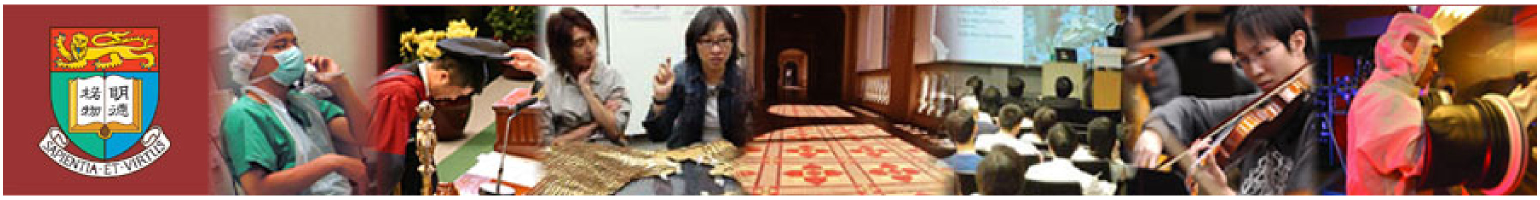

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Recursive Filtering for a Class of Nonlinear Systems with Missing Measurements

Jun Hu
Research Institute of
Intelligent Control and Systems,
Harbin Institute of Technology,
Harbin 150001, China.

Zidong Wang
Department of Information Systems
and Computing, Brunel University,
London, UB8 3PH, UK.
Email: Zidong.Wang@brunel.ac.uk

Bo Shen
Institute for Automatic
Control and Complex Systems,
University of Duisburg-Essen,
D-47057 Duisburg, Germany.

Chenxiao Cai
School of Automation, Nanjing
University of Science and Technology,
Nanjing 210094, China.

James Lam
Department of Mechanical Engineering,
The University of Hong Kong,
Hong Kong, China.

***Abstract*—This paper is concerned with the finite-horizon recursive filtering problem for a class of nonlinear time-varying systems with missing measurements. The missing measurements are modeled by a series of mutually independent random variables obeying Bernoulli distributions with possibly different occurrence probabilities. Attention is focused on the design of a recursive filter such that, for the missing measurements, an upper bound for the filtering error covariance is guaranteed and such an upper bound is subsequently minimized by properly designing the filter parameters at each sampling instant. The desired filter parameters are obtained by solving two Riccati-like difference equations that are of a recursive form suitable for online applications. A simulation example is exploited to demonstrate the effectiveness of the proposed filter design scheme.**

I. Introduction

The past few decades have seen a surge of research interest on the filtering or state estimation theories due to their extensive applications in a variety of practical areas including weather forecasting, economics, radar tracker and global positioning system. Up to now, a great deal of efforts has been delivered to the design issues of various kinds of filters, for example, Kalman filters [2], [16], [22], extended Kalman filters [9], [11], [24], [25] and H_∞ filters [1], [6], [10], [12], [15], [23], [28]. Among them, the traditional Kalman filter has been shown to be an optimal one in the sense of minimum variance for the linear systems, and the extended Kalman filter has been developed to serve as an effective way for handling the nonlinear estimation problems. Recently, the robust extended Kalman filtering problem has been tackled in [24] for a class of nonlinear systems, and a filtering algorithm has been presented in a recursive form suitable for online applications.

Most traditional filter design approaches rely on the assumption that the measurement signals are perfectly transmitted. Such an assumption, however, is conservative in many engineering practice presented with unreliable communication channels. For example, due to temporal sensor failures or network congestions, the system measurements may contain noise only at certain time points and the true signals are simply missing. As such, the control and filtering problems with missing measurements have received considerable research attention, see e.g. [5], [8], [13], [14], [18]–[21], [27]. A common way for modeling the data missing is to introduce a random variable satisfying the Bernoulli binary distribution taking values on either 1 or 0, where 1 is for the perfect signal delivery and 0 represents the measurement missing. Most of the aforementioned results have been based on the hypothesis that all sensors have identical failure characteristics [8]. However, such a hypothesis may not be true in the case that the signals are observed by multiple sensors and each individual sensor may have different failure rate.

It is worth mentioning that most existing results regarding the missing measurements have concentrated on *linear* systems. It is well known that the nonlinearity is a ubiquitous feature in almost all practical systems, and the occurrence of the nonlinearity inevitably degrades the system performance and even leads to instability [25], [26]. However, so far, the filtering problem for *general nonlinear* stochastic systems with missing measurements has not been thoroughly investigated yet, not to mention the case where multiple sensors undergo probabilistic missing measurements. It is, therefore, our aim of this paper to shorten the gap by initiating a study on such a challenging issue.

Motivated by the above discussions, we aim to investigate the recursive filtering problem for a class of nonlinear time-varying systems with missing measurements. A series of mutually independent random variables are introduced to describe the phenomenon of missing measurements where individual sensor is allowed to have different missing probability. The finite-horizon filter is designed such that an upper bound on the filtering error covariance is guaranteed and such an upper bound is subsequently minimized by the designed filter at each sampling instant. The proposed filter scheme is given in terms of the solutions to two Riccati-like difference equations, and therefore the algorithm is suitable for recursive computations.

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Notations. The notations used throughout the paper are standard. R^n and $R^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ matrices, respectively. For a matrix P, P^T and P^{-1} represent its transpose and inverse, respectively. $\mathrm{tr}(\cdot)$ stands for the trace of a matrix. $\circ$ is the Hadamard product defined as $[A \circ B]_{ij} = A_{ij} \cdot B_{ij}$. $E\{x\}$ stands for the expectation of the stochastic variable x. I and 0 represent the identity matrix and the zero matrix with appropriate dimensions, respectively. $\mathrm{diag}\{X_1, X_2, \ldots, X_n\}$ stands for a block-diagonal matrix with matrices $X_1, X_2, \ldots, X_n$ on the diagonal. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminaries

Consider the following class of time-varying nonlinear systems:

$$x_{k+1} = f(x_k) + D_k\omega_k \quad (1)$$
$$y_k = \Xi_k C_k x_k + \nu_k \quad (2)$$

where $x_k \in R^n$ is the system state to be estimated, the initial value x_0 has mean $\bar{x}_0$ and covariance $P_{0|0}$, $y_k \in R^m$ is the output vector, $\omega_k \in R^r$ is the process noise with zero-mean and covariance $Q > 0$, and $\nu_k \in R^m$ is the zero-mean measurement noise with covariance $V > 0$. The nonlinear function $f(x_k)$ is analytic everywhere with known form, C_k and D_k are known and bounded matrices with appropriate dimensions. $\Xi_k = \mathrm{diag}\{\xi_k^1, \xi_k^2, \ldots, \xi_k^m\}$ is to account for the missing measurements where the mutually uncorrelated (in k and i) random variables $\xi_k^i \in R$ $(i = 1, 2, \ldots, m)$ take values of 1 and 0 with

$$\mathrm{Prob}\left\{\xi_k^i = 1\right\} = E\left\{\xi_k^i\right\} := \vartheta_k^i, \quad (3)$$
$$\mathrm{Prob}\left\{\xi_k^i = 0\right\} = 1 - E\left\{\xi_k^i\right\} := 1 - \vartheta_k^i. \quad (4)$$

Here, $\vartheta_k^i \in [0, 1]$ is a known constant, ξ_k^i is assumed to be independent with ω_k, ν_k and x_0. Also, the noise signals mentioned above are uncorrelated with each other.

In this paper, we design a filter of the following form:

$$\hat{x}_{k+1|k} = f\left(\hat{x}_{k|k}\right), \quad (5)$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - \bar{\Xi}_{k+1}C_{k+1}\hat{x}_{k+1|k}\right) \quad (6)$$

where $\hat{x}_{k|k}$ is the estimate of x_k at time k with $\hat{x}_{0|0} = \bar{x}_0$, $\hat{x}_{k+1|k}$ is the one-step prediction at time k, K_{k+1} is the filter parameter to be determined, and $\bar{\Xi}_{k+1} := E\{\Xi_{k+1}\} := \mathrm{diag}\{\vartheta_{k+1}^1, \vartheta_{k+1}^2, \ldots, \vartheta_{k+1}^m\}$.

The objective of this paper is twofold. First, we aim to design a finite-horizon filter of form (5)-(6) such that, for the missing measurements, an upper bound for the filtering error covariance is guaranteed, i.e., there exists a sequence of positive-definite matrices $\Sigma_{k+1|k+1}$ $(0 \le k \le N)$ satisfying

$$E\left\{\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)^T\right\} \le \Sigma_{k+1|k+1} \quad (7)$$

Second, we shall minimize such an upper bound $\Sigma_{k+1|k+1}$ by appropriately designing the filter parameter at each sampling instant.

Remark 1 In the model (2), $C_k x_k$ represents the measurement output subject to probabilistic information loss characterized by the matrix Ξ_k, and ν_k is the random exogenous noise acting on the measurement output. In other words, the model (2) is comprehensive to include the practical cases of probabilistic missing measurements and external additive disturbances, thereby reflecting the engineering practice in a more realistic way.

Remark 2 In this paper, the phenomena of measurements missing is considered. Owing to the sensors aging and/or sensor temporal failure, the missing measurements may occur intermittently. In (2), Ξ_k is introduced to characterize the missing measurements where the random variable ξ_k^i $(i = 1, 2, \ldots, m)$ corresponds to the i sensor operating at the kth sampling time point. For different sensors, it would be more reasonable to allow multiple sensors to have different missing probabilities (or failure rates [8]).

Before ending this section, we recall the following lemmas which will be frequently used in subsequent developments.

Lemma 1 [7] Let $A = [a_{ij}]_{n\times n}$ be a real-valued matrix and $B = \mathrm{diag}\{b_1, b_2, \ldots, b_n\}$ be a diagonal stochastic matrix. Then

$$E\{BAB^T\} = \begin{bmatrix} E\{b_1^2\} & E\{b_1b_2\} & \cdots & E\{b_1b_n\} \\ E\{b_2b_1\} & E\{b_2^2\} & \cdots & E\{b_2b_n\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{b_nb_1\} & E\{b_nb_2\} & \cdots & E\{b_n^2\} \end{bmatrix} \circ A$$

where $\circ$ is the Hadamard product.

Lemma 2 [22] Given matrices A, H, G and F with appropriate dimensions such that $FF^T \le I$. Let X be a symmetric positive definite matrix and γ be an arbitrary positive constant such that $\gamma^{-1}I - GXG^T > 0$. Then the following inequality holds

$$(A + HFG)X(A + HFG)^T \le A\left(X^{-1} - \gamma G^TG\right)^{-1}A^T + \gamma^{-1}HH^T. \quad (8)$$

Lemma 3 [17] For $0 \le k \le N$, suppose that $X = X^T > 0$, $S_k(X) = S_k^T(X) \in R^{n\times n}$ and $H_k(X) = H_k^T(X) \in R^{n\times n}$. If

$$S_k(Y) \ge S_k(X), \ \forall \ X \le Y = Y^T \quad (9)$$

and

$$H_k(Y) \ge S_k(Y), \quad (10)$$

then the solutions M_k and N_k to the following difference equations

$$M_{k+1} = S_k(M_k), \ N_{k+1} = H_k(N_k), \ M_0 = N_0 > 0 \quad (11)$$

satisfy $M_k \le N_k$.

III. Main results

In this section, a sufficient condition for the design of filter parameters is established by solving two Riccati-like difference equations.

To proceed, denote the one-step prediction error as $\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$ and the filtering error as $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$. Subtracting (5) from (1), we obtain

$$\tilde{x}_{k+1|k} = f(x_k) - f(\hat{x}_{k|k}) + D_k\omega_k. \tag{12}$$

By using the Taylor series expansion around $\hat{x}_{k|k}$, we linearize $f(x_k)$ as follows:

$$f(x_k) = f(\hat{x}_{k|k}) + A_k\tilde{x}_{k|k} + o(|\tilde{x}_{k|k}|) \tag{13}$$

where

$$A_k = \frac{\partial f(x_k)}{\partial x_k}|_{x_k=\hat{x}_{k|k}},$$

and $o(|\tilde{x}_{k|k}|)$ stands for the high-order terms of the Taylor series expansion. For presentation convenience, along the similar line of [3], [25], the high-order terms are transformed into the following easy-to-handle formulation:

$$o(|\tilde{x}_{k|k}|) = B_k\Omega_k L_k\tilde{x}_{k|k}, \tag{14}$$

where B_k is a bounded problem-dependent scaling matrix, L_k is a bounded matrix for providing an extra degree of freedom to tune the filter, and Ω_k is an unknown time-varying matrix accounting for the linearization errors of the dynamical model and satisfies

$$\Omega_k\Omega_k^T \le I. \tag{15}$$

Remark 3 In traditional extended Kalman filter algorithms, the Taylor series expansion is employed to linearize the nonlinearity $f(x_k)$, and the linearization errors are simply neglected which would inevitably lead to conservatism in certain cases. Recently, a new approach has been proposed in [3] to describe the higher-order terms in the Taylor series in terms of parameter uncertainties. In this paper, as in [3], [24], we use the deterministic matrix Ω_k and the scaling matrix B_k in (14)-(15) to account for the linearization errors in obtaining the matrix A_k. For more details we refer the reader to Appendix C of [3] where a nice interpretation has been given. It is worthwhile to further mention that, in practice, the high-order terms in the Taylor series expansion are commonly bounded and it is reasonable to regard them as deterministic uncertainties affecting the system matrix A_k.

It follows from (12)-(14) that the one-step prediction error is given by

$$\tilde{x}_{k+1|k} = (A_k + B_k\Omega_k L_k)\,\tilde{x}_{k|k} + D_k\omega_k. \tag{16}$$

On the other hand, it follows from (6) that the filtering error $\tilde{x}_{k+1|k+1}$ can be described by

$$\begin{aligned}&\tilde{x}_{k+1|k+1}\\ &= \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\tilde{x}_{k+1|k} - K_{k+1}\nu_{k+1}\\ &\quad - K_{k+1}\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)C_{k+1}x_{k+1}\end{aligned} \tag{17}$$

Based on (16) and (17), we are ready to present the following lemmas which give the recursion of the one-step prediction error covariance and filtering error covariance, respectively.

Lemma 4 The one-step prediction error covariance $P_{k+1|k}$ obeys the following recursion:

$$\begin{aligned}P_{k+1|k} =& \left(A_k + B_k\Omega_k L_k\right)P_{k|k}\left(A_k + B_k\Omega_k L_k\right)^T\\ &+ D_kQD_k^T\end{aligned} \tag{18}$$

where $P_{k|k} = E\{\tilde{x}_{k|k}\tilde{x}_{k|k}^T\}$ is the filtering error covariance.

Proof Since (18) follows from (16) directly, the proof is omitted for brevity.

Lemma 5 The filtering error covariance $P_{k+1|k+1}$ is given as follows:

$$\begin{aligned}&P_{k+1|k+1}\\ &= \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)P_{k+1|k}\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T\\ &\quad + K_{k+1}(J_{k+1} + V)K_{k+1}^T\end{aligned} \tag{19}$$

where

$$\begin{aligned}J_{k+1} :=& \check{\Xi}_{k+1} \circ \left(C_{k+1}E\left\{x_{k+1}x_{k+1}^T\right\}C_{k+1}^T\right),\\ \check{\Xi}_{k+1} :=& \mathrm{diag}\{\vartheta_{k+1}^1\left(1 - \vartheta_{k+1}^1\right), \vartheta_{k+1}^2\left(1 - \vartheta_{k+1}^2\right),\\ &\ldots, \vartheta_{k+1}^m\left(1 - \vartheta_{k+1}^m\right)\}.\end{aligned} \tag{20}$$

Proof According to (17), we have

$$\begin{aligned}&P_{k+1|k+1}\\ &= \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)P_{k+1|k}\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T\\ &\quad + K_{k+1}E\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)C_{k+1}x_{k+1}x_{k+1}^TC_{k+1}^T\\ &\quad \times \left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\}K_{k+1}^T + K_{k+1}VK_{k+1}^T.\end{aligned} \tag{21}$$

Next, applying Lemma 1 and together with the property of conditional expectation, we obtain

$$\begin{aligned}&E\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)C_{k+1}x_{k+1}x_{k+1}^T\\ &\times C_{k+1}^T\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\}\\ =&\check{\Xi}_{k+1} \circ \left(C_{k+1}E\left\{x_{k+1}x_{k+1}^T\right\}C_{k+1}^T\right)\end{aligned} \tag{22}$$

where $\check{\Xi}_{k+1}$ is defined in (20). Therefore, (19) follows directly from (21) and (22), and the proof of this Lemma is complete.

Remark 4 It can be seen that the linearization has been enforced to facilitate the recursive filtering algorithm developments. From Lemmas 4-5, the filtering error covariance can be obtained for the missing measurements provided that the matrix equations (18) and (19) are solvable. Unfortunately, due to the consideration of the nonlinearity, (18) and (19) are contaminated by some uncertain terms Ω_k and $E\left\{x_{k+1}x_{k+1}^T\right\}$, which lead to essential difficulty in determining the accurate value of the filtering error covariance $P_{k+1|k+1}$. In the following, an alternatively way is employed to design an appropriate filter parameter K_{k+1} such that there exists an upper bound for the filtering error covariance.

Now, we are in a position to present our main results. In view of Lemmas 2-5, the filter parameter is designed such that an optimized upper bound for the filtering error covariance is achieved at each sampling instant.

Theorem 1 Consider the one-step prediction error covariance and the filtering error covariance in (18)-(19), respectively. Assume that (15) holds. Let γ_k and ε be positive scalars.

If the following two Riccati-like difference equations

$$\begin{aligned}&\Sigma_{k+1|k}\\&=A_k\left(\Sigma_{k|k}^{-1}-\gamma_k L_k^T L_k\right)^{-1}A_k^T+\gamma_k^{-1}B_k B_k^T\\&\quad+D_k Q D_k^T,\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\Sigma_{k+1|k+1}\\&=\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\Sigma_{k+1|k}\\&\quad\times\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T+K_{k+1}\\&\quad\times\left[\breve{\Xi}_{k+1}\circ\left(C_{k+1}\Phi_{k+1|k}C_{k+1}^T\right)+V\right]K_{k+1}^T\end{aligned} \tag{24}$$

with initial condition $\Sigma_{0|0}=P_{0|0}>0$ have positive-definite solutions $\Sigma_{k+1|k}$ and $\Sigma_{k+1|k+1}$ such that, for all $0\le k\le N$, the following constraint

$$\gamma_k^{-1}I-L_k\Sigma_{k|k}L_k^T>0, \tag{25}$$

are satisfied where

$$\Phi_{k+1|k}:=(1+\varepsilon)\Sigma_{k+1|k}+\left(1+\varepsilon^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T, \tag{26}$$

then with the filter parameter K_{k+1} given by

$$\begin{aligned}&K_{k+1}\\&=\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Big[\bar{\Xi}_{k+1}C_{k+1}\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\\&\quad+\breve{\Xi}_{k+1}\circ\left(C_{k+1}\Phi_{k+1|k}C_{k+1}^T\right)+V\Big]^{-1}\end{aligned} \tag{27}$$

the matrix $\Sigma_{k+1|k+1}$ is an upper bound for $P_{k+1|k+1}$, i.e.,

$$P_{k+1|k+1}\le\Sigma_{k+1|k+1}. \tag{28}$$

Moreover, the filter parameter K_{k+1} given by (27) minimizes the upper bound $\Sigma_{k+1|k+1}$.

Proof Note that the covariance matrices $P_{k+1|k}$ and $P_{k+1|k+1}$ can be rewritten as the functions of $P_{k|k}$ and $P_{k+1|k}$, respectively. Then, it is not difficult to verify that the condition (9) in Lemma 3 is satisfied.

Now, we are ready to deal with the terms of the right-hand side of (19). Considering the following elementary inequality

$$\left(\varepsilon^{\frac{1}{2}}\tilde{x}_{k+1|k}-\varepsilon^{-\frac{1}{2}}\hat{x}_{k+1|k}\right)\left(\varepsilon^{\frac{1}{2}}\tilde{x}_{k+1|k}-\varepsilon^{-\frac{1}{2}}\hat{x}_{k+1|k}\right)^T\ge 0,$$

we can obtain the following inequality

$$\begin{aligned}&\tilde{x}_{k+1|k}\hat{x}_{k+1|k}^T+\hat{x}_{k+1|k}\tilde{x}_{k+1|k}^T\\&\le\varepsilon\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T+\varepsilon^{-1}\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T\end{aligned}$$

with $\varepsilon>0$ being a scalar, which yields

$$\begin{aligned}&E\left\{x_{k+1}x_{k+1}^T\right\}\\&\le E\left\{(1+\varepsilon)\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T+\left(1+\varepsilon^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T\right\}\\&=(1+\varepsilon)P_{k+1|k}+\left(1+\varepsilon^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T.\end{aligned} \tag{29}$$

Then, the last term of the right-hand side of (19) can be determined as

$$\begin{aligned}&K_{k+1}(J_{k+1}+V)K_{k+1}^T\\&\le K_{k+1}\left[\breve{\Xi}_{k+1}\circ\left(C_{k+1}M_{k+1|k}C_{k+1}^T\right)+V\right]K_{k+1}^T\end{aligned} \tag{30}$$

where where

$$M_{k+1|k}:=(1+\varepsilon)P_{k+1|k}+\left(1+\varepsilon^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T.$$

It then follows from (19) and (30) that

$$\begin{aligned}&P_{k+1|k+1}\\&\le\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)P_{k+1|k}\\&\quad\times\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T+K_{k+1}\\&\quad\times\left[\breve{\Xi}_{k+1}\circ\left(C_{k+1}M_{k+1|k}C_{k+1}^T\right)+V\right]K_{k+1}^T\end{aligned} \tag{31}$$

Combining (23), (24) and (31), we can show that the condition (10) in Lemma 3 is satisfied. Therefore, it follows directly from Lemmas 2-3 that

$$P_{k+1|k+1}\le\Sigma_{k+1|k+1}.$$

Having determined the upper bound $\Sigma_{k+1|k+1}$, we are now ready to show that the filter parameter given by (27) is optimal in the sense that it minimizes the upper bound $\Sigma_{k+1|k+1}$. Taking the partial derivative of (24) with respect to K_{k+1} and letting the derivative be zero, we have

$$\begin{aligned}&\frac{\partial\mathrm{tr}\left(\Sigma_{k+1|k+1}\right)}{\partial K_{k+1}}\\&=-2\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\\&\quad+2K_{k+1}\left[\breve{\Xi}_{k+1}\circ\left(C_{k+1}\Phi_{k+1|k}C_{k+1}^T\right)+V\right]\\&=0.\end{aligned} \tag{32}$$

From (32), and through straightforward algebraic manipulations, the optimal filter parameter K_{k+1} can be determined as follows:

$$\begin{aligned}&K_{k+1}\\&=\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Big[\bar{\Xi}_{k+1}C_{k+1}\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\\&\quad+\breve{\Xi}_{k+1}\circ\left(C_{k+1}\Phi_{k+1|k}C_{k+1}^T\right)+V\Big]^{-1}\end{aligned} \tag{33}$$

Obviously, the filter parameter K_{k+1} in (33) is identical to (27). To this end, the optimal filter gain K_{k+1} is designed in the sense of minimizing the upper bound $\Sigma_{k+1|k+1}$ for the filtering error covariance and, therefore, the proof of this theorem is complete.

Remark 5 At each sampling instant, the filter parameter K_{k+1} is designed in Theorem 1 to minimize the upper bound of filtering error covariance. The consideration of the multiple missing measurements constitutes the main difference between our work and the work of [24]. In our main results, the constants ϑ_k^i $(i=1,2,\ldots,m)$ are there for the missing measurements where all sensors are allowed to have different missing probabilities. Furthermore, the proposed filter is derived in terms of the solutions to two Riccati-like difference equations, which is recursive and therefore suitable for online applications.

IV. An Illustrative Example

Consider the following nonlinear system with missing measurements:

$$\begin{cases} x_{k+1} = f(x_k) + D_k \omega_k \\ y_k = \Xi_k C_k x_k + \nu_k \end{cases}$$

where

$$\begin{aligned} f(x_k) &= \begin{bmatrix} 0.8x_{1,k} + x_{1,k}x_{2,k} \\ 1.5x_{2,k} - x_{1,k}x_{2,k} \end{bmatrix}, \\ D_k &= \begin{bmatrix} 0.06 \\ 0.03 + 0.5e^{-5k} \end{bmatrix}, \\ C_k &= \begin{bmatrix} 0.85 & 0 \\ 0 & -1.5 \end{bmatrix} \end{aligned}$$

and $x_k = \begin{bmatrix} x_{1,k} & x_{2,k} \end{bmatrix}^T$ is the state vector with $x_{i,k}$ $(i = 1, 2)$ being the i-th element of the system state, $\omega_k \in R$ and $\nu_k \in R^2$ are zero-mean Gaussian white noises with covariances 0.5 and $0.02I_2$, respectively.

In the simulation, set the initial value of estimation as $\hat{x}_{0|0} = \bar{x}_0 = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}^T$ and $\Sigma_{0|0} = 10I$. Assume that $\bar{\Xi}_k = \text{diag}\{0.95, 0.90\}$. The other parameters are chosen as $B_k = \text{diag}\{0.1, 0.2\}$, $L_k = 0.1I_2$, $\gamma_k = 0.005$, and $\varepsilon = 0.35$. By solving (23) and (24), the filter parameter can be obtained recursively and the simulation results are shown in Figs. 1-4. Here, MSE-i $(i = 1, 2)$ denotes the mean square error (MSE) for the estimation of the state.

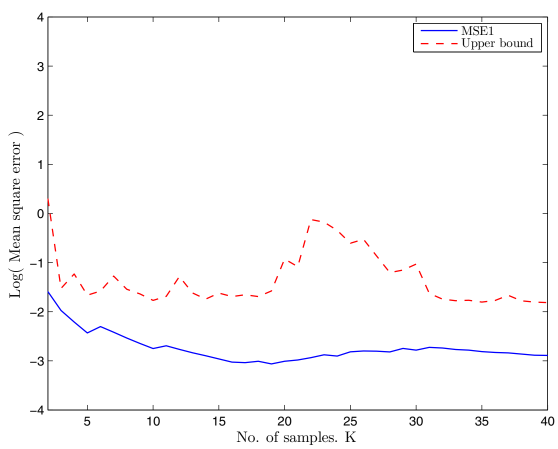

Fig. 1. MSE1 and its upper bound

In the figures, Figs. 1-2 show the upper bounds $\Sigma^{11}_{k|k}$ and $\Sigma^{22}_{k|k}$ as well as the MSE for the states $x_{1,k}$ and $x_{2,k}$, which confirm that the MSE stay below their upper bounds. The trajectories of the actual states $x_{i,k}$ and their estimates $\hat{x}_{i,k}$ $(i = 1, 2)$ are plotted in Figs. 3-4, which illustrate that the

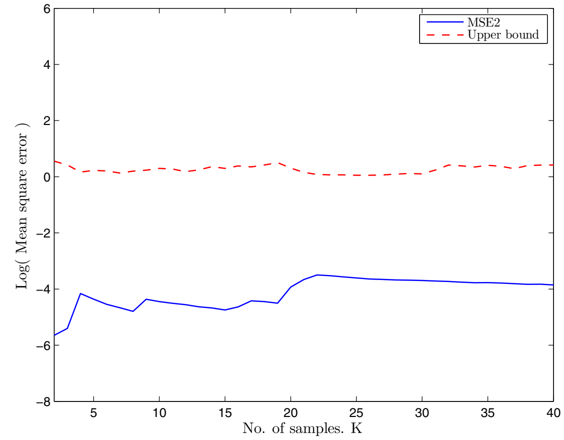

Fig. 2. MSE2 and its upper bound

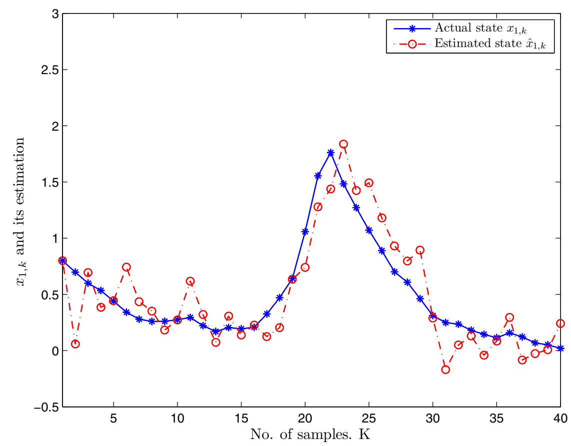

Fig. 3. The actual state $x_{1,k}$ and its estimation $\hat{x}_{1,k}$

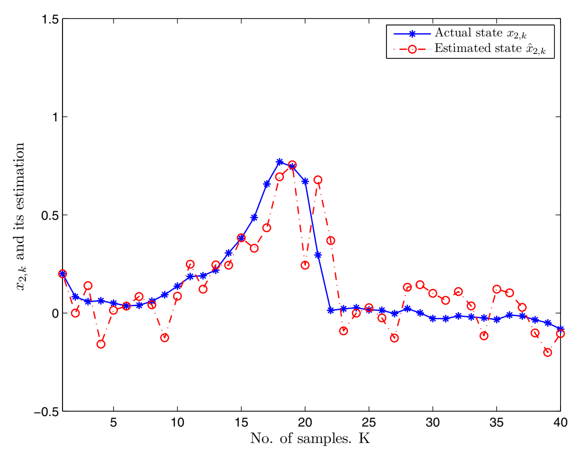

Fig. 4. The actual state $x_{2,k}$ and its estimation $\hat{x}_{2,k}$

presented scheme can perform well to estimate the system states.

V. Conclusions

In this paper, the finite-horizon filter design problem has been investigated for a class of time-varying nonlinear systems with missing measurements. A series of mutually independent random variables that obeys Bernoulli distribution has been introduced to describe the missing measurement phenomenon. A filter has been designed to guarantee an optimized upper bound on the filtering error covariance by means of solving two Riccati-like difference equations. Finally, a numerical example has been provided to illustrate the effectiveness of the main results. Further research topics include the extension of the main results to the recursive filtering problem for general nonlinear stochastic systems, to the finite-horizon H_∞ filtering problem with fading measurements, and so on.

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