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Estimation of Time-varying Autocorrelation and its Application to Time-frequency Analysis of Nonstationary Signals

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***Abstract*—This paper introduces a new method for adaptively estimating the time-varying autocorrelation (TV-AC) of nonstationary signals and studies its application to time-frequency analysis. The proposed method employs local estimation with a sliding window having a certain bandwidth to estimate the TV-AC locally. The window bandwidths are selected adaptively by a local plug-in rule to address the bias and variance tradeoff problem. Further, based on the proposed adaptive TV-AC estimation, a new time-frequency analysis method called adaptive windowed minimum variance spectral estimation (AWMVSE) is developed. Simulation results show that the proposed adaptive TV-AC estimation method and AWMVSE method have improved performances over conventional estimators with a fixed window.**

Keywords—time varying autocorrelation; adaptive window selection; nonstationary signal; minimum variance spectral estimation; time-frequency analysis

I. INTRODUCTION

Autocorrelation is a fundamental tool in signal processing for finding repeated patterns and characterizing signals [1]. The autocorrelation of a random process with lag k is the expectation of the product of its value at time t and the conjugate of its value at time t-k, i.e. $r(t,t-k) = E[x(t)x^{\mathrm{H}}(t-k)]$. Therefore, the autocorrelation is a function of both time t and the lag k. Due to this complication, it is usually assumed that the random process is wise-sense stationary (WSS), i.e. independent of t above, or quasi-WSS so that the estimation process can be considerably simplified since the autocorrelation is now a function of lag only. However, many practical signals encountered in biomedical engineering and other physical phenomena are non-stationary in nature due to the time-varying information they carry. Consequently, the autocorrelation is also time-varying. For non-stationary signals, a useful approach is to employ local estimation where the TV autocorrelation (TV-AC) is estimated by combining appropriately autocorrelation estimates around at the given time instant using a sliding window. Usually larger weights will be given to neighboring data and smaller weights to remote data. In order words, the random process is assumed to be locally stationary [2, 3].

The selection of the window size or bandwidth is however extremely critical to such local or TV autocorrelation estimation. A long time window is desirable for estimating slowly varying autocorrelation, while a small window is preferred for estimating fast-varying autocorrelation. Basically, a window with a suitable bandwidth or size can help avoid excessive bias for autocorrelation coefficients and reduce the estimation variance. However, adaptive bandwidth selection for estimating TV-autocorrelation (TV-AC) is still an open and difficult problem [4].

In this paper, we adopt a plug-in rule to adaptively select the window bandwidth in TV-AC estimation to address the bias-variance tradeoff problem mentioned above. The basic idea behind the plug-in rule is to find an optimal bandwidth which minimizes the mean squared error (MSE) of the estimated TV-MSE. We further apply this adaptive TV-AC estimation to time-frequency analysis of nonstationary signals and develop a new adaptive windowed MVSE (AWMVSE) method. The conventional minimum variance spectral estimator (MVSE) method is a minimum-variance unbiased frequency estimator [2] and it requires the estimation of the autocorrelation matrix. The proposed AWMVSE method extends the conventional MVSE method by employing adaptive windows to estimate the TV-AC more accurately. Simulation results show that the adaptive TV-AC estimation method and the AWMVSE method have improved performances over conventional methods.

The rest of this paper is organized as follows. The TV-AC estimation and its adaptive bandwidth selection method using plug-in rule are introduced in Section II. In Section III, we apply this TV-AC estimator in MVSE to estimate the time-varying spectrum of nonstationary signals. Simulation results are presented in Section IV. Finally, conclusion is drawn in Section V.

II. ADAPTIVE TV-ACM

A. TV-AC Matrix (TV-ACM)

Given a realization of a stationary discrete-time random process $x(n)$, $n = 1,2,\cdots,N$. The conventional estimate of the autocorrelation matrix with a maximum lag of p-1 is:

$$\mathbf{R}_x = \frac{1}{N}\sum_{n=1}^{N}\begin{bmatrix} |x(n)|^2 & \cdots & x(n)x^{\mathrm{H}}(n-p+1) \\ \vdots & \ddots & \vdots \\ x(n)x^{\mathrm{H}}(n-p+1) & \cdots & |x(n-p+1)|^2 \end{bmatrix}. \quad (1)$$

For nonstationary signals, its time-varying autocorrelation matrix $\mathbf{R}_x(n)$ can be estimated by local estimation using a sliding-window approach. Suppose the maximum order is p, a p-variates signal $\mathbf{X}(n) = [x(n), x(n-1), \cdots, x(n-p+1)]^{\mathrm{H}}$, $n = p, \cdots, N$, can be generated from $x(n)$ with different lags from 0 to $(p-1)$. The TV-ACM at each time instant n is:

$$\mathbf{R}_x(n) = E[\mathbf{X}(n)\mathbf{X}^H(n)]. \quad (2)$$

Since the distribution of $\mathbf{X}(n)$ is unknown, it is difficult to calculate the TV-ACM analytically. In local estimation, the TV-ACM $\mathbf{R}_x(n)$ can be calculated from the local data segment, which is assumed to be stationary, using a sliding window which assigns large weights on local samples and small weights on remote samples. The window or kernel is generally selected as a positive value function over time with finite second moment and symmetric around the origin. The corresponding TV-ACM $\mathbf{R}_x(n)$ is:

$$\mathbf{R}_x(n;h) = \sum_{m=-h}^{h} K_h(m)\mathbf{X}(n+m)\mathbf{X}^{\mathrm{H}}(n+m), \quad (3)$$

where $K_h(u) = \frac{1}{h}K(u/h)$ is the window function whose effective size is controlled by a bandwidth h. As mentioned before, the selection of the bandwidth is crucial to the estimation bias and variance and thus local and adaptive bandwidths are desired [5].

B. Adaptive Window Selection

Considering each entry of the $\mathbf{R}_x$ matrix as a nonstationary process, the following model is assumed:

$$R_{i,j}(n) = r(n) + \varepsilon_n,\ i,j = 1,2,\cdots,p, \quad (4)$$

where $R_{i,j}(n) = x(n-i+1)x(n-j+1)$ is the observed (i,j)-th entry of $\mathbf{R}_x$, $r(n)$ is a regression function, and ε is a zero mean Gaussian process with variance σ^2. By employing local estimation [6] on the data, the estimator $\hat{r}(n)$ can be obtained locally as:

$$\hat{r}(n,h(n)) = \frac{1}{h(n)}\sum_{s=1}^{N} K\left(\frac{s-n}{h(n)}\right) R_{i,j}(s), \quad (5)$$

where $h(n)$ is the selected bandwidth. The optimal bandwidth should minimize the mean integrated squared error (MISE) of the windowed data, which is the sum of mean squared error (MSE) $\mathrm{MSE}(r(n;h_n)) = E(\hat{r}(n;h_n) - r(n))^2$ for n included in the window. It has been shown that the asymptotically optimal bandwidth of (5) is given by:

$$h_{ASY}(n) = \left(\frac{\sigma^2\psi\sum_{s=1}^{N} v(s)}{N\mu_2^2\sum_{s=1}^{N} v(s)r''(s)^2}\right)^{1/5}, \quad (6)$$

where $\psi = \int (K(x))^2 dx$ and $\mu_2 = \int K(x)x^2 dx > 0$. Further, we employed an iterative algorithm proposed in [7-9] to estimate local optimal bandwidths for TV-AC. The details of the algorithm are given in Table 1.

Table 1

Local plug-in estimator for TV-AC

At each time instant n
Step 1. Set initial bandwidth $\hat{h}_0(n) = N^{-1}$.
Step 2. Iterate $\hat{h}_i(n) = \left(\frac{\hat{\sigma}^2\psi\sum_{s=1}^{N} v(s)}{N\mu_2^2\sum_{s=1}^{N} v(s)\hat{r}''(s;\hat{h}_{i-1}(n)N^{1/10})}\right)^{1/5}$, for $i = 1,\ldots,8$.
Step 3. Iterate $\hat{h}_i(n) = \left(\frac{\hat{\sigma}^2\psi\sum_{s=1}^{N} v(s)}{N\mu_2^2\sum_{s=1}^{N} K\left(\frac{s-n}{\hat{h}_{i-1}(n)}\right)\hat{r}''(s;\hat{h}_{i-1}(n)N^{1/10})}\right)^{1/5}$, for $i = 9, 10$.
Step 4. Let $\hat{h}_{10}(n)$ be the optimal bandwidth at n.

The first eight iteration steps are necessary to stabilize the estimator and yield the correct rate of $n^{1/5}$ [9]. In Table 1, σ^2 can be estimated as:

$$\hat{\sigma}^2 = \frac{1}{2(N-1)}\sum_{n=1}^{N-1}(R_{i,j}(n+1) - R_{i,j}(n))^2. \quad (7)$$

The second derivative of $\hat{r}$ is estimated by a kernel estimator with bandwidth $\tilde{h}$:

$$\hat{r}''(n;\tilde{h}) = \frac{1}{N\tilde{h}}\sum_{s=1}^{N}\tilde{K}\left(\frac{s-n}{\tilde{h}}\right)R_{i,j,s}, \quad (8)$$

where $\tilde{K}$ is a kernel satisfying the condition in [9] and $\tilde{h} = h_i(n)N^{1/10}$.

The above algorithm can estimate the local adaptive bandwidth for every entry in the TV-AC matrix. However, because different entries in the TV-ACM are independently estimated with different bandwidths, the autocorrelation matrix may not be positive definite. To solve this problem, a universal bandwidth for all entries of the TV-ACM is needed at each time instant. Here, we propose to approximate the universal bandwidth as the average of the optimal local bandwidths of all entries of the TV-ACM, and such approximation works well in our simulations.

III. ADAPTIVE WINDOWED MVSE

The proposed adaptive TV-ACM estimator can find numerous applications in signal processing. Here, we consider its application to time-frequency analysis. In particular, a new time-frequency analysis method based on MVSE will be developed below based on the proposed adaptive TV-ACM method. In MVSE, the input discrete-time signal $x(n)$ is passed through a filter of order p, $a_0(\omega), a_1(\omega), \cdots, a_{p-1}(\omega)$, so as to extract the component in $x(n)$ with frequency ω. The corresponding output $y(n)$ is then given by:

$$y(n) = \sum_{i=0}^{p-1} a_i(\omega) x(n-i) = \mathbf{a}^{\mathrm{H}}(\omega)\mathbf{x}(n) . \quad (9)$$

To extract the desired component and hence estimate the power spectrum for different values of ω, we constrain the filter to have a unit gain at the desired frequency ω, i.e. [10]:

$$\sum_{k=0}^{p\text{-}1} a_k(\omega) e^{-jk\omega} = 1, \; j = \sqrt{-1} , \quad (10)$$

while minimizing its output variance:

$$\sigma_{\mathrm{y}}^2 = r_y(0) = \mathbf{a}^{\mathrm{H}}(\omega)\mathbf{R}_x \mathbf{a}(\omega) , \quad (11)$$

where $\mathbf{R}_x$ is the $p \times p$ autocorrelation matrix. The constrained optimization problem can be solved to obtain the following analytical formula for $\mathbf{a}(\omega)$:

$$\mathbf{a}(\omega) = \frac{\mathbf{R}_x^{-1}\mathbf{e}(\omega)}{\mathbf{e}^{\mathrm{H}}(\omega)\mathbf{R}_x^{-1}\mathbf{e}(\omega)} , \quad (12)$$

where $\mathbf{e}(\omega) = [1, e^{-j\omega/F_s}, ..., e^{-j\omega(p-1)/F_s}]^{\mathrm{H}}$ and F_s is the sampling rate. Substituting (12) into (11), the power spectrum can be estimated as:

$$P(\omega) = \sigma_{y,\min}^2 = \frac{1}{\mathbf{e}^{\mathrm{H}}(\omega)\mathbf{R}_x^{-1}\mathbf{e}(\omega)} . \quad (13)$$

It was shown in our previous work [11] that the conventional MVSE cannot effectively reveal the time-varying spectral information of nonstationary signals because it is calculated from a fixed autocorrelation matrix $\mathbf{R}_x$. For a signal with time-varying spectrum, its autocorrelation matrix also varies with time. The proposed adaptive TV-ACM estimation method can be used to track the time-varying spectrum. With the TV-ACM $\mathbf{R}_x(n)$ estimated adaptively by the method in Section II, the adaptive windowed MVSE (AWMVSE) is given by:

$$\mathbf{P}(n,\omega) = \frac{1}{\mathbf{e}^{\mathrm{H}}(\omega)\mathbf{R}_x^{-1}(n)\mathbf{e}(\omega)} . \quad (14)$$

In this AWMVSE method, when the autocorrelation and the spectrum change rapidly, a small bandwidth will be automatically selected to achieve a good time resolution for identifying onset of different events. On the other hand, when the spectrum and the autocorrelation vary slowly, a large bandwidth will be chosen to obtain a better frequency resolution for the slowly-varying spectrum.

IV. SIMULATION RESULTS

In this section, a discrete-time sinusoidal signal $x(n) = \sin(2\pi f_n n)$, $1 \le n \le 600$ with a sampling rate of 100 Hz is used to evaluate the performance of the proposed and conventional algorithms. The frequency of $x(n)$ is chosen as:

$$f_n = 10, \quad 1 \le n \le 200,$$
$$f_n = 5, \quad 200 < n \le 400,$$
$$f_n = 30, \quad 400 < n \le 600.$$

Hence, the signal has three parts, each having a constant frequency and autocorrelation matrix. The Epanechnikov kernel $K(h) = \frac{3}{4}(1-h^2)$ is chosen as the kernel function in the adaptive TV-ACM method.

Firstly, a zero mean Gaussian noise with a SNR of 20dB is added to the simulated signal to test the performance of the proposed adaptive TV-ACM method. The order is set as $p = 20$. TV-ACMs are also estimated from two fix bandwidths $h = 5$ or 80 for comparison. Fig. 1 shows an example autocorrelation entry ($[\mathbf{R}_x(n)]_{1,10}$) in the TV-ACM estimated by different methods and the bandwidth selected by the local plug-in rule. It can be seen that when a small bandwidth is used, the autocorrelation estimate shows a very large variability. On the other hand, when a large bandwidth is used, the autocorrelation estimated around the jump discontinuities is blurred. By using adaptively selected local windows (small windows around jump discontinuities and large windows at flat areas), the proposed adaptive TV-ACM method can achieve more accurate estimation of autocorrelation than conventional estimator with fix bandwidths.

Secondly, this simulated signal is used to test the performance of the AWMVSE method. Three filter orders $p = 12, 16, \text{or } 20$ are tested. The other parameters are the same as those in previous simulations. Fig. 2 compares the AWMVSE and WMVSE with fixed window sizes. For WMVSE, when the window size is small, the time resolution is high but the frequency resolution is low. When the window size is large, the WMVSE has degraded time resolution but improved frequency resolution. The AWMVSE can achieve good tradeoff between time and frequency resolution at the same time. Moreover, we compare these MVSE-based methods quantitatively by computing the average relative power (ARP) under Gaussian noises with different SNRs (10dB, 20dB, and 30dB) and different filter orders (p=12 or 20). The ARP is calculated as the ratio between the power at the real frequency and the total power of the spectrum and it is averaged across all time samples. A higher ARP means the spectral estimate is more accurate. Table 2 is obtained from averages of 100 independent simulations. It justifies that the AWMVSE method achieves a better time-frequency resolution than conventional WMVSE with constant bandwidths under different testing scenarios.

Table 2

Average relative power percentages (ARP) of different methods

	SNR=10 *p*=12	SNR=10 *p*=20	SNR=20 *p*=12	SNR=20 *p*=20	SNR=30 *p*=12	SNR=30 *p*=20
h =5	12.71%	13.56%	42.73%	45.19%	77.11%	77.46%
h = 80	10.51%	18.67%	27.96%	42.59%	44.03%	53.43%
Adaptive *h*	13.12%	22.87%	47.42%	71.86%	86.04%	86.43%

CONCLUSION

A new adaptive TV-AC (and TV-ACM) estimator and a new time-frequency analysis method, AWMVSE, are presented. The adaptive TV-AC method is based on local estimation and employs a sliding window with adaptively-selected local bandwidths to address the bias-variance tradeoff problem encountered in nonstationary signal estimation. The proposed adaptive TV-ACM method may find numerous practical applications and its application to MVSE leads to a new AWMVSE, which offers much better time-frequency resolution than conventional windowed MVSE with fixed windows.

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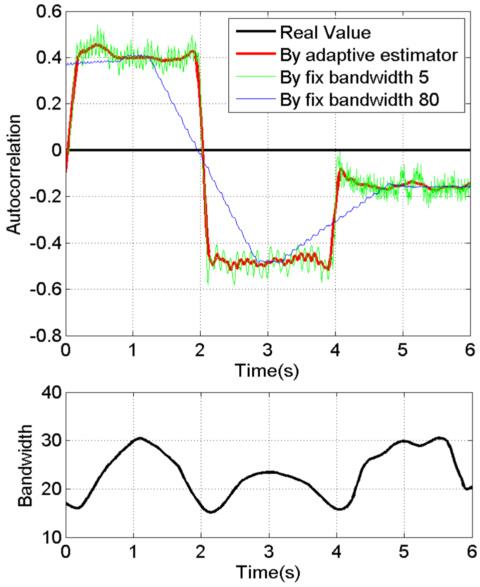

Fig. 1. Upper panel: an example entry of the autocorrelation (row 1, column 10) in the TV-ACM estimated by different methods; lower panel: adaptive bandwidths selected. The filter order is $p = 20$ and SNR = 20dB.

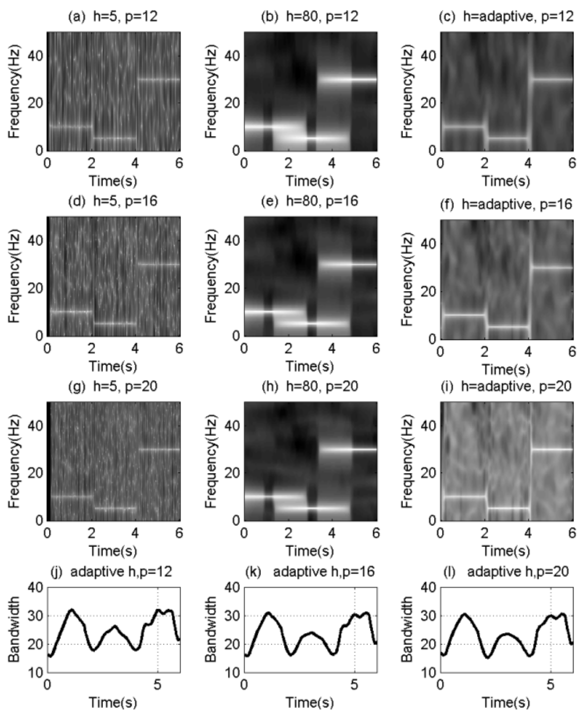

Fig. 2. Time-frequency distributions of different methods (under the SNR of 20dB).