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Realizability of *n*-Port Resistive Networks with 2*n* Terminals

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Abstract—**In this paper, we consider the realizability problem of *n*-port resistive networks containing 2*n* terminals. A necessary and sufficient condition for any real symmetric matrix to be realizable as the admittance of an *n*-port resistive network containing 2*n* terminals is obtained. The condition is based on the existence of a parameter matrix. We then focus on a three-port resistive network containing six terminals. A necessary and sufficient condition is derived for any real symmetric matrix to be realizable as the admittance of a three-port resistive network containing six terminals and at most five positive elements, whose topological structure is properly restricted.**

Keywords: **Network synthesis, resistive network, inerter.**

I. INTRODUCTION

Passive network synthesis is an important subject in systems theory and experienced a "golden era" in the 1930–1970s [1], [2], [3], [19]. Although there have been many elegant and useful results in this field, there are still many unsolved important problems such as minimal realizations.

Recently, a new mechanical element named "inerter" [13], [26] has been introduced, in which the force applied at its two terminals is proportional to the relative acceleration between them. One of the main motivations for the inerter research is passive mechanical network synthesis. The spring, damper, and inerter are analogous to the inductor, resistor, and capacitor, respectively [26]. The impedance (admittance) of any passive network is positive-real [19] and any positive-real impedance (admittance) is realizable with a finite number of resistors (dampers), inductors (springs), and capacitors (inerters) following the Bott-Duffin procedure [3]. Hence, one can use dampers, springs, and inerters to construct any passive mechanical network in a systematic manner. Since mechanical systems require as few elements as possible, interest in passive network synthesis has been renewed lately [10], [11], [12], [14], [15], [16], [17], [18], [30]. In particular, there was an independent call for a renewed investigation by Kalman [22].

Realization problems of multi-port transformerless networks have not been fully addressed, not even for the *n*-port resistive networks. Noticeably, the investigation on *n*-port resistive networks is the first step of solving realization problems of multi-port transformerless networks and can provide guidance on further investigations. Besides, several results of minimal realizations of one-port networks can be derived based on the results of the *n*-port resistive networks [14], [15]. Therefore, as one of the important topics in passive network synthesis, it is quite essential to have a new endeavor on the realization problem of *n*-port resistive networks. For any *n*-port resistive network, using the generalized star-mesh transformation [29], the number of terminals is ranged from $n+1$ to 2*n*. Networks containing $n+1$ terminals have been determined [7], [20], while networks containing $n+p$ terminals with $p > 1$ remain unsolved. Among them, networks containing 2*n* terminals form a particularly important class, which is investigated here.

This paper considers the realizability condition for an *n*-port resistive network containing 2*n* terminals. A necessary and sufficient condition is obtained for the realizability of any *n*-port resistive network containing 2*n* terminals, which is based on the existence of a parameter matrix and presented in a unified form. Besides, the values of the elements are also parameterized. Furthermore, an explicit condition is derived for a three-port resistive network containing six terminals and at most five elements in a restricted topological structure.

The remaining part of this paper is organized as follows. Section II formulates the problem to be solved. In Section III, the main results are presented. Section III-A provides some preliminaries; Section III-B derives a necessary and sufficient condition for the realization of an *n*-port resistive network with 2*n* terminals; and Section III-C focuses on the realizability conditions for a three-port resistive network. The conclusion is drawn in Section IV.

II. PROBLEM FORMULATION

Consider an $n \times n$ real symmetric matrix in the form of

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ y_{12} & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ y_{13} & y_{23} & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{1,n-1} & y_{2,n-1} & y_{3,n-1} & \cdots & y_{n-1,n-1} & y_{n-1,n} \\ y_{1n} & y_{2n} & y_{3n} & \cdots & y_{n-1,n} & y_{nn} \end{bmatrix}. \tag{1}$$

It is well known that Y is necessarily paramount [8] for any resistive n-port network consisting of only non-negative resistors. Furthermore, it is also known from [28] that paramountcy is a necessary and sufficient condition for realization when $n \leq 3$. From the results in [4], it can be derived that the port graph of a realization must be made part of a tree. The internal vertices (the vertices which are not terminals) can be eliminated by the generalized star-mesh transformation [29]. Therefore, only the networks whose vertices are all terminals need to be investigated. In this setting, the number of vertices (terminals) of the network is from $n+1$ to $2n$. The realization problem for the n-port resistive network with $n+1$ vertices has been solved [5], [6], [7], [20]. The problem with $n+p$ vertices where $p > 1$ was studied in [21], [23], [24], [25]. However, this problem has not been completely solved.

This paper is concerned with the realization problem of $2n$-terminal resistive networks consisting of only non-negative resistors. We first consider the general n-port networks, and then investigate the three-port case. In this study, no negative resistors or internal vertices are permitted. The basic notations and theorems in graph theory are referred to [27].

III. MAIN RESULTS

This section contains the main results of this paper, which is divided into three subsections.

A. Basic Concepts and Results

This subsection presents some definitions and lemmas that will be used in the following discussion.

Definition 3.1: A connected graph in which each pair of vertices has one and only one edge is called a *complete graph*.

It is noted that *complete graph* was first defined in [7], which confines the number of vertices to $n+1$, as [7] only considers the realization of n-port resistive networks containing $n+1$ vertices. In this paper, we generalize this term to one for $n+p$ with $p \geq 1$. If the network graph of any n-port resistive network is regarded as a complete graph, then the conductance of the resistor in each edge is non-negative with zero conductance corresponding to an open-circuit element.

Definition 3.2: [9], [28] A real symmetric matrix, with each main-diagonal element not less than the sum of the absolute values of all the other elements in the same row, is called a *dominant matrix*.

Definition 3.3: [9], [28] A real symmetric matrix, with each principle minor not less than the absolute value of any

non-principle minor formed by the same rows, is called a *paramount matrix*.

It is obvious that when $n = 2$, the dominant matrix and the paramount matrix are equivalent.

Definition 3.4: [9] A real $m \times m$ symmetric matrix $\overline{Y}$ is called a *uniformly tapered matrix* if its entries satisfy

$$\overline{y}_{i,j} - \overline{y}_{i,j+1} \geq \overline{y}_{i-1,j} - \overline{y}_{i-1,j+1}$$

for $j \geq i$, where $\overline{y}_{0,i} = \overline{y}_{i,m+1} = 0$ for all i and $i, j = 1, 2, \ldots, m$.

From the above definition, it can be verified that all the entries of a uniformly tapered matrix must be non-negative.

Lemma 3.1: [7], [9] A real symmetric matrix can be realized as the admittance of a resistive network, as shown in Fig. 1, whose port graph is a linear and ordered tree if and only if it is uniformly tapered.
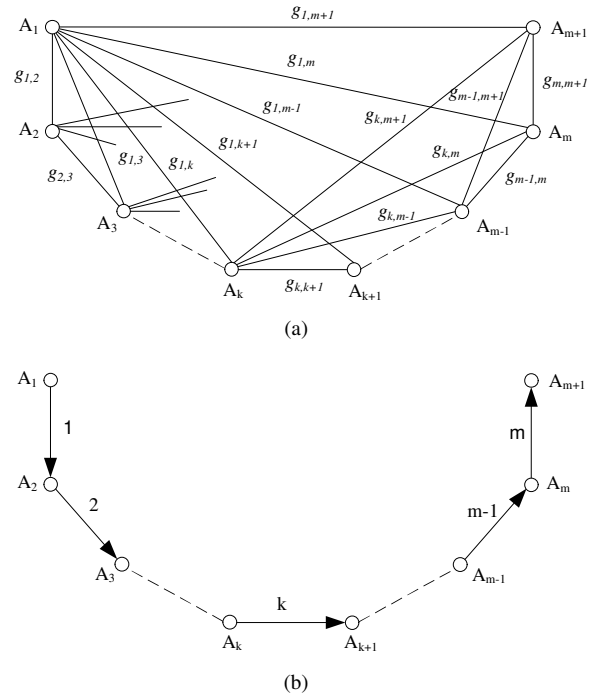

(a)

(b)

Fig. 1. (a) The complete graph for an m-port resistive network with $m+1$ vertices. (b) The port graph which is a linear and ordered tree.

Lemma 3.2: [7] If the condition of Lemma 3.1 is satisfied, then the values of the elements of the realization can be expressed as

$$\overline{g}_{i,j} = (\overline{y}_{i,j-1} - \overline{y}_{i,j}) - (\overline{y}_{i-1,j-1} - \overline{y}_{i-1,j})$$

for $j \geq i$, where $\overline{y}_{0,i} = \overline{y}_{i,m+1} = 0$ for all i and $i, j = 1, 2, \ldots, m$. The realization is shown in Fig. 1.

For any k-port resistive network N_k with $k + p$ vertices where $p > 1$, its admittance is denoted as Y_k. Adding a port to the network N_k without changing its other properties, a new network N_{k+1} with $k+1$ ports and $k+p$ vertices is obtained. To guarantee the existence of the admittance Y_{k+1} of N_{k+1}, the port graph of N_{k+1} also forms a set of trees. The relationship between Y_k and Y_{k+1} is shown in the following lemma.

Lemma 3.3: [25] If the $(k+1)$th row and column of Y_{k+1} correspond to the new port of N_{k+1} and the other correspondences are unchanged, then the relationship between Y_k and Y_{k+1} is expressed as

$$Y_{k+1} = (Y_k \dot{+} 0) + \alpha_k \alpha_k^T, \tag{2}$$

where $\alpha_k = [p_1, p_2, \ldots, p_k, p_{k+1}]^T$.

It should be noted that if any other row of Y_{k+1} is used for the new added port, a similar result can be obtained.

Lemma 3.4: [25] Consider two real symmetric matrices Y_k and Y_{k+1}, which satisfy the relationship (2). Then Y_k can be realized as a k-port resistive network N_k if and only if Y_{k+1} is realizable as a $(k+1)$-port resistive network N_{k+1}.

B. A Necessary and Sufficient Condition for n-Port Networks

The complete graph of an n-port resistive network with $2n$ terminals (vertices) is shown in Fig. 2, where the orientations of the edges of port graph are arbitrary. Theorem 3.1 presents a necessary and sufficient condition for any real symmetric $n \times n$ matrix Y in the form of (1) to be realizable as an n-port resistive network with $2n$ terminals.
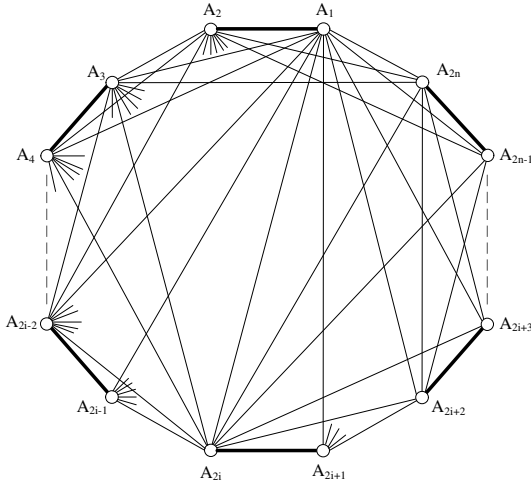
Fig. 2. The complete graph of an n-port resistive network with $2n$ terminals, where the bold line segments correspond to the edges of the port graph.

Theorem 3.1: Consider a real symmetric $n \times n$ matrix Y in the form of (1). It can be realized as an n-port resistive network N with $2n$ terminals if and only if there exists a real $(2n-1) \times (n-1)$ matrix P in the form of

$$P = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \end{bmatrix} \tag{3}$$

with

$$\gamma_k := \begin{bmatrix} p_1^{(k)} & p_{n+1}^{(k)} & \cdots & p_k^{(k)} & p_{n+k}^{(k)} & p_{k+1}^{(k)} & 0 & \cdots p_{n-1}^{(k)} & 0 & p_n^{(k)} \end{bmatrix}^T \tag{4}$$

for $1 \leq k \leq n-1$ such that

$$\max\{-(\alpha_i - \beta_{i-1})^T(\alpha_j - \beta_j), \; -(\alpha_i - \beta_i)^T(\alpha_j - \beta_{j-1})\} \leq y_{ij}$$
$$\leq \min\{-(\alpha_i - \beta_{i-1})^T(\alpha_j - \beta_{j-1}), \; -(\alpha_i - \beta_i)^T(\alpha_j - \beta_j)\} \tag{5}$$

for $1 \leq i < j \leq n$ and

$$y_{ii} \geq -(\alpha_i - \beta_{i-1})^T(\alpha_i - \beta_i), \qquad 1 \leq i = j \leq n, \tag{6}$$

where α_l^T is the $(2l-1)$th row of P with $1 \leq l \leq n$, β_m^T is the $2m$th row of P with $1 \leq m \leq n-1$, and $\beta_0 = \beta_n := 0_{(n-1)\times 1}$.

Proof: Since the network graph can be regarded as a complete graph, the required network can be converted into one whose port graph is a linear ordered tree by adding new ports step by step. Hence, the theorem can be proven by making use of Lemma 3.1, Lemma 3.3, and Lemma 3.4. The details are omitted for brevity. ∎

Furthermore, the element values are presented in Theorem 3.2.

Theorem 3.2: Consider a real symmetric $n \times n$ matrix Y in the form of (1). If it can be realized as an n-port resistive network with $2n$ terminals as shown in Fig. 2, where the orientations of the ports are from vertex A_{2k-1} to A_{2k} with $1 \leq k \leq n$, that is, Y satisfies the condition of Theorem 3.1, then the values of the conductance of the resistor of each edge are given by

$$g_{2r-1,2s} = y_{r,s} + (\alpha_r - \beta_{r-1})^T(\alpha_s - \beta_s), \quad 1 \leq r \leq s \leq n,$$
$$g_{2r-1,2s-1} = -y_{r,s} - (\alpha_r - \beta_{r-1})^T(\alpha_s - \beta_{s-1}), \; 1 \leq r < s \leq n,$$
$$g_{2r,2s-1} = y_{r,s} + (\alpha_r - \beta_r)^T(\alpha_s - \beta_{s-1}), \quad 1 \leq r < s \leq n,$$
$$g_{2r,2s} = -y_{r,s} - (\alpha_r - \beta_r)^T(\alpha_s - \beta_s), \quad 1 \leq r < s \leq n, \tag{7}$$

where α_k and β_k are obtained from the parameter matrix P as defined in Theorem 3.1, and $g_{h,l}$ denotes the conductance of the element connecting vertices A_h and A_l for $1 \leq h < l \leq 2n$.

Proof: This theorem can be proven by using Lemma 3.2 and Theorem 3.1. The details are omitted for brevity. ∎

C. Realizability Conditions for Three-Port Networks

In the previous subsection, a necessary and sufficient condition is given, which is based on the existence of a parameter matrix. In this subsection, we discuss the three-port resistive networks such that the realizability condition is given directly based on the entries of the admittance matrix. It is known in [31, pg. 372] that any 3×3 paramount admittance can be realized by a canonical network containing six elements and five terminals. Hence, this subsection considers the realization problem of a six-terminal network with at most five elements, for which the topological structure is properly restricted.

For a 2×2 symmetric matrix, the next lemma can be deduced from [28].

Lemma 3.5: A real symmetric 2×2 matrix Y can be realized as a two-port resistive network with four terminals if and only if it is paramount. Furthermore, at most four elements are required.

A third-order symmetric matrix is expressed in the form of

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & y_{23} \\ y_{13} & y_{23} & y_{33} \end{bmatrix}. \tag{8}$$

Since we discuss the realizability problem of a special class of this kind of networks, where no internal vertices are permitted,

to better describe the restriction on this kind of networks, the following definition is introduced.

Definition 3.5: For the complete graph of a resistive network, an edge whose corresponding resistor has positive conductance is called a *positive edge*.

The positive edge is defined to distinguish the one corresponding to an open-circuit element.

Definition 3.6: Consider an *n*-port network whose port graph contains p parts, where $p > 1$. If any two parts of the port graph are directly connected with each other by a positive edge, then the network is a called *terminal well-connected network*.

In what follows, we discuss the realization problem of a three-port resistive network containing six terminals and at most five positive elements that is *not* terminal well-connected.

Lemma 3.6: Consider a third-order real symmetric matrix Y in the form of (8). If Y is a dominant matrix with at least two of y_{12}, y_{13}, and y_{23} being zero, then it can be realizable as the admittance matrix of a three-port resistive network with six terminals and at most five positive elements, which is not terminal well-connected.

Proof: After a proper rearrangement of rows and corresponding columns, Y in (8) can be written in the form of $Y_1 \dotplus y$ with Y_1 being a 2×2 paramount matrix and $y \geq 0$. Thus, this lemma follows immediately from Lemma 3.5. ∎

Lemma 3.7: Consider a third-order real symmetric matrix Y in the form of (8) that does not satisfy the condition of Lemma 3.6. If it can be realized as the admittance matrix of a three-port resistive network with six terminals, which is not terminal well-connected, then it can be realized by a network from the one shown in Fig. 3 by judiciously swapping the ports and switching the polarities of them, where all the elements denoted by the bold lines always have positive conductance.
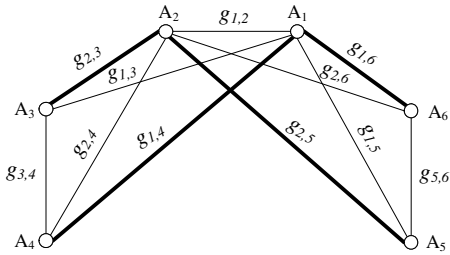

Fig. 3. The graph of a three-port resistive network with six terminals, which is not terminal well-connected. A_1 and A_2 are the terminals of port 1; A_3 and A_4 are the terminals of port 2; and A_5 and A_6 are the terminals of port 3.

Proof: Assume that i, j, and k are the integers among 1, 2, and 3. Suppose that there are no positive edges directly connecting port i and port j. By swapping the ports properly, it can be assumed that $i = 2$, $j = 3$, and $k = 1$. Moreover, it is not difficult to show that for port k and port i (j), there must be at least one positive edge connecting a terminal of port k and a terminal of port i (j), and the other positive edge connecting the other terminal of port k and the other terminal of port i (j). Otherwise, Y will satisfy the condition of Lemma 3.6. Therefore, at least four positive elements must

be required, and the remaining possible positive edges must be parallel with the ports or connecting the terminals of port k and i (j). After switching the polarities of ports properly, the network is shown in Fig. 3. ∎

When $n = 3$, the realizability condition shown in Theorem 3.1 reduces to the existence of

$$P = \begin{bmatrix} p_{11} & p_{41} & p_{21} & 0 & p_{31} \\ p_{12} & p_{42} & p_{22} & p_{52} & p_{32} \end{bmatrix}^T \quad (9)$$

such that the following conditions hold:

$$L_1 \leq y_{11}, \quad L_2 \leq y_{22}, \quad L_3 \leq y_{33}, \quad (10)$$

$$\max\{N_{12}^{(1)}, N_{12}^{(2)}\} \leq y_{12} \leq \min\{M_{12}^{(1)}, M_{12}^{(2)}\}, \quad (11)$$

$$\max\{N_{13}^{(1)}, N_{13}^{(2)}\} \leq y_{13} \leq \min\{M_{13}^{(1)}, M_{13}^{(2)}\}, \quad (12)$$

$$\max\{N_{23}^{(1)}, N_{23}^{(2)}\} \leq y_{23} \leq \min\{M_{23}^{(1)}, M_{23}^{(2)}\}, \quad (13)$$

where $L_1 = p_{11}p_{41} + p_{12}p_{42} - (p_{11}p_{11} + p_{12}p_{12})$, $L_2 = p_{21}p_{41} + p_{22}p_{42} + p_{22}p_{52} - (p_{21}p_{21} + p_{22}p_{22} + p_{42}p_{52})$, $L_3 = p_{32}p_{52} - (p_{31}p_{31} + p_{32}p_{32})$, $N_{12}^{(1)} = p_{12}p_{52} - (p_{11}p_{21} + p_{12}p_{22})$, $N_{12}^{(2)} = p_{11}p_{41} + p_{21}p_{41} + p_{12}p_{42} + p_{22}p_{42} - (p_{41}p_{41} + p_{11}p_{21} + p_{42}p_{42} + p_{12}p_{22})$, $M_{12}^{(1)} = p_{11}p_{41} + p_{12}p_{42} - (p_{11}p_{21} + p_{12}p_{22})$, $M_{12}^{(2)} = p_{21}p_{41} + p_{22}p_{42} + p_{12}p_{52} - (p_{11}p_{21} + p_{12}p_{22} + p_{42}p_{52})$, $N_{13}^{(1)} = -(p_{11}p_{31} + p_{12}p_{32})$, $N_{13}^{(2)} = p_{31}p_{41} + p_{12}p_{52} + p_{32}p_{42} - (p_{11}p_{31} + p_{12}p_{32} + p_{42}p_{52})$, $M_{13}^{(1)} = p_{12}p_{52} - (p_{11}p_{31} + p_{12}p_{32})$, $M_{13}^{(2)} = p_{31}p_{41} + p_{32}p_{42} - (p_{11}p_{31} + p_{12}p_{32})$, $N_{23}^{(1)} = p_{31}p_{41} + p_{32}p_{42} - (p_{21}p_{31} + p_{22}p_{32})$, $N_{23}^{(2)} = p_{22}p_{52} + p_{32}p_{52} - (p_{21}p_{31} + p_{22}p_{32} + p_{52}p_{52})$, $M_{23}^{(1)} = p_{31}p_{41} + p_{22}p_{52} + p_{32}p_{42} - (p_{21}p_{31} + p_{22}p_{32} + p_{42}p_{52})$, and $M_{23}^{(2)} = p_{32}p_{52} - (p_{21}p_{31} + p_{22}p_{32})$. By Theorem 3.2, the values of the conductances $g_{i,j}$ can be calculated as $g_{1,2} = y_{11} - L_1$, $g_{3,4} = y_{22} - L_2$, $g_{5,6} = y_{33} - L_3$, $g_{1,3} = M_{12}^{(1)} - y_{12}$, $g_{2,4} = M_{12}^{(2)} - y_{12}$, $g_{1,4} = y_{12} - N_{12}^{(1)}$, $g_{2,3} = y_{12} - N_{12}^{(2)}$, $g_{1,5} = M_{13}^{(1)} - y_{13}$, $g_{2,6} = M_{13}^{(2)} - y_{13}$, $g_{1,6} = y_{13} - N_{13}^{(1)}$, $g_{2,5} = y_{13} - N_{13}^{(2)}$, $g_{3,5} = M_{23}^{(1)} - y_{23}$, $g_{4,6} = M_{23}^{(2)} - y_{23}$, $g_{3,6} = y_{23} - N_{23}^{(1)}$, and $g_{4,5} = y_{23} - N_{23}^{(2)}$.

Definition 3.7: A row of a matrix Y is *marginally dominant* if the elements of the row satisfy $y_{ii} = \sum\limits_{j=1, \ j\neq i}^{n} |y_{ij}|$.

Theorem 3.3: A real symmetric matrix Y in the form of (8) can be realized as the admittance matrix of a three-port resistive network with six terminals and at most five positive elements, which is not terminal well-connected, if and only if Y is dominant and one of the following three conditions holds:

1. at least two of y_{12}, y_{13}, and y_{23} are zero;
2. one of y_{12}, y_{13}, and y_{23} is zero, and at least two of the three rows are marginally dominant;
3. one of y_{12}, y_{13}, and y_{23} is zero, denoted by $y_{ij} = 0$, and only one of the three rows is marginally dominant, which is either the ith row or the jth row.

Proof: Necessity. Consider a real symmetric matrix Y in the form of (8). If Y does not satisfy the condition of Lemma 3.6, then by Lemma 3.7 it can be realized by the

network shown in Fig. 3 via judiciously swapping the ports and switching the polarities of them, where all the elements denoted by the bold lines have positive conductance, and there is at most one positive element among the edges denoted by the solid lines. In this figure, A_1, A_3, and A_5 are at higher potentials; A_2, A_4, and A_6 are at lower ones. It suffices to consider the network shown in Fig. 3 because the admittances of other cases can be obtained from the network in this figure by proper arrangement of rows and corresponding columns and multiplication of the rows and columns with -1, which does not violate the condition.

Since Y can be realized by the network in the form of Fig. 3, there must exist a parameter matrix P in the form of (9) such that (10)–(13) hold. It is noted that $g_{3,5} = g_{3,6} = g_{4,5} = g_{4,6} = 0$, which is equivalent to

$$N_{23}^{(1)} - y_{23} = 0, \tag{14}$$
$$N_{23}^{(2)} - y_{23} = 0, \tag{15}$$
$$M_{23}^{(2)} - y_{23} = 0, \tag{16}$$
$$M_{23}^{(1)} - y_{23} = 0. \tag{17}$$

This implies that

$$p_{22} = p_{42} = p_{52} \neq 0, \quad p_{31}p_{41} = 0, \quad y_{23} = -p_{21}p_{31}.$$

For the edges represented by the lighter lines in Fig. 3, it suffices to discuss the following two cases.

Case 1: $g_{1,3} = g_{2,4} = g_{1,5} = g_{2,6} = 0$. In this case,

$$M_{12}^{(1)} - y_{12} = 0, \tag{18}$$
$$M_{12}^{(2)} - y_{12} = 0, \tag{19}$$
$$M_{13}^{(1)} - y_{13} = 0, \tag{20}$$
$$M_{13}^{(2)} - y_{13} = 0. \tag{21}$$

Together with $g_{1,6}, g_{2,5}, g_{1,4}, g_{2,3} > 0$, it can be derived that Y is dominant and at least two of the three rows are marginally dominant.

Case 2: One of $g_{1,3}$, $g_{2,4}$, $g_{1,5}$, and $g_{2,6}$ is positive. It can be derived that Y is dominant, one of y_{12}, y_{13}, and y_{23} is zero, denoted by $y_{ij} = 0$, and only one of the three rows is marginally dominant, which is either the ith row or the jth row.

Sufficiency. Suppose that Y in the form of (8) satisfies the given condition. If it is dominant and satisfies Condition 1, then it must be realizable as the admittance of the required network by Lemma 3.6.

If it is dominant and satisfies Condition 2, then after proper arrangement of the rows and the corresponding columns, and multiplication of the rows and the corresponding columns with -1, the matrix satisfies $y_{23} = 0$, $y_{12} > 0$, and $y_{13} > 0$. Choose the parameter matrix P such that its elements satisfy

$$p_{31} = 0, \; p_{11} = p_{21}, \; 0 < p_{11}p_{41} < p_{41}p_{41},$$
$$p_{12} = p_{32}, \; p_{22} = p_{42} = p_{52}, \; 0 < p_{12}p_{22} < p_{22}p_{22},$$

and

$$p_{11}p_{41} - p_{11}p_{11} = y_{12},$$
$$p_{12}p_{22} - p_{12}p_{12} = y_{13}.$$

It is obvious that this choice is always possible. It can be verified that (10)–(13) hold. Furthermore, $g_{1,3} = g_{2,4} = g_{1,5} = g_{2,6} = g_{3,5} = g_{4,6} = g_{3,6} = g_{4,5} = 0$, $g_{1,4} = p_{11}p_{41} > 0$, $g_{2,3} = -p_{11}p_{41} + p_{41}p_{41} > 0$, $g_{1,6} = p_{12}p_{22} > 0$, and $g_{2,5} = -p_{12}p_{22} + p_{22}p_{22} > 0$. Since $g_{1,2} = y_{11} - L_1 = y_{11} - |y_{12}| - |y_{13}|$, $g_{3,4} = y_{22} - L_2 = y_{22} - |y_{12}| - |y_{23}|$, and $g_{5,6} = y_{33} - L_3 = y_{33} - |y_{13}| - |y_{23}|$, one has $g_{1,2}, g_{3,4}, g_{5,6} \geq 0$, and at least two of them equal zero.

If it is dominant and satisfies Condition 3, then after proper arrangement of the rows and the corresponding columns, and multiplication of the rows and the corresponding columns with -1, the matrix satisfies $y_{23} = 0$, $y_{12} > 0$, $y_{13} > 0$, and the third row is marginally dominant. Choose the parameter matrix P such that its elements satisfy

$$p_{31} = 0, \; 0 < p_{21}p_{41} < p_{11}p_{41} < p_{41}p_{41},$$
$$p_{12} = p_{32}, \; p_{22} = p_{42} = p_{52}, \; 0 < p_{12}p_{22} < p_{22}p_{22},$$

and

$$p_{21}p_{41} - p_{11}p_{21} = y_{12},$$
$$p_{12}p_{22} - p_{12}p_{12} = y_{13},$$
$$p_{21}p_{41} - p_{21}p_{21} = y_{22},$$
$$p_{11}p_{41} - p_{11}p_{11} + y_{13} = y_{11}.$$

It is obvious that this choice is always possible. It can be verified that (10)–(13) hold. Furthermore, $g_{2,4} = g_{1,5} = g_{2,6} = g_{3,5} = g_{4,6} = g_{3,6} = g_{4,5} = g_{1,2} = g_{3,4} = g_{5,6} = 0$, $g_{1,3} = p_{11}p_{41} - p_{21}p_{41} > 0$, $g_{1,4} = p_{21}p_{41} > 0$, $g_{2,3} = -p_{11}p_{41} + p_{41}p_{41} > 0$, $g_{1,6} = p_{12}p_{22} > 0$, and $g_{2,5} = -p_{12}p_{22} + p_{22}p_{22} > 0$. ∎

IV. CONCLUSION

This paper has investigated the realizability problem of an n-port resistive network containing $2n$ terminals. A general result was derived, which is a necessary and sufficient condition for any real symmetric matrix to be realizable as an n-port resistive network containing $2n$ terminals. The condition is in a unified form, obtained based on the existence of a parameter matrix. To make the condition be directly dependent on the entries of the given matrix, the case of $n = 3$ was discussed in detail. A new concept named the terminal well-connected network was introduced. Consequently, a necessary and sufficient condition for any real symmetric matrix to be realizable as the admittance of a three-port six-terminal resistive network, which is not terminal well-connected and contains at most five positive elements, was derived.

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