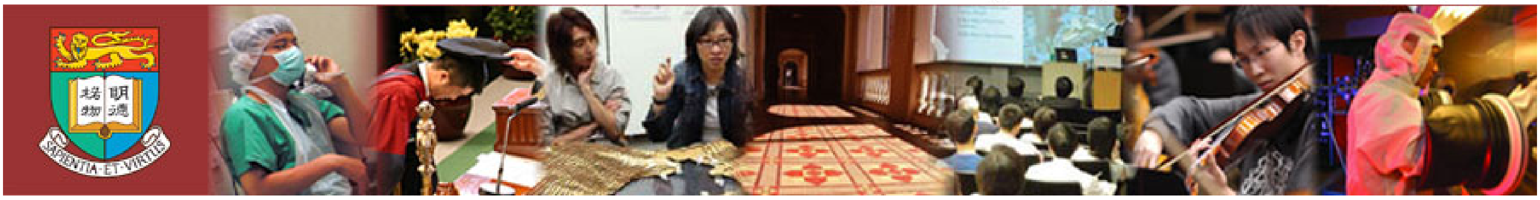

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Stability-Preserving Model Order Reduction for Nonlinear Time Delay Systems

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Abstract: Delay elements are needed to model physical, industrial and engineering systems as action and reaction always come with latency. In this paper, we present an algorithm to obtain the reduced-order models (ROMs) while preserving the stability of nonlinear time delay systems (TDSs), which are approximated first by the piecewise-linear TDSs. One contribution is the derivation of the input-output stability of piecewise-linear TDSs, for the first time. The other is the preservation of the input-output stability of the ROMs. The system matrices are obtained by the left projection matrix from the solution of linear matrix inequalities (LMIs) for the input-output stability test of the original piecewise-linear TDSs and the right projection matrix from matching the estimated moments. An application example then verifies the effectiveness of the proposed method.

Key Words: Model order reduction, nonlinear system, time delay, linear matrix inequality (LMI).

1 Introduction

Although the reduced-order models (ROMs) of linear time-invariant (LTI) systems provide very powerful design and analysis tools for the higher LTI systems [1, 2], the modeling of complex physical systems always results in high order nonlinear systems, which are much more complex but useful than LTI systems. A lot of model order reduction (MOR) work about nonlinear systems have been reported recently [3–14]. The problem of locally balancing nonlinear systems about a given point in the state space over an infinite interval of time is first addressed by Scherpen in [3] and later generalized in [4]. While the approach in [3] is arguably the most natural extension of the well known balanced truncation for LTI systems, it does not lend itself directly to a numerical solution. The empirical balanced truncation is first introduced in [7] to overcome the issues in [3] and the extension is found in [8]. Algebraic Grammians of bilinear systems, which are used to approximate nonlinear systems first, are used to do the truncation in [5] instead of reducing them directly. The moments of the bilinear systems are matched for reduction in [6]. Roychowdhury presents an MOR method called time-varying Padé (TVP) for reducing large time-varying linear and nonlinear systems described by Volterra series in [9] by matching the moments of the first-order transfer function. A compact nonlinear MOR method is studied in [10] by matching the moments of the first, second and even higher order transfer functions, which is more suitable to a class of weakly nonlinear systems. Recently, the trajectory piecewise-linear (TPWL) approach is proposed in [11] via approximating the original nonlinear system by a piecewise-linear system first and then reducing each linear system by matching its moments. A modification by the approximating the original nonlinear systems by piecewise-polynomial systems has been done in [12]. The stability is enforced for certain classes of nonlinear descriptor systems in [13].

In many physical, industrial and circuit systems, time delays are inherent due to the finite capabilities of information processing, data transmission among various parts of the systems and some essential simplification of the corresponding process models [15, 16]. The delaying effect is often detrimental to the performance, and even leads to instability. The presence of time delays often substantially complicates the analytical and theoretical aspects of system design. In the past few decades, researchers have paid great attention to the MOR of linear time delay systems (TDSs) [15, 17–23] but little attention has been made to nonlinear TDSs. This gives the strong impetus to study the MOR of nonlinear TDSs.

In this paper, the MOR of nonlinear TDSs is investigated by preserving the input-output stability of the ROMs, which is inspired by the work in [13]. The rest of the paper is organized as follows. The input-output stability of piecewise-linear TDSs is given in Section 2. Section 3 shows the ROMs with input-output stability. An example is used to illustrate the main results in Section 4. Finally, Section 5 draws the conclusion.

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2 MOR of Linear TDSs

Notations: $\mathbb{R}^n$ is the set of all column vectors with n real entries. For $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n$, $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$ is the Euclidean norm of the vector x. $\|X\|$ means the 2 norm of matrix X. A function $u(t)$ is said to be essentially bounded if $ess\sup_{t\geq 0} |u(t)| < \infty$. For given times $0 \leq T_1 \leq T_2$, $u_{[T_1,T_2]} : [0, +\infty) \to \mathbb{R}^m$ denotes the function given by

$$u_{[T_1,T_2]} = \begin{cases} u(t) & t \in [T_1, T_2] \\ 0 & \text{otherwise} \end{cases} .$$

A continuous function $\lambda : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if it is strictly increasing and $\lambda(0) = 0$. λ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A function $\beta : [0, \infty)^2 \to [0, \infty)$ is of class $\mathcal{KL}$ if for each fixed t the function $s \to \beta(s, t)$ is of class $\mathcal{K}$ and for each fixed s the function $t \to \beta(s, t)$ is non-increasing and goes to zero as $t \to \infty$.

2.1 Background and Preliminaries

A linear TDS is formulated as

$$\begin{aligned}\Sigma_{tds}: E\dot{x}(t) &= Ax(t) + A_d x(t-d) + Bu(t),\\ y(t) &= Cx(t),\\ x_{t_0}(\theta) &= x(t_0+\theta) = \psi(t) \in \mathcal{C}([-d,0],\mathbb{R}^n),\\ \forall\theta &\in [-d,0], \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^l$ is the output. E, A, A_d, B and C are properly dimensioned real constant matrices. $d > 0$ is the constant delay and for a given $d > 0$, $\mathcal{C}([-d,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence and designates the norm of an element ψ in $\mathcal{C}([-d,0],\mathbb{R}^n)$ by $\|\psi\|_c = \sup_{\theta\in[-d,0]}|\psi(\theta)|$. E is assumed to be nonsingular.

Under the assumption $x(0) = \psi(0) = 0$, the linear TDS Σ_{tds} is also characterized by its transfer function

$$G_{tds}(s) = C\left(sE - A - A_d e^{-sd}\right)^{-1} B, \quad (2)$$

by performing Laplace transform for the linear TDS Σ_{tds}. Due to the appearance of exponential term e^{-sd}, direct Taylor series expansion of $G_{tds}(s)$ is impossible. The approximation of the term e^{-sd} gives an exponential-free approximation of the Taylor series expansion of $G_{tds}(s)$

$$G_{tds}(s) \approx G_0 + G_1 s + \cdots + G_n s^n + \cdots, \quad (3)$$

where constant matrices G_i, $i = 0,1,\dots,$ are called approximated moments of linear TDS Σ_{tds}. A natural thought is to expand e^{-sd} by its Taylor series expansion $e^{-sd} = \sum_{k=0}^{\infty}\frac{(-d)^k}{k!}s^k$, which is proposed in [22, pp 834] and modified in [24].

Lemma 1 *[24] The approximated moments G_i, $i = 0,1,\dots,$ in (3) are given by*

$$G_i = CL_i\Gamma_0^{-1}B,\ i = 0,1,\dots, \quad (4)$$

where

$$\begin{aligned} L_k &= -\sum_{j=0}^{k-1}\Gamma_0^{-1}\Gamma_{k-j}L_j,\ L_0 = I, && (5)\\ \Gamma_1 &= E + dA_d, && (6)\\ \Gamma_k &= -\frac{(-d)^k}{k!}A_d,\ k \geq 2. && (7)\end{aligned}$$

The MOR problem of the linear TDS Σ_{tds} by moment matching method is to find a projection matrix $V_{tds} \in \mathbb{R}^{n\times\hat{n}}$ such that the resulting ROM

$$\begin{aligned}\hat{\Sigma}_{tds}: \quad \hat{E}\dot{z}(t) &= \hat{A}z(t) + \hat{A}_d z(t-d) + \hat{B}u(t),\\ y(t) &= \hat{C}z(t),\end{aligned}$$

with system matrices given by

$$\begin{aligned}\hat{E} &= V_{tds}^T E V_{tds}, \quad \hat{A} = V_{tds}^T A V_{tds},\\ \hat{A}_d &= V_{tds}^T A_d V_{tds}, \quad \hat{B} = V_{tds}^T B, \quad \hat{C} = CV_{tds}. \end{aligned} \quad (8)$$

match the first approximated moments of $G_{tds}(s)$. The projection matrix V_{tds} to derive the reduced-order linear TDS $\hat{\Sigma}_{tds}$ is given by

$$\mathrm{colspan}(V_{tds}) \supseteq \mathrm{colspan}\{G_0,\ G_1,\ \dots,\ G_{\hat{n}-1}\} \quad (9)$$

with $V_{tds}^T V_{tds} = I$ [24].

However, if we choose V_{tds} in (9) by matching the approximated moments G_i, obtained above, the stability of the reduced-order linear TDS $\hat{\Sigma}_{tds}$ is still indeterminate, which is shown in the following numerical example.

Example 1 *Consider a linear TDS Σ_{tds} with system matrices $E = I_4$, $A_d = I_4$, $d = 1$, $B = \begin{bmatrix} 1 & 1 & 0 & 0\end{bmatrix}^T$, $C = \begin{bmatrix} 1 & 0 & 1 & 1\end{bmatrix}$ and*

$$A = \begin{bmatrix} -33.9561 & 1.0782 & -2.0588 & -10.4103\\ -30.5568 & -0.1473 & -7.1396 & -8.3009\\ 2.9693 & 16.5070 & -2.9161 & -17.0315\\ -27.6949 & 0.9044 & 2.4510 & -14.9708 \end{bmatrix},$$

It is asymptotically stable as linear matrix inequality (LMI) (14) has solution $P > 0$ and $Q > 0$. By using the projection matrix

$$V_{tds} = \begin{bmatrix} -0.2868 & 0.0702\\ 0.7141 & -0.3092\\ 0.0950 & 0.9166\\ 0.6314 & 0.2436 \end{bmatrix},$$

obtained in (9), system matrices of the reduced-order system $\hat{\Sigma}_{tds}$ are given by

$$\begin{aligned}\hat{E} &= V_{tds}^T E V_{tds} = I_2,\\ \hat{A} &= V_{tds}^T A V_{tds} = \begin{bmatrix} 0.4755 & -8.9711\\ -0.8251 & -8.8262\end{bmatrix},\\ \hat{A}_d &= V_{tds}^T A_d V_{tds} = I_2,\\ \hat{B} &= V_{tds}^T B = \begin{bmatrix} 0.4273\\ -0.2390\end{bmatrix},\\ \hat{C} &= CV_{tds} = \begin{bmatrix} 0.4397 & 1.2304\end{bmatrix}.\end{aligned}$$

However, the above reduced-order system is unstable as eigenvalues of $\hat{A}$ are 1.2128 and -9.5635.

The above Example gives the conclusion that the stability of the reduced linear TDS $\hat{\Sigma}_{tds}$ is indefinite by the traditional moment matching method, though the original linear TDS is asymptotically stable. We will extend the idea in [13] for deriving an asymptotically stable ROM by introducing a left projection matrix for nonlinear system without delay to linear and nonlinear TDSs in later sections.

2.2 Stability-Preserving MOR for Linear TDSs

Consider the nonlinear TDS

$$\begin{aligned} E\dot{x}(t) &= f(x(t)) + A_d x(t-d),\\ x_{t_0}(\theta) &= x(t_0+\theta) = \psi(t) \in \mathcal{C}([-d,0],\mathbb{R}^n),\\ \forall\theta &\in [-d,0], \end{aligned} \quad (10)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $f(\cdot)$ is continuously differentiable nonlinear function relating to state $x(t)$. All derivations in this paper can straightforwardly be extended to time varying delay case by assuming d as the upper bound of the time varying delay. Assume that the equilibrium point of the nonlinear TDS (17) is $x(t) = 0$.

Definition 2 *[25, 26] If there exist $\mathcal{K}_\infty$-functions α, β and $\mathcal{K}$-function γ such that the Lyapunov-Krasovskii functional $L : \mathcal{C}([-d, 0], \mathbb{R}^n) \to \mathbb{R}$ satisfies*

$$\begin{aligned} \alpha(\|\psi(0)\|) &\leq L(t, \psi) \leq \beta(\|\psi\|_c), \quad (11) \\ \dot{L}(t, \psi) &\leq -\gamma(\|\psi(0)\|), \quad (12) \end{aligned}$$

Then, the solution $x(t) = 0$ of the nonlinear TDS (10) is uniformly stable. If $\gamma(r) > 0$ for $r > 0$, then, the solution $x(t) = 0$ is uniformly asymptotically stable.

For the linear TDS Σ_{tds}, the functions $\alpha(r)$, $\beta(r)$ and $\gamma(r)$ in (11) and (12) can be reduced to be continuous, non-negative and nondecreasing functions with $\alpha(r), \quad \beta(r) > 0 \quad$ for $r \neq 0$ and $\alpha(0) = 0, \quad \beta(0) = 0$. One possible choice of Lyapunov-Krasovskii functional $L(t, \psi)$ for linear TDS $\hat{\Sigma}_{tds}$ is

$$x^T(t) E^T P E x(t) + \int_{t-d}^{t} x^T(s) Q x(s) ds. \quad (13)$$

This will render that the LMI, shown in (14) is independent of the delay d. Other types of Lyapunov–Krasovskii functionals in [27] could lead to less conservative and delay-dependent conditions at the expense of larger size LMIs. Normally we will assume that the solution $x(t) = 0$ of the original linear TDS Σ_{tds} is asymptotically stable to guarantee that the LMI in (14) has solution. The following theorem provides a way to find a right projection matrix in order to ensure the stability of reduced linear TDS $\hat{\Sigma}_{tds}$. And the choice of right projection matrix is V_{tds} in (9) to match the approximated moments, which is the same as in the traditional moment matching method.

Theorem 3 *Let $P > 0$ and $Q > 0$ be the solution of LMI*

$$\Gamma = \begin{bmatrix} A^T P E + E^T P A + Q & E^T P A_d \\ A_d^T P E & -Q \end{bmatrix} < 0. \quad (14)$$

If the left projection matrix U is defined as

$$U_{tds}^T = \left(V_{tds}^T E^T P E V_{tds}\right)^{-1} V_{tds}^T E^T P, \quad (15)$$

and the right orthonormal projection matrix V_{tds} is given in (9), the reduced-order linear TDS $\hat{\Sigma}_{tds}$ in (19) with system matrices

$$\begin{aligned} \hat{E} &= U_{tds}^T E V_{tds}, \quad \hat{A} = U_{tds}^T A V_{tds}, \\ \hat{A}_d &= U_{tds}^T A_d V_{tds}, \quad \hat{B} = U_{tds}^T B, \quad \hat{C} = C V_{tds} \quad (16) \end{aligned}$$

is asymptotically stable.

Proof. See the appendix for the proof. ■

3 MOR of Nonlinear TDS

3.1 Problem Formulation

By introducing a linear input term $Bu(t)$, the unforced nonlinear TDS described in (10) becomes

$$\begin{aligned} E\dot{x}(t) &= f(x(t)) + A_d x(t-d) + Bu(t), \\ y(t) &= Cx(t), \quad (17) \end{aligned}$$

where $B \in \mathbb{R}^{n \times m}$ is a real constant matrix. As there is no direct counterpart of the transfer function for the nonlinear TDS Σ, the MOR by moment matching method in linear TDS Σ_{tds} cannot be applied to the nonlinear TDS Σ directly. One likely way to use the established MOR technique for the linear TDS in Section 2.2 is to approximate the nonlinear system by a piecewise-linear TDS first [11–13]. From [28, Theorem 4.2.2], the nonlinear TDS (17) is approximated by a piecewise-linear TDS

$$\begin{aligned} \Sigma : E\dot{x}(t) &= \sum_{i=1}^{p} w_i(x(t)) A_i x(t) + \begin{bmatrix} l & B \end{bmatrix} \begin{bmatrix} 1 \\ u(t) \end{bmatrix}, \\ y(t) &= Cx(t), \quad (18) \end{aligned}$$

where

$$\begin{aligned} A_i &= \left. \frac{\partial f(x(t))}{x(t)} \right|_{x(t_i) = x_i}, \\ l &= \sum_{i=1}^{p} w_i(x(t)) f(x_i) - A_i x_i, \\ w_i(x(t)) &= \frac{\exp\left(\frac{-\beta \|x(t) - x_i\|_2^2}{\min_k \|x(t) - x_k\|_2^2}\right)}{\sum_j^p \exp\left(\frac{-\beta \|x(t) - x_j\|_2^2}{\min_k \|x(t) - x_k\|_2^2}\right)} \in [0, 1], \end{aligned}$$

with $\sum_{i=1}^{p} w_i(x(t)) = 1$. Other types of weighting functions $w_i(x(t))$ can be found in [11, 12]. Here we treat the constant vector l as an additional input vector.

The objective of this paper is to find a right projection matrix V and a left projection matrix U, where $x(t) = Vz(t)$, $z(t) \in \mathbb{R}^q$ and $q \ll n$, such that the reduced-order piecewise-linear TDS

$$\begin{aligned} \hat{\Sigma} : \quad \hat{E}\dot{z}(t) &= \sum_{i=1}^{p} w_i(x(t)) \hat{A}_i z(t) + \hat{A}_d z(t-d) \\ &\quad + \begin{bmatrix} \hat{l} & \hat{B} \end{bmatrix} \begin{bmatrix} 1 \\ u(t) \end{bmatrix}, \\ \hat{y}(t) &= \hat{C} z(t) \quad (19) \end{aligned}$$

is input-output stable with

$$\begin{aligned} \hat{E} &= U^T E V, \quad \hat{A}_i = U^T A_i V, \quad \hat{A}_d = U^T A_d V, \\ \hat{l} &= U^T l, \quad \hat{B} = U^T B, \quad \hat{C} = CV. \quad (20) \end{aligned}$$

The definition of input-output stability was given in Definition 5.

The process of MOR for nonlinear TDS (17) is first approximated by the piecewise-linear TDS Σ. Then it is reduced to a lower-order piecewise-linear TDS $\hat{\Sigma}$ by reducing the number of state variables $x(t)$ and guaranteeing stability.

3.2 Input-Output Stability of Piecewise-Linear TDSs

Assumption 1: There exists a very small scalar μ such that

$$|l + Bu(t)| \leq \mu \|\psi\|_c, \quad (21)$$

The definitions about the stability of nonlinear TDS (17) are listed below.

Definition 4 *[29, 30] The nonlinear TDS (17) is input-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function λ such that, for any locally essentially bounded input $u(t)$, the solution exists for all $t \geq 0$ and furthermore satisfies*

$$|x(t)| \leq \beta(\|\psi\|_c, t) + \lambda\left(\left\|u_{[0,t]}\right\|_\infty\right).$$

Definition 5 *[31, Definition 6.3] The nonlinear TDS (17) is input-output stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function λ such that, for any locally essentially bounded input $u(t)$, the solution exists for all $t \geq 0$ and furthermore satisfies*

$$|y(t)| \leq \beta(\|\psi\|_c, t) + \lambda\left(\left\|u_{[0,t]}\right\|_\infty\right).$$

The Lyapunov-Krasovskii functional method to derive the input-state stability for the nonlinear TDS (17) is given.

Lemma 2 *[29, 30] If there exist $\mathcal{K}_\infty$-function α, β and $\mathcal{K}$-function χ and γ such that the Lyapunov-Krasovskii functional $V : \mathcal{C}([-d,0], \mathbb{R}^n) \to \mathbb{R}$ satisfies*

$$\begin{aligned} \alpha(\|\psi(0)\|) &\leq V(\psi) \leq \beta(\|\psi\|_c), \\ \dot{V}(\psi, u) &\leq -\gamma(\|\psi\|_c), \ \forall \|\psi\|_c \geq \chi(|u|), \end{aligned}$$

Then, the nonlinear TDS (17) is input-state stable.

The stability of the piecewise-linear TDS Σ is tested by the following LMIs.

Theorem 6 *The piecewise-linear TDS Σ is asymptotically stable, if there exist $P > 0$ and $Q > 0$ such that*

$$\Gamma_i = \begin{bmatrix} A_i^T PE + E^T PA_i + Q & E^T PA_{d_i} \\ A_{d_i}^T PE & -Q \end{bmatrix} < 0, \quad (22)$$

Proof. See the appendix for the proof. ■

The construction of the right projection matrix V for the piecewise-linear TDS Σ is similar to the method in [11].

Algorithm 7 *Generation of the right orthonormal projection matrix for the piecewise-linear TDS Σ.*

1) *Let $\hat{V} = []$, $i = 0$. Set x_0 to be the initial state.*
2) *While $i < p$ do*
   a) *Consider linearization of nonlinear TDS Σ_i about $x(t_i) = x_i$*

   $$\begin{aligned} \Sigma_i : E\dot{x}(t) &= A_i x(t) + A_d x(t-d) \\ &\quad + l_i + Bu(t), \\ y &= Cx(t), \end{aligned}$$

   where A_i is the Jacobian of $f(x(t))$, evaluated at $x(t_i) = x_i$ and $l_i = f(x_i) + A_i x_i$.
   b) *Construct an orthogonal basis $\bar{V}_i$ from (9) by replacing $A = A_i$, $B = B_i$, to match the first r_ith approximated moments.*
   c) *Orthonormalize the initial state vector x_i with respect to the columns of $\bar{V}_i$ and obtain vector w_i.*
   d) *Take $\tilde{V}_i$ as a union of $\bar{V}_i$ and w_i : $\tilde{V}_i = \begin{bmatrix} \bar{V}_i & w_i \end{bmatrix}$.*
   e) *Take $\hat{V} = \begin{bmatrix} V & \tilde{V}_i \end{bmatrix}$, set $i = i + 1$.*
3) *Orthogonalize the columns of the aggregated basis $\hat{V}$ using the singular value decomposition (SVD) algorithm and construct a new basis V, which constrains orthogonalized columns of $\hat{V}$ corresponding to singular values larger than a given $\varepsilon > 0$.*

3.3 Reduced-Order Piecewise-Linear TDSs with Input-Output Stability

Theorem 8 *let V be an orthonormal projection matrix from Algorithm 7. The reduced-order piecewise-linear TDS $\hat{\Sigma}$ in (19) is asymptotically stable, if U is defined as*

$$U^T = \left(V^T E^T PEV\right)^{-1} V^T E^T P, \quad (23)$$

where $P > 0$ satisfying (22). Moreover, if $\mu \left\|U^T\right\|$ is also small enough, the reduced-order piecewise-linear TDS $\hat{\Sigma}$ is input-output stable.

Proof. It is easy to get that

$$\begin{aligned} &\begin{bmatrix} V^T & \\ & V^T \end{bmatrix} \Gamma_i \begin{bmatrix} V & \\ & V \end{bmatrix} \\ = &\begin{bmatrix} \hat{A}_i^T Y + Y\hat{A}_i + V^T QV & Y\hat{A}_{d_i} \\ \hat{A}_{d_i}^T Y & -V^T QV \end{bmatrix} < 0 \quad (24) \end{aligned}$$

where

$$Y = V^T E^T PEV.$$

From Theorem 6 and $\hat{E} = U^T EV = I \geq 0$, reduced-order piecewise-linear TDS $\hat{\Sigma}$ in (19) is asymptotically stable. From (21), it is easy to get that

$$\left|\hat{l} + \hat{B}u(t)\right| \leq \left\|U^T\right\| |l + Bu(t)| \leq \mu \left\|U^T\right\| \|\psi\|_c. \quad (25)$$

As $\mu \left\|U^T\right\|$ is small enough, the input-output stability of the ROMs is preserved. ■

4 Numerical Example

Example 9 *The following example is cited from [32] and [33, pp.214] for a longitudinally single mode semiconductor laser subject to lateral carrier diffusion and weak convention optical feedback. We take $\alpha = 3$, $\theta = 0.1$, $\phi = 0$, $d = 2.3203 \times 10^{-2}$ and $\eta = 1.9926 \times 10^{-2}$ and use a second-order central difference formula on a uniform mesh, with $\rho = 128$ intervals. The order of resulting nonlinear delay differential equation (DDE) is $\rho + 1$ by removing two boundary conditions. The input $B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ and output is*

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{bmatrix} x(t).$$

By Theorem 8, a 5th-order ROM is obtained and is input-output stable. The comparison of outputs is given in Figure 1 (a) for the original system (solid black line) and the 5th-order stable reduced model (star red line) trained by the sinusoidal inputs of amplitude 1 and frequency 100Hz. The output errors between the original system and the ROMs and relative errors are given in Figure 1(b) and Figure 1(c), where $\hat{y}(t)$ represents the reduced output.

5 Conclusions

The ROMs generating from the reduction of nonlinear TDSs has been guaranteed to be input-output stable. The input-output stability is achieved by left projection matrix from the solution of LMIs and right projection matrix from the estimated moments. Numerical example has confirmed the effectiveness of the proposed method.

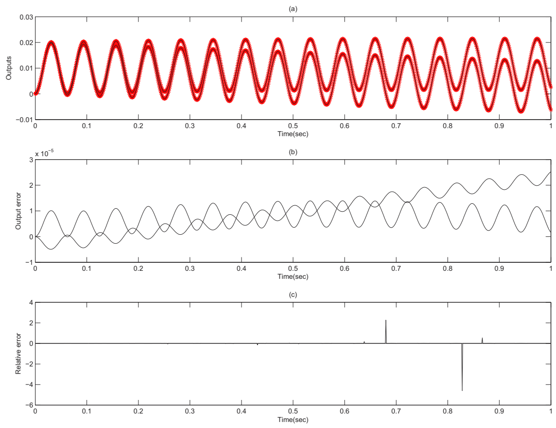

Fig. 1: Outputs, output error and relative error

6 Acknowledgment

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7 Appendix

7.1 Proof of Theorem 3

Proof. Consider the Lyapunov-Krasovskii functional in (13), it is easy to get

$$\dot{L}(t,x) \leq 0, \tag{26}$$

where $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-d) \end{bmatrix}^T$. This results in

$$\begin{aligned} L(t,x) &\leq L(0,x(0)) \\ &\leq \left(\lambda_{\max}\left(E^T PE\right) + \lambda_{\max}(Q)\, d\right) \|\psi\|_c^2 \\ &= \beta(\psi), \end{aligned} \tag{27}$$

where $\lambda_{\max}\left(E^T PE\right)$ denotes the maximum eigenvalue of $E^T PE$. Furthermore, we get

$$\begin{aligned} L(t,x) &\geq \lambda_{\min}\left(E^T PE\right) |x(t)|^2 \\ &\quad +\lambda_{\min}(Q) \int_{t-d}^{t} |x(s)|^2\, ds \\ &= \alpha(|x|), \end{aligned} \tag{28}$$

where $\lambda_{\min}\left(E^T PE\right)$ denotes the minimum eigenvalue of $E^T PE$. Combining (27) and (28), we have

$$\alpha(|x|) \leq L(t,x) \leq V(0) \leq \beta(\psi),$$

which further results in

$$\begin{aligned} |x(t)| &\leq \sqrt{r_v}\, \|\psi\|_c, \\ r_v &= \frac{\left(\lambda_{\max}\left(E^T PE\right) + \lambda_{\max}(Q)\, d\right)}{\lambda_{\min}\left(E^T PE\right)}. \end{aligned} \tag{29}$$

Considering (26) and (29), there exists a scalar $\varepsilon = 2r_v \lambda_{\min}(-\Gamma) > 0$ such that

$$\dot{L}(t,x) \leq -\varepsilon \|\psi\|_c^2, \tag{30}$$

From Definition 2, the original linear TDS Σ_{tds} is asymptotically stable.

Now we want to prove that the reduced-order linear TDS $\hat{\Sigma}$ in (19) with system matrices is stable. By considering Lyapunov-Krasovskii functional

$$\hat{L}(t,z) = L(t, V_{tds} z)$$

with $x(t) = V_{tds} z(t)$, we have $\dot{\hat{L}}(t,z) = \dot{L}(t, V_{tds} z)$. It follows that

$$\begin{aligned} \dot{\hat{L}}(t,z) &= \dot{L}(t, V_{tds} z) \leq -\varepsilon \|\psi\|_*^2, \\ \alpha(|V_{tds} z|) &\leq \hat{L}(t,z) = L(t, V_{tds} z) \leq \beta(\psi), \end{aligned}$$

where $\beta(\cdot)$, $\alpha(\cdot)$ and ε are given in (27), (28) and (30), respectively. This implies that the reduced-order linear TDS $\hat{\Sigma}_{tds}$ is asymptotically stable from Definition 2. ∎

7.2 The proof of Theorem 6

Proof. By considering the Lyapunov–Krasovskii functional

$$V(x) = x^T(t) E^T PEx(t) + \int_{t-d}^{t} x^T(s) Qx(s)\, ds,$$

with $P > 0$, $Q > 0$ and considering the unforced system of the piecewise-linear TDS Σ, from (22), we get $\frac{dV(x)}{dt} \leq 0$, which leads to

$$V(x) \leq V(0) \leq \left(\lambda_{\max}\left(E^T PE\right) + \lambda_{\max}(Q)\, d\right) \|\psi\|_c^2. \tag{31}$$

Furthermore,

$$V(x) \geq W(x) = \alpha(|x|), \tag{32}$$

where

$$W(x) = \lambda_{\min}\left(E^T PE\right) |x(t)|^2 + \lambda_{\min}(Q) \int_{t-d}^{t} |x(s)|^2\, ds.$$

From (31)–(32), we have

$$W(x) \leq \left(\lambda_{\max}\left(E^T PE\right) + \lambda_{\max}(Q)\, d\right) \|\psi\|_c^2 = \beta(\psi), \tag{33}$$

which results in

$$|x(t)| \leq \sqrt{r_v}\, \|\psi\|_c, \quad r_v = \frac{\lambda_{\max}\left(E^T PE\right) + \lambda_{\max}(Q)\, d}{\lambda_{\min}\left(E^T PE\right)}. \tag{34}$$

From Definition 2, the piecewise-linear TDS Σ is asymptotically stable. (33) also implies that $V(x)$ satisfies (1) of Lemma 2. By adding the inputs and considering (21) and (34), there exists a scalar ε such that

$$\frac{dV(x,u)}{dt} \leq \varsigma \|\psi\|_c^2, \tag{35}$$

where $\varsigma = \theta r_v + \varepsilon^{-1} \lambda_{\max}\left(E^T PE\right) r_v + \varepsilon \mu^2$, $\theta = \max\{\lambda_{\min}(\Gamma_i)\} < 0$ and

$$\tilde{B}(l,u) = \begin{bmatrix} l & B \end{bmatrix} \begin{bmatrix} 1 \\ u(t) \end{bmatrix}.$$

Due to the small value of μ, we get

$$\frac{-\theta^2 r_v^2 + \lambda_{\max}\left(E^T PE\right) r_v \mu^2}{\mu^2} < 0$$

Then, if we choose $\varepsilon = -\frac{\theta r_v}{\mu^2} > 0$, which renders

$$\varsigma = \varepsilon^{-1} \frac{-\theta^2 r_v^2 + \lambda_{\max}\left(E^T P E\right) r_v \mu^2}{\mu^2} < 0, \tag{36}$$

By taking $\chi(|u|) = \frac{1}{\mu}|u|$, from (21), we have $\frac{1}{\mu}|u| \leq \frac{1}{\mu}|l + Bu| \leq \|\psi\|_c$, together (35) and (36) which show that (2) of Lemma 2 holds. From Lemma 2, the piecewise-linear TDS Σ is input-state stable. Therefore, it is input-output stable as $y(t)$ is linear function of $x(t)$ and $u(t)$ [31, Theorem 6.3]. ■

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