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Automatic Adaptive Multi-point Moment Matching for Descriptor System Model Order Reduction

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***Abstract*—We propose a novel automatic adaptive multi-point moment matching algorithm for model order reduction (MOR) of descriptor systems. The algorithm implements both adaptive frequency expansion point selection and automatic moment order control via a transfer function based error metric. Without a *priori* information of the system response, the proposed algorithm guarantees a much higher global accuracy compared with standard multi-point moment matching without adaptation. The moments are computed via a generalized Sylvester equation which is subsequently solved by a newly proposed generalized alternating direction implicit (GADI) method. Numerical examples then confirm the efficacy of the proposed schemes.**

I. Introduction

With the increasing scale of VLSI circuits, it becomes extremely expensive, sometimes impossible, to analyze and simulate the original large circuit system. Model order reduction (MOR) [1], [2] comes to the rescue by reducing the complexity and size of those large systems with the goals of preserving the input-output responses and important physical properties.

One mainstream for MOR is multi-point moment matching [3], [4], which performs better than single-point moment matching in terms of global accuracy. However, there still exists the following issues. First, it is very hard to place proper (Laplace domain) expansion points without prior information of the system response. Second, the order of moments at a particular expansion point is usually chosen in an ad hoc manner or heuristically. Increasing the order of moments or the number of expansion points is likely to reduce the global error, but this increases the dimension of the reduced order model (ROM) and thus defeats the purpose of MOR.

There are some work on adaptive sample points selection in the framework of truncated balanced realization (TBR) [5], [6] or sample-based MOR [7]. However, little work has been done to tackle the issues mentioned above in the framework of moment matching based MOR, especially for the more general descriptor systems. The TBR-based method in [5] introduces a statistical re-sampling scheme which suffers high computation cost. The sample-based MOR method in [7] presents an adaptive scheme which requires a huge number of candidate sample points and this method does not involve higher order moments, so the ROM may lose accuracy if the sample points are insufficient or not well-placed. Consequently, we propose a novel and fully automatic adaptive multi-point moment matching algorithm for descriptor system MOR which fills the gap in the literature.

The main contributions of this paper are twofold. First, we propose a transfer function based error metric to adaptively select expansion points from a prescribed initial frequency expansion point set and automatically determine the order of moments per expansion point. Adaptive selection of expansion points can place more points in the region with fast-changing frequency response and fewer points in the region with a smooth response. If the initial points are insufficient in some region, adaptive determination of order of moments helps to solve this problem and guarantee the global accuracy. Second, MOR is performed by moment matching through solving a generalized Sylvester equation corresponding to a descriptor system. The equation is subsequently solved by a newly proposed generalized alternating direction implicit (GADI) scheme with an error-control terminating condition.

This paper is structured as follows. Section II reviews moment matching. Section III introduces the descriptor system moment matching via a generalized Sylvester equation and the GADI method. Section IV presents the proposed adaptive moment matching scheme. Numerical results are reported in Section V. Finally, Section VI concludes the paper.

II. Moment Matching

Consider a linear, time-invariant, continuous-time descriptor system described by

$$E\dot{x} = Ax + Bu,\ y = Cx + Du, \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. E is generally singular. The transfer function of (1) is $H(s) = D + C(sE - A)^{-1}B$. Moment matching is to find a projection matrix $V \in \mathbb{R}^{n\times q}$ by computing the moments of the transfer function. The ROM is

$$E_r\dot{x_r} = A_r x_r + B_r u,\ y_r = C_r x_r + Du, \tag{2}$$

where $E_r = V^T EV, A_r = V^T AV, B_r = V^T B$ and $C_r = CV$. The transfer function of the ROM, $H_r(s) = D + C_r(sE_r - A_r)^{-1}B_r$ approximates $H(s)$. Expand $H(s)$ at an expansion point s_i by s-domain Taylor series, we can get

$$H(s) = D + \sum_{k=0}^{\infty} \underbrace{(-1)^k C[(s_iE - A)^{-1}E]^k (s_iE - A)^{-1}B}_{\eta_k(s_i)} (s - s_i)^k, \tag{3}$$

where $\eta_k(s_i)$ is called the kth moment at s_i. If we generate a projection matrix of order qm at s_i, it means the reduced transfer function $H_r(s)$ matches the first q moments of $H(s)$ at s_i [8].

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III. Extension and Improvement

A. Moment matching via Sylvester equation

We utilize a recently developed method which computes the moments of $H(s)$ through solving a Sylvester equation [9], and extend it from regular systems (without E in the state equation) to descriptor systems. As noted, the kth moment at s_i is $\eta_k(s_i) = (-1)^k C[(s_iE - A)^{-1}E]^k(s_iE - A)^{-1}B$, denoted as $\eta_k(s_i) = C\pi_k$, where $\pi_k \in \mathbb{R}^{n\times m}$. Then, the first q moments of $H(s)$, denoted as $C\Pi$, can be computed from the following Sylvester equation,

$$A\Pi + BL = E\Pi S, \tag{4}$$

with $\Pi = [\pi_0, \cdots, \pi_{q-1}] \in \mathbb{R}^{n\times qm}$, $L = [I_m, O_m, \cdots, O_m] \in \mathbb{R}^{m\times qm}$ and

$$S = \begin{bmatrix} s_iI_m & I_m & O_m & \cdots & O_m \\ O_m & s_iI_m & I_m & \cdots & O_m \\ \vdots & & \ddots & \ddots & \vdots \\ O_m & \cdots & O_m & s_iI_m & I_m \\ O_m & \cdots & O_m & O_m & s_iI_m \end{bmatrix} \in \mathbb{R}^{qm\times qm},$$

where $I_m \in \mathbb{R}^{m\times m}$ is an identify matrix and $O_m \in \mathbb{R}^{m\times m}$ is a zero matrix.

B. Generalized ADI for descriptor system Sylvester equation

The alternating direction implicit (ADI) iterative method in [10] for solving Sylvester equation is only applicable to regular state spaces and without a terminating condition. Here we adapt it to generalized ADI (GADI) for solving the generalized Sylvester equation, offer a new choice of shift parameters and develop an economic terminating condition for GADI.

1) Extension to descriptor systems: The Sylvester equation derived from descriptor systems in the previous section is

$$A\Pi + BL = E\Pi S \tag{5}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $L \in \mathbb{R}^{m\times qm}$, $S \in \mathbb{R}^{qm\times qm}$. E is generally singular. A and $-S$ are assumed stable (i.e. their eigenvalues have negative real parts). Given two sets of shift parameters $\{\alpha_i\}$ and $\{\beta_i\}$, the GADI consists of two half-steps,

$$E\Pi_{i+\frac{1}{2}}(S - \alpha_iI) = (A - \alpha_iE)\Pi_i + BL, \tag{6}$$

$$(A - \beta_iE)\Pi_{i+1} = E\Pi_{i+\frac{1}{2}}(S - \beta_iI) - BL. \tag{7}$$

It can be verified that

$$\begin{aligned}\Pi_{i+1} = {} & (\alpha_i - \beta_i)(A - \beta_iE)^{-1}BL(S - \alpha_iI)^{-1} \\ & + (A - \beta_iE)^{-1}(A - \alpha_iE)\Pi_i(S - \beta_iI)(S - \alpha_iI)^{-1}.\end{aligned} \tag{8}$$

Let $\Pi_i = Z_iD_iY_i$, we get a low-rank version of (8) as

$$\begin{aligned}\Pi_{i+1} = {} & \begin{bmatrix}(A - \beta_iE)^{-1}B & (A - \beta_iE)^{-1}(A - \alpha_iE)Z_i\end{bmatrix} \times \\ & \begin{bmatrix}(\alpha_i - \beta_i)I_m & \\ & D_i\end{bmatrix} \times \begin{bmatrix}L(S - \alpha_iI)^{-1} \\ Y_i(S - \beta_iI)(S - \alpha_iI)^{-1}\end{bmatrix} \\ = {} & Z_{i+1}D_{i+1}Y_{i+1}.\end{aligned} \tag{9}$$

2) Choice of shift parameters: The ADI shift parameters are usually chosen according to a minmax problem [10], which is very expensive to compute. Here we give a new choice of shifts. From (8) we can see if the shifts can achieve

$$\left\| \prod_{j=1}^{k} \frac{(\lambda_A - \alpha_j)}{(\lambda_A - \beta_j)} \right\| \cdot \left\| \prod_{j=1}^{k} \frac{(\lambda_S - \beta_j)}{(\lambda_S - \alpha_j)} \right\| < 1, \tag{10}$$

where λ_A, λ_S are the eigenvalues of (A, E) and S, respectively, then (8) will always converge. Therefore, we can choose the shifts α and β as the real part of minimum spectral radii of (A, E) and S, respectively. As a result, $\|\lambda_A - \alpha_j\| < \|\lambda_A - \beta_j\|$ and $\|\lambda_S - \beta_j\| < \|\lambda_S - \alpha_j\|$, then the left part of (10) would be less than 1. The new shifts are easy to compute and often lead to fast convergence.

3) Terminating condition: It can be observed that within each iteration, Z_i and Y_i in (9) are only increased by one block column or one block row. $\Pi_{i+1} = \Pi_i + Z^{(i+1)}D^{(i+1)}Y^{(i+1)}$. This allows us to derive a very cheap residual error monitor involving only block vector-block vector multiplication,

$$errR = \|\Pi_{i+1} - \Pi_i\| = \|(\alpha_{i+1} - \beta_{i+1})Z^{(i+1)}Y^{(i+1)}\|, \tag{11}$$

where $Z^{(i+1)}$ is the $(i+1)$th block column of Z and $Y^{(i+1)}$ is the $(i+1)$th block row of Y. GADI is terminated when the $errR$ drops below a certain threshold. The algorithm is shown in Algorithm 1 where tol denotes the threshold. In order to be more general, we present the algorithm with multi-shift parameters.

Algorithm 1 GADI

Input: E, A, S, B, L, $\{\alpha_1, \alpha_2, \cdots\}$, $\{\beta_1, \beta_2, \cdots\}$, tol
Output: Solution Π

$Z^{(1)} = (A - \beta_1E)^{-1}B$, $Y^{(1)} = L(S - \alpha_1I)^{-1}$
for i=1:k **do**
  $Z^{(i+1)} = Z^{(i)} + (\beta_{i+1} - \alpha_i)(A - \beta_{i+1}E)^{-1}Z^{(i)}$
  $Y^{(i+1)} = Y^{(i)} + (\alpha_{i+1} - \beta_i)Y^{(i)}(S - \alpha_{i+1}I)^{-1}$
  $D^{(i+1)} = (\alpha_{i+1} - \beta_{i+1})I_m$
  $errR = \|(\alpha_{i+1} - \beta_{i+1})Z^{(i+1)}Y^{(i+1)}\|$
  if $errR < tol$ **then**
    BREAK
  end if
end for
$Z_{i+1} = (Z^{(1)}, Z^{(2)}, \cdots, Z^{(i+1)})$
$Y_{i+1} = (Y^{(1)}, Y^{(2)}, \cdots, Y^{(i+1)})$
$D_{i+1} = diag((\alpha_1 - \beta_1)I_m, \ldots, (\alpha_{i+1} - \beta_{i+1})I_m)$
$\Pi = Z_{i+1}D_{i+1}Y_{i+1}$

IV. Automatic Adaptive Moment Matching

A. Transfer function based error metric

1) Error metric for moment order control: For certain selected expansion point, a good error metric to monitor the order of moment is the absolute error between transfer functions of the original system and the reduced system, namely,

$$errM(s_i) = \left\| H(s_i) - H_r(s_i) \right\|. \tag{12}$$

However, it is very expensive to explicitly compute the original system's transfer function, as it contains the expensive computation of the inverse of (s_iI-A). Instead of computing the actual error, we derive an error monitor from (12) which avoids the explicit inverse of (s_iI-A). It can be verified

$$\begin{aligned} errM(s_i) &= \left\|C(s_iE-A)^{-1}B - CV_{s_i}(s_iE_r-A_r)^{-1}B_r\right\| \\ &= \left\|C(s_iE-A)^{-1}[B-(s_iE-A)V_{s_i}(s_iE_r-A_r)^{-1}B_r]\right\| \end{aligned} \tag{13}$$

where V_{s_i} is the orthogonal projection matrix according to expansion point s_i, $E_r = V_{s_i}^T E V_{s_i}$, $A_r = V_{s_i}^T A V_{s_i}$, $B_r = V_{s_i}^T B$. Thus, we use

$$RM(s_i) = \left\|B-(s_iE-A)V_{s_i}(s_iE_r-A_r)^{-1}B_r\right\| \tag{14}$$

as the error monitor to indicate the error between the original system and the ROM. This error monitor is highly economic as it only contains (block) matrix-vector multiplication and a *small-size inverse*.

2) Error metric for expansion point selection: Assume we have already got a projection matrix containing the projection bases with respect to all the already selected expansion points, the following error metric can be derived as above to indicate whether to continue selecting expansion point and which point should be selected.

$$errS(s_t) = \left\|Z_v - Z_t\right\|, \tag{15}$$

where $Z_v = C(s_tE-A)^{-1}B$ and $Z_t = C_t(s_tE_t-A_t)^{-1}B_t$, $E_t = V_t^T E V_t$, $A_t = V_t^T A V_t$, $B_t = V_t^T B$, V_t is the orthogonal projection matrix with respect to all the already selected expansion points, s_t is the candidate point needed to be tested. Similarly, we can derive the error monitor from (15) as

$$RS(s_t) = \left\|B-(s_tE-A)V_t(s_tE_t-A_t)^{-1}B_t\right\|, \tag{16}$$

to indicate whether the ROM is accurate in the test point.

B. Adaptive moment matching

1) Automatic moment order control: The adaptive control scheme works in an iterative manner. At each expansion point, we iteratively increase the order of moments, i.e. the order of Π, S and L in (4), until $RM(s_i)$ of (14) drops below a tolerance (denoted as $tolM$). The projection basis V_{s_i} is obtained via solving (4) by GADI (see Algorithm 1). When $RM(s_i)$ drops below the tolerance, it means the current ROM is sufficiently accurate at s_i, and therefore the order of moments at s_i is sufficient and iteration should end. The procedure is summarized in Algorithm 2.

2) Adaptive expansion point selection: For the expansion point selection, if the reduced transfer function generated by the current projection matrix is not accurate at a certain frequency point, then this point should be added as a new expansion point. Based on this idea, we prescribed an initial expansion point set containing the candidate expansion points distributed in the frequency range of interest and use $RS(s_t)$ of (16) to test each candidate point in the set. The point which has the maximum value of RS should be selected as the next expansion point (denoted as s_{new}) in that it contains more new information which the current projection matrix cannot provide than any other points in the expansion point set. Afterwards, this point is removed from the set. The maximum value of RS is denoted as $MaxRS$ which is used to control the projection matrix expansion flow.

Algorithm 2 Adaptive Moment Order Control

Input: E, A, B, s_i, $tolM$
Output: new projection basis expanded at s_i: V_{s_i}

$S_1 = [s_iI_m]$, $L_1 = [I_m]$
for $q = 2 : k$ **do**
  $S_q = \begin{bmatrix} S_{q-1} & \\ & s_iI_m \end{bmatrix}$, $L_q = [L_{q-1}, O_m]$
  solve $A\Pi_q + BL_q = E\Pi_q S_q$ by Algorithm 1 and get a new projection basis V_{s_i} ($V_{s_i} = qr(\Pi_q)$)
  if $RM < tolM$ **then**
    BREAK
  end if
end for

3) Adaptive moment matching flow: The complete flow of adaptive moment matching is summarized in Algorithm 3.

Algorithm 3 Adaptive Moment Matching

Input: E, A, B, initial expansion point set, tolerance $tolS$
Output: Multi-point multi-moment projection matrix V

initial: expand at initial point s_0 and get V_{S_0}
for $k = 1 : M$ (total number of points in initial set) **do**
  compute the error monitor (16) of all the candidate points in the set and select the point which has maximum RS as the next candidate expansion point s_{new}
  if $MaxRS > tolS$ **then**
    do moment matching at s_{new} by Algorithm 2
    update the projection matrix V by adding the new basis $V_{s_{new}}$: $V = \left[V\ V_{s_{new}}\right]$, $V = qr(V)$
    remove s_{new} from initial expansion points set
  else
    BREAK
  end if
end for

The flow will stop when $MaxRS$ falls below the tolerance (denoted as $tolS$), which means the current projection matrix has already contained all the information the points in the initial set can provide.

V. Numerical Results

The proposed algorithm is verified by four RLC examples with linearly distributed initial expansion points. E is singular in the state equation. We will compare the CPU runtime and global accuracy in the frequency domain between adaptive moment matching and standard multi-point moment matching without adaption. The moments are generated via Sylvester equation and solved by GADI in both methods. The error is the relative error between the transfer function of the original

system and that of the ROM, namely,

$$error(jw) = \frac{\|H(jw) - H_r(jw)\|}{\|H(jw)\|}, \tag{17}$$

at all the frequency points in simulation. All experiments are performed in Matlab R2012b on a platform of Intel Core i5-2400 with 3.10GHz CPU and 16GB RAM.

In our proposed method, the expansion points in the initial set are evenly distributed within the frequency range of interest in a linear scale. The number of expansion points in the initial set is 1000 in all the experiments. To be fair, if we generate a ROM of order R and expand at k points in the adaptive method, then in the standard method we set the number of expansion points as k (evenly linearly distributed in the same frequency range) and the number of moments per point as R/k (fix it as an integer). Then the order of ROMs and number of expansion points in the two methods will be the same.

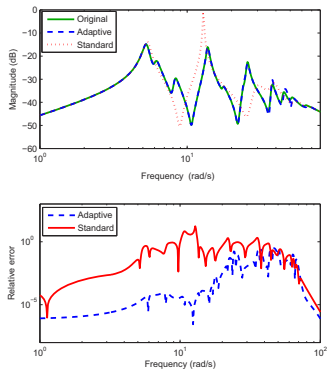

Fig. 1. Frequency response and relative error comparison of example 2.

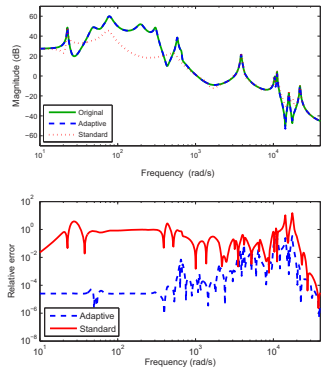

Fig. 2. Frequency response and relative error comparison of example 3.

Fig. 1 and Fig. 2 show the comparison of frequency response and relative error of examples 2 and 3, respectively. It can be observed from both examples that the frequency response of the ROM generated by adaptive method cannot be distinguished from the original one, whereas the ROM generated by standard method can only match the original response in the smooth region but loses accuracy in the range with fast-changing responses. The relative error of adaptive method is much lower than standard method throughout the whole frequency range. Observations from examples 1 and 4 are similar. Table I gives the details of four examples, t is the runtime (in seconds). It can be seen that with the same order of ROM, the proposed method enjoys much higher accuracy than the non-adaptive method, though at the expense of higher CPU times.

TABLE I
COMPARISON WITH LINEARLY DISTRIBUTED EXPANSION POINTS

Example	Order		Adaptive			Standard		
	Original	ROM	t/s	Error		t/s	Error	
				Average	Max		Average	Max
1	540	65	1.91	0.003	0.29	0.20	0.21	21.6
2	1536	44	3.65	0.007	0.35	0.90	0.22	17.1
3	1920	93	17.6	0.082	0.40	7.68	0.58	15.1
4	4863	100	13.4	0.010	1.80	1.88	0.19	48.1

VI. Conclusion

In linear circuit simulation, standard moment matching method is faced with the problem of determining the expansion points and the order of moments. This paper has proposed an efficient automatic solver (GADI) for generalized Sylvester equation and a new automatic adaptive multi-point moment matching algorithm, adapted to the general settings of descriptor systems. The experimental results have shown that the proposed algorithm provides much higher global accuracy than the multi-point moment matching method without adaption.

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