i.e. the quantities c_{sn} and c_h in (26)–(27). For c_{sn}, we solve the eigenvalue problem (33), finding the lower bound

$$\hat{c}_{sn} = 5.021.$$

For c_h, we solve the eigenvalue problem (38), finding the lower bound

$$\hat{c}_h = 0.524.$$

Figure 1 shows the curves $\Re(f_2(\omega,p)) = 0$, $\Im(f_2(\omega,p)) = 0$ and $g(p) = \hat{c}_h$ in the plane p-ω.

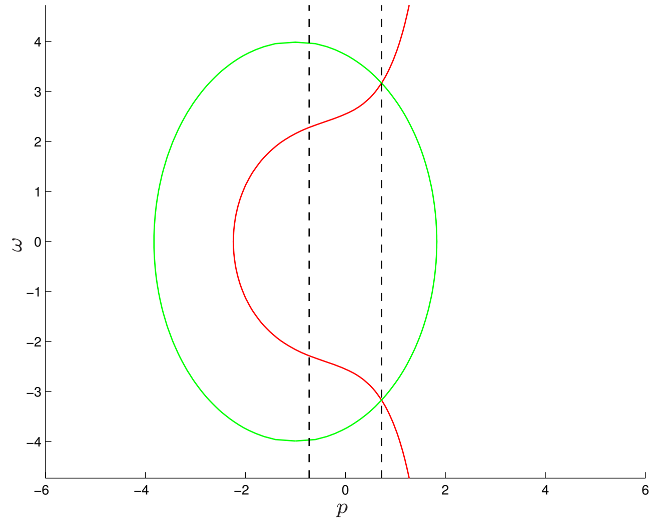

Fig. 1. Example 1. Curves $\Re(f_2(\omega,p)) = 0$ (red line), $\Im(f_2(\omega,p)) = 0$ (green line) and $g(p) = \hat{c}_h$ (black dashed line) in the plane p-ω.

B. Example 2

Let us consider (1) with

$$f(x,p) = \begin{pmatrix} x_1(1+p_1) - 6x_2 + x_1x_3 - x_2x_3 \\ 4x_1 - x_3(2p_1 - p_2 + 3) - p_1x_1^2 \\ -3x_1 + (3-p_2)x_2 - x_3 + p_2x_2^2 \end{pmatrix},$$

and

$$x^*(p) = (0,0,0)'.$$

First, let us use Theorems 3 and 4 to obtain a description of the sets $\mathcal{B}_{sn}$ and $\mathcal{B}_h$. From Theorem 3 it follows that $\mathcal{B}_{sn}$ is given by (22) with

$$f_0(p) = -21p_1 - 8p_1p_2 + 12p_2 + p_2^2 - 69 + 6p_1^2 - 2p_1^2p_2 + p_1p_2^2,$$

and

$$f_1(p) = 5p_1 - 2p_1p_2 - 6p_2 + p_2^2 + 32.$$

From Theorem 4 it follows that $\mathcal{B}_h$ is given by (25) with

$$\begin{aligned} f_2(\omega,p) \quad = \quad & 69 + 21p_1 - 12p_2 - 6p_1^2 + 8p_1p_2 - p_2^2 \\ & + 2p_1^2p_2 - p_1p_2^2 + \omega^2 p_1 + j\,(32\omega + 5\omega p_1 \\ & - 6\omega p_2 - 2\omega p_1 p_2 + \omega p_2^2 - \omega^3), \end{aligned}$$

and

$$f_3(\omega,p) = 32 + 5p_1 - 6p_2 - 2p_1p_2 + p_2^2 - 3\omega^2 + j\left(-2\omega p_1\right).$$

Second, let us estimate the largest sublevel set of the function

$$g(p) = p_1^2 + p_2^2,$$

that does not contain values of p leading to bifurcations, i.e. the quantities c_{sn} and c_h in (26)–(27). For c_{sn}, we solve the eigenvalue problem (33), finding the lower bound

$$\hat{c}_{sn} = 3.574.$$

Figure 2 shows the curves $f_0(p) = 0$ and $g(p) = \hat{c}_{sn}$ in the plane p_1-p_2.

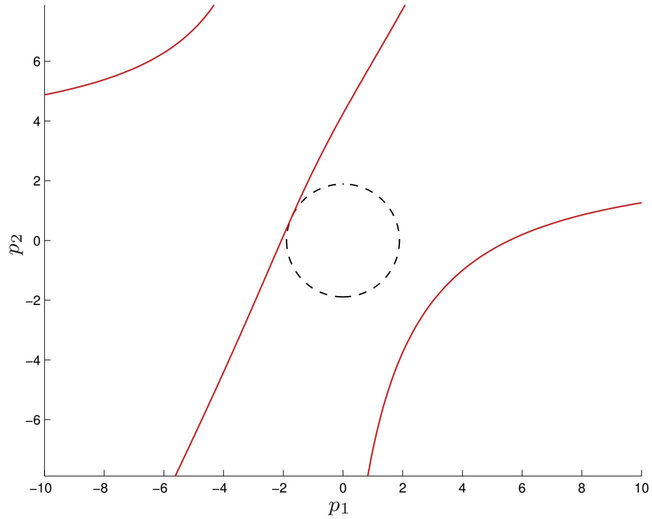

Fig. 2. Example 2. Curves $f_0(p) = 0$ (red line) and $g(p) = \hat{c}_{sn}$ (black dashed line) in the plane p_1-p_2.

For c_h, we solve the eigenvalue problem (38), finding the lower bound

$$\hat{c}_h = 1.510.$$

Figure 3 shows the surfaces $\Re(f_2(\omega, p)) = 0$, $\Im(f_2(\omega, p)) = 0$ and $g(p) = \hat{c}_h$ in the space p_1-p_2-ω.

V. Conclusions

We have investigated the sets of parameters that lead to saddle-node bifurcations and Hopf bifurcations at an equilibrium point of interest for a class of nonlinear dynamical systems. A characterization of these sets has been provided in terms of the zeros of some functions. This characterization has been exploited for the case of polynomial dynamical systems to derive conditions based on convex programming for establishing whether a sublevel set of a given polynomial does not contain parameters that lead to bifurcations. By using these conditions, the largest of such sublevel sets can be estimated by solving an eigenvalue problem.

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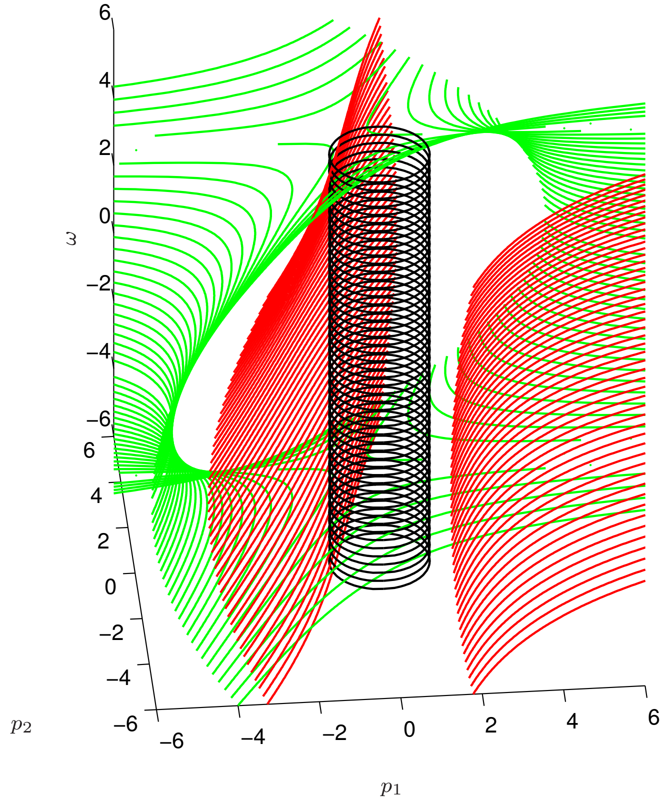

Fig. 3. Example 2. Surfaces $\Re(f_2(\omega, p)) = 0$ (red line), $\Im(f_2(\omega, p)) = 0$ (green line) and $g(p) = \hat{c}_h$ (black dashed line) in the space p_1-p_2-ω.

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